The semantics of the canonical commutation relation

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March 15, 2016

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Abstract

We treat the canonical commutation relations and the conventional calculus based on it as an algebraic syntax of quantum mechanics and establish a geometric semantics of this syntax. This leads us to a geometric model, the space of states with the action of time evolution operators, which is a limit of finite models. The finitary nature of the space allows to give a precise meaning and calculate various classical quantum mechanical quantities.

1   Introduction

1.1 This paper is a part of a broader project which aims to establish, for a typical 'co-ordinate algebra' $\mathcal{A}$ in the sense of non-commutative geometry, or a scheme, a geometric counterpart $\mathbf{V}_{\mathcal{A}}$. Here "geometric" should also be understood in some broad but well-defined sense. We believe this is possible in the setting of model theory, where a crucial new experience in generalisation of semantics of algebraic and analytic geometry, and applications of this, has been accumulated in the last 50 years.

In fact, we aim to establish a rigorous mathematical duality

$$\mathcal{A}_{\mathbf{V}} \;\longleftrightarrow\; \mathbf{V}_{\mathcal{A}} \tag{1}$$

between algebras $\mathcal{A}_{\mathbf{V}}$ and structures $\mathbf{V}_{\mathcal{A}}$ of geometric flavour.

Let us recall that a project with similar aims, but mainly concentrated around classical structures of mathematical physics, has been suggested by C.Isham and J.Butterfield and is being developed by C.Isham, A.Döring and others, see e.g. [8] and later publications. In that approach $\mathbf{V}_{\mathcal{A}}$ is supposed to be a topos.

Our approach is different in many details: $\mathbf{V}_{\mathcal{A}}$ is a *multi-sorted* structure, each sort of which is a *Zariski geometry* in the sense of [18]. An essential component of the multi-sorted structure $\mathbf{V}_{\mathcal{A}}$ is the set of morphisms between sorts functorially agreeing with embeddings between certain sublagebras of $\mathcal{A}$, which makes the left-to right arrow a functor between a category of those subalgebras A and sorts $\mathbf{V}_A$ of $\mathbf{V}_{\mathcal{A}}$. This functor, in fact, defines a quite rich sheaf over the category of sublagebras. This makes an interesting point of contact of our approach with the topos-theoretic approach to foundations of physics.

1.2 At this stage of our project we decided to keep the general context to minimum and instead concentrate on developing and illustrating the approach in a very special case of Heisenberg-Weyl algebras and the *canonical commutation* relation. And even there we quite quickly drop the generality of the n-th Heisenberg-Weyl algebras for the sake of developing rather sophisticated calculus with one position and one momentum operators.

1.3 The n-th Heisenberg-Weyl algebra $\mathcal{A}^{(n)}$ has generators $P_1, \ldots, P_n, Q_1, \ldots, Q_n$, the "co-ordinates" of n-dimensional quantum mechanics, with commutation relations

$$Q_l P_k - P_k Q_l = 2\pi i \delta_{lk} I, \tag{2}$$

where the P_k and Q_l are seen as self-adjoint operators, I is the "identity operator" and δ_{kl} the Kronecker symbol. (We have chosen the multiplier $2\pi i$ for convenience of further notations and to mean that $\exp 2\pi i I = I$.)

The infinite-dimensional Heisenberg algebra is

$$\mathcal{A}^{(\omega)} = \cup_n \mathcal{A}^{(n)}.$$

The representation theory of $\mathcal{A}^{(n)}$ is quite complicated, even for $n = 1$, due to the fact that the algebra can not be represented as an algebra of bounded operators on a Hilbert space, not to speak about finite-dimensional representations.

However, as suggested by Herman Weyl, we may instead consider the representation theory of algebras generated by the *Weyl operators* which can be formally defined as

$$U_k^{a_k} = \exp i a_k Q_k, \quad V_l^{b_l} = \exp i b_l P_l \tag{3}$$

for $a_k, b_l \in \mathbb{R}$. These are unitary (and so bounded!) operators if the P_k and Q_l are self-adjoint, and the following commutation relation holds:

$$\begin{aligned} U_k^{a_k} V_l^{b_l} &= q_{kl} V_l^{b_l} U_k^{a_k} \\ \text{where } q_{kl} &= \exp 2\pi i a_k b_l \delta_{kl} \end{aligned} \tag{4}$$

Given a choice of real $a_1, \ldots, a_n, b_1, \ldots, b_n$ the complex C^*-algebra generated by Weyl operators $V_1^{\pm a_1}, \ldots, V_n^{\pm a_n}, U_1^{\pm b_1}, \ldots, U_n^{\pm b_n}$,

$$A(a_1, \ldots, a_n, b_1, \ldots, b_n) = \mathbb{C}[V_1^{\pm a_1}, \ldots, V_n^{\pm a_n}, U_1^{\pm b_1}, \ldots, U_n^{\pm b_n}]$$

is called (in this paper) a Weyl algebra.

The all-important Stone – von Neumann theorem states that the representation theory of all the Weyl algebras $A(a_1, \ldots, a_n, b_1, \ldots, b_n)$ together is equivalent to the Heisenberg commutation relation (2). One can say that $\mathcal{A}^{(n)}$ can be fully replaced by the entirety of its Weyl subalgebras $A(a_1, \ldots, a_n, b_1, \ldots, b_n)$ which have a good Hilbert space representation theory.

Especially nice finite-dimensional representations have Weyl algebras for which the multipliers $e^{2\pi i a_k b_l}$ are roots of unity. This is the case when all the a_k, b_l are rational. We call such algebras **rational Weyl algebras**.

In a certain sense, made precise in the paper, the rational Weyl algebras approximate the full Heisenberg-Weyl algebra just as rational points approximate points of the n-dimensional Euclidean space. In fact, the paper uses the correspondence between rational Weyl algebras A and certain geometric objects $\mathbf{V}_A$. The structure $\mathbf{V}_A$ encapsulates the representation theory of A and is at the same time a well-understood object of model-theoretic studies, *a quantum Zariski geometry*, see [16].

The quoted paper established the duality (1) between quantum algebras A at roots of unity (rational Weyl algebras in our case) and corresponding quantum Zariski geometries $\mathbf{V}_A$, which extends the classical duality between commutative affine algebras and affine algebraic varieties. The new step implemented in the present work is to extend the duality on algebras A approximated by rational Weyl algebras. In order to achieve this we need to develop a notion of approximation on the side of structures $\mathbf{V}_A$. Such a notion, of *structural approximation*, has essentially been developed in [17] (quite similar to the context of *positive model theory* developed by I. Ben-Yaacov [2] and others).

1.4 A few introductory words about structural approximation:

We work with the category whose objects are rational Weyl algebras $A(a, b)$, $a, b \in \mathbb{Q}$, and morphism are embeddings. It is useful to have in mind that the embedding $A(a, b) \subset A(c, d)$ can be determined in terms of divisibility relations between denumerators of the fractions of integers representing a, b, c and d.

We consider an ultrafilter $\mathcal{D}$ on $\mathbb{Q}^2$ to be defined more specifically below. Consider the ultraproduct of algebras $\{A(a, b) : \langle a, b \rangle \in \mathbb{Q}^2\}$ modulo $\mathcal{D}$, call it $\tilde{\mathrm{A}}$. This can be seen as a *non-standard* algebra "generated" by operators $U^a V^b$, for $a, b \in {}^*\mathbb{Q}$, the field of *non-standard rational numbers*.

We consider then the corresponding ultraproduct modulo $\mathcal{D}$ of Zariski geometries $\mathbf{V}_{A(a,b)}$, denote it $\mathbf{V}_{\tilde{A}}$.

Now we are ready to define a limit object, *the space of states*

$$\mathbb{S} = \lim{}_{\mathcal{D}} \mathbf{V}_{A(a,b)}$$

and the procedure of taking the limit. By definition in [17] the limit structure is a *homomorphic* image of $\mathbf{V}_{\tilde{A}}$ and $\lim$ is a surjective *homomorphism*

$$\lim \, : \mathbf{V}_{\tilde{A}} \twoheadrightarrow \mathbb{S},$$

which means a map preserving basic relations of the language.

The canonical equivalence on $\mathbf{V}_{\tilde{A}}$ associated with $\lim$,

$$x_1 \approx x_2 \Leftrightarrow \lim x_1 = \lim x_2$$

should be read as "x_1 is infinitesimally close to x_2".

This definition is very sensitive to the choice of the basic relations (primitives) of the language and is not preserved under the transition to an interdefinable language. Effectively, the choice of the "right" primitives amounts to the choice of what physicists would consider as **observables**.

The fact that we construct our structures as Zariski geometries plays a key role here: the notion of homomorphism in the above definition agrees with the notion of morphisms in the category of respective Zariski geometries, structures with (a generalised) Zariski topology on them.

Further choices of primitives of the language lead to the introduction of probability (equal to the absolute value of the inner product) as an observable as opposed to the inner product, which is well-defined for each $\mathbf{V}_{A(a,b)}$ but can not be defined uniformly without breaking the symmetries required by Zariski structure.

1.5 The ultrafilter $\mathcal{D}$ of 1.4 is chosen to be a Fréchet ultrafilter with respect to the partial ordering between the $A(a,b)$, that is:

for each $A(a,b)$ the set $\{\langle c,d \rangle \in \mathbb{Q}^2 : A(a,b) \subset A(c,d)\}$ is in $\mathcal{D}$.

We also add a condition on $\mathcal{D}$ which depends on a choice of a rational number h and which results in the fact that $\tilde{A}$ is "generated" by a pair of operators $U^{\frac{1}{\mu}}$ and $V^{\frac{1}{\nu}}$, for non-standard integers μ and ν such that

$$\mathrm{h} = \frac{\mu}{\nu}.$$

The fact that $\mathcal{D}$ was Fréchet has an important consequence: both μ and ν are *divisible by all the standard integers.*

Consequently we will get that $A(a,b) \subset \tilde{A}$ for any (standard) rational a, b which is one of the desirable properties; $\tilde{A}$ is universal for the family of all rational algebras.

1.6 Let us turn now to more detailed description of the objects $\mathbf{V}_A$ and their limit, the space of states $\mathbb{S}$.

For a rational Weyl algebra A the object $\mathbf{V}_A$ is defined, following [16], as a bundle (not locally trivial!) of irreducible A-modules over an algebraically closed field, say $\mathbb{C}$. In the current paper we identify a more interesting structure of what we call "algebraic-Hilbert" modules on $\mathbf{V}_A$. "Algebraicity" of the modules first of all reflects the fact that the modules are of finite dimension n_A. More serious algebraic feature of $\mathbf{V}_A$ is in the fact that the submodule over a much smaller subfield, $\mathbb{R}_0[\sqrt{-1}] \subset \mathbb{C}$, where $\mathbb{R}_0$ is the field of algebraic totally real numbers, has an inner product structure with values in $\mathbb{R}_0[\sqrt{-1}]$. To express the sesquilinearity of the inner product we use the formal complex conjugation on $\mathbb{R}_0[\sqrt{-1}]$, which does not require definability of the reals. This is crucial since we want to keep the structure $\mathbf{V}_A$ a Zariski structure (hence ω-stable). In fact, for each particular A we need just a finitely generated subfield $\mathbb{Q}[q]$ of $\mathbb{R}_0[\sqrt{-1}]$, where q is a root of unity introduced in (4).

With the inner product and a formal complex conjugation defined we can also speak of adjoint operators, unitarity and the related notions.

1.7 Each algebraic-Hilbert space $\mathbf{V}_{A(a,b)}$ is acted upon by respective Weyl operators U^a and V^b. In the limit procedure the rational parameters a and b become infinitesimals and for this reason can not be observed in the limit structure. The same can be said about U^a and V^b.

Instead of U^a and V^b we introduce in the language of $\mathbf{V}_{A(a,b)}$ the new names P and Q which can be used for all values of $a, b \in \mathbb{Q}$:

$$\mathrm{Q} := \frac{U^a - U^{-a}}{2ia}, \quad \mathrm{P} := \frac{V^b - V^{-b}}{2ib}$$

(compare with (3)). Note that in each $A(a,b)$ the operators Q and P are inter-definable with U^a and V^b respectively (but by different formulas!) The effect of redefining the basic operators is that the new operators Q and P have a meaning throughout the whole construction and in the limit of it. So

the operators Q *and* P *are observables.*

Crucially, we prove:

Theorem (see 9.3) *The canonical commutation relation (2) holds in* $\mathbb{S}$.

This can be seen as another version of Stone - von-Neumann Theorem.

1.8 Other observable operators can be constructed as e.g. linear combinations $c\mathrm{Q} + d\mathrm{P}$, $c, d \in \mathbb{R}$. To each such operator in the conventional quantum mechanical setting one associates a Lagrangian subspace of the phase space. A symplectomorphism between two such Lagrangian subspaces can be identified as a (generalised) Fourier transform (see e.g. [12]).

In our finitary setting this picture translates into considering commutative subalgebras of $\tilde{A}$ of the form $A(c, d)$ where c, d are rationals (eventually approximating real values). To each such commutative subalgebra one can associate the space of eigenvectors, a subset of $\mathbf{V}_{\tilde{A}}$, which is an equivalent of the lagrangian, or rather the lagrangian subspace with a line bundle with connection. In place of the Fourier transforms we define a class of Zariski morphisms with special properties which we call *regular unitary transformations* since they preserve the orthonormality.

Theorem. *The regular unitary transformations survive to the limit and form a group which has* $\mathrm{SL}(2, \mathbb{R})$ *as its homomorphic image.*

In n-dimensional case $\mathrm{SL}(2, \mathbb{R})$ should become the symplectic group $\mathrm{Sp}(2n, \mathbb{R})$, but we do not do this case in the current version of the paper.

1.9 The fact that the construction of the limit structure involves nonstandard (infinite) integers μ and ν (as in 1.5) is one of the causes of emergence of non-observable values. More concretely, the absolute value of the inner product of norm 1 eigenvectors $|x\rangle$ and $|p\rangle$ of operators Q and P is

$$|\langle x\,|p\rangle| = \frac{1}{\sqrt{\mu\nu}}$$

and so is an infinitesimal. This can be used to define the *probability density* but for more subtle calculations one needs a *renormalisation*, a systematic procedure which produces a finite value. We introduce such a universal procedure in terms of our construction. This is called in the paper the *Dirac rescaling*. As an intermediate stage the Dirac rescaling introduces a "discrete" Dirac delta-function on each lagrangian and in the above example

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gives us

$$\langle x \,| p \rangle_{Dir} = \frac{1}{\sqrt{2\pi\hbar}} \mathrm{e}^{ixp},$$

where the right-hand side is defined up to a phase factor.

1.10 It is well-known that some (not all) of the generalised Fourier transforms can be identified as *time evolution operators* for certain particles. The same is true for regular unitary transformations. We consider such transformations K_{free}^t and K_{QHO}^t corresponding to time evolution operators for the free particle and the quantum harmonic oscillator and calculate the Feynman propagator, which coincides with the well-known results:

$$\langle x_1 | K_{\mathrm{free}}^t x_2 \rangle_{Dir} = \frac{1}{\sqrt{2\pi i \hbar t}} \mathrm{e}^{\frac{i(x_1 - x_2)^2}{2t\hbar}}$$

and

$$\langle x_2 | K_{\mathrm{QHO}}^t | x_1 \rangle_{Dir} = \sqrt{\frac{1}{2\pi i \hbar \sin t}} \, \exp i \frac{(x_1^2 + x_2^2)\cos t - 2x_1 x_2}{2\hbar \sin t},$$

where t are arbitrary positive real numbers such that $\hbar = 2\pi\mathrm{h} > 0$ and $\sin t \neq 0$.

1.11 The most interesting example we calculated is the formula

$$\mathrm{Tr}(K_{\mathrm{QHO}}^t) = \frac{1}{i|\sin\frac{t}{2}|}$$

for the trace of the evolution operator for the quantum harmonic oscillator. A naive approach to calculating the trace would be to consider the sum of eigenvalues of K_{QHO}^t which are well known

$$\mathrm{Tr}(K_{\mathrm{QHO}}^t) =^? \sum_{n=1}^{\infty} \mathrm{e}^{i(n-\frac{1}{2})t}.$$

But this does not make sense since $|\mathrm{e}^{i(n-\frac{1}{2})t}| = 1$. Our calculation takes place for arbitrary rational values of the parameter $\arcsin t$ in big enough finite-dimensional algebraic-Hilbert spaces $\mathbf{V}_A$. In each such space the trace can be calculated and the result has the same form as above (for the given value of the parameter). Note that the easy of obtaining the formula is compensated by the difficulty of calculating eigenvalues of K_{QHO}^t in the finite-dimensional space. In fact, we have not calculated these eigenvalues.

1.12 Of course, discrete models have been in permanent use throughout the history of quantum physics as a heuristic tool, see the reference [14] or more advanced treatment in similarly defined spaces in [9]. The analysis of such a discrete model usually concludes with the phrase "now we pass to continuous limit", which actually is an ill-defined notion and a source of main troubles of quantum physics.

Perhaps the main achievement of this paper is the approximation procedure lim suggesting a rigorous interpretation of the above.

Moreover, the mathematical model developed in this paper may have a much stronger physical interpretation. The pseudo-finite-dimensional Hilbert space which is associated with our main object $\mathbf{V}_{\tilde{A}}$ corresponds to a pseudo-finite space of states. Here "pseudo" refers to the non-standard finite (that is infinite) size of the space. This is a good mathematical substitute for the notion of a *"huge finite" universe*. The limit $\mathbb{S}$ of this space should then be considered a "continuous approximation" to the huge finite universe. The claim of the paper is that by assuming this as a model for reality we get the correct mathematical theory of quantum mechanics.

1.13 The role of logic and model theory. We see canonical commutation relations and more general formulas of algebraic quantum mechanics as rules of a syntax that speaks correctly about quantum mechanics since the time of Heisenberg and Dirac. However, unlike the case of the syntax of classical Hamiltonian mechanics (the calculus of differential manifolds), no geometric semantic interpretation of this syntax has been established. And this is arguably the main cause of troubles in quantum physics, see [7]. From our point of view this is also a true challenge to logicians and, more concretely, model-theorists.

The quoted paper [7] suggest toposes as possible semantical interpretations for similar syntactic construction (general von Neumann algebras), and there have been a few other constructions of category-theoretic origin.

The construction we use in [16], in the present paper and in other ongoing joint works has many common features with the above mentioned approaches, but the role of category theory is taken by model theory. The advantage is that the semantical interpretation is supposed to take shape of a concrete structure with crucial model-theoretic characteristics. These characteristics supposed also to justify the adjective *geometric* in the definition *geometric semantics*. Model theory has tools for distinguishing structures of this desired type, tools developed in *stability theory*. In particular, the notion of Zariski

geometry (and its generalisations) proved to be a very convenient generalisation of objects of algebraic geometry, including non-commutative one. The insistence that the model of quantum mechanics must be a Zariski geometry severely restricts our choices but also brings in strong parallels with classical geometric structures. Zariski geometries come with an internally defined notion of *dimension*. Another key property is *homogeneity*: any two subobjects which are indiscernible by the language are conjugated by an automorphism of the structure. In the current setting this property explains the inherent symmetries of quantum mechanics and exhibits their common nature with Galois automorphisms of algebraic geometry.

In the end the approach works, and this paper is here to prove the statement.

2 Preliminaries

2.1 Weyl operators (3) generate so called Heisenberg group $H(a_1, \ldots, a_n, b_1, \ldots, b_n)$. Most of the time will be interested in the case $n = 1$ and the Heisenberg group $H(a, b)$, which is a group of rank 2. More generally a Heisenberg group of rank 2 is a subgroup H of $H(a, b)$ such that the quotient $H/[H, H]$ is non-cyclic.

2.2 Lemma. *A Heisenberg group of rank 2 is isomorphic to* $H(c, d)$ *for some c, d. If $H \subseteq H(a, b)$ for rational a, b, then c, d are rational as well.*
    Proof. Easy. $\square$

2.3 A general Heisenberg group of rank $2n$ is by definition a group of the form

$$H_1 \times \ldots \times H_n$$

where the H_i are Heisenberg groups of rank 2. In this definition we also assume that the generators of the H_i are of the form $U_i^{g_{11}} V_i^{g_{12}}$ and $U_i^{g_{21}} V_i^{g_{22}}$, for real $g_{11}, g_{12}, g_{21}, g_{22}$. We write, correspondingly,

$$H_i = \langle U_i^{g_{11}} V_i^{g_{12}}, U_i^{g_{21}} V_i^{g_{22}} \rangle.$$

We also will consider an algebraically closed field $\mathbb{F}$ of characteristic 0 (e.g. the field of complex numbers $\mathbb{C}$) and the Weyl algebras, which are defined as a group algebras $\mathbb{F}H$ of the respective Heisenberg groups. In particular,

$$A(a_1, \ldots, a_n, b_1, \ldots, b_n) := \mathbb{F}H(a_1, \ldots, a_n, b_1, \ldots, b_n),$$

where

$$H(a_1, \ldots, a_n, b_1, \ldots, b_n) = \langle U^{a_1}, \ldots, U^{a_n}, V^{b_1}, \ldots, V^{b_n} \rangle.$$

2.4 By 2.2 any Heisenberg group of rank $2n$ is isomorphic to one of the form $H(a_1, \ldots, a_n, b_1, \ldots, b_n)$ and every Weyl algebra is isomorphic to one of the form $A(a_1, \ldots, a_n, b_1, \ldots, b_n)$.

2.5 A symplectic structure associated with the Heisenberg group.
    Recall that H is a nilpotent group of class 2 and hence the commutator operation $[h, g] := hgh^{-1}g^{-1}$ determines an alternating $\mathbb{Z}$-bilinear form on $H/[H, H]$,

$$[\cdot, \ \cdot] : \ H/[H, H] \times H/[H, H] \to [H, H] \cong \mathbb{Z}.$$

For the case of real Heisenberg group this is called the symplectic structure on H. The same terminology can be applied to the integer Heisenberg group without the risk of ambiguity.

2.6 A **rational Weyl $\mathbb{F}$-algebra of rank** $2n$ is a Weyl algebra isomorphic to one of the form $A(a_1, \ldots, a_n, b_1, \ldots, b_n)$ with rational non-zero $a_1, \ldots, a_n$, $b_1, \ldots, b_n$.

In this case the q_{kk} in (4) are roots of unity. Such algebras, and more generally *quantum algebras at roots of unity*, can be seen as "coordinate algebras" of corresponding Zariski geometries. This has been studied in [16].

2.7 Theorem. *Let $A = A(a_1, \ldots, a_n, b_1, \ldots, b_n)$ be a rational Weyl $\mathbb{F}$-algebra. Then*

 (i) *A and the centre $Z(A)$ are prime noetherian PI rings. Let n_A be the PI degree of A.*

 (ii) *Every maximal ideal of A is of the form $\tilde{\alpha} = A\alpha$, for $\alpha \subseteq Z(A)$, $\alpha \in \mathrm{Spec}(Z(A))$, a maximal ideal of the centre.*

 (iii) *The unique irreducible A-module $\mathrm{V}_A(\alpha)$ with annihilator $\mathrm{Ann}(\mathrm{V}) = \tilde{\alpha}$ is of dimension n_A over $\mathbb{F}$. The irreducible representations of A are in bijective correspondence with points $\alpha \in \mathrm{Spec}(Z(A))$, the maximal spectrum of $Z(A)$, which can be identified with ($\mathbb{F}$-points of) an affine irreducible variety.*

 (iv) *There exists an isomorphism $A/\tilde{\alpha} \cong \mathrm{M}(n_A, \mathbb{F})$, the full $n_A \times n_A$-matrix algebra.*

 (v) *The irreducible A-module $\mathrm{V}_A(\alpha)$ is isomorphic to a tensor product of corresponding irreducible A_k-modules $\mathrm{V}_{A_k}(\alpha_k)$,*

$$\mathrm{V}_A(\alpha) \cong \bigotimes_{k=1}^{n} \mathrm{V}_{A_k}(\alpha_k)$$

 for $A_k = A(a_k, b_k)$ and some maximal ideals α_k of $Z(A_k)$.

Proof. For (i)-(iv) see [1], III.1.1 and III.1.6.
(v) follows from the fact that

$$\mathrm{H}(a_1, \ldots, a_n, b_1, \ldots, b_n) = \prod_{k=1}^{n} \mathrm{H}(a_k, b_k). \tag{5}$$

$\square$

2.8 Note that the isomorphism in (iv) is given by a representation $j_a : A \to$ $M(n_A, \mathbb{F})$, which in its own right is determined by a choice of a maximal ideal $a \subset Z(A)$ and a basis $\mathbf{e}$ in the module $V_A(a)$, so $j = j_{a,e}$.

2.9 Definition. Given the Heisenberg group $H(a,b)$ with rational a, b, it is easy to describe its automorphism group Aut $H(a,b)$. It consists of automorphisms $\xi_{g,n,m}$, for $g = (g_{ij}) \in \mathrm{SL}(2, \mathbb{Z})$, $n, m \in 0, 1, \ldots, N-1$ where N is the order of defined on the generators of $H(a,b)$ the root of unity $q = e^{2\pi i a b}$:

$$\xi_{g,n,m} : \begin{matrix} U^a \mapsto q^n U^{g_{11}a} V^{g_{12}b} \\ V^b \mapsto q^m U^{g_{21}a} V^{g_{22}b} \end{matrix}$$

Note that by isomorphism the commutator of the image of U^a and V^b is equal to the same number q, hence $\det(g_{ij}) = 1$.

Set $\Gamma_{H(a,b)}$ to be the subgroup of $\mathrm{Aut}(H(a,b))$ which fixes $H(a,b)/Z(H(a,b))$, the quotient of the Heisenberg group modulo its centre.

Given a rational Weyl algebra $A = A(a_1, \ldots, a_n, b_1, \ldots, b_n)$ we set

$$\Gamma_A = \Gamma_{A_1} \times \ldots \times \Gamma_{A_n}$$

for $A_k = A(a_k, b_k)$, $k = 1 \ldots, n$, with the action on $A = A_1 \oplus \ldots \oplus A_n$ coordinate-wise.

2.10 Lemma. *For $A = A(a,b)$ the group Γ_A is generated by the two automorphisms μ and ν defined on the generators of A as follows*

$$\mu : \begin{matrix} U^a \mapsto qU^a \\ V^b \mapsto V^b \end{matrix} \qquad \nu : \begin{matrix} U^a \mapsto U^a \\ V^b \mapsto qV^b \end{matrix} \qquad (6)$$

Γ_A and Σ_A fix the centre $Z(A)$ point-wise.

$$\Gamma_A \cong (\mathbb{Z}/N\mathbb{Z})^2.$$

Proof. By definition any $\gamma \in \Gamma_A$ acts by $U^a \mapsto q^k U^a$ and $V^b \mapsto q^m V^b$, for some integers k and m. Then $\gamma : U^a V^b U^{-a} V^{-b} \mapsto U^a V^b U^{-a} V^{-b}$, that is $\gamma(qI) = qI$ for all $\gamma \in \Gamma_A$.

Clearly, $\mu, \nu \in \Gamma_A$. Also $\gamma^{-1}\mu^k\nu^m$ acts as identity on U^a, V^b, hence $\gamma = \mu^k\nu^m$. The transformation ϕ fixes qI by (4).

Finally, it is immediate from (6) that the generators μ, ν of Γ_A commute and are of order N. $\square$

2.11 In what follows we will extensively use (5) and reduce the study of n-th Weyl algebras to the study of 1-st Weyl algebras.

2.12 Lemma. *The centre $Z(A)$ of rational $A = A(a_1, \ldots, a_n, b_1, \ldots, b_n)$ is of the form $A(N_1 a_1, \ldots, N_n a_n, N_1 b_1, \ldots, N_n b_n)$, where N_k is the order of the root of unity $\exp 2\pi i a_k b_k$, that is equal to the denominator of the reduced rational number $a_k b_k$.*
    Proof. Immediate. $\square$

2.13 Lemma. *Suppose*

$$A(a'_1, \ldots, a'_n, b'_1, \ldots, b'_n) \subseteq A(a_1, \ldots, a_n, b_1, \ldots, b_n)$$

and $A(a'_1, \ldots, a'_n, b'_1, \ldots, b'_n)$ is commutative and maximal among commutative.
    Then
    (i) $a'_k = M_k a_k$, $b'_k = L_k b_k$, for some $M_k, L_k \in \mathbb{Z}$ such that $M_k L_k$ is equal to the denominator of the reduced fraction $a_k b_k$, $k = 1, \ldots, n$.
    (ii) In particular, $A(N_1 a_1, \ldots, N_n a_n, b_1, \ldots, b_n)$ and $A(a_1, \ldots, a_n, N_1 b_1, \ldots, N_n b_n)$ are maximal commutative subalgebras of $A(a_1, \ldots, a_n, b_1, \ldots, b_n)$.
    (iii) Maximal commutative subalgebras $A(N_k a_k, b_k)$ and $A(a_k, N_k b_k)$ of $A(a_k, b_k)$ are 1-generated by V^{b_k} and U^{a_k}, respectively, over the centre of the algebra.
    Proof. (i) We may assume that $n = 1$ and we consider $A(a', b') \subseteq A(a, b)$ maximal commutative. Then $a' = Ma$, $b' = Lb$, for some $M, L \in \mathbb{Z}$ by the embedding assumption. Commutativity requires that $MLab \in \mathbb{Z}$. Maximality implies the last of the conditions.
    (ii) Immediate from (i).
    (iii) Just note that $U^{N_k a_k}$ and $V^{N_k b_k}$ are in the centre of the algebra. $\square$

2.14 Lemma. *Given a rational commutative n-th algebra $A = A(a_1, \ldots, a_n, b_1, \ldots, b_n)$, there is finitely many commutative rational n-th Weyl algebras B extending A.*
    Proof. Any element of B is of a product of some $U_k^{a'_k}$ and $V_k^{b'_k}$, $k = 1, \ldots, n$. Commutativity implies that $a'_k b_k, a_k b'_k \in \mathbb{Z}$ for all k. This puts bounds on the denominators of a'_k and b'_k. On the other hand, since $U_k^{a_k}, V_k^{b_k} \in B$, we can reduce every product in B modulo A so that $|a'_k| \leq |a_k|$ and $|b'_k| \leq |b_k|$. There are finitely many such a'_k and b'_k with the bound on denominators. $\square$

2.15 Lemma. *Given a commutative rational n-th Weyl algebra*
$C = A(c_1, \ldots, c_n, d_1, \ldots, d_n)$, let $C^\uparrow$ be the algebra generated by all the
commutative rational n-th Weyl algebras B extending C. Then

(i) $C^\uparrow = A(\frac{c_1}{N_1}, \ldots, \frac{c_n}{N_n}, \frac{d_1}{N_1}, \ldots, \frac{d_n}{N_n})$, where $N_k = c_k d_k$, $k = 1, \ldots, n$.

(ii) $C^\uparrow$ is generated by two maximal commutative subalgebras
$A(\frac{c_1}{N_1}, \ldots, \frac{c_n}{N_n}, d_1, \ldots, d_n)$ and $A(c_1, \ldots, c_n, \frac{d_1}{N_1}, \ldots, \frac{d_n}{N_n})$.

(iii) If C is the centre of $A = A(a_1, \ldots, a_n, b_1, \ldots, b_n)$, then $C^\uparrow = A$.

Proof. (i) It is clear that $C^\uparrow$ contains $A(\frac{c_1}{N_1}, \ldots, \frac{c_n}{N_n}, d_1, \ldots, d_n)$ and
$A(c_1, \ldots, c_n, \frac{d_1}{N_1}, \ldots, \frac{d_n}{N_n})$ as the two subalgebras are commutative and contain
C.

(ii) It is easy to see that both are maximal commutative subalgebras of
$C^\uparrow$ as presented in (i). It is obvious from the explicitly given parameters that
the two sublagebras generate $C^\uparrow$.

(iii) By 2.12 $C = A(N_1 a_1, \ldots, N_n a_n, N_1 b_1, \ldots, N_n b_n)$, where the N_k are
denominators of the reduced rational numbers $a_k b_k$. Hence, by setting $c_k = N_k a_k$ and $d_k = N_k b_k$ we get by (i) $C^\uparrow = A$. $\square$

2.16 Corollary. *The functors $A \mapsto Z(A)$ and $C \mapsto C^\uparrow$ between categories*
of rational n-th Weyl algebras and commutative rational n-th Weyl algebras
are inverse to each other.

2.17 Lemma.
The family of all the commutative Weyl subalgebras C of a rational Weyl
algebra A containing the centre $Z(A)$ of A is finite.
Proof. It is enough to note that the generators $U_k^{a_k}, V_k^{b_k}$ of A are periodic
modulo the centre. Hence the images of canonical generators $U_k^{c_k}, V_k^{d_k}$ of C
modulo $Z(A)$ take only finitely many values. $\square$

2.18 Definition. Given a rational Weyl algebra A $O(A)$ be the set of
maximal commutative Weyl subalgebras $C \subseteq A$.

2.19 Proposition. *There is a natural bijection between $O(A)$ and the set*
of Lagrangian subgroups of the Heisenberg group H such that $A = A(H)$.
Proof. Immediate by definition. $\square$

2.20 Lemma. *Suppose now* $A = A(a, b)$ *and* $C \in O(A)$. *Then* $C = A(a, Nb)^\gamma$ *for some* $\gamma \in \mathrm{Aut}H(a, b)$.

Proof. Note that $A(a, Nb)$ is maximal commutative in A. By 2.13(iii) C is generated by one element $W \in A$ over $Z(A)$. Then $W = U^{g_{11}a}V^{g_{12}b}$, $g_{11}, g_{12} \in \mathbb{Z}$, co-prime modulo N. Choose $g_{21}, g_{22} \in \mathbb{Z}$ such that $g_{11}g_{22} - g_{12}g_{21} = 1$ and consider $W' := U^{g_{12}a}V^{g_{21}b}$. We see that

$$\gamma: \ U^a \mapsto W \ V^b \mapsto W'$$

is an automorphism of $H(a, b)$. $\square$

2.21 Lemma. *For a rational* n*-th Weyl algebra* $A = A(a_1, \ldots, a_n, b_1, \ldots, b_n)$,

$$O(A) := \{C_1 \times \ldots \times C_n : C_k \in O(A(a_k, b_k)), \ k = 1, \ldots, n\}.$$

Proof. We may consider maximal commutative subgroups $C \subset \mathrm{H}$ of a rank-$2n$ Heisenberg group instead. Recall that

$$\mathrm{H} = \prod_{i=1}^{n} \mathrm{H}_i$$

where H_i are Heisenberg groups of rank 2. The projection C_i of C to the i-th subgroup must be commutative. It also must be maximal, since otherwise there is an element h_i with $h_i \notin C_i$, $[h_i, C_i] = 1$, on the i-th co-ordinate and equal to 1 on other co-ordinates, and such an h commutes with C, contradicting maximality.

Since $C \subseteq \prod_{i=1}^{n} C_i$ and the product is commutative we have the equality. $\square$

2.22 Corollary. *There is a canonical bijection between the set* $O(A)$ *of maximal commutative subalgebras of* A *and the set of all Lagrangian subspaces on the symplectic space associated with the respective Heisenberg group* H *by 2.5.*

3   Categories $\mathcal{A}_{fin}$ and $\mathcal{C}_{fin}$.

3.1 Definition. We fix $\mathbb{F}$ to be an algebraically close field of characteristic zero.

$\mathcal{A}_{fin}$ is the category of all the rational Weyl $\mathbb{F}$-algebras with canonical embeddings as morphisms.

$\mathcal{A}_{fin}^{(n)}$ is the subcategory of n-th Weyl algebras of $\mathcal{A}_{fin}$

$\mathcal{C}_{fin}$ and $\mathcal{C}_{fin}^{(n)}$ are the subcategories of commutative algebras of $\mathcal{A}_{fin}$ and $\mathcal{A}_{fin}^{(n)}$ respectively.

3.2 Properties.

It is easy to check:

(i) $\mathcal{A}_{fin}$ and $\mathcal{C}_{fin}$ are small categories of unital $\mathbb{F}$-algebras with at most one morphism between two objects.

(ii) The functor $A \mapsto Z(A)$ is an isomorphism between $\mathcal{A}_{fin}^{op}$ and $\mathcal{C}_{fin}$. We will describe the inverse functor $Z(A) \mapsto A : Z \mapsto Z^{\uparrow}$ below.

(iii)
$$\bigcap_{A \in \mathcal{A}_{fin}} Z(A) = \mathbb{F}.$$

Proof. (i) and (iii) are immediate from definitions. (ii) is just 2.16.□

3.3 Subfield $\mathbb{F}_0$ with complex conjugation.

For most of our purposes it is suficient to work within a subfield $\mathbb{F}_0$ of $\mathbb{F}$.

We set $\mathbb{R}_0 \subset \mathbb{F}$ to the subfield of the **totally real** algebraic numbers and $\mathbb{F}_0 = \mathbb{R}_0[i]$, the extension of R_0 by $i = \sqrt{-1}$. In particular, $\mathbb{F}_0 \subseteq \mathbb{Q}^{alg}$, the algebraic closure of $\mathbb{Q}$.

Note that μ, be the group of all roots of unity in $\mathbb{F}$, is a subset of $\mathbb{F}_0$. Note also that $q + q^{-1}$ is a totally real number for $q \in \mu$.

We will think of μ as the group of "elements of modulus 1" and introduce an **formal (complex) conjugation** $x \mapsto x^*$, an automorphism of the field $\mathbb{F}_0$, defined as
$$^* : \ r + is \mapsto r - is, \ \text{for } r, s \in \mathbb{R}_0.$$
In particular, $u^* = u^{-1}$, for $u \in \mu$ and $r^* = r$ for $r \in \mathbb{R}_0$.

In particular, given $n \times n$ matrix X over $\mathbb{F}_0$, we define an adjoint X^* as the matrix obtained by transposition and formal conjugation. We call X $\mathbb{F}_0$**-unitary** if $X^*X = 1$.

3.4 Remark. Suppose $\mathbb{F}$ is of cardinality continuum (which may be assumed without loss of generality). Then $\mathbb{F}$ is abstractly isomorphic to the field of complex numbers. So, in this case $\mathbb{F}_0 \subset \mathbb{C}$ and by construction the involution $x \mapsto x^*$ coincides on $\mathbb{F}_0$ with the complex conjugation. This is immediate from the definition.

3.5 Proposition. *An $A \in \mathcal{A}_{fin}$ is an affine prime algebra over $\mathbb{F}$, finitely generated over its centre $Z(A)$ as a module. A is finitely generated as an $\mathbb{F}$-algebra with the field of definition $\mathbb{Q}[q_1, \ldots, q_n]$, for $q_k = \exp 2\pi i a_k b_k$ $k = 1, \ldots, n$.*
   Every maximal ideal of A is regular.
   Proof. Immediate.

Such algebras are considered in [1] which contains in particular the following statement.

3.6 Proposition. Let $A, B \in \mathcal{A}_{fin}$. Then $Z(A) \cap Z(B) \in \mathcal{C}_{fin}$ and so $(Z(A) \cap Z(A))^\uparrow = \langle A, B \rangle \in \mathcal{A}_{fin}$.
   Proof. Immediate by 2.15. $\square$

3.7 Notation. Given $A \in \mathcal{A}_{fin}$ we write Sp_A for $\mathrm{SpecMax}(Z(A))$.

3.8 Lemma. *Suppose $A, B \in \mathcal{A}_{fin}$, $B \subseteq A$, so $Z(A) \subseteq Z(B)$. Given $\beta \in \mathrm{Sp}_B$, a maximal ideal of $Z(B)$, we get a maximal ideal $\beta \cap Z(A)$ of $Z(A)$. The map*

$$b \mapsto \beta \cap Z(A), \quad \pi_{BA} : \mathrm{Sp}_B \to \mathrm{Sp}_A$$

is surjective.
   Proof. Follows from 2.7(ii). $\square$

3.9 Lemma. *Let $A \in \mathcal{A}_{fin}$ and $C \in O(A)$. Let $B = C^\uparrow$. Then*

$$\mathrm{Sp}_C = \mathrm{Sp}_B \text{ and } \pi_{BA} : \mathrm{Sp}_B \to \mathrm{Sp}_A$$

is unramified of order n_A.
   Proof. Since $C = Z(B)$, we have by definition $\mathrm{Sp}_C = \mathrm{Sp}_B$.
   Let $n = n_A$. Recall that by 3.2 the image of $Z(A)$ in $A/\tilde{\alpha}$ is $\mathbb{F} \cdot \mathbf{1}$, the subalgebra of scalar matrices. By 2.18(ii) the image of C in the algebra of matrices is the diagonal subalgebra, that is of the form $\mathbb{F} \cdot \mathbf{i}_1 + \ldots + \mathbb{F} \cdot \mathbf{i}_n$, where $\mathbf{i}_1, \ldots, \mathbf{i}_n$ are orthogonal idempotents.

Note that every maximal ideal of $C/\tilde{\alpha} \cap C$ is of the form $\sum_{j \neq k} \mathbb{F} \cdot \mathbf{i}_j$, for some $k = 1, \ldots, n$. It follows that every maximal ideal of C is of the form

$$\xi_k = \sum_{j \neq k} \mathbb{F} \cdot i_k + \tilde{\alpha} \cap C, \text{ where } i_k \in C, \ \mathbf{i}_k = i_k + \tilde{\alpha} \cap C.$$

On the other hand, $\xi_k \cap Z(A) = \alpha$. Indeed, $(\xi_k \cap Z(A))/(\tilde{\alpha} \cap C \cap Z(A)) = 0$ since $\mathbb{F} \cdot 1 \cap \sum_{j \neq k} \mathbb{F} \cdot \mathbf{i}_k = 0$. Hence $\xi_k \cap Z(A) \supseteq \tilde{\alpha} \cap C \cap Z(A) = \alpha$. The inverse inclusion is obvious by maximality of α.

Hence $\pi_{BA} : c \mapsto \alpha = c \cap Z(A)$ is an $n-$to-1 map $\mathrm{Sp}_B \to \mathrm{Sp}_A$. $\square$

3.10 Corollary. *Suppose $B \subseteq A$, $A, B \in \mathcal{A}_{fin}$. Then*

(i)
$$\pi_{BA} : \mathrm{Sp}_B \to \mathrm{Sp}_A$$

is an unramified projection of order $n_A : n_B$.

(ii) for a generic $\alpha \in \mathrm{Sp}_A$ and $\beta \in \pi_{BA}^{-1}(\alpha)$ the extension $(\mathbb{Q}(\beta) : \mathbb{Q}(\alpha))$ is Galois, and the Galois group of the extension acts transitively on $\pi_{BA}^{-1}(\alpha)$.

Proof. (i) is immediate from 3.9.

(ii) follows from the fact that Sp_B is irreducible (see 3.2(ii)) and so $\pi_{BA}^{-1}(a)$ is irreducible over $\mathbb{Q}(\alpha)$. $\square$

3.11 Remark. The projection π_{BA} can be seen in a graphical way if we take into account that $Z(B)$ is an extension of $Z(A)$ by a finite set of operators X. Correspondingly, any element of Sp_B can be determined by $\alpha^\frown \xi$ (concatenation) where $\alpha \in \mathrm{Sp}_A$ and ξ correspond to eigenvalues of operators $X \in Z(B)$. Now

$$\pi_{BA} : \alpha^\frown \xi \mapsto \alpha. \tag{7}$$

3.12 Now suppose $C \in O(A)$, $\alpha \in \mathrm{Sp}_A$. Then $\alpha = Z(A) \cap \tilde{\alpha} \subseteq C \cap \tilde{\alpha}$. Consider the irreducible A-module $V_A(\alpha)$ introduced in 2.7(iii). Let $n = n_A = \dim_{\mathbb{F}} V_A(\alpha)$. So there is a basis $\{e_1(\alpha), \ldots, e_n(\alpha)\}$ of $V_A(\alpha)$ consisting of common eigenvectors of all the operators in C.

On the other hand, given such a basis $\{e_1(\alpha), \ldots, e_n(\alpha)\}$, $\alpha \in \mathrm{Sp}_A$, we will have

$$C = \{X \in A : \forall \alpha \in \mathrm{Sp}_A \ \exists x_i \in \mathbb{F} \ \bigwedge_{i=1}^{n_A} X_\alpha e_i(\alpha) = x_i e_i(\alpha)\},$$

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where X_α denotes the image of X in the representation $A/\tilde{\alpha}$.

We call such a basis a C-basis.

3.13 Now suppose A is the Weyl algebra $A(a, b)$ and $C = A(a, Nb)$, a maximal commutative Weyl subalgebra as defined in 2.13(ii). Recall that $A(a, Nb)$ is generated over the centre by U^a. Maximal ideals α of the centre $A(Na, Nb)$ of $A(a, b)$ are generated by two elements $U^{aN} - \hat{u}I$ and $V^{bN} - \hat{v}I$ of the centre, where $\hat{u}, \hat{v} \in \mathbb{F}^\times$ and I is the unit of $A(a, b)$.

Choose two elements of $\mathbb{F}^\times$, suggestively denoted u^a and v^b, such that $\hat{u} = (u^a)^N$ and $\hat{v} = (v^b)^N$. Let $q = \exp 2\pi i a b$, and let N be the order of the root of unity q.

We define a **canonical $A(a, Nb)$-basis** $\{\mathbf{u}\}_\alpha$ for $V_A(\alpha)$, $\alpha = \langle U^{aN} - u^{aN}I, V^{bN} - v^{bN}I \rangle$,

$$\mathbf{u}^{a,b}_{V_A(\alpha)} = \{\mathbf{u}^{a,b}(u^a, v^b), \mathbf{u}^{a,b}(qu^a, v^b), ..., \mathbf{u}^{a,b}(q^{N-1}u^a, v^b)\}$$

to satisfy

$$\begin{aligned}
U^a &: \mathbf{u}^{a,b}(q^k u^a, v^b) \mapsto q^k u^a \mathbf{u}^{a,b}(q^k u^a, v^b) \\
V^b &: \mathbf{u}^{a,b}(q^k u^a, v^b) \mapsto v^b \mathbf{u}^{a,b}(q^{k-1} u^a, v^b)
\end{aligned} \tag{8}$$

We often abbreviate the terminology and call such a basis a U^a-basis.

We also introduce symmetrically a canonical $A(Na, b)$-basis, or V^b-basis, for the same module,

$$\mathbf{v}^{a,b}_{V_A(\alpha)} = \{\mathbf{v}^{a,b}(v^b, u^a), \mathbf{v}^{a,b}(qv^b, u^a), ..., \mathbf{v}^{a,b}(q^{N-1}v^b), u^a\}$$

to satisfy

$$\begin{aligned}
V^b &: \mathbf{v}^{a,b}(q^k v^b, u^a) \mapsto q^k v^b \mathbf{v}^{a,b}(q^k v^b, u^a) \\
U^a &: \mathbf{v}^{a,b}(q^k v^b, u^a) \mapsto u^a \mathbf{v}^{a,b}(q^{k+1} v^b, u^a)
\end{aligned} \tag{9}$$

3.14 Suppose now $A = A(a, b)$ and $C \in O(A)$. Then by 2.20 there is an automorphism γ of $H(a, b)$ such that $U^{a\gamma} = S$, $V^{b\gamma} = T$ and S generates C over $Z(A)$. A canonical C-basis $\{\mathbf{s}\}_\alpha$ of $V_A(\alpha)$ is defined following (3.13) as

$$\begin{aligned}
S &: \mathbf{s}(q^k s, t) \mapsto q^k s \, \mathbf{s}(q^k s, t) \\
T &: \mathbf{s}(q^k s, t) \mapsto t \, \mathbf{s}(q^{k-1} s, t)
\end{aligned} \tag{10}$$

3.15 In the general case of $A = A(a_1, \ldots, a_n, b_1, \ldots, b_n)$ we will also consider maximal commutative subalgebras of the form $C = A(a_1, \ldots, a_n, N_1 b_1, \ldots, N_n b_n)$, generated by $U^{a_1}, \ldots, U^{a_n}$ over the centre. Using the fact that an irreducible

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A-module V_A can be represented as a tensor product of modules $V_{A_k(\alpha_k)}$ for $A_k = A(a_k, b_k)$ and the corresponding α_k.

A canonical $A(a_1, \ldots, a_n, N_1 b_1, \ldots, N_n b_n)$**-basis** (or $\langle U^{a_1}, \ldots, U^{a_n} \rangle$**-basis**) is defined correspondingly as the tensor product of the $A(a_k, N_k b_k)$-bases.

3.16 Below, unless indicated otherwise, we assume $A = A(a, b)$, $\check{U} = U^a$, $\check{V} = V^b$ and the A module $V_A(\alpha)$ is determined by parameters $u = u^a$, $v = v^b$. We abbreviate the notation for canonical $\check{U}$-bases as $\mathbf{u} = \mathbf{u}^{a,b}$.

3.17 Lemma. (i) Any two canonical bases $\{\mathbf{u}(u, v), \mathbf{u}(qu, v), \ldots \mathbf{u}(q^{N-1}u, v)\}$ and $\{\mathbf{u}'(u, v), \mathbf{u}'(qu, v), \ldots \mathbf{u}'(q^{N-1}u, v)\}$ given by the condition (8) can only differ by a scalar multiplier, that is for some $c \neq 0$,

$$\mathbf{u}'(q^k u, v) = c\,\mathbf{u}(q^k u, v), \quad k = 0, \ldots, N-1.$$

(ii) Given also a canonical base $\{\mathbf{u}(u, vq^m), \mathbf{u}(qu, vq^m), \ldots \mathbf{u}(q^{N-1}u, vq^m)\}$, there is a $c \neq 0$ such that

$$\mathbf{u}(q^k u, vq^m) = cq^{km}\mathbf{u}(q^k u, v), \quad k = 0, \ldots, N-1.$$

(iii) Let $\hat{V} \in \mathrm{H}(a, b)$ be such that

$$\check{U}\hat{V} = q\hat{V}\check{U}$$

and $\hat{v}$ an $\hat{V}$-eigenvalue in the module $V_A(\alpha)$ and let $\{\hat{\mathbf{u}}(u, \hat{v}), \hat{\mathbf{u}}(qu, \hat{v}), \ldots, \hat{\mathbf{u}}(q^{N-1}u, \hat{v})\}$ be a canonical $\check{U}$-basis in $V_A(\alpha)$ with regards to $\check{U}$ and $\hat{V}$. Then there is a $c \neq 0$ and an integers n such that

$$\hat{\mathbf{u}}(q^k u, \hat{v}) = cq^{-n\frac{k(k+1)}{2}}\mathbf{u}(q^k u, v), \quad k = 0, \ldots, N-1.$$

Proof. (i) We will have $\mathbf{u}'(u, v) = c\,\mathbf{u}(u, v)$ for some c since the space of $\check{U}$-eigenvectors with a given eigenvalue u in an irreducible $(\check{U}, \check{V})$-module is one-dimenional. The rest follows from the definition (8) of the action by $\check{V}$ on the bases.

(ii) We have $\mathbf{u}(q^k u, vq^m) = c\,\mathbf{u}(q^k u, v)$ and so by induction on k, applying $\check{V}^{-1}$ and (8) to the both parts we get the desired formula.

(iii) First note that from assumptions $\hat{V} = q^l \check{V} \check{U}^n$ for some integers n and l. In particular, it follows $\hat{v} = q^{r+l}u^n v$ for some integer r.

Assume the equality holds for a given k. Apply $\hat{V}^{-1}$ to both sides:

$$\hat{V}^{-1} : \hat{\mathbf{u}}(q^k u, \hat{v}) \mapsto \hat{v}^{-1}\hat{\mathbf{u}}(q^{k+1}u, \hat{v})$$

$$\hat{V}^{-1} : cq^{-n\frac{k(k+1)}{2}}\mathbf{u}(q^k u, v) \mapsto cq^{-n\frac{k(k+1)}{2}}q^{-n(k+1)}\hat{v}^{-1}\mathbf{u}(q^{k+1}u, v).$$

Comparing we get the required. $\square$

3.18 Lemma - definition. *Set*

$$\mathbf{v}(vq^m, u) := \frac{1}{\sqrt{N}}\sum_{k=0}^{N-1} q^{-mk}\mathbf{u}(uq^k, v), \ \ m = 0, \dots, N-1.$$

The system

$$\{\mathbf{v}(v, u), \dots, \mathbf{v}(vq^{N-1}, u)\}$$

is a canonical $\check{V}$-basis of $V_{A(a,b)}$ satisfying (9)
   Proof. One checks directly that the system satisfies (9). $\square$

3.19 Groups $\Gamma_A(\alpha)$.
   For $A \in \mathcal{A}_{fin}$, $\alpha \in \mathrm{Sp}_A$, we define the $\mathbb{F}$-linear transformations of $V_A(\alpha)$ given in a canonical $\check{U}$-basis as:

$$\mu_\alpha : \mathbf{u}(q^m u, v) \mapsto \mathbf{u}(q^{m-1}u, v),$$

and

$$\nu_\alpha : \mathbf{u}(uq^m, v) \mapsto q^{-m}\mathbf{u}(uq^m, v).$$

We define $\Gamma_A(\alpha)$ to be the group generated by μ_α and ν_α.

3.20 Lemma. *The definition of μ_α and ν_α depend on the choice of an $\hat{U}$-basis $\{\mathbf{u}(q^m u, v)\}$ in the following way:*
   (i) any other basis is of the form

$$\mathbf{u}(q^m u', v') := c \cdot \gamma\, \mathbf{u}(q^m u, v), \ \ m = 0, 1 \dots, N-1$$

for $\gamma \in \Gamma_A(\alpha)$ and $c \in \mathbb{F}$;
   (ii) for μ'_α and ν'_α corresponding to the basis $\{\mathbf{u}(q^m u', v')\}$ we will have

$$\mu'_\alpha = \mu_\alpha^\gamma, \ \ \nu'_\alpha = \nu_\alpha^\gamma;$$

(iii) the group $\Gamma_A(\alpha)$ does not depend on the choice of the canonical $\hat{U}$-basis, that is the group generated by μ'_α, ν'_α is $\Gamma_A(\alpha)$;

(iv) moreover, $\Gamma_A(\alpha) \cong \Gamma_A(\beta)$ for any $\beta \in \mathrm{Sp}_A$.

Proof. (i) is by 3.17. (ii) is immediate and (iii) follows from the fact that the change of basis corresponds to the inner automorphism of the groups.

(iv) For any $\beta \in \mathrm{Sp}_A$ a vector space isomorphism $V_A(\beta) \to V_A(\alpha)$ sending an $\hat{U}$-basis to $\hat{U}$-basis induces, by (ii), a group isomorphism $\Gamma_A(\alpha) \to \Gamma_A(\beta)$. $\square$

Given $\xi \in \Gamma_A(\alpha)$ and $X \in A$ let X_α be the image of X in $A(\alpha)$, an operator acting on $V_A(\alpha)$, and

$$X_\alpha^\xi := \xi X_\alpha \xi^{-1}.$$

3.21 Lemma. *We have*

$$\check{U}_\alpha^{\mu_\alpha} = q\check{U}_\alpha \qquad \check{U}_\alpha^{\nu_\alpha} = \check{U}_\alpha$$
$$\check{V}_\alpha^{\mu_\alpha} = \check{V}_\alpha \qquad \check{V}_\alpha^{\nu_\alpha} = q\check{V}_\alpha$$

3.22 Corollary. *The map $h_\alpha : \Gamma_A \to \Gamma_A(\alpha)$ defined on the generators as*

$$h_\alpha : \begin{matrix} \mu \mapsto \mu_\alpha \\ \nu \mapsto \nu_\alpha \end{matrix}$$

extends to a surjective homomorphism of groups.

3.23 Remarks. (i) For A commutative we have $\Gamma_A(\alpha) = \mathbf{1}$, for all $\alpha \in \mathrm{Sp}_A$. Indeed, $n_A = 1$ in this case.

(ii) In a canonical $\check{U}$-basis the group $\Gamma_A(\alpha)$ is represented by a subgroup $\mathrm{SU}(n_A, \mathbb{F}_0)$ of $\mathbb{F}_0$-unitary matrices. This is immediate by definition.

3.24 Lemma. (i) *If $\xi \in \Sigma_A(\alpha)$ and $X_\alpha^\gamma = X_\alpha$ for all $X \in A$, then $\xi = gI$, for some $g \in \mathbb{F}[n_A]$, root of unity of order n_A.*

(ii) *$\Sigma_A(\alpha)$ is finite.*

Proof. (i) Suppose $X_a\xi = \xi X_a$, for all $X \in A$. Let $e \in V_A(a)$ be an eigenvector of ξ with eigenvalue g. Then $\xi' := \xi - gI$ is a linear transformation of $V_A(a)$ which commutes with all the X_a and

$$\xi' V_A(a) = W \subset V_A(a), \; W \neq V_A(a).$$

But $X_a W = X_a \xi' V_A(a) = \xi' X_a V_A(a) = \xi' V_A(a) = W$, that is W is an A-submodule of $V_A(a)$. By irreducibility of the latter, $W = 0$ and so $\xi = gI$. But since $\det \xi = 1$, $g^{n_A} = 1$.

(ii) By 2.17 Σ_A contains a subgroup Σ_A^0 of finite index which fixes setwise $O(A)$.

By definition elements of $\Sigma_A^0(\alpha)$ send a C-basis to a C-basis, for any $C \in O(A)$. So, a possibly smaller subgroup of finite index preserves C-eigenvalues, for all $C \in O(A)$. We may assume $\Sigma_A^0(\alpha)$ has this property.

Hence any $\xi \in \Sigma_A^0(\alpha)$ commutes with any X_α, for $X \in C$. By 2.7(ii) ξ commutes with $A/\tilde{\alpha}$. By (i) $\xi = gI$, $g \in \mathbb{F}[n_A]$. $\square$

3.25 U-**frames** $\mathrm{E}_A^U(\alpha)$. Given $\alpha \in \mathrm{Sp}_A$, choose a $\check{U}$-eigenvector $e \in V_A(\alpha)$ and set
$$\mathrm{E}_A^U(\alpha) := \Gamma_A(\alpha) \cdot e.$$

We call $\mathrm{E}_A^U(\alpha)$ an U-frame of $V_A(\alpha)$.

Clearly, e can be included in a canonical basis $\{\mathbf{u}(uq^m, v) : m = 0, \ldots, N-1\}$ as, say, $e = \mathbf{u}(uq^k, v)$. Then

$$\{\mathbf{u}(uq^m, v) : m = 0, \ldots, N-1\} = \{\mu_\alpha^m e : m = 0, \ldots, N-1\} \subseteq \mathrm{E}_A^U(\alpha).$$

By 3.17 and the fact that $\nu_\alpha \in \Gamma_A(\alpha)$, any other canonical $\check{U}$-basis is a subset of $\mathrm{E}_A^U(\alpha)$ of the form $\Delta(\alpha)e$ for a cyclic subgroup $\Delta(\alpha) \subset \Gamma_A(\alpha)$.

3.26 Corollary. Assuming that the bases in 3.17 are all in $\mathrm{E}_A^U(\alpha)$, the scalar c is of the form q^l for some $l \in \{0, 1, \ldots, N-1\}$.

3.27 Inner product. Let $\mathbb{F}_0[\mathrm{E}_A^u(\alpha)]$ stand for the $\mathbb{F}_0$-linear span of $\mathrm{E}_A^U(\alpha)$, that is the $\mathbb{F}_0$-vector subspace of $V_A(\alpha)$ span over $\mathrm{E}_A^U(\alpha)$. By above $\{\mathbf{u}(uq^m, v) : m = 0, \ldots, N-1\}$ is a basis of the vector space $\mathbb{F}_0[\mathrm{E}_A^U(\alpha)]$.

We introduce an **inner product** on $\mathbb{F}_0[\mathrm{E}_A^U(\alpha)]$ by declaring the basis **orthonormal**. By 3.23(ii) the action of $\Gamma_A(\alpha)$ on $\mathbb{F}_0[\mathrm{E}_A^U(\alpha)]$, and so on $V_A(\alpha)$, is given by **unitary** $\mathbb{F}_0$-matrices. Since any other canonical basis can be obtained by a transformation in $\Gamma_A(\alpha)$, the inner product structure does not depend on the choice of the initial basis. We denote the inner product between two elements $f, g \in \mathbb{F}_0[\mathrm{E}_A^U(\alpha)]$ by

$$\langle f | g \rangle_\alpha.$$

We usually omit the subscript.

Following physics tradition we write the property of $\mathbb{F}_0$-sesquilinearity as

$$\langle f|\tau g\rangle = \tau\langle f|g\rangle = \langle \tau^*f|g\rangle$$

for $\tau \in \mathbb{F}_0$.

3.28 Example. The definition of a V basis in terms of U-basis in 3.18 gives

$$\langle \mathbf{u}(uq^k, v)|\mathbf{v}(vq^m, u)\rangle = \frac{1}{\sqrt{N}}q^{km}.$$

3.29 We call each instance of $V_A(\alpha)$ endowed with an U-frame $\mathrm{E}_A^U(\alpha)$ **an algebraic-Hilbert space** and denote it $\mathbf{V}_A(\alpha)$. Note $\mathrm{E}_A^U(\alpha)$ gives rise to the unique inner product on $\mathbb{F}_0[\mathrm{E}_A^U(\alpha)]$.

3.30 Remark. As in the remark in 3.3 assume $\mathbb{F} = \mathbb{C}$. Then we may use the same canonical basis to introduce an inner product on $\mathbb{C}[\mathrm{E}_A^U(\alpha)] = V_A(\alpha)$. This gives $V_A(\alpha)$ a structure of a Hilbert space which extends the algebraic-Hilbert space structure on $\mathbb{F}_0[\mathrm{E}_A^U(\alpha)]$.

3.31 We call $S \in A(a_1, \ldots, a_n, b_1, \ldots, b_n)$ a **pseudo-unitary operator** if $S \in H(a_1, \ldots, a_n, b_1, \ldots, b_n)$. We set a **formal adjoint of** S to be S^{-1}.

3.32 We call $P \in A(a_1, \ldots, a_n, b_1, \ldots, b_n)$ a **pseudo-selfadjoint operator** if

$$P = \sum_{i=1}^n a_iS_i, \quad \text{some pseudo-unitary } S_i \text{ and } a_i \in \mathbb{F}_0$$

such that

$$P = P^* := \sum_{i=1}^n a_i^*S_i^*,$$

for formal conjugates a_i^* and adjoints S_i^* of a_i and S_i respectively.
   $a_1, \ldots, a_n \in k$.

3.33 *Suppose $S \in A = A(a_1, \ldots, a_n, b_1, \ldots, b_n)$ is pseudo-unitary, $\alpha \in \mathrm{Sp}_A$ and s is an eigenvalue of S acting on $V_A(\alpha)$. Then inner product in $\mathbb{F}_0[\mathrm{E}_A^U(\alpha)]$ is invariant under the action of $s^{-1}S$, that is for any $f, g \in \mathbb{F}_0[\mathrm{E}_A^U(\alpha)]$*

$$\langle s^{-1}Sf|s^{-1}Sg\rangle_\alpha = \langle f|g\rangle_\alpha.$$

In particular, the matrix of $s^{-1}S$ in a canonical U-basis is $\mathbb{F}_0$-orthogonal.

Proof. We may assume without loss of generality that $n = 1$. By skew-linearity we just need to check the identity for f, g belonging to a given canonical U-basis, so we may assume $f = \mathbf{u}(uq^m, v)$ and $g = \mathbf{u}(uq^k, v)$. Clearly, if the statement is true for $S = \check{U}$ and for $S = \check{V}$, then it is true for any group word of these, so is true for S. The lemma follows. $\square$

3.34 For each rational Weyl algebra A one particular module corresponding to the the ideal $\alpha = \langle 1, 1 \rangle =: \mathbf{1}$ (i.e. $u^N = 1 = v^N$) will play at the end a special role. We call the module $\mathrm{V}_A(\mathbf{1})$ and the respective algebraic-Hilbert space $\mathbf{V}_A(\mathbf{1})$ **the principal A-module** and **the principal algebraic-Hilbert space.** The main property of this module is that the eigenvalues of U^a and V^b are roots of unity.

Again, the notions introduced above have natural extension to rational Weyl n-algebras.

3.35 Proposition. *Let $\mathbf{V}_A(\mathbf{1})$ be the principal algebraic-Hibert space, S a pseudo-unitary and P a pseudo-selfadjoint operators. Then*

(i) there is an orthonormal basis of S-eigenvectors of the space;

(ii) eigenvalues of S are in $\mathbb{F}_0[N]$ for some N (i.e. roots of unity);

(iii) there is an orthonormal basis of P-eigenvectors of the space;

(iv) eigenvalues of P are totally real algebraic numbers.

Proof. (i) and (ii) is immediate by 3.33.
Note also that by the argument in 3.33

$$\langle Pf | g \rangle = \langle f | Pg \rangle$$

for any $f, g \in \mathbb{F}_0[\mathrm{E}_A(\mathbf{1})]$. Thus P is a self-adjoint operator in the (formal) Hilber space over field $\mathbb{F}_0 \subset \mathbb{C}$ with complex conjugation. (iii) and (iv) follow. $\square$

3.36 *Suppose $A = A(a, b)$, $C \in O(A)$ and $\mathbf{e} = \{e_0, \ldots, e_{N-1}\}$ is a canonical C-basis of $\mathrm{V}_A(\alpha)$ such that $\langle e_0 | e_0 \rangle = 1$. Then the basis is orthonormal and the transition matrix from a canonical U-basis to $\mathbf{e}$ is $\mathbb{F}_0$-unitary.*

Proof. By 3.14 there are pseudo-unitary generators S, T of A such that C is generated by a pseudo-unitary S and the basis $\mathbf{e}$ is of the form $\{\mathbf{s}(sq^m, t) : m = 0, \ldots, N-1\}$ satisfying (10). Hence it is orthonormal.

By definition $S = q^l \check{U}^k \check{V}^m$ for some integer l, k and m such that k and m are co-prime modulo N. We may also assume without loss of generality that $l = 0$. Then by (8) we may set $s = u^k v^m$.

Denote

$$w_{n,p} = \langle \mathbf{s}(sq^n, t) | \mathbf{u}(uq^p, v) \rangle, \ p = 0, 1, \ldots, N-1.$$

Apply $s^{-1}S$ to both parts of the pairing. Then by 3.36 we get

$$w_{n,p} = \langle \mathbf{s}(sq^n, t) | \mathbf{u}(uq^p, v) \rangle = \langle q^n \mathbf{s}(sq^n, t) | q^{(p-m)k} \mathbf{u}(uq^{p-m}, v) \rangle =$$

$$= q^{(p-m)k-n} \langle \mathbf{s}(sq^n, t) | \mathbf{u}(uq^{p-m}, v) \rangle = q^{(p-m)k-n} \cdot w_{n,p-m}.$$

Using the fact that $\check{U} = q^l S^c T^d$, for some integer l, c and d, and applying $u^{-1}\check{U}$ to the initial pairing one gets similarly

$$w_{n,p} = q^r w_{o,p}$$

for some integer r depending on l, c, d, n, p and o.

We thus get an $\mathbb{F}_0$-linear homogeneous system of equation for the transition matrix $W = \{w_{np}\}$. We may thus assume that W is an $\mathbb{F}_0$-matrix, so conjugation is applicable, and since it takes an $\mathbb{F}_0$-orthonormal basis to an $\mathbb{F}_0$-orthonormal basis, W is $\mathbb{F}_0$-orthogonal. $\square$

3.37 Remark. It is not hard to write down the matrix W in terms of $\mathbb{F}_0$.

3.38 Corollary. *The binary relation $e \in E_A^U(\alpha)$ (between e and α) is definable if and only if the $N + 1$-ary relation "$\{e_0, \ldots, e_{N-1}\}$ is a canonical orthonormal U-basis of $V_A(\alpha)$" is definable.*

3.39 Definitions. Let $C \in O(A)$. Set $E^C(\alpha)$ for each $\alpha \in Sp_A$ to be the set of C-eigenvectors in $\mathbb{F}_0[E_A^U(\alpha)]$ of norm 1. Set

$$E_A(\alpha) := \bigcup_{C \in O(A)} E_A^C(\alpha).$$

3.40 Lemma. *Let $C \in O(A)$ and suppose that the binary relation $e \in \mathrm{E}_A^U(\alpha)$ is definable.*

Then the binary relation $e \in \mathrm{E}_A^C(\alpha)$ (between e and α) is definable.

Proof. By 3.14 and 3.36 we can pick a canonical U-basis $\mathbf{u}(\alpha)$ in $\mathrm{E}_A^U(\alpha)$, for each $\alpha \in \mathrm{Sp}_A$, and a $\mathbb{F}_0$ unitary matrix R such that $\mathbf{e}(\alpha) = R\mathbf{u}(\alpha)$ is a canonical orthonormal C-basis. So, $\mathbf{e}(\alpha) \subseteq \mathrm{E}_A^C(\alpha)$ for each α. By 3.17 there is a finite group of $\mathbb{F}_0$-unitary transformations (independent on α) such that any element in $\mathrm{E}_A^C(\alpha)$ can be obtained by applying these transformations to $\mathbf{e}(\alpha)$. $\square$

3.41 Corollary. *" $e \in \mathrm{E}_A(\alpha)$ " is definable if and only if " $e \in \mathrm{E}_A^U(\alpha)$ " is definable.*

3.42 Lemma-definition *For all $\alpha \in \mathrm{Sp}_A$, the algebraic-Hilbert spaces $\mathbf{V}_A(\alpha) = (\mathrm{V}_A(\alpha), \mathrm{E}_A(\alpha))$ are of the same isomorphism type. We denote it*

$$\mathbf{V}_A = (\mathrm{V}_A, \mathrm{E}_A).$$

Proof. Immediate by 3.20 and definitions. $\square$

3.43 Lemma. *Suppose $\mathbb{F} = \mathbb{C}$ and consider a Hilbert space structure on $\mathrm{V}_A(\alpha)$ extending the algebraic-Hilbert space structure on $\mathbb{F}_0[\mathrm{E}_A(\alpha)]$ (see 3.30).*

Let $C \in O(A)$, C_α be the image of C in the representation $A/\tilde{\alpha}$ of A on $\mathrm{V}_A(\alpha)$ and let $X \in C_\alpha$. Then the adjoint $X_\alpha^* \in \mathbb{C}_\alpha$.

Proof. By 2.7(iii) in any canonical C-basis the algebra C_α is represented as the algebra of the diagonal complex matrices. This is closed under taking adjoints. $\square$

3.44 Lemma. *Let $B \subseteq A$, $B \in \mathcal{A}_{fin}$ and let $B_\alpha \subseteq A/\tilde{\alpha}$ be the image of B in $A/\tilde{\alpha}$. Then B_α is a semi-sipmle algebra.*

Proof. Without loss of generality we may assume that $\mathbb{F} = \mathbb{C}$. Then each $C \in O(B) \subseteq O(A)$, C_α is closed under taking adjoints. Since B_α is generated by all such C_α (see 2.7(ii)), B_α is closed under taking adjoints, in particular the involution $X_\alpha \mapsto X_\alpha^*$ is an anti-automorphism of the ring B_α. This implies that the Jacobson radical of B_α is closed under the involution.

It remains to invoke a well-known argument that under the above condition the Jacobson radical J of the matrix algebra B_α is trivial, that is B_α is

semisimple. Indeed, suppose $Y \in J$. Then $Y^*Y \in J$ and is nilpotent, that is $(Y^*Y)^k = 0$ for some k. By positive-semidefiniteness of Y^*Y, this implies $Y^*Y = 0$. So the inner product $\langle Yv | Yv \rangle = 0$, for any vector v. Hence $Y = 0$.
$\square$

3.45 The bundle of algebraic-Hilbert A-modules

We treat the family
$$\{(\mathbf{V}_A(\alpha) : \alpha \in \mathrm{Sp}_A\}$$

of A-modules as a bundle
$$\mathrm{V}_A := \bigcup \{\mathrm{V}_A(\alpha) : \alpha \in \mathrm{Sp}_A\}$$

with the projection map $\mathrm{ev} : \mathrm{V}_A \to \mathrm{Sp}_A$, $\mathrm{ev}(v) = \alpha$ iff $v \in \mathrm{V}_A(\alpha)$, the algebraic-Hilbert space structure and the A-module structure on each fibre.

Note that as a bundle of algebraic-Hilbert spaces it is trivial, but as a bundle of A-modules, in general it can be highly non-trivial. See [16] for a statement and a proof of the fact that the bundle of A-modules is not definable in $\mathbb{F}$ even when E_A is omitted.

3.46 Let $A, B \in \mathcal{A}_{fin}$, $B \subseteq A$. Hence $\mathrm{V}_A(\alpha)$ can also be considered a B-module, in general, reducible. By 3.44 $\mathrm{V}_A(\alpha)$ splits into a direct sum of irreducible B_α-modules, so irreducible B-modules. The number of the irreducible components is determined by dimensions n_A and n_B, so is equal to $n_A : n_B$. Invoking 3.10 we obtain,

$$\mathrm{V}_A(\alpha) = \bigoplus_{\pi_{BA}(\beta)=\alpha} \mathrm{V}_{AB}(\beta), \quad \mathrm{V}_{AB}(\beta) \cong \mathrm{V}_B(\beta),$$

for uniquely determined B-submodules $\mathrm{V}_{AB}(\beta) \subseteq \mathrm{V}_A(\alpha)$ for each maximal ideal $\beta \subseteq Z(B)$, $\alpha \subseteq \beta$.

3.47 We will describe the embeddings of 3.46 in terms of the canonical bases.

Let as before A be generated by $\check{U} = U^a$ and $\check{V} = V^b$, $n_A = N$ and $\{\mathbf{u}(uq^m, v) : m = 0, \dots, N-1\}$ be a canonical orthonormal $\check{U}$ basis in a A-module $\mathrm{V}_A(\alpha)$, where α is determined by u^N and v^N.

Then $B \subset A$ is the algebra $\langle \check{U}^n, \check{V}^k \rangle$ generated by $\check{U}^n$ and $\check{V}^k$ for some $k, n \in \mathbb{N}$. In fact, we may assume that $B = \langle \check{U}^n, \check{V} \rangle$ or $B = \langle \check{U}, \check{V}^n \rangle$, since any B can be reached by such two steps.

In case $B = \langle \check{U}^n, \check{V} \rangle$ a B-submodules $\mathrm{V}_{AB}(\beta)$ can be constructed by choosing an $\check{U}^n$-eigenvector $\mathbf{u}^{n,1}(u^n, v)$ of modulus 1 and then generating a whole canonical $\check{U}^n$-basis by applying $\check{V}$ according to (8). Since $\check{U}^N$ acts on the module as a scalar, we may assume that n divides N.

It is easy to see that a canonical $\check{V}$-basis in $\mathrm{V}_{AB}(\beta)$ can be identified as follows:

$$\mathbf{v}^{1,n}(vq^{pn}, u^n) = \mathbf{v}(vq^{pn}, u), \ p = 0, \ldots, \frac{N}{n} - 1, \tag{11}$$

where on the right we have elements of the $\check{V}$-basis in $\mathrm{V}_A(\alpha)$.

Set, for $\ell = 0, \ldots, n-1$, $m = 0, \ldots, \frac{N}{n} - 1$, $\xi = \mathrm{e}^{\frac{2\pi i}{n}}$,

$$\mathbf{u}^{n,1}(u^n q^{mn}, vq^\ell) := \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} \mathbf{u}(uq^m \xi^k, vq^\ell) = \frac{q^{m\ell}}{\sqrt{n}} \sum_{k=0}^{n-1} \xi^{k\ell} \mathbf{u}(uq^m \xi^k, v) \tag{12}$$

The second equality follows from 3.17(ii).

One checks immediately that for a fixed ℓ this is a canonical orthonormal $\check{U}^n$-basis for the B-submodule corresponding to the maximal ideal β_ℓ of $Z(B) = \langle \check{U}^N, \check{V}^{\frac{N}{n}} \rangle$ generated by $\check{U}^N - u^N I$ and $\check{V}^{\frac{N}{n}} - v^{\frac{N\ell}{n}} I$. There are n distinct such β and respective non-isomorphic B-modules.

In case $B = \langle \check{U}, \check{V}^n \rangle$ set, for $\ell = 0, \ldots, n-1$,

$$\mathbf{u}^{1,n}(uq^{nm+\ell}, v^n) := \mathbf{u}(uq^{nm+\ell}, v), \ m = 0, \ldots, \frac{N}{n} - 1 \tag{13}$$

This is a canonical orthonormal $\check{U}^n$-basis for the B-submodule corresponding to the maximal ideal β_ℓ of $Z(B) = \langle \check{U}^{\frac{N}{n}}, \check{V}^N \rangle$ generated by $\check{U}^{\frac{N}{n}} - u^{\frac{N\ell}{n}} I$ and $\check{V}^N - v^N I$.

3.48 Lemma. *Suppose as above $A = \langle \check{U}, \check{V} \rangle$, $B = \langle \check{U}^n, \check{V}^k \rangle$. Let $\mathrm{V}_{AB}(\beta) \subset \mathrm{V}_A(\alpha)$ be a B-submodule and $\{\mathbf{u}^B(u'q_B^m, v') : 0 \le m < n_B\}$ and $\{\mathbf{v}^B(v'q_B^p, u) : 0 \le p < n_B\}$ be canonical $\check{U}^n$- and $\check{V}^k$-bases in $\mathrm{V}_{AB}(\beta)$. Then*

$$\langle \mathbf{u}^B(u'q_B^m, v') | \mathbf{v}^B(v'q_B^p, u) \rangle_{\mathbf{V}_A(\alpha)} = \langle \mathbf{u}^B(u'q_B^m, v') | \mathbf{v}^B(v'q_B^p, u) \rangle_{\mathbf{V}_{AB}(\beta)}$$

In particular, the inner product is preserved in the embedding $\mathbf{V}_{AB}(\beta) \to \mathbf{V}_A(\alpha)$.

Proof. As above it is enough to check the case $k = 1$. Then $q_B = q^n$, where now $n = n_A : n_B$. We can rewrite

$$\mathbf{u}^B((u'q_B^m, v') = \mathbf{u}^{1,n}(u^n q^{mn}, v)$$

and

$$\mathbf{v}^B((u'q_B^m, v') = \mathbf{v}^{n,1}(vq^{mn}, u^n).$$

By 3.28

$$\langle \mathbf{u}^B(u'q_B^m, v') | \mathbf{v}^B(v'q_B^p, u') \rangle_{\mathbf{V}_{AB}(\beta)} = \frac{1}{\sqrt{n_B}} q_B^{mp} = \frac{1}{\sqrt{n_B}} q^{nmp}.$$

By (12) and (11) we can calculate

$$\langle \mathbf{u}^{1,n}(u^n q^{nm}, v) | \mathbf{v}^{n,1}(vq^{np}, u^n) \rangle_{\mathbf{V}_A(\alpha)} = \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} \langle \mathbf{u}(uq^{m+\frac{n_A}{n}k}, v) | \mathbf{v}(vq^{np}, u) \rangle =$$

$$= \frac{1}{\sqrt{n}} \cdot n \frac{1}{\sqrt{n_A}} q^{nmp} = \frac{\sqrt{n}}{\sqrt{n_A}} \cdot q^{nmp}$$

(taking into account that q is of order n_A.)

It remains to note that

3.49 Keeping the notation and assumptions of 3.46 we will consider B-module isomorphisms p_{BA}^{β}, abbreviated to p^{β},

$$p^{\beta} : V_B(\beta) \to V_{AB}(\beta) \subset V_A(\alpha)$$

such that

$$p^{\beta}(E_B(\beta)) \subseteq E_A(\alpha) \cap V_{AB}(\beta).$$

In case $B = \langle \check{U}^n, \check{V} \rangle$ we define the maps on the bases, according to (12), as

$$p^{\beta} : \mathbf{u}^{n,1}(u^n q^{mn}, vq^{\ell}) \mapsto \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} \mathbf{u}(uq^{m+\frac{N}{n}k}, vq^{\ell}), \ m = 0, \dots, \frac{N}{n} - 1, \quad (14)$$

where $\mathbf{u}^{n,1}(u^n q^{mn}, vq^{\ell})$ is now naming elements of a canonical basis of $V_B(\beta)$.

In case $B = \langle \check{U}, \check{V}^n \rangle$

$$p^{\beta} : \mathbf{u}^{1,n}(uq^{nm+\ell}, v^n) \mapsto \mathbf{u}(uq^{nm+\ell}, v), \ m = 0, \dots, \frac{N}{n} - 1 \quad (15)$$

Note that in terms of the $\check{V}$ bases (14) can be rewritten as

$$p^{\beta} : \mathbf{v}^{n,1}(vq^{nm+\ell}, u^n) \mapsto \mathbf{v}(vq^{nm+\ell}, u), \ m = 0, \dots, \frac{N}{n} - 1 \quad (16)$$

(just use the symmetry between $\check{V}$ and $\check{U}$).

Finally, we would like to remark that the definition of p^β does depend on the choice of the canonical orthonormal bases in $V_B(\beta)$ and $V_A(\alpha)$.

3.50 Lemma. Let p^β and p'^β, $\beta \in \mathrm{Sp}_B$ be two isomorphisms $V_B(\beta) \to V_{AB}(\beta)$. Then there is a $g_\beta \in \mathbb{F}^*[n_B]$ such that $p^\beta = p'^\beta g_\beta$. In particular, there are exactly n_B choices for p^β for a given $\beta \in \mathrm{Sp}_B$.

Proof. Consider $(p'^\beta)^{-1}p^\beta$. This is a linear transformation commuting with operators from $B/\tilde{\beta}$, since p^β and p'^β are B-module isomorphisms. By the argument in 3.24(i) $(p'^\beta)^{-1}p^\beta = g_\beta I$ for some $g_\beta \in \mathbb{F}^*[n_B]$. $\square$

3.51 Definition A map p^β satisfying 3.49 will be called a **local morphism at** β of $\mathbf{V}_B \to \mathbf{V}_A$.

We call the collection $\mathbf{p}^\beta$ of all the local morphism at β a fibre of a morphism $\mathbf{V}_B \to \mathbf{V}_A$ at β.

We call the family of fibres

$$\mathbf{p}_{BA} = \{\mathbf{p}^\beta : \ \beta \in \mathrm{Sp}_B\}$$

the morphism $\mathbf{V}_B \to \mathbf{V}_A$.

3.52 Remark. Lemma 3.50 implies that $\mathbf{V}_A$ is determined by A uniquely up to isomorphism, that is given $\mathbf{V}_A$ and $\mathbf{V}'_A$ both satisfying the definition in 3.49, there is a family

$$p^\alpha : \mathbf{V}_A(\alpha) \to \mathbf{V}'_A(\alpha), \ \alpha \in \mathrm{Sp}_A$$

of local isomorphisms establishing an isomorphism between the two structures.

3.53 Continuing the notation of 3.46, set

$$\Gamma_{AB}(\beta) = \{\gamma \in \Gamma_A(\alpha) : \gamma V_{AB}(\beta) = V_{AB}(\beta)\}.$$

Let

$$\gamma \mapsto \check{\gamma}, \ \Gamma_{AB}(\beta) \to \mathrm{GL}\,(V_B(\beta))$$

be the group homomorphism defined as follows. For $\gamma \in \Gamma_{AB}(\beta)$ set $\check{\gamma}$ to be the unique $\mathcal{A}_{fin}$-transformation $\check{\gamma}$ of $V_B(\beta)$ such that

$$\gamma p^\beta v = p^\beta \check{\gamma}\, v, \ \ \text{for all } v \in V_B(\beta)$$

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(the transformation on $V_B(\beta)$ induced from $V_{AB}(\beta)$ by the isomorphism p^β).

In general, $\det \check{\gamma}$ does not have to be equal to 1 (in fact, $(\det \check{\gamma})^{\frac{n_A}{n_B}} = 1$), so we define

$$\pi_{BA}^*(\beta) : \gamma \mapsto (\det \check{\gamma})^{-\frac{1}{n_B}} \cdot \check{\gamma}, \quad \check{\gamma} = (p^\beta)^{-1} \gamma p^\beta$$

a 1-n_B-correspondence, a multivalued map into $\Gamma_B(\beta)$, or a homomorphism

$$\pi_{BA}^*(\beta) : \Gamma_A(\alpha) \to \Gamma_B(\beta)/\mathbb{F}^*[n_B] \subseteq \mathrm{PGL}\left(V_B(\beta)\right),$$

with the domain $\Gamma_{AB}(b) \subset \Gamma_A(\alpha)$. Here $\mathbb{F}^*[n_B]$ is the group of roots of unity of order n_B, which is also the centre of $\Gamma_B(\beta)$. We write $\mathrm{P}\Gamma_B(\beta)$ for $\Gamma_B(\beta)/\mathbb{F}^*[n_B]$ below for any $B \in \mathcal{A}_{fin}$.

Note that by definition, for $\alpha \in \mathbb{F}^*[n_A]$, $\check{\alpha\gamma} = \alpha\check{\gamma}$, so in fact we have equivalently defined

$$\pi_{BA}^*(\beta) : \mathrm{P}\Gamma_A(\alpha) \to \mathrm{P}\Gamma_B(\beta),$$

a homomorphism of the projective groups.

3.54 Remark. Recall that the action of $\Gamma_A(\alpha)$ on the algebra $A/\tilde{\alpha}$ reduces to the action of $\mathrm{P}\Gamma_A(\alpha)$ on the algebra. So, in this regard we do not lose any information switching to the projective groups.

3.55 V_A as structures in the language $\mathcal{L}_A$.

For each $A \in \mathcal{A}_{fin}$ the bundle $\mathbf{V}_A$ can be described as a two sorted structure in a language $\mathcal{L}_A$.

Sorts. First sort, named F will stand for (the universe of) the field $\mathbb{F}$.

The language $\mathcal{L}_A$ will have names for all Zariski closed subsets of $\mathbb{F}^n$, all n, including the ternary relations for graphs of $+$ and $\cdot$, the field operations. One of the closed subsets of $\mathbb{F}^m$, where m is the number of generators of the affine algebra A, will correspond to the spectrum $\mathrm{Sp}_A = \mathrm{Spec}(Z(A))$ of the centre $Z(A)$ of A : for each of the generating operators $X_1, \ldots, X_m$ of $Z(A)$ and for each maximal ideal α of $Z(A)$ we put in correspondence the m-tuple $\langle x_1, \ldots, x_m \rangle \in \mathbb{F}^m$ such that $X_1 - x_1 I, \ldots, X_m - x_m I$ generate the ideal α. We call this closed subset Sp_A.

The second sort named V will stand for V_A, that is as a universe (set of points) it is the set $\bigcup\{V_A(\alpha) : \alpha \in \mathrm{Sp}_A\}$ described in 3.45.

There is in $\mathcal{L}_A$ a symbol ev for a map $\text{ev} : V_A \to \text{Sp}_A \subseteq \mathbb{F}^m$ which is interpreted as described in 3.45. In particular, for each $\alpha \in \text{Sp}_A$, $\text{ev}^{-1}(a) = V_A(a)$ is definable.

Algebraic structure. The language $\mathcal{L}_A$ contains a binary relation $R_X(v, w)$ for each $X \in A$, which is interpreted on sort V_A as $X \cdot v = w$ for $v, w \in V_A(\alpha)$, for $\alpha \in \text{Sp}_A$.

There is also a ternary relation $S(\xi, v, w)$ between $\xi \in \mathbb{F}$, $v, w \in V_A(a)$ allowing us to say that $\xi \cdot v = w$.

A ternary relation $v +_A u = w$ on V_A will be saying that there exists α such that $u, v, w \in V_A(\alpha)$, and that $v + u = w$ in the sense of the module structure on $V_A(\alpha)$.

Canonical bases. $\mathcal{L}_A$ contains a binary symbol E^S, for each pseudo-unitary $S \in A$, which distinguishes in V_A a relation

$$E^S(e, \alpha) \equiv \quad e \in \mathrm{E}_A^S(\alpha) \ \& \ \alpha \in \text{Sp}_A$$

as well as the binary symbols IP_r, for non-zero $r \in \mathbb{F}_0$, which are interpreted on V_A as

$$\text{IP}_r(e_1, e_2) \equiv \quad \exists \alpha \in \text{Sp}_A \ e_1, e_2 \in \mathrm{E}_A(\alpha) \ \& \ \langle e_1 | e_2 \rangle_\alpha = r,$$

the value of the **inner product** (see 3.25). Here $\mathrm{E}_A(\alpha)$ stands for $\bigcup \{\mathrm{E}_A^S(\alpha) : S \in A \text{ pseudo-unitary}\}$. This is definable in $\mathcal{L}_A$ since there are only finitely many E_A^S for each A.

3.56 Lemma. (i) *The binary relation " $e \in \mathrm{E}_A(\alpha)$" is definable in the language $\mathcal{L}_A$.*

(ii) *The $N + 1$-ary relation " $\{e_0, \ldots, e_{N-1}\}$ is a canonical orthonormal U-basis of $\mathbf{V}_A(\alpha)$" is definable in the language $\mathcal{L}_A$.*

Proof. (i) is immediate by 3.41. (ii) is by 3.38. $\square$

Note that defining $e \in \mathrm{E}_A(\alpha)$ from $e \in \mathrm{E}_A^U(\alpha)$ may require formulas which depend on A.

3.57 Remark. The language $\mathcal{L}_A$ has no symbols for Γ_A or group actions. The group plays a role in the theory of $\mathbf{V}_A$ through axioms describing canonical bases in $\mathrm{E}_A(a)$ and the matrices of linear transformations from one base to another, see subsection 2.1 of [16].

3.58 Definition. Let $\mathbf{N}$ and $\mathbf{M}$ be structures in languages L_N and L_M, correspondingly. We say that a structure $\mathbf{N}$ **is almost interpretable (definable) in M** if,

(i) there is a structure $\mathbf{N}^0$ in a language L_N^0, which is definable in $\mathbf{N}$ and interpretable in $\mathbf{M}$;

(ii) $\mathbf{N}$ is prime over $\mathbf{N}^0$ (in particular, any automorphism of $\mathbf{N}^0$ can be lifted to an automorphism of $\mathbf{N}$);

(iii) $N \subseteq \mathrm{acl}_{\mathbf{N}}(N^0)$, that is the universe N of $\mathbf{N}$ is in the algebraic closure of N^0, in the sense of $\mathbf{N}$.

Remark. [10], [16] and [13] discuss examples of Zariski structures which are almost interpretable but not interpretable in an algebraically closed field $\mathbb{F}$.

3.59 Theorem. *Given an $A \in \mathcal{A}_{fin}$,*

1. $\mathbf{V}_A$ is almost interpretable in the field $\mathbb{F}$, in particular, the first order theory of the structure $\mathbf{V}_A$ is categorical in uncountable cardinalities.

2. $\mathbf{V}_A$ is a Zariski geometry in the topology given by the positive formulas with all quantifiers restricted to the predicate E.

Proof. 1. The field $\mathbb{F}$ is definable in $\mathbf{V}_A$ as the sort F by definition, see 3.55. It is also clear by the construction that $\mathrm{acl}_{\mathbf{V}_A}(F) = V_A$. Theorem A of [16] proves that $\mathbf{V}_A$ is categorical in uncountable cardinalities and that $\mathbf{V}_A$ is prime over $\mathbb{F}$.

2. This is a special case of Theorem B of [16]. $\square$

3.60 Corollary. *Given $\mathcal{A}_{fin}$, $A, B \in \mathcal{A}_{fin}$, $\mathbf{V}_B$ is almost interpretable in $\mathbf{V}_A$. .*

3.61 Proposition. *Assume $B \subset A$. Then $\mathbf{V}_B$ is definable in $\mathbf{V}_A$.*

We may assume that A is the Weyl algebra generated by $\check{U}$ and $\check{V}$, $N = n_A$, and B is generated by $\check{U}^n$ and $\check{V}^m$, n, m divide N. Let $\langle \check{u}, \check{v} \rangle \in \mathbb{F}^2$ define α as in 3.13. We deduce that $\check{u} = u^N$, $\check{v} = v^N$, for some $u, v \in \mathbb{F}^\times$ (following notations of 3.15) and β is determined by $\langle u^{\frac{N}{m}}, v^{\frac{N}{n}} \rangle$.

The subbundle $\{\mathbf{V}_{AB}(\beta) : \beta \in \mathrm{Sp}_B\}$, of the bundle $\{\mathbf{V}_A(\alpha) : \alpha \in \mathrm{Sp}_A\}$ as $\alpha = \pi_{BA}(\beta)$ is defined by the formula

$$\text{``}\mathbf{w} \in \mathrm{V}_B(u^{\frac{N}{m}}, v^{\frac{N}{n}})\text{''} := \check{U}^{\frac{N}{m}}\mathbf{w} = u^{\frac{N}{m}}\mathbf{w} \ \& \ \check{V}^{\frac{N}{n}}\mathbf{w} = v^{\frac{N}{n}}\mathbf{w}.$$

Since $\mathrm{V}_A(\alpha)$ are B-modules, the $\mathrm{V}_{AB}(\beta)$ get the structure of B-modules too.

Finally, we define

$$\mathrm{E}_{AB}(\beta) := \mathrm{E}_A(\alpha) \cap \mathrm{V}_{AB}(\beta) \text{ for } \alpha = \pi_{BA}(\beta).$$

Thus we get the bundle of algebraic-Hilbert B-modules on $\{\mathrm{V}_{AB}(\beta) : \beta \in \mathrm{Sp}_B\}$.

This is canonically (but not uniquely) isomorphic to the bundle of algebraic-Hilbert B-modules on $\{\mathrm{V}_B(\beta) : \beta \in \mathrm{Sp}_B\}$ by fibrewise isomorphisms $\mathbf{p}_{BA}^{\beta}$ between $\mathrm{V}_B(\beta)$ and the corresponding submodules of $\mathrm{V}_{AB}(\beta)$ as defined in 3.49. $\square$

3.62 Remark. The image of a 0-definable relation on V_B under $\mathbf{p}_{BA}$ does not depend on the choice of a representative in $\mathbf{p}_{BA}$ since definable relations are invariant under isomorphisms.

Moreover, $\mathbf{p}_{BA}$ sends Zariski-closed relations on V_B to Zariski-closed ones on V_A, that is this is a Zariski-continuous Zariski-closed morphism.

3.63 Proposition. *The categories $\mathcal{A}_{fin}(\mathbb{F})$ and $\mathcal{V}_{fin}(\mathbb{F})$ are isomorphic. In other words, the functor*

$$A \mapsto \mathbf{V}_A$$

is invertible.

Proof. In fact, even the weaker functor $A \mapsto \mathrm{V}_A$ is invertible. We only need to show how to recover A from V_A. Since V_A has the structure of a bundle of A-modules, we are already given the action of A on each module $\mathrm{V}_A(\alpha)$. It remains to see that the annulator of all the modules $\mathrm{V}_A(\alpha)$, $\alpha \in \mathrm{Sp}_A$, is trivial. This is immediate from the classification of irreducible representations of A, see [1]. $\square$

4 The category $\mathcal{V}_{fin}$ and the sheaf on $\mathcal{A}_{fin}$

4.1 Category $\mathcal{V}_{fin}$. The category $\mathcal{V}_{fin}$ consists of objects $\mathbf{V}_A$, $A \in \mathcal{A}_{fin}$, and morphisms $\mathbf{p}_{BA} : \mathbf{V}_B \to \mathbf{V}_A$, $B \subseteq A$.

Note that $\mathbf{p}_{BA}$ determines both

$$\pi_{BA} : \mathrm{Sp}_B \to \mathrm{Sp}_A$$

and

$$\pi_{BA}^* : \mathrm{P\Gamma}_A \to \mathrm{P\Gamma}_B.$$

It is immediate by definition that the functor $A \to \mathbf{V}_A$ from $\mathcal{A}_{fin}$ to $\mathcal{V}_{fin}$ is a presheaf on the category $\mathcal{A}_{fin}$ and so a presheaf on the category $\mathcal{C}_{fin}^{op}$.

We use the same name $\mathcal{V}_{fin}$ for the functor.

4.2 Remark. We have associated with an $A \in \mathcal{A}_{fin}$ the (finite) set $O(A)$ of maximal $\mathcal{C}_{fin}$-algebras.

It is appropriate to think about the collection $O(\mathcal{A}_{fin})$ of all maximal $C \in \mathcal{C}_{fin}$ as a Grothendieck site with subsets $O(A)$ providing "open subsets" of the Grothendieck topology. Any finite covering will be considered admissible.

4.3 Remark. Given $A, B \in \mathcal{A}_{fin}$, $B \subseteq A$ there is a unique morphism

$$\mathbf{p}_{BA} : \mathbf{V}_B \to \mathbf{V}_A.$$

This follows from Lemma 3.50.

4.4 Theorem. *$\mathcal{V}_{fin}$ is a sheaf over the site $O(\mathcal{A}_{fin})$.*

Proof. Let $B_1, \ldots, B_k \in \mathcal{A}_{fin}$ and $\mathbf{V}_{B_1}, \ldots, \mathbf{V}_{B_k}$, the corresponding objects of the presheaf. By the remark above the morphisms between the object are uniquely determined. By 3.6 and 3.2 there is $A \in \mathcal{A}_{fin}$, $Z(A) \in \mathcal{C}_{fin}$ such $Z(A) = Z(B_1) \cap \ldots \cap Z(B_k)$, equivalently, $O(A) = O(B_1) \cup \ldots \cup O(B_k)$.

We need to prove that there is a unique object $\mathbf{V}_A$ in the category with the corresponding morphisms $\mathbf{p}_{B_i A} : \mathbf{V}_{B_i} \to \mathbf{V}_A$. For this just take the uniquely defined $\mathbf{V}_A$ and the unique morphisms $\mathbf{p}_{B_i A}$. $\square$

4.5 Sheaf $\mathcal{V}_{fin}$ as a structure. We define the structure $\mathcal{V}_{fin}$ as a multi-sorted structure with sorts $\mathbf{V}_A$, for each $A \in \mathcal{A}_{fin}$, in the language $\mathcal{L}_A$ expanded by unary predicates for each of the sorts $\mathbf{V}_A$.

In this definition, each sort $\mathbf{V}_A$ is a multi-sorted structure, and it is assumed that all the sorts have a common field $\mathbb{F}$.

4.6 Theorem.

(i) *The theory of $\mathcal{V}_{fin}$ is categorical in uncountable cardinalities. Moreover, every two models over a field $\mathbb{F}$ are isomorphic over $\mathbb{F}$.*

(ii) *For any $A, B \in \mathcal{A}_{fin}$, $B \subseteq A$, the morphism $\mathbf{p}_{BA}$ is definable. More precisely, there is a definable sort $\mathbf{P}_{BA}$ and a definable relation*

$$P_{BA} \subseteq \mathbf{P}_{BA} \times \mathrm{Sp}_B \times \mathrm{V}_B \times \mathrm{V}_A$$

in $\mathcal{V}_{fin}$, such that for every $p \in \mathbf{P}_{BA}$ and $\beta \in \mathrm{Sp}_B$ the binary relation on $\mathrm{V}_B \times \mathrm{V}_A$,

$$P_{BA}(p, \beta, v_1, v_2)$$

is the graph of a local morphism p_{BA}^{β}, and every local morphism can be obtained in this way.

Proof. (i) Immediate from 3.59.

(ii) By construction. $\square$

4.7 A symmetric pairing. Given $B, D \in \mathcal{A}_{fin}$ we introduce a pairing

$$[\cdot \mid \cdot]: \ \mathrm{E}_B \times \mathrm{E}_D \to \mathbb{R}_0.$$

First, we recall that the algebra $A = \langle B \cup D \rangle$ is in $\mathcal{A}_{fin}$ and so there are morphisms $\mathbf{p}_{BA}: \mathbf{V}_B \to \mathbf{V}_A$ and $\mathbf{p}_{DA}: \mathbf{V}_D \to \mathbf{V}_A$ as defined in 3.51.

Let $\beta \in \mathrm{Sp}_B$, $\delta \in \mathrm{Sp}_D$. For $e \in \mathrm{E}_B(\beta)$ and $f \in \mathrm{E}_D(\delta)$ set

$$[e|f] = \begin{cases} |\langle \mathbf{p}^{\beta}(e) | \mathbf{p}^{\delta}(f) \rangle_{\alpha}|^2, & \text{if } \pi_{BA}(\beta) = \alpha = \pi_{DA}(\delta) \\ 0, & \text{otherwise} \end{cases}$$

(the absolute value of the inner product).

4.8 Lemma. *The pairing $[e|f]$ is well-defined and does not depend on the choice of $\mathbf{p}^{\beta}$ and $\mathbf{p}^{\delta}$.*

The pairing is symmetric.

$$[se|f] = |s|^2 \cdot [e|f] \ \text{ and } \ [e|tf] = |t|^2 \cdot [e|f]$$

for any $s, t \in \mathbb{F}_0$, roots of unity of orders n_B and n_D respectively.

$$[e|f] = s \cdot s^* \ \text{for} \ s = \langle \mathbf{p}^{\beta}(e) | \mathbf{p}^{\delta}(f) \rangle_{\alpha}$$

Proof. Note that if $\pi_{BA}(\beta) = \alpha = \pi_{DA}(\delta)$, we will have $\mathbf{p}^{\beta}(e) \in \mathrm{E}_A(\alpha)$, $\mathbf{p}^{\delta}(f) \in \mathrm{E}_A(\alpha)$ and so the inner product $\langle \mathbf{p}^{\beta}(e) | \mathbf{p}^{\delta}(f) \rangle_a$ is defined.

It does not depend on $\mathbf{p}^{\beta}$ and $\mathbf{p}^{\delta}$ by 3.50.

Clearly,

$$[e|f] = [f|e].$$

The last two statement are also by definition. $\square$

4.9 Commentaries. (i) The physicists interpretation of the pairing $[\cdot \mid \cdot]$ would be "**probability**", whereas the inner product would correspond to "amplitude". We have means to define amplitude locally, on each module $V_A(\alpha)$, but globally we have to use probability because of non-uniqueness in the choice of embedding $\mathbf{p}^\beta$. The latter forced upon us if we want uniqueness for our whole construction (Theorem 4.6), in fact, by model-theoretic stability assumptions.

(ii) A geometric interpretation of the pairing should be that it provides a **gluing correspondence** between E_B and E_D, and more specifically between E_B^S and E_D^T for chosen pseudo-unitary S and T (equivalently, maximal commutative subalgebras). Namely, the relation between $e \in E_B^S$ and $f \in E_D^T$ given by the definable condition $[e|f] \neq 0$ is a finite-to-finite correspondense.

Note that the correspondence $[e|f] \neq 0$ is Zariski closed, which is easy to work out from definitions.

4.10 We call **an orthonormal canonical basis of V_A** a collection

$$\mathbf{e} \subseteq \bigcup \{E_A(\alpha) : \alpha \in \mathrm{Sp}_A\}$$

with the property that $\mathbf{e} \cap E_A(\alpha)$ is an orthonormal canonical C-basis in $\mathbf{V}_A(\alpha)$ for some $C \in O(A)$.

4.11 Lemma. *Let $B, D \in \mathcal{A}_{fin}$ and $\mathbf{e}$ be an orthonormal basis of $\mathbf{V}_B$. Then, for any $\delta \in \mathrm{Sp}_D$ and $f \in E_D(\delta)$,*

(i) there are finitely many $e \in \mathbf{e}$ such that $[e|f] \neq 0$ and

(ii) $\quad \sum_{e \in \mathbf{e}} [e|f] = 1$.

Proof. Note that $\mathbf{g} := \mathbf{p}_{BA}(\mathbf{e})$ is an orthonormal basis of $\mathbf{V}_A$, more precisely,

$$\mathbf{g} \subset \bigcup \{V_{AB}(\alpha) : \alpha \in \mathrm{Sp}_A\}.$$

Now (i) follows by definition.

For (ii) note that, assuming $\alpha = \pi_{DA}(\delta)$ and using the fact that $\mathrm{p}^\delta(f) \in E_A(\alpha)$,

$$\sum_{e \in \mathbf{e}} [e|f] = \sum_{g \in \mathbf{g}} |\langle g|\mathrm{p}^d(f)\rangle_a|^2 = \sum_{g \in \mathbf{g} \cap E_A(a)} \langle \mathrm{p}^d(f)|g\rangle_a \cdot \langle g|\mathrm{p}^d(f)\rangle_a = \langle \mathrm{p}^d(f)|\mathrm{p}^d(f)\rangle_a = 1.$$

$\square$

5   The categories $\mathcal{A}^*_{fin}$ and $\mathcal{V}^*_{fin}$.

5.1 Define the category $\mathcal{A}^*_{fin}(\mathbb{C})$ as consisting of algebras $A \in \mathcal{A}_{fin}(\mathbb{C})$ (that is $\mathbb{F} = \mathbb{C}$) endowed with an involution $X \mapsto X^*$, which is defined on the pseudo-unitary operators by the rule

$$* : \ U^a V^b \mapsto V^{-b} U^{-a},$$

and on $\mathbb{C}$ is defined as complex conjugation. The morphisms of $\mathcal{A}^*_{fin}(\mathbb{C})$ will be the same as of $\mathcal{A}_{fin}(\mathbb{C})$, which of course preserve the involution by definition.

Given $A \in \mathcal{A}_{fin}(\mathbb{C})$, we will write A_* for the algebra in $\mathcal{A}^*_{fin}(\mathbb{C})$ obtained by introducing the involution. We consider the functor

$$F_{\mathcal{A}} : A \mapsto A_*; \ \ \mathcal{A}_{fin}(\mathbb{C}) \to \mathcal{A}^*_{fin}(\mathbb{C}).$$

5.2 Lemma. *The functor $F_{\mathcal{A}}$ is an isomorphism of the categories $\mathcal{A}_{fin}(\mathbb{C})$ and $\mathcal{A}^*_{fin}(\mathbb{C})$.*

Proof. Immediate. $\square$

5.3 The objects of the category $\mathcal{V}^*_{fin}(\mathbb{C})$ will be defined as the **real-coordinate parts** of the Zariski geometries $\mathbf{V}_A$, for $A = A(a,b) \in A^*_{fin}(\mathbb{C})$. We define the **real-coordinate part** of Sp_A to be

$$\mathrm{Sp}_A(\mathbb{R}) = \{\langle u^N, v^N \rangle : |u| = |v| = 1\},$$

that is the eigenvalues of the generators U^a and V^b of the algebras on the module $\mathbf{V}_A(\alpha)$ are of modulus 1 when $\alpha \in \mathrm{Sp}_A(\mathbb{R})$.

Note that for $\alpha \in \mathrm{Sp}_A(\mathbb{R})$, pseudo-unitary operators act in the algebraic-Hilbert module $\mathbf{V}_A(\alpha)$ as unitary, if one considers $\mathbf{V}_A(\alpha)$ as a Hilbert space as described in

Now define the object

$$\mathbf{V}_A(\mathbb{R}) = \{\mathbf{V}_A(\alpha) : \ \alpha \in \mathrm{Sp}_A(\mathbb{R})\}$$

as the bundle of Hilbert spaces $\mathbf{V}_A(\alpha)$ as defined in 3.30, with the distinguished bases $\mathrm{E}_A(\alpha)$.

Clearly, $\mathbf{V}_A(\mathbb{R})$ is a substructure of $\mathbf{V}_A$.

The morphisms in the category $\mathcal{V}^*_{fin}(\mathbb{C})$ are given by the same maps as in $\mathcal{V}_{fin}(\mathbb{C})$ restricted to the substructures $\mathbf{V}_A(\mathbb{R})$.

We consider the functor of taking the real co-ordinate part

$$F_{\mathcal{V}} : \mathbf{V}_A \mapsto \mathbf{V}_A(\mathbb{R}); \ \ \mathcal{V}_{fin}(\mathbb{C}) \to \mathcal{V}_{fin}(\mathbb{C})$$

5.4 Lemma. *$F_{\mathcal{V}}$ is an isomorphism of the categories $\mathcal{V}_{fin}(\mathbb{C})$ and $\mathcal{V}^*_{fin}(\mathbb{C})$.*
   Proof. By 5.2 and 3.63 it suffices to prove that the functor

$$A \mapsto \mathbf{V}_A(\mathbb{R})$$

is invertible.

Suppose not. Then there is a non-zero operator $X \in A$ which annihilates all the modules $\mathbf{V}_A(\alpha)$ with $\alpha \in \mathrm{Sp}_A(\mathbb{R})$.

Note that the set

$$\mathrm{Null}(X) = \{\alpha \in \mathrm{Sp}_A : \ \forall v \in \mathbf{V}_A(\alpha) \, Xv = 0\}$$

is definable in the Zariski structure V_A. Moreover, it is Zariski closed since

$$\mathrm{Null}(X) = \{\alpha \in \mathrm{Sp}_A : \exists u, v \, \langle u^N, v^N \rangle = \alpha \ \& \ \bigwedge_{k=0}^{N-1} \mathbf{u}(uq^k, v) \in \mathrm{E}_A^U \ \& \ X\mathbf{u}(uq^k, v) = 0\}$$

is given by a core-formula (see [16]).

Our assumption implies that $\mathrm{Sp}_A(\mathbb{R}) \subseteq \mathrm{Null}(X)$. But the real 2-torus $\mathrm{Sp}_A(\mathbb{R})$ is Zariski dense in the complex 2-torus $\mathrm{Sp}_A = \mathbb{C}^\times \times \mathbb{C}^\times$, so

$$\mathrm{Null}(X) = \mathrm{Sp}_A$$

that is X annihilates all the irreducible modules. Then $X = 0$. The contradiction. $\square$

5.5 Model-theoretic commentary. Note that $\mathrm{Sp}_A(\mathbb{R})$ is a real 2-torus and as such can be defined in the field $\mathbb{R}$. Also each Hilbert space $\mathbf{V}_A(\alpha)$ with the distinguished bases $\mathrm{E}_A(\alpha)$ is definable in $\mathbb{R}$ using parameters (needed to fix the frame, the family of distinguished bases). It can be shown that the bundle $\mathbf{V}_A(\mathbb{R})$ is not interpretable in $\mathbb{R}$. However, one can see that $\mathbf{V}_A(\mathbb{R})$ is prime over its definable substructure $\mathrm{Sp}_A(\mathbb{R})$.

From applications points of view it is category $\mathcal{A}^*_{fin}(\mathbb{R})$ and objects of the form $\mathbf{V}_A(\mathbb{R})$ rather than $\mathcal{A}_{fin}(\mathbb{C})$ and $\mathbf{V}_A(\mathbb{C})$ which are of interest. However, there is a certain gain in viewing the real structures as the real part of the complex ones, and conversely, to see the complex structure as "complexifications" of the real ones. The property exhibited by our real structures is that their complexifications are nice Zariski geometries.

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6 Regular unitary transformations.

In this section we work with a (temporarily) fixed Weyl algebra $A = A(a,b)$ generated by $\hat{U} = U^a$ and $\hat{V} = V^b$.

We denote, for $\alpha \in \mathrm{Sp}_A$,

$\mathrm{E}_N^U(\alpha) :=$ the set of $\mathbf{u} = \langle e_0, \ldots, e_{N-1} \rangle$, canonical orthonormal $\hat{U}$-bases of $\mathbf{V}_A(\alpha)$, $N = N_A$,

and denote $\mathrm{E}_N^U := \{\{\alpha\} \times \mathrm{E}_N^U(\alpha) : \alpha \in \mathrm{Sp}_A\}$.

6.1 Let $B = \langle S, T \rangle$ and $B' = \langle S', T' \rangle$ be subalgebras of A, with S, T and S', T' pseudo-unitary generators.

Suppose that

(i) There is an isomorphism

$$\sigma : B \to B'; \quad S^\sigma = S', \ T^\sigma = T'.$$

(ii) There is a Zariski closed relation $\Lambda \subset \mathrm{E}_N^U \times \mathrm{E}_N^U$ defining a finite-to-finite correspondence and there is a Zariski regular map

$$\lambda : \Lambda \to \mathrm{E}_{N_B}^S \times \mathrm{E}_{N_B}^{S'}; \ \langle \alpha, \mathbf{u}, \alpha', \mathbf{u}' \rangle \mapsto \langle \beta, \mathbf{s}, \beta', \mathbf{s}' \rangle, \ \alpha = \pi_{BA}(\beta), \alpha' = \pi_{B'A}(\beta');$$

λ is surjective on the co-ordinates $\langle \beta, \mathbf{s} \rangle \in \mathrm{E}_{N_B}^S$.

This data for each choice of $\langle \alpha, \mathbf{u}, \alpha', \mathbf{u}' \rangle$ determines the unique $\mathbb{F}$-linear transformation

$$L_{\alpha,\alpha'}^{\mathbf{u},\mathbf{u}'} : \mathbf{V}_B(\beta) \to \mathbf{V}_{B'}(\beta')$$

which sends $\mathbf{s}$ to $\mathbf{s}'$.

We assume that the transformation preserves inner product and the action, that is: for any $e \in \mathbf{V}_B(\beta)$, $X \in B$ and $g, f \in \mathbf{s}$

$$\langle g|f \rangle = \langle L_{\alpha,\alpha'}^{\mathbf{u},\mathbf{u}'}(g)|L_{\alpha,\alpha'}^{\mathbf{u},\mathbf{u}'}(f) \rangle \text{ and } L_{\alpha,\alpha'}^{\mathbf{u},\mathbf{u}'}(Xe) = X^\sigma L_{\alpha,\alpha'}^{\mathbf{u},\mathbf{u}'}(e). \qquad (17)$$

(iii) There is a positive integer M such that, given that

$$\langle \alpha, \mathbf{u}_1, \alpha', \mathbf{u}_1' \rangle \mapsto \langle \beta, \mathbf{s}_1, \beta', \mathbf{s}_1' \rangle \text{ and } \langle \alpha, \mathbf{u}_2, \alpha', \mathbf{u}_2' \rangle \mapsto \langle \beta, \mathbf{s}_2, \beta', \mathbf{s}_2' \rangle,$$

there is a $\zeta \in \mathbb{F}$, $\zeta^M = 1$ and

$$L_{\alpha,\alpha'}^{\mathbf{u}_1,\mathbf{u}_1'}(\mathbf{b}) = \zeta \cdot L_{\alpha,\alpha'}^{\mathbf{u}_2,\mathbf{u}_2'}(\mathbf{b})$$

for every $\mathbf{b} \in \mathbf{V}_B(\beta)$.

(iv) For $\langle \alpha, \mathbf{u}, \alpha', \mathbf{u}' \rangle \in \Lambda$, if $\alpha = \mathbf{1}$ (that is $\mathbf{V}_A(\alpha)$ is a principal algebraic-Hilbert space) then $\alpha' = \alpha$.

We call the family

$$L = \{ L_{\alpha,\alpha'}^{\mathbf{u},\mathbf{u}'} : \; \mathbf{V}_B(\beta) \to \mathbf{V}_{B'}(\beta'), \; \langle \alpha, \mathbf{u}, \alpha', \mathbf{u}' \rangle \in \Lambda \}$$

a **regular unitary transformation of $\mathbf{V}_A$ (associated with B and σ)** and write

$$L : \mathbf{V}_A \to \mathbf{V}_A.$$

We call $\mathbf{V}_B$ the **domain** and $\mathbf{V}_{B'}$ the **range** of L.

6.2 Commentary. Later we will apply certain meaning of a limit and assume that the index B in A is small relative to the dimension of the algebraic-Hilbert space $\mathbf{V}_A(\alpha)$. This will allow a meaning to the notion that "$\mathbf{V}_B(\beta)$ is dense in $\mathbf{V}_A(\alpha)$" and that "L can be continuously extended from $\mathbf{V}_B(\beta)$ to $\mathbf{V}_A(\alpha)$".

In the second half of the paper we will only work with principal modules $\mathbf{V}_A(\mathbf{1})$ and its submodules $\mathbf{V}_B(\mathbf{1})$ and $\mathbf{V}_{B'}(\mathbf{1})$.

6.3 Lemma. (i) *(17) can be extended to any $g, f \in \mathrm{span}_{\mathbb{F}_0}(\mathbf{s})$.*
(ii) *β' is determined by β.*
Proof. (i) is obvious by definition.
(ii) (17) implies that for $s, t \in \mathbb{F}_0$ we have

$$L_{\alpha,\alpha'}^{\mathbf{u},\mathbf{u}'} : \mathbf{s}(s,t) \mapsto \mathbf{s}'(s,t'),$$

where $(t')^{N_B} = t^{N_B}$ (and the same eigenvalues s on both sides). But $\langle s^{N_B}, t^{N_B} \rangle$ is the invariant of the B-module, that is determines β, and $\langle s^{N_B}, (t')^{N_B} \rangle$ is the invariant of the B'-module, tha is determines β'.

It follows that the set of $\beta \in \mathrm{Sp}_B$ for which Λ and λ determine β' uniquely is Zariski dense in Sp_B. Hence this is true for all β. $\square$

6.4 Lemma *Suppose $C \subset B$, a subalgebra generated by a pair of pesudo-unitaries, $C = \langle X, Y \rangle$. Then σ of 6.1(i) acts on C and we get $C' = C^\sigma = \langle X', Y' \rangle$. Let σ_C be the restriction of σ to C.*

We claim that L as given above is also a regular unitary transformation of $\mathbf{V}_A$ associated with C and σ_C.

Proof. We start with the same Λ and need to construct a

$$\lambda_C : \Lambda \to \mathrm{E}_{N_C}^X \times \mathrm{E}_{N_C}^{X'}; \ \langle \alpha, \mathbf{u}, \alpha', \mathbf{u}' \rangle \mapsto \langle \gamma, \mathbf{x}, \gamma', \mathbf{x}' \rangle, \ \alpha = \pi_{CA}(\gamma), \alpha' = \pi_{C'A}(\gamma');$$

We do it by constructing a Zariski regular map

$$\langle \beta, \mathbf{s}, \beta', \mathbf{s}' \rangle \mapsto \langle \gamma, \mathbf{x}, \gamma', \mathbf{x}' \rangle \tag{18}$$

Recall that for $\mathbf{V}_B(\beta)$ splits into the direct sum of irreducible C-submodules. Let γ be the invariant for one of them, $\mathbf{V}_C(\gamma) \subset \mathbf{V}_B(\beta)$. Let $\mathbf{x}$ be a canonical X-basis of $\mathbf{V}_C(\gamma)$. Then there is a $N_B \times N_C$-matrix M over $\mathbb{F}_0$ such that $\mathbf{x} = \mathbf{s}$M. Since any two irreducible C-submodules are conjugated by an automorphism of $\mathbf{V}_B(\beta)$, as we ran through all canonical S-bases $\mathbf{s}$ of $\mathbf{V}_B(\beta)$, the N_C-tuple $\mathbf{x} = \mathbf{s}$M will run through all canonical X-basis for all irreducible C-submodules. Moreover, by definition L will take $\mathbf{s}$M to $\mathbf{s}'$M, and once $\mathbf{s}'$ is known the module $\mathbf{V}_C(\gamma')$ and γ' are determined. Thus M determines the right-hand side of (18) for the given β and $\mathbf{s}$. It remains to note that the same matrix M will define $\mathbf{x}$, $\mathbf{x}'$, γ and γ' for any β and $\mathbf{s}$. $\square$

6.5 Lemma. *Let $\bar{\mathbf{V}}_B(\beta)$ and $\bar{\mathbf{V}}_{B'}(\beta')$ stand for the quotients of $\mathbf{V}_B(\beta)$ and $\mathbf{V}_{B'}(\beta')$, respectively, by the equivalence relation*

$$g \approx g' \ iff \ \exists \zeta \ \zeta^M = 1 \ \& \ g' = \zeta \, g.$$

Then the formula

$$L_{\alpha,\alpha'}(\bar{\mathbf{b}}) = \bar{\mathbf{b}}' :\equiv \quad \exists \mathbf{u}, \mathbf{u}' \ \Lambda(\alpha, \mathbf{u}, \alpha', \mathbf{u}') \ \& \ L_{\alpha,\alpha'}^{\mathbf{u},\mathbf{u}'}(\mathbf{b}) = \mathbf{b}' \tag{19}$$

determines for every $\beta \in \mathrm{Sp}_B$ a well-defined map

$$\bar{\mathbf{V}}_B(\beta) \to \bar{\mathbf{V}}_{B'}(\beta')$$

for a corresponding unique β'.
 Proof. Immediate from (iii) of the definition.$\square$

6.6 Lemma. *A regular unitary transformation sends a canonical S-basis (with respect to T) to a canonical S'-basis (with respect to T') preserving eigenvalues:*

$$L_{\alpha,\alpha'}^{\mathbf{u},\mathbf{u}'} : \mathbf{s}(sq^{ck}, t) \mapsto \mathbf{s}'(sq^{ck}, t), \ k = 0, \ldots, N_B - 1, \tag{20}$$

were c is the index of B in A and $N_B = \frac{N}{c}$.

Proof. Let $\mathbf{s}(sq^{ck}, t)$ be in an S-basis with respect to T, that is satisfying (10). Let $\mathbf{s}'_k = L^{\mathbf{u},\mathbf{u}'}_{\alpha,\alpha'}(\mathbf{s}(sq^{ck}, t))$. Then by (ii) and (i) of the definition $\mathbf{s}'_k$ $S'\mathbf{s}'_k = sq^{ck}\mathbf{s}'_k$, that is $\mathbf{s}'_k = sq^{ck}\mathbf{s}'(sq^{ck}, t)$ for the k-th element of a canonical S'-basis $\{\mathbf{s}'(sq^{cm}, t) : m = 0, 1, \ldots, N_B - 1\}$. Using (ii) and (10) we get

$$L\left(\mathbf{s}(sq^{c(k-1)}, t)\right) = L\left(t^{-1}T\mathbf{s}(sq^{ck}, t)\right) = t^{-1}T'\mathbf{s}'(sq^{ck}, t) = \mathbf{s}'(sq^{c(k-1)}, t),$$

which extents the equality to all k and proves (20). $\square$

6.7 Let L be regular on $\mathbf{V}_A$. Let $\{\mathbf{s}(sq^{ck}, t), 0 \le k < N_B\}$ and $\{\mathbf{s}'(sq^{ck}, t), 0 \le k < N_B\}$ be as in (20).

We have

$$\mathbf{s}(sq^{ck}, t) = \sum_{n=0}^{N-1} \lambda_{n,k} \cdot \mathbf{u}_n$$

and

$$\mathbf{s}'(sq^{ck}, t) = \sum_{n=0}^{N-1} \lambda'_{n,k} \cdot \mathbf{u}'_n$$

for some coefficients $\lambda_{n,k}, \lambda'_{n,k} \in \mathbb{F}_0$ which are determined by λ of 6.1, and since the latter is Zariski regular we have that $\lambda_{n,k}, \lambda'_{n,k} \in \mathbb{F}_0$ and do not depend on $\alpha, \alpha', \mathbf{u}, \mathbf{u}'$.

Now for arbitrary $\mathbf{b} \in \mathbf{V}_B(\beta)$ we have

$$\mathbf{b} = \sum_{k=0}^{N_B-1} b_k \mathbf{s}(sq^{ck}, t)$$

for some $b_1, \ldots, b_k \in \mathbb{F}$, and hence

$$L^{\mathbf{u},\mathbf{u}'}_{\alpha,\alpha'}(\mathbf{b}) = \mathbf{b}' = \sum_{k=0}^{N_B-1} b_k \mathbf{s}'(sq^{ck}, t).$$

Or

$$\mathbf{b} = \sum_{k=0}^{N_B-1} \sum_{n=0}^{N-1} \lambda_{n,k} \cdot b_k \cdot \mathbf{u}_k$$

and

$$\mathbf{b}' = \sum_{k=0}^{N_B-1} \sum_{n=0}^{N-1} \lambda'_{n,k} \cdot b_k \cdot \mathbf{u}'_k$$

Thus

$$L_{\alpha,\alpha'}^{\mathbf{u},\mathbf{u}'}(\mathbf{b}) = \mathbf{b}' \Leftrightarrow \exists\, b_1, \ldots, b_k \in \mathbb{F}$$
$$\mathbf{b} = \sum_{k=0}^{N_B-1} \sum_{n=0}^{N-1} \lambda_{n,k} \cdot b_k \cdot \mathbf{u}_k \,\,\&\,\, \mathbf{b}' = \sum_{k=0}^{N_B-1} \sum_{n=0}^{N-1} \lambda'_{n,k} \cdot b_k \cdot \mathbf{u}'_k \tag{21}$$

Finally, $L_{\alpha,\alpha'}(\bar{\mathbf{b}}) = \bar{\mathbf{b}}'$ of (19) becomes

$$\exists\, \mathbf{u} \in \mathrm{E}_N^U(\alpha) \,\, \exists\, \mathbf{u}' \in \mathrm{E}_N^U(\alpha') \,\, \exists\, b_1, \ldots, b_k \in \mathbb{F} :$$
$$\mathbf{b} = \sum_{k=0}^{N_B-1} \sum_{n=0}^{N-1} \lambda_{n,k} \cdot b_k \cdot \mathbf{u}_k \,\,\&\,\, \mathbf{b}' = \sum_{k=0}^{N_B-1} \sum_{n=0}^{N-1} \lambda'_{n,k} \cdot b_k \cdot \mathbf{u}'_k \tag{22}$$

6.8 Theorem. *Given a regular unitary transformation L,*
(i) the formula $L_{\alpha,\alpha'}(\bar{\mathbf{b}}) = \bar{\mathbf{b}}'$ defines a family of maps

$$L_{\alpha,\alpha'} : \bar{\mathbf{V}}_B(\beta) \to \bar{\mathbf{V}}_{B'}(\beta'), \quad \bar{\mathbf{V}}_B(\beta) \subseteq \mathbf{V}_A(\alpha), \bar{\mathbf{V}}_B(\beta') \subseteq \mathbf{V}_A(\alpha').$$

(ii) The family L is Zariski closed, that is the relation (between $\bar{\mathbf{b}}, \bar{\mathbf{b}}', \alpha, \alpha', \beta, \beta'$) given by (22) is Zariski closed.
(iii) There is an inverse regular transformation L^{-1} associated to B' and σ^{-1}. It defines, in notations of (i) above, the family of maps

$$L_{\alpha',\alpha}^{-1} : \bar{\mathbf{V}}_{B'}(\beta') \to \bar{\mathbf{V}}_B(\beta)$$

such that the composition

$$L_{\alpha',\alpha}^{-1} \circ L_{\alpha,\alpha'} : \bar{\mathbf{V}}_B(\beta) \to \bar{\mathbf{V}}_B(\beta)$$

defines a family of isomorphisms.
(iv) Let $L^{(1)}$ and $L^{(2)}$ be both regular unitary transformations on $\mathbf{V}_A$ associated with B_1, σ_1 and B_2, σ_2 respectively and $C \subset B_1$ be a subalgebra generated by pseudo-unitaries such that $C^{\sigma_1} \subseteq B_2$. Let σ_C be the restriction of $\sigma_2 \circ \sigma_1$ to C.
Then the composition $L^{(2)} \circ L^{(1)}$ given by the formula

$$\exists\, \alpha' \,\, \exists\, \mathbf{b}' \,\, L_{\alpha,\alpha'}^{(1)}(\bar{\mathbf{b}}) = \bar{\mathbf{b}}' \,\,\&\,\, L_{\alpha',\alpha''}^{(2)}(\bar{\mathbf{b}}') = \bar{\mathbf{b}}''$$

defines a regular unitary transformation associated to C and σ_C..
Proof. (i) is just 6.5.
(ii) follows from the fact that (19) in the form (22) is a canonical form of a Zariski closed relation (a *core formula*) as defined in [16], section 3.

(iii) For L^{-1} take Λ^{-1} (inverting the order of 4-tuples) in place of Λ. Define λ_{-1} for L^{-1} in place of λ accordingly:

$$\lambda_{-1} : \langle \alpha', \mathbf{u}', \alpha, \mathbf{u} \rangle \mapsto \langle \beta', \mathbf{s}', \beta, \mathbf{s} \rangle.$$

By definition this satisfies the required properties.

(iv) Follows by definition from 6.4. $\square$

6.9 Lemma. *To every regular $L : \mathbf{V}_A \to \mathbf{V}_A$ one can associate a rational 2×2-matrix $g_L = (g_{lk})$ of determinant 1 and an automorphism of the rational Heisenberg groups $\mathrm{H}(\mathbb{Q}, \mathbb{Q})$.*

Proof. Given L, by definition we are given $S, T, S', T' \in \mathrm{H}(a, b)$ and the isomorphism σ between subgroups of $\mathrm{H}(a, b)$ generated by S, T and S', T'. This isomorphism corresponds to an isomorphism of Lie subalgebras of the Lie algebra $\langle P, Q \rangle$ over the rationals with generators corresponding to S, T and S', T'.

This can be naturally extended to the automorphism of the Lie algebra $\langle P, Q \rangle$ which induces an automorphism

$$\check{\sigma} : \mathrm{H}(\mathbb{Q}, \mathbb{Q}) \to \mathrm{H}(\mathbb{Q}, \mathbb{Q})$$

of the rational Heisenberg groups of rank 2. We thus will have

$$\check{\sigma} : \begin{array}{c} S \mapsto S' \\ T \mapsto T' \end{array} ; \begin{array}{c} \hat{U} \mapsto c_U \hat{U}^{g_{11}} \hat{V}^{g_{12}} \\ \hat{V} \mapsto c_V \hat{U}^{g_{21}} \hat{V}^{g_{22}} \end{array}$$

for some rational matrix (g_{kl}) and $c_S, c_T, c_U, c_V \in Z(\mathrm{H}(\mathbb{Q}, \mathbb{Q}))$.

The fact that $\check{\sigma}$ is an automorphism implies that $\det(g_{kl}) = 1$ (see a remark in 2.9). $\square$

6.10 Proposition. *To every regular $L : \mathbf{V}_A \to \mathbf{V}_A$ one can associate an automorphism $\check{\sigma}_L$ of the category $\mathcal{A}_{fin}^{(1)}$ such that for any non-zero rational r, s,*

$$\check{\sigma}_L : \langle \hat{U}^r, \hat{V}^s \rangle \to \langle q^{-\frac{r(r-1)}{2}} c_U \hat{U}^{rg_{11}} \hat{V}^{rg_{12}}, q^{-\frac{s(s-1)}{2}} c_V \hat{U}^{sg_{21}} \hat{V}^{sg_{22}} \rangle.$$

The same determines an automorphism of the algebra

$$\mathcal{A}^{(1)} = \bigcup_{A \in \mathcal{A}_{fin}^{(1)}} A.$$

Proof. This follows by 6.9. The formula for $\check{\sigma}$ is obtained by using the Campbell - Hausdorff formula for the Lie exponentiation. $\square$

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6.11 Problem. Prove that, conversely to 6.9, given A, any matrix $(g_{kl}) \in$ $\mathrm{SL}(2, \mathbb{Q})$ is associated with a regular unitary transformation L of $\mathbf{V}_A$, provided that A is of the form $A = \langle U^{\frac{1}{\mu}}, V^{\frac{1}{\nu}} \rangle$, $\mu, \nu \in \mathbb{Z}$ and both divisible by the product of denumerators of rational numbers g_{kl}.

Note that by 6.13 and 6.19 below it suffices to prove that under the above assumptions the composition of regular unitary transformation is again a regular unitary transformation.

6.12 Let q, N and the invariants of A-modules be determined in agreement with 3.16. In particular, $\alpha = \langle u^N, v^N \rangle$ and, conversely, u, v are defined by α up to roots of unity of order N. Denote $\alpha' = \langle v^{-N}, u^{-N} \rangle$.

The Fourier transform $\Phi_A = \Phi : \mathbf{V}_A(\alpha) \to \mathbf{V}_A(\alpha')$ is defined on the A-modules as

$$\Phi : \mathbf{u}(uq^m, v) \mapsto \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} q^{-mk} \mathbf{u}(v^{-1}q^k, u^{-1}), \tag{23}$$

or, according to 3.18,

$$\Phi : \mathbf{u}(uq^m, v) \mapsto \mathbf{v}(u^{-1}q^m, v^{-1}), \tag{24}$$

and the domain and range of Φ is the whole of $\mathbf{V}_A(\alpha)$, that is $B = A = B'$, $\beta = \alpha$, $\beta' = \alpha'$.

Applying Φ twice we get

$$\Phi^2 : \mathbf{u}(uq^m, v) \mapsto \mathbf{u}(uq^{-m}, v).$$

Note that Φ restricted to the principal A-module is a transformation of the module

$$\Phi : \mathbf{V}_A(\mathbf{1}) \to \mathbf{V}_A(\mathbf{1}).$$

6.13 Proposition. Φ *is a regular unitary transformation associated with the matrix*

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and corresponds to the automorphism of the category

$$\hat{\sigma} : \begin{array}{ll} U^r \mapsto & V^r \\ V^s \mapsto & U^{-s} \end{array}.$$

Proof. (i),(ii) and (iv) of the definition 6.1 are obvious. In order to prove (iii) we consider the effect of replacing u by uq and v by vq in the formula (23). Applying these operations several times one gets an arbitrary case of (iii).

Substituting $u := uq$ in (23) and using 3.17 we get $\mathbf{u}(uq^{m+1}, v)$ on the left of $\mapsto$, and

$$\frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} q^{-mk}\mathbf{u}(v^{-1}q^k, u^{-1}q^{-1}) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} q^{-mk}q^{-k}\xi\mathbf{u}(v^{-1}q^k, u^{-1}) = \xi\mathbf{v}(u^{-1}q^{m+1}, v^{-1})$$

on the right, for some scalar ξ. According to 3.26 $\xi^N = 1$. This satisfies (iii).

Substituting $v := vq$ we get $\mathbf{u}(uq^m, vq) = \xi q^m\mathbf{u}(uq^m, v)$ on the left, and

$$\frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} q^{-mk}\mathbf{u}(v^{-1}q^{k-1}, u^{-1}) = q^{-m}\mathbf{v}(u^{-1}q^m, v^{-1})$$

on the right, which again agrees with the requirement (iv). $\square$

6.14 Gaussian transformation. We introduce a special transformation $\mathbf{G}_A = \mathbf{G}$ of the bundle $\mathbf{V}_A$. We set $q^{\frac{1}{2}}$ to be a root of order 2 of q. We assume $\alpha = \langle u^N, v^N \rangle$ and $\alpha' = \langle (uv)^N, v^N \rangle$.

Set

$$\mathbf{G} : \mathbf{u}(uq^m, v) \mapsto \frac{\mathfrak{a}}{\sqrt{N}} \sum_{l=0}^{N-1} q^{\frac{(l-m)^2}{2}}\mathbf{u}(uvq^l, v), \quad \mathbf{V}_A(\alpha) \to \mathbf{V}_A(\alpha') \qquad (25)$$

where $\mathfrak{a} \in \mathbb{F}_0$ is a constant of modulus 1 to be determined later and $\mathbf{u}(uq^m, v)$ and $\mathbf{u}(uvq^l, v)$ are assumed to be from E_A^U. By definition $\beta = \alpha$ and $\beta' = \alpha'$.

6.15 Lemma. G *is unitary. Moreover,*

$$\{\mathbf{G}\,\mathbf{u}(uq^m, v) : m = 0, 1, \ldots, N-1\}$$

is a canonical S-basis, for

$$S := q^{-\frac{1}{2}}\check{U}\check{V}^{-1}.$$

More precisely, for $\mathbf{s}(uq^m, v) := \mathbf{G}\,\mathbf{u}(uq^m, v)$ *we have*

$$\begin{aligned} S : \mathbf{s}(uq^m, v) &\mapsto uq^m \mathbf{s}(uq^m, v) \\ V : \mathbf{s}(uq^m, v) &\mapsto v\mathbf{s}(uq^{m-1}, v) \end{aligned} \tag{26}$$

and

$$S\mathbf{Gu}(uq^m, v) = uq^m \mathbf{Gu}(uq^m, v) = \mathbf{G}\check{U}\mathbf{u}(uq^m, v). \tag{27}$$

Proof. (26) is by direct application of S to (25). (27) is a corollary of (26).

Since S is unitary by construction, the basis is orthogonal, so Lemma follows. $\square$

6.16 We can calculate the action of $\mathbf{G}$ on the basis $\{\mathbf{v}(vq^n, u) : n = 0, 1, \ldots, N - 1\}$.

Substituting (25) in formula (3.18) for $\mathbf{v}(vq^n, u)$ we get

$$\mathbf{Gv}(vq^n, u) = \frac{1}{\sqrt{N}} \sum_{m=0}^{N-1} q^{nm} \mathbf{Gu}(uq^m, v) = \frac{\mathfrak{a}}{N} \sum_{m=0}^{N-1} q^{nm} \sum_{l=0}^{N-1} q^{\frac{(l-m)^2}{2}} \mathbf{u}(uvq^l, v) =$$

$$= \frac{\mathfrak{a}}{N} \sum_{l=0}^{N-1} \left\{ \sum_{m=0}^{N-1} q^{\frac{(l-m)^2}{2} + nm} \right\} \mathbf{u}(uvq^l, v)$$

Note that $(l-m)^2 + 2nm = [-n^2 + 2nl] + (m+n-l)^2$. Hence we may continue the above

$$= \frac{\mathfrak{a}}{N} \sum_{l=0}^{N-1} q^{-\frac{n^2}{2} + nl} \left\{ \sum_{m=0}^{N-1} q^{\frac{(m+n-l)^2}{2}} \right\} \mathbf{u}(uvq^l, v) = \frac{\mathfrak{a}}{N} G(N) q^{-\frac{n^2}{2}} \sum_{l=0}^{N-1} q^{nl} \mathbf{u}(uvq^l, v)$$

where

$$G(N) = \sum_{m=0}^{N-1} q^{\frac{(m+n-l)^2}{2}} = \sum_{m=0}^{N-1} q^{\frac{m^2}{2}}$$

is a *quadratic Gauss sum*, see [11]. The last expression by (3.18) can be rewritten as

$$= \frac{\mathfrak{a}}{\sqrt{N}} G(N) q^{-\frac{n^2}{2}} \mathbf{v}(vq^n, uv).$$

We have proven that

$$\mathbf{G} : \mathbf{v}(vq^n, u) \mapsto \frac{\mathfrak{a}}{\sqrt{N}} G(N) q^{-\frac{n^2}{2}} \mathbf{v}(vq^n, uv),$$

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in particular, for $v = 1$ the $\mathbf{v}(vq^n, u)$ are eigenvectors of $\mathbf{G}$ with eigenvalues $\frac{\mathfrak{a}}{\sqrt{N}} G(N) q^{-\frac{n^2}{2}}$.

6.17 We set

$$\mathfrak{a} = \frac{\sqrt{N}}{G(N)}.$$

By 6.16 this is equivalent to the assumption that

$$\mathbf{G}\mathbf{v}(vq^n, u) = q^{-\frac{n^2}{2}} \mathbf{v}(vq^n, uv) \tag{28}$$

Moreover, reversing the calculation in 6.16 we see that (28) is equivalent to (25).

6.18 Proposition. *Assume that N is even. Then*

$$\mathfrak{a} = \mathrm{e}^{-i\frac{\pi}{4}}.$$

Proof. Recall the Gauss formula, for integers a and L,

$$\sum_{n=0}^{L-1} \mathrm{e}^{\pi i \frac{a}{L} n^2} = \sqrt{\left|\frac{L}{a}\right|} \cdot \mathrm{e}^{\pi i \frac{|aL|}{4aL}} \cdot \sum_{n=0}^{a-1} \mathrm{e}^{\pi i \frac{L}{a} n^2}, \tag{29}$$

see [11]. The statement follows, if we take $a = 1$ and $L = N$ even. $\square$

6.19 A general form of Gaussian transformation more useful in applications is given by the associated matrix defined in 6.9, of the form

$$\begin{pmatrix} 1 & -\frac{b}{d} \\ 0 & 1 \end{pmatrix}$$

for some positive integers b and d.

This corresponds to

$$B = \langle \hat{U}^d, \hat{V}^b \rangle = B' = \langle \hat{U}^d \hat{V}^{-b}, \hat{V}^b \rangle.$$

$$\hat{\sigma} : \quad \begin{array}{l} \hat{U}^d \to \hat{U}^d \hat{V}^{-b} \\ \hat{V}^b \to \quad \hat{V}^b \end{array}$$

6.20 Proposition. $\mathbf{G}_B$ *is a regular unitary transformation.*

Proof. For simplicity (but without loss of generality) we consider the example with $b = 1$.

We assume $A = \langle \hat{U}, \hat{V} \rangle$ and correspondingly

$$B = \langle \hat{U}^d, \hat{V} \rangle = B' = \langle \hat{U}^d \hat{V}^{-1}, \hat{V} \rangle.$$

Let $\beta \in \mathrm{Sp}_B$ and α be an element of Sp_A such that $\mathbf{V}_B(\beta) \subseteq \mathbf{V}_A(\alpha)$. We may identify α with a pair $\langle u^N, v^N \rangle \in \mathbb{F}^\times \times \mathbb{F}^\times$, and according to 3.46-3.47, $\beta = \langle u^N, v^{\frac{N}{d}} \rangle$, so we have exactly d distinct B-submodules in the A-module.

A canonical $\hat{U}^d$ basis of $\mathbf{V}_B(\beta)$ can be represented in the form

$$\{\mathbf{u}^{d,1}(u^d q^{dm}, v) : m = 0, \ldots, \frac{N}{d} - 1\}$$

defined by the formula (12). In this notation the definition (25) becomes

$$\mathbf{G}_B \mathbf{u}^{d,1}(u^d q^{dm}, v) = \mathfrak{a}\sqrt{\frac{d}{N}} \sum_{l=0}^{\frac{N}{d}-1} q^{d\frac{(l-m)^2}{2}} \mathbf{u}^{d,1}(u^d v q^{dl}, v) =: \mathbf{s}(u^d q^{dm}, v) \quad (30)$$

where $\mathbf{s}(u^d q^{dm}, v)$ is an element of a canonical $\hat{U}^d \hat{V}^{-1}$-basis as described in (26).

So, the formula (30) defines, for the given choice of parameters u and v, the map

$$\mathbf{V}_B(\beta) \to \mathbf{V}_{B'}(\beta'), \quad \beta' = \beta$$

between the submodules of $\mathbf{V}_A(\alpha)$. This verifies conditions (i),(ii) and (iii) of the definition 6.1.

To check (iv) note that $\beta(u,v) = \beta(u',v')$ if and only if $u' = uq^k$ and $v' = vq^{dn}$ for some $k, n \in \mathbb{Z}$.

Suppose $u = u'$ and consider the substitution $v' = vq^{dn}$ for v in (30). By 3.17 $\mathbf{u}^{d,1}(u^d q^{dm}, vq^{dn}) = c_1 q^{dnm} \mathbf{u}^{d,1}(u^d q^{dm}, v)$ and $\mathbf{u}^{d,1}(u^d v q^{dl}, vq^{dn}) = c_2 q^{dnl} \mathbf{u}^{d,1}(u^d v q^{dl}, v)$ with c_1, c_2 roots of unity of order N (since the choice of the bases is within E_B). Write $\mathbf{G}'_B$ for the transformation defined by $\Lambda(\mathbf{b}, \mathbf{b}', u', v')$.

We will have

$$\mathbf{G}'_B \mathbf{u}^{d,1}(u^d q^{dm}, vq^{dn}) := \mathfrak{a}\sqrt{\frac{d}{N}} \sum_{l=0}^{\frac{N}{d}-1} q^{d\frac{(l-m)^2}{2}} \mathbf{u}^{d,1}(u^d v q^{d(l+n)}, vq^{dn})$$

$$= c_2 \mathfrak{a} \sqrt{\frac{d}{N}} \sum_{l=0}^{\frac{N}{d}-1} q^{d\frac{(l-m)^2}{2}} q^{dn(l+n)} \mathbf{u}^{d,1}(u^d v q^{dl}, v)$$

and

$$\mathbf{G}'_B \mathbf{u}^{d,1}(u^d q^{dm}, vq^{dn}) = c_1 q^{dnm} \mathbf{G}'_B \mathbf{u}^{d,1}(u^d q^{dm}, v).$$

Hence

$$\mathbf{G}'_B \mathbf{u}^{d,1}(u^d q^{dm}, v) = c_1^{-1} c_2 \mathfrak{a} \sqrt{\frac{d}{N}} \sum_{l=0}^{\frac{N}{d}-1} q^{d\frac{(l-m)^2}{2}} q^{dn(l+n-m)} \mathbf{u}^{d,1}(u^d v q^{d(l+n)}, v) =$$

$$= c_1^{-1} c_2 q^{\frac{n^2}{2}} \mathfrak{a} \sqrt{\frac{d}{N}} \sum_{l=0}^{\frac{N}{d}-1} q^{d\frac{(l+n-m)^2}{2}} \mathbf{u}^{d,1}(u^d v q^{d(l+n)}, v) = c_1^{-1} c_2 q^{\frac{n^2}{2}} = c_1^{-1} c_2 q^{\frac{n^2}{2}} \mathbf{s}(u^d q^{dm}, v).$$

Set $\zeta = c_1^{-1} c_2 q^{\frac{n^2}{2}}$. Clearly, $\zeta^{2N} = 1$. $\square$

6.21 Remark. If we only require the coefficient ζ to be an element of $\mathbb{F}_0$ of modulus 1, this can be shown by a simpler argument:

Note that the substitution $u := u'$ and $v := v'$ by 3.17 transforms the basis $\{\mathbf{u}^{d,1}(u^d q^{dl}, v) : 0 \le l < \frac{N}{d}\}$ by the application of a transformation of the form $c\gamma$, where γ is a $\mathbb{F}_0$-unitary matrix and c a non-zero constant. It follows that the basis $\{\mathbf{G}_B \mathbf{u}^{d,1}(u^d q^{dl}, v) : 0 \le l < \frac{N}{d}\}$ under the substitution transforms by the application of a $\mathbb{F}_0$-unitary matrix γ' to a basis $\{\mathbf{s}'(u^d q^{dl}, v) : 0 \le l < \frac{N}{d}\}$ satisfying (26). Now 3.17 imply that $\mathbf{s}'(u^d q^{dl}, v) = \zeta \cdot \mathbf{s}(u^d q^{dl}, v)$ for some $\zeta \in \mathbb{F}_0$ of modulus 1. $\square$

6.22 Finally, we consider **diagonal transformations** D_m on $\mathbf{V}_A$ which correspond to the matrices of the form

$$\begin{pmatrix} m & 0 \\ 0 & \frac{1}{m} \end{pmatrix}.$$

We assume that N is divisible by m.

We construct a respective regular unitary transformation by considering the subalgebras $B = \langle \hat{U}, \hat{V}^m \rangle$ and $B' = \langle \hat{U}^m, \hat{V} \rangle$ and the isomorphism

$$\sigma : \hat{U} \mapsto \hat{U}^m, \quad \hat{V}^m \mapsto \hat{V}.$$

A canonical U-base for a B-module is of the form

$$\{\mathbf{u}(uq^{mk}, v) : k = 0, 1 \ldots, \frac{N}{m} - 1\}$$

and that of B'

$$\{\mathbf{u}^m(u^m q^{mk}, v) : k = 0, 1 \ldots, \frac{N}{m} - 1\}$$

where by the definition (12)

$$\mathbf{u}^m(u^m q^{mk}, v^m) = \frac{1}{\sqrt{m}} \sum_{l=0}^{m-1} \mathbf{u}(uq^k \zeta^l, v^m),$$

for ζ a root of 1 of order m, and respectively,

$$\begin{aligned} U^m &: \mathbf{u}^m(u^m q^{mk}, v^m) \mapsto u^m q^{mk} \mathbf{u}^m(uq^{mk}, v^m), \\ V &: \mathbf{u}^m(u^m q^{mk}, v^m) \mapsto v^m \mathbf{u}^m(u^m q^{mk-m}, v) \end{aligned} .$$

In terms of bases the transformation is

$$L_{\alpha,\alpha'}^{u^m,u} : \mathbf{u}(u^m q^{mk}, v) \to \mathbf{u}^m(u^m q^{mk}, v^m)$$

where for $\alpha = \langle u^{mN}, v^N \rangle$ the correspondence Λ determines $\alpha' = \langle u^N, v^{mN} \rangle$.

7 The limit coordinate system

7.1 Consider $E_A(\alpha)$ defined in 3.55 as the union of $E_A^S(\alpha)$, canonical $\langle S, R \rangle$-bases (see 3.38).

We note first that each $E_A^S(\alpha)$ is in a natural way coordinatised by the residue ring $\mathbb{Z}_{/N}$.

The group $\Sigma_A(\alpha) = \langle \mu \rangle$ of definable permutations of a basis $\{ \mathbf{s}(sq^n, r) : n \leq N - 1 \}$ can be realised as a group of 0-defined operators R^n, or equivalently as pairs $\mathbf{s}(sq^k, r) \mapsto \mathbf{s}(sq^{k+n}, r)$ of elements of $E_A^S(\alpha)$, up to the obvious equivalence.

Using parameters s and r the inner product formula

$$\langle \mathbf{s}(sq^{n_1}, r) | \mathbf{r}(rq^{m_1}, s) \rangle = q^{n_1 m_1} \cdot \langle \mathbf{s}(s, r) | \mathbf{r}(r, s) \rangle = \langle \mathbf{s}(sq^{n_2}, r) | \mathbf{r}(rq^{m_2}, s) \rangle$$

allows to recognise the 4-ary relation

$$n_1 m_1 \equiv n_2 m_2 \bmod N.$$

Since we can choose and fix $n_2 = 1$ we obtain the relation $n_1 m_1 = m_2$.

7.2 Here we start the analysis of how the above coordinatisation behaves under *passing to an infinite limit,* which we formalise through considering a nonstandard model of integers and an infinite value of N in it.

Let $^*\mathbb{Z}$ be the non-standard model of the ring of integers and $\mu \in {}^*\mathbb{Z}$ divisible by all standard integers. Let $N = \mu^2 h^{-1}$, for some $h \in \mathbb{Q}$. Let $\bar{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ and $\mathbb{S} = \mathbb{R}/\mathbb{Z}$. We construct a surjective map

$$\mathrm{st}_\mu : {}^*\mathbb{Z}_{/N} \to \bar{\mathbb{R}}.$$

We identify an element $\bar{m}$ of the residue ring $^*\mathbb{Z}_{/N}$ with the unique (non-standard) integer $m \in [-\frac{N}{2}, \frac{N}{2}) \cap {}^*\mathbb{Z}$. Note that $\frac{m}{N}$ and $\frac{m}{\mu}$ are elements of $^*\mathbb{Q}$.

Set

$$\mathrm{st}_\mu : \bar{m} \mapsto \mathrm{st}(\frac{m}{\mu})$$

where st is the standard part map $^*\mathbb{Q} \to \bar{\mathbb{R}}$.

7.3 Lemma.

(i) st_μ is an additive (semi)group homomorphism;

(ii) st_μ *preserves the linear pre-order on* $[-\frac{N}{2}, \frac{N}{2}) \cap {}^*\mathbb{Z}$, *that is*

$$m_1 \leq m_2 \Rightarrow \mathrm{st}_\mu(m_1) \leq \mathrm{st}_\mu(m_2);$$

(iii) *if* $m_1 n_1 = m_2 n_2$ *does hold in* ${}^*\mathbb{Z}_{/N}$ *and the images of* m_1, n_1, m_2 *and* n_2 *are finite then*

$$\mathrm{st}_\mu m_1 \cdot \mathrm{st}_\mu n_1 \equiv \mathrm{st}_\mu m_2 \cdot \mathrm{st}_\mu n_2 \bmod \mathbb{Z}$$

Proof. (i) and (ii) are obvious.

(iii). We have the unique representation $m_1 = a_1\mu + c_1$, $m_2 = a_2\mu + c_2$, $n_1 = b_1\mu + d_1$, $n_1 = b_2\mu + d_2$, and

$$|a_1|, |a_2|, |b_1|, |b_2|, c_1, c_2, d_1, d_2 \in {}^*\mathbb{Z} \cap [0, \mu).$$

The assumption of finiteness implies that $a_1, a_2, b_1, b_2 \in \mathbb{Z}$ (standard). Now

$$m_1 n_1 - m_2 n_2 = (a_1 b_1 - a_2 b_2)\mu^2 + (a_1 d_1 + b_1 c_1 - a_2 b_2 - b_2 c_2)\mu + c_1 d_1 - c_2 d_2$$

Note that $a_1 b_1 - a_2 b_2 \in \mathbb{Z}$, $|a_1 d_1 + b_1 c_1 - a_2 b_2 - b_2 c_2| < k\mu$ for some positive $k \in \mathbb{Z}$ and $|c_1 d_1 - c_2 d_2| < \mu^2$. It follows that

$$m_1 n_1 - m_2 n_2 \equiv 0 \bmod \mu^2 \;\; \text{iff} \;\; m_1 n_1 - m_2 n_2 = (a_1 b_1 - a_2 b_2)\mu^2.$$

Hence the condition in (ii) implies that

$$\frac{m_1}{\mu} \frac{n_1}{\mu} - \frac{m_2}{\mu} \frac{n_2}{\mu} = a_1 b_1 - a_2 b_2 \in \mathbb{Z}.$$

It is now immediate from the definition

$$\mathrm{st}_\mu m_1 \cdot \mathrm{st}_\mu n_1 - \mathrm{st}_\mu m_2 \cdot \mathrm{st}_\mu n_2 = a_1 b_1 - a_2 b_2$$

thus the statement follows. $\square$

For $\alpha, \beta \in {}^*\mathbb{Z}$ let $|\alpha| << |\beta|$ stand for "$\frac{\alpha}{\beta}$ is an infinitesimal". And let $\alpha << \beta$ mean $\alpha < 0$ & $|\beta| << |\alpha|$ $\vee$ $\beta > 0$ & $|\alpha| << |\beta|$.

7.4 Corollary. st_μ *is a homomorphism of the additive ordered group* $\{m \in {}^*\mathbb{Z} : -\frac{\mu^2}{2} << m << \frac{\mu^2}{2}\}$ *onto the additive ordered group* $\mathbb{R}$ *and*

$$m_1 n_1 = m_2 n_2 \Rightarrow \mathrm{e}^{2\pi i x_1 y_1} = \mathrm{e}^{2\pi i x_2 y_2}$$

for $x_1 := \mathrm{st}_\mu m_1$, $y_1 := \mathrm{st}_\mu n_1$, $x_2 := \mathrm{st}_\mu m_2$ *and* $y_2 := \mathrm{st}_\mu n$.

In particular, the image of the partial ring structure on $\{m \in {}^*\mathbb{Z} : 0 \leq m < \mu\}$ *induces via* st_μ *the partial addition and multiplication on the real interval* $[0, 1)$.

7.5 Definition. The **weak ring** structure $\mathbb{R}_w$ on the reals $\mathbb{R}$ is given by the addition $+$ and the ternary relation

$$P^4(x_1, y_1, x_2, y_2) :\equiv \mathrm{e}^{2\pi i x_1 y_1} = \mathrm{e}^{2\pi i x_2 y_2}.$$

The compactified weak ring structure $\bar{\mathbb{R}}_w$ on the compactified reals $\bar{\mathbb{R}} = \mathbb{R}\{-\infty, +\infty\}$ is given by the extension of $+$ to $\bar{\mathbb{R}}$ in the natural way as the ternary relation $S^3(x, y, z)$ (in particular, $S^3(-\infty, +\infty, z)$ holds for any $z\mathbb{R}$) and the extension of P^4 to $\bar{\mathbb{R}}$ defined by the condition

$$\bar{\mathbb{R}}_w \vDash P^4(a_1, b_1, a_2, b_2) \text{ if } \pm\infty \in \{a_1, b_1, a_2, b_2\}.$$

Clearly, one can always consider the 4-ary relation $a_1 b_1 - a_2 b_2 = 0$ on an arbitrary ring R. For a ring with 1 this relation is 0-definably equivalent to the multiplication. In this context any ring R is a weak ring.

7.6 Corollary to 7.4. st_μ *is a surjective homomorphism of the weak ring* ${}^*\mathbb{Z}/N$ *onto the compactified weak ring structure* $\bar{\mathbb{R}}_w$.

7.7 Dirac rescaling. By assuming the (practically) infinite dimensionality of the algebraic-Hilbert spaces we deal with, we encounter an immediate problem of vanishing values of inner product since the absolute value of $\langle e|f\rangle$ in $\mathbf{V}_A(\alpha)$ is proportional to $\frac{1}{\sqrt{N}}$, for $N = n_A$.

In order to remedy this problem we rescale the definition of the inner product and introduce another pairing, **Dirac inner product** $\langle e|f\rangle_{Dir}$. The rescaling depends on the number N as well as on a "parametrisation" by a variable x based on the following data.

Let $B, D \subseteq A$ be two Weyl subalgebras of A with pseudo-unitary generators $B = \langle R, R_\dagger\rangle$ and $D = \langle S, S_\dagger\rangle$.

Suppose also that there is a regular unitary transformation L of the bundle $\mathbf{V}_A(\alpha)$ such that for each $\alpha \in \mathrm{Sp}_A$ the domain of L_α is a B-submodule $\mathbf{V}_B(\alpha)$ and the range of L_α is a D-submodule $\mathbf{V}_D(\alpha)$ of $\mathbf{V}_A(\alpha)$, both of dimension n_B, the domain is spanned by the R-eigenvectors

$$\{\mathbf{r}(rq^{dk}, r_\dagger) : k = 0, 1, \ldots, n_B - 1\}$$

and the range by S-eigenvectors

$$\{\mathbf{s}(sq^{dm}, s_\dagger) : m = 0, 1, \ldots, n_B - 1\}$$

canonical R and S bases of $\mathbf{V}_B(\alpha)$ and $\mathbf{V}_D(\alpha)$ respectively, such that

$$L_\alpha : \mathbf{r}(rq^{dk}, r_\dagger) \mapsto \mathbf{s}(sq^{dk}, \; s_\dagger), \; k = 0, 1, \ldots, n_B - 1. \tag{31}$$

Our next step is to define a rescaling parameter called Δk.

If R and S commute, then $\mathbf{r} = \mathbf{s}$ and we define

$$\Delta k = \mathfrak{c}\sqrt{\frac{d}{N}},$$

where $\mathfrak{c}$ is a fixed parameter to be defined later, independent of A and L.

Otherwise we assume that $RS \neq SR$.

Let $b < N$ be the minimal positive integer with the property

$$RS = q^b SR \text{ or } SR = q^b RS.$$

Let a_R, a_S be the maximal positive integers with the property that

$$R^{\frac{1}{a_R}} \in A \text{ and } S^{\frac{1}{a_S}} \in A.$$

Set

$$\Delta k = \frac{b\mathfrak{c}}{a_R a_S \sqrt{N}}.$$

We can now define the **Dirac delta-function**, which depends on the parametrisation, that is on A and L.

$$\delta(p) = \begin{cases} \frac{1}{\Delta k}, & \text{if } p = 0 \\ 0, & \text{otherwise.} \end{cases}$$

Further set the rescaled product, dependent on the chosen parametrisation:

$$\langle \mathbf{r}(rq^{dk}, r_\dagger) | \mathbf{s}(sq^{dm}, s_\dagger) \rangle_{Dir} = \frac{1}{\Delta k} \langle \mathbf{r}(rq^{dk}, r_\dagger) | \mathbf{s}(sq^{dm}, s_\dagger) \rangle \tag{32}$$

7.8 An example. Let $A = \langle \check{U}, \check{V} \rangle$.

L be the Fourier transform $\Phi : \mathbf{V}_A \to \mathbf{V}_A$, which by definition is a bijection of $\mathbf{V}_A$-modules. That is $B = D = A$.

We respectively will have $R = \check{U} = S_\dagger$, $S = \check{V} = R_\dagger$, and $\langle B, D \rangle = A$.

Hence $\Delta k = \frac{1}{\mathfrak{c}\sqrt{N}}$

$$\langle \mathbf{u}(uq^k, v) | \mathbf{v}(vq^m, u) \rangle_{Dir} = \mathfrak{c} \cdot \sqrt{N} \langle \mathbf{u}(uq^k, v) | \mathbf{v}(vq^m, u) \rangle = \mathfrak{c}q^{km}$$

and does not depend on N.

7.9 A non-example. Consider the transform

$$\Phi_{B(e,f)} : \mathbf{u}^{e,f}(u^e q^{ek}, v^f) \mapsto \mathbf{v}^{f,e}(v^f q^{fk}, u^e).$$

This is not a restriction of Φ_A, so 7.10 is not applicable. Moreover, condition (31) for parametrisation is not satisfied.

7.10 Proposition. *Let e, f be positive integers and*

$$B' = \langle R^e, R_\dagger^f \rangle, \quad D' = \langle S^e, S_\dagger^f \rangle,$$

subalgebras of B and D of 7.7 respectively. Consider the restriction L' of L sending B'-submodules onto D'-submodules

$$L'_\alpha : \mathbf{r}^{e,f}(r^e q^{dek}, r_\dagger^f) \mapsto \mathbf{s}^{e,f}(s^e q^{dek}, \ s_\dagger^f), \ k = 0, f, 2f, \ldots, \frac{N}{def} - f.$$

Then

$$\langle \mathbf{r}^{e,f}(rq^{dek}, r_\dagger^f) | \mathbf{s}^{e,f}(s^e q^{dem}, s_\dagger^f) \rangle_{Dir} = \langle \mathbf{r}(rq^{dk}, r_\dagger) | \mathbf{s}(sq^{dm}, s_\dagger) \rangle_{Dir},$$

where the left-hand side is rescaled with respect to L', and in the case $RS = SR$ both sides of the equality take value equal to $\delta(k - m)$, with respective interpretations of this symbol on the left- and on the right-hand side.

Proof. It is enough to prove the statement in the two cases:

Case 1. $e = 1$.

and

Case 2. $f = 1$.

The relation between the $\mathbf{r}^{e,f}$ and $\mathbf{r}$ (and between $\mathbf{s}^{e,f}$ and $\mathbf{s}$) has been analysed in 3.47.

In the first case, the respective bases of the submodules are just subsets of the initial bases, so nothing changes and we have the required equality.

In the second case, the new bases are different, however calculating with formula (12) we get

$$\langle \mathbf{r}^{e,f}(rq^{dek}, r_\dagger^f) | \mathbf{s}^{e,f}(s^e q^{dem}, s_\dagger^f) \rangle = \langle \mathbf{r}(rq^{dk}, r_\dagger) | \mathbf{s}(sq^{dm}, s_\dagger) \rangle.$$

Now we look at the new Δk. Clearly, $b' = e^2 b$, $a'_R = e a_R$ and $a'_S = e a_S$. Hence Δk remains the same and so the required equality holds. $\square$

7.11 We can now introduce the **Dirac rescaled pairing** (probability measure) just replacing the inner product in 4.7 by the Dirac rescaled inner product.

$$[e|f]_{Dir} := (\frac{1}{\Delta k})^2 \cdot [e|f].$$

7.12 Lemma – definition. *Let $B \subseteq A$, $A, B \in \mathcal{A}_{fin}$ and $C \in O(A)$. Then the commutative subalgebra $C_B = \langle C \cap B, Z(B) \rangle$ (generated by $C \cap B$ and $Z(B)$) is in $O(B)$. Moreover, the map*

$$\pi^*_{BA} : O(A) \to O(B), \quad C \mapsto C_B,$$

is surjective.

7.13 Lemma – definition. *For a pseudo-unitary S define*

$$C(S) := \langle S^a : a \in \mathbb{Q} \rangle,$$

that is the subalgebra of $A^{(1)}$ generated by all elements $S^{\frac{1}{k}}$, $k \in \mathbb{Z}$.

$C(S)$ is a maximal commutative subalgebra of $A^{(1)}$.

This has natural generalisation to maximal commutative subalgebra of $A^{(n)}$.

Proof. Immediate. $\square$

8 Limit objects

Our aim here is to extend the categories $\mathcal{A}^*_{fin}(\mathbb{C})$ and $\mathcal{V}^*_{fin}(\mathbb{C})$ by adding some limit objects. This follows the idea that objects in $\mathcal{A}^*_{fin}(\mathbb{C})$ and $\mathcal{V}^*_{fin}(\mathbb{C})$ *approximate* objects from a larger categories $\mathcal{A}^*(\mathbb{C})$ and $\mathcal{V}^*(\mathbb{C})$ in a functorial

correspondence to the way the rational Weyl algebras approximate (in the sence of normed algebras) subalgebras of the whole Weyl algebra.

The approximation will be carried out in terms of *structural approximation* described in [17], sections 3 and 4.

One particular limit object of $\mathcal{V}^*_{fin}(\mathbb{C})$ will be called the **space of states** and will take the role of the rigged Hilbert space of mathematical physics.

We assume $\mathbb{F}_0 \subset \mathbb{C}$ when $\mathbb{F}_0$ is mentioned and will mostly drop referring to $\mathbb{C}$ below.

8.1 We fix an ultrapower $^*\mathbb{Z}$ (equivalently, a saturated enough model) of the ring of integers.

Fix a non-standard integer $\mu \in {}^*\mathbb{Z}$ which in addition to assumptions in 7.2 we assume to have the **high divisibility property**:

$$\mu \text{ is divisible by any standard } m \in \mathbb{Z} \tag{33}$$

We also fix for the rest of the paper a (standard) positive rational number h. [1]

The definitions and statements of previous sections are applicable in the new context, with 'rational' and 'integer' replaced by more general 'non-standard rational' and 'non-standard integer'. In particular, in the definitions we consider 'non-standard' Heisenberg group $\tilde{\mathrm{H}}(\frac{1}{\mu}, \frac{\mathrm{h}}{\mu})$ consisting of non-standard pseudo-unitary operators generated (in the non-standard sense) by $U^{\frac{1}{\mu}}$ and $V^{\frac{\mathrm{h}}{\mu}}$. More precisely, $\tilde{\mathrm{H}}(\frac{1}{\mu}, \frac{\mathrm{h}}{\mu})$ can be identified with the ultraproduct of the groups $\mathrm{H}(\frac{1}{m}, \frac{\mathrm{h}}{m})$, $m \in \mathbb{N}$, along an ultrafilter $\mathcal{D}_{div}$ which for every positive integer n contains the set

$$\{m \in \mathbb{N} : n | m\}.$$

We use, for a positive $d \in {}^*\mathbb{Z}$ the notation "much less"

$$d \prec \mu$$

with the meaning that $\frac{d}{\mu}$ is an infinitesimal and d divides μ.

---

[1]The theory which follows is also applicable to a non-standard rational $\mathrm{h} \in {}^*\mathbb{Q}_{fin}$ provided the numerator of h is "much less" than μ and divides μ.

8.2 A universal nonstandard-rational Weyl algebra. Define the algebra $\tilde{\mathrm{A}}$ to be the ultraproduct of the algebras $A(\frac{1}{m}, \frac{\mathrm{h}}{m})$, $m \in \mathbb{N}$, along the ultrafilter $\mathcal{D}_{div}$. One may think of $\tilde{\mathrm{A}}$ as being generated, in a pseudo-finite sense, by $U^{\frac{1}{\mu}}$ and $V^{\frac{\mathrm{h}}{\mu}}$. A pseudo-unitary operator W in $\tilde{\mathrm{A}}$ is by definition any element of $\tilde{\mathrm{H}}(\frac{1}{\mu}, \frac{\mathrm{h}}{\mu})$.

8.3 We call a pseudo-unitary $W \in \tilde{\mathrm{A}}$ **finite** if it can be presented in the form

$$W = \mathrm{e}^{\frac{2\pi i \mathrm{h} \nu}{\mu^2}} U^{\frac{\rho}{\mu}} V^{\frac{\mathrm{h}\tau}{\mu}} \tag{34}$$

for nonstandard integers ν, ρ, τ such that $|\nu| < \mu^2$ and $\frac{\rho}{\mu}, \frac{\tau}{\mu} \in {}^*\mathbb{Q}_{fin}$.

Define $\tilde{\mathrm{A}}_{fin}$ to be the subalgebra of $\tilde{\mathrm{A}}$ generated by all the finite pseudo-unitary W.

Note that

$$\mathrm{A} \subset \tilde{\mathrm{A}} \tag{35}$$

and so $\tilde{\mathrm{A}}$ is an upper bound for $\mathcal{A}_{fin}$.

8.4 We set

$$\hbar := 2\pi\mathrm{h} \tag{36}$$

and the constant introduced in 7.7

$$\mathfrak{c} := \sqrt{2\pi\hbar}. \tag{37}$$

8.5 We are interested in the physically meaningful $\tilde{\mathrm{A}}$-module corresponding to $U^{\frac{1}{\mu}}$- and $V^{\frac{\mathrm{h}}{\mu}}$-eigenvalues $u = 1$ and $v = 1$, which is the principal module $\mathbf{V}_{\tilde{\mathrm{A}}}(\mathbf{1})$ (see also 3.34). We abbreviate the latter to $\mathbf{V}_*(\mathbf{1})$.

It follows from definitions that $\mathbf{V}_*(\mathbf{1})$ is the ultraproduct of $\mathbf{V}_A(\mathbf{1})$, $A = \langle U^{\frac{1}{m}}, V^{\frac{\mathrm{h}}{m}} \rangle$ along the ultrafilter $\mathcal{D}_{div}$. In order to understand $\mathbf{V}_*(\mathbf{1})$ we study some construction on the $\mathbf{V}_A(\mathbf{1})$.

8.6 Let N be the denominator of the reduced fraction $\frac{\mathrm{h}}{\mu^2}$. By definitions

$$\mathrm{N} = \dim \mathbf{V}_*(\mathbf{1}).$$

By divisibility of μ we have

$$\mathrm{N} = \frac{\mu^2}{\mathrm{h}} = \frac{2\pi\mu^2}{\hbar}. \tag{38}$$

If not stated otherwise, we denote

$$q = \mathrm{e}^{\frac{2\pi i \mathrm{h}}{\mu^2}} = \mathrm{e}^{\frac{i\hbar}{\mu^2}}.$$

8.7 The global language $L_{\mathbf{V}}^{Glob}$.

We are interested in switching to a language which is, on each $\mathbf{V}_{A(\frac{1}{m},\frac{h}{m})}$, interdefinable with our original language but has the advantage not to refer to the parameter m, (equivalently N), or similar parameters which change with the "size" of $\mathbf{V}_A$.

For each $C \in O(A)$ the language $L_{\mathbf{V}}^{Glob}$ contains the unary predicate E^C, which we also may denote E^S if $C = C(S)$. In each $\mathbf{V}_A(\alpha)$ the interpretation of $e \in \mathrm{E}^C$ is $e \in \mathrm{E}_A^{C_A}(\alpha)$.

The unitary predicate E is interpreted in each $\mathbf{V}_A$ as

$$\mathrm{E} \equiv \bigvee_{C \in O(A)} \mathrm{E}^C.$$

We are going to "forget" the names of pseudo-unitary predicates such as $U^{\frac{1}{m}}$ and $V^{\frac{h}{m}}$ and instead introduce to $L_{\mathbf{V}}^{Glob}$ names for pseudo-selfadjoint operators $R = R_W$, for each $W \in \tilde{\mathrm{A}}_{fin}$, or more generally of the form

$$W = \mathrm{e}^{\frac{2\pi i h n}{m^2}} U^{\frac{r}{m}} V^{\frac{ht}{m}}$$

where n, m, r and t can be standard or non-standard integers.

Interpret R_W in $\mathbf{V}_A$ as a linear operator

$$R_W = \frac{W - W^{-1}}{2i\{|\frac{r}{m}| + |\frac{t}{m}|\}}. \tag{39}$$

In particular, we use special names for operators

$$\mathrm{Q} = \frac{U^a - U^{-a}}{2ia} \quad \text{and} \quad \mathrm{P} = \frac{V^{ah} - V^{-ah}}{2ia}, \quad a = \frac{1}{m}.$$

Note R_W are in $\tilde{\mathrm{A}}_{fin}$.

Clearly, for each A, the Q is interdefinable with U^a and P with V^{ah}. However the interpretation is given by different formulas depending on A.

A concrete algebraic Hilbert space $\mathbf{V}_A(\alpha)$ is given in terms of global language $L_{\mathbf{V}}^{Glob}$ as the two sorted structure $(\mathrm{E}, \mathbb{F})$ with

- unary predicates E^C, for each $C \in O(A)$;

- pseudo-selfadjoint operators R_C for each $C \in \mathcal{C}_{fin}$;

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- regular unitary transformations L interpreted for each A by the formula in (21) and a fixed associated matrix g_L (see 6.9).

As before $\mathbb{F}$ will stand in $L_{\mathbf{V}}^{Glob}$ for the field sort with addition and multiplication defined on it.

The formal linear equalities $\text{Eq}(x_1, \ldots, x_n, e_1, \ldots, e_n)$:

$$x_1 e_1 + \ldots + x_n e_n = 0, \ x_1, \ldots, x_n \in \mathbb{F}, \ e_1, \ldots, e_n \in \mathrm{E}$$

are expressible in $L_{\mathbf{V}}^{Glob}$, and are interpreted in each $\mathbf{V}_A(\alpha)$ according to the values taken by the linear combination in the $\mathbb{F}$-vector space. Note that one can interpret arbitrary elements of the vector space $\mathrm{V}_A(\alpha)$ in terms of Eq.

Later we will add to the global language predicates encoding the Dirac inner product (see 7.7 and 7.11).

We call these globally defined operators and predicates.

8.8 Below we standardise our notation for the generators of $\tilde{\mathrm{A}}$,

$$\hat{U} := U^{\frac{1}{\mu}}, \ \hat{V} = V^{\frac{\mathrm{h}}{\mu}}.$$

More generally, an important role will be played by pairs of pseudo-unitary generators of $\tilde{\mathrm{A}}$, $\hat{S}$ and $\hat{T}$ conjugated to $\hat{U}$ and $\hat{V}$ by a $\gamma \in \mathrm{Aut}(\mathrm{H}(\frac{1}{\mu}, \frac{\mathrm{h}}{\mu})$:

$$\hat{S} = \hat{U}^\gamma \text{ and } \hat{T} = \hat{V}^\gamma$$

(see 3.14). Canonical $\hat{U}$, $\hat{V}$ and generally $\hat{S}$-bases for $\mathbf{V}_*(\mathbf{1})$ will be denoted $\mathbf{u}(q^k)$, $\mathbf{v}(q^k)$ and $\mathbf{s}(q^k)$, $0 \le k < N$, respectively. (See the definition (10) for $\mathbf{s}$).

We will also often work in subalgebras of $\tilde{\mathrm{A}}$ of the form $B(a, b) = \langle \hat{U}^a, \hat{V}^b \rangle$, with $a, b, c, d \in {}^*\mathbb{Z}$, $0 < a, b < \mu$. The corresponding canonical bases for the principal module $\mathbf{V}_{B(a,b)}(\mathbf{1})$ will be referred to as $\mathbf{u}^{a,b}(q^m)$, $\mathbf{v}^{b,a}(q^m)$, and $\mathbf{s}^{a,b}(q^m)$, respectively.

8.9 We define a pseudo-metric (that is the distance between distinct point can be 0) on the group $\tilde{\mathrm{H}}(\frac{1}{\mu}, \frac{\mathrm{h}}{\mu})_{fin}$ of all the finite pseudo-unitary elements of $\tilde{\mathrm{A}}_{fin}$.

For W and W' in the form (34) define the pseudo-distance,

$$\text{dist}(W, W') := \lim \{|\frac{\rho}{\mu} - \frac{\rho'}{\mu}| + |\frac{\tau}{\mu} - \frac{\tau'}{\mu}| + |\frac{\nu}{\mu^2} - \frac{\nu'}{\mu^2}|\},$$

where ν', ρ' and τ' are the respective parameters of W' in the representation (34) and lim is just the standard part map ${}^*\mathbb{C} \to \mathbb{C}$.

Note that
$$\mathrm{dist}(\hat{U}, 1) = 0 = \mathrm{dist}(\hat{V}, 1). \qquad (40)$$

Let
$$\tilde{\mathrm{A}}_{fin}^0 = \{R_W \in \tilde{\mathrm{A}}_{fin} : \nu = 0 \ \& \ \frac{\rho}{\mu}, \frac{\tau}{\mu} \in {}^*\mathbb{Q}_{fin}\}$$

which we call the **basic** pseudo-selfadjoint operators.

The pseudo-metric on $\tilde{\mathrm{A}}_{fin}^0$ is given by

$$\mathrm{dist}(R_W, R_{W'}) := \mathsf{lim} \ \{|\frac{\rho}{|\rho| + |\tau|} - \frac{\rho'}{|\rho'| + |\tau'|}| + |\frac{\tau}{|\rho| + |\tau|} - \frac{\tau'}{|\rho'| + |\tau'|}|\} + \mathrm{dist}(W, W').$$

8.10 Define
$$W \approx W' \ \text{iff} \ \mathrm{dist}(W, W') = 0,$$

an equivalence relation on the group $\tilde{\mathrm{H}}(\frac{1}{\mu}, \frac{\mathrm{h}}{\mu})_{fin}$. It is immediate from the definition that $\approx$ is invariant under the group operation. Set

$$\mathrm{H}_{\mathbb{R}} = \tilde{\mathrm{H}}(\frac{1}{\mu}, \frac{\mathrm{h}}{\mu})_{fin}/\approx.$$

This is a real Heisenberg group.

Define on $\tilde{\mathrm{A}}_{fin}^0$
$$R \approx R' \ \text{iff} \ \mathrm{dist}(R, R') = 0.$$

The elements of the quotient
$$\tilde{\mathrm{A}}_{fin}^0/\approx$$

will be called the **basic selfadjoint operators**. The pseudo-metric becomes a metric on the set of basic selfadjoint operators. [2]

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[2]The basic self-adjoint operators can be interpreted as the elements of the real Lie algebra generated by the classical P and Q (the Heisenberg algebra)

It is possible in principle to consider in the similar way the associative algebra of operators generated by these operators (the Weyl algebra). This, however, in our setting becomes rather messy business. Instead, we will study in later section operators which correspond in mathematical terms to automorphisms of the Lie algebra and in physical terms to time evolution operators, without trying to identify these operators with elements of the Weyl algebra.

8.11 Test-equivalence on E. Let $a, b, c, d \in {}^*\mathbb{Z}$, $0 < a, b, c, d \prec \mu$.

For $e \in {}^*\mathrm{E}(\mathbf{1})_{B(a,b)}(\mathbf{1})$ we define W_e to be the unique element of $\tilde{\mathrm{H}}(\frac{1}{\mu}, \frac{\mathrm{h}}{\mu})_{fin}$ with the properties:

- $\nu = 0$ in the form (34) for W_e,

- e is an eigenvector of W_e,

- $\rho > 0$ or $\rho = 0$ and $\tau > 0$,

- $|\rho| + |\tau|$ is minimal.

Given $e, e' \in \mathrm{E}_{B(c,d)}(\mathbf{1})$ let $W = W_e$, and $W' = W_{e'}$ and $R = R_W$, $R' = R_{W'}$ be the corresponding pseudo-selfadjoints.

Define the two elements to be **test-equivalent**,

$$e \approx e'$$

if

$$R \approx R' \text{ and } \langle e | Re \rangle \sim \langle e' | R' e' \rangle.$$

8.12 Note that
$$e \approx re,$$

for r a root of unity. In other words, $\approx$ identifies elements representing the same physical state.

Also, crucially with the use of regular unitary transformations, $\approx$ agrees with the relation introduced in 6.5.

8.13 Lemma. $f_1 \approx f_2$ *is an equivalence relation.*
Proof. (i) Immediate by definition. $\square$

8.14 Extending the global language by $\approx_\epsilon$. We introduce for each positive $\epsilon \in \mathbb{Q}$ a new binary predicate $e_1 \approx_\epsilon e_2$, interpreted in each E_A as a statement:

"$e_1, e_2 \in \mathrm{E}$ and there are pseudo-selfadjoint operators $R_1, R_2 \in A$ such that for some $w_1, w_2 \in \mathbb{F}$,

$$R_1 e_1 = w_1 e_1 \ \& \ R_2 e_2 = w_2 e_2 \ \& \ |w_1 - w_2| < \epsilon \text{ ''.}$$

This is definable by a positive quantifier-free formula since for every choice of A the existential quantifier runs through finitely many choices.

Note that the relation $\approx$ on $^*\mathrm{E}(\mathbf{1})$ is definable by a type consisting of positive formulas (positively type-definable):

$$e_1 \approx e_2 \equiv \bigwedge_{n \in \mathbb{N}} e_1 \approx_{\frac{1}{n}} e_2.$$

8.15 Lemma. *Let $R = R_W \in \tilde{\mathrm{A}}_{fin}$ be a selfadjoint operator and $e \in {}^*\mathrm{E}(\mathbf{1})$ its eigenvector. Then*

$$e \approx \hat{U}e \text{ and } e \approx \hat{V}e.$$

Proof. Clearly, e is a W-eigenvector too. That is $We = we$ for some $w \in {}^*\mathbb{F}_0^\times$.

Let $W' := \hat{U}W\hat{U}^{-1}$. Then, using the commutation identity (4),

$$W' = \mathrm{e}^{2\pi \mathrm{i} h \frac{\tau}{\mu^2}} W$$

and

$$W'e' = \hat{U}W\hat{U}^{-1}\hat{U}e = we'.$$

It follows that

$$R_{W'}e' = \frac{1}{i\sigma} \left(\mathrm{e}^{2\pi \mathrm{i} h \frac{\tau}{\mu^2}} w - \mathrm{e}^{-2\pi \mathrm{i} h \frac{\tau}{\mu^2}} w^{-1} \right) e',$$

for $\sigma = |\frac{\rho}{\mu}| + |\frac{\tau}{\mu}|$ as defined in (34).

We claim that

$$\frac{1}{\sigma} \left(\mathrm{e}^{2\pi \mathrm{i} h \frac{\tau}{\mu^2}} w - \mathrm{e}^{-2\pi \mathrm{i} h \frac{\tau}{\mu^2}} w^{-1} \right) \sim \frac{1}{\sigma}(w - w^{-1})$$

that is the $R_{W'}$-eigenvalue of e' is infinitesimally close to the R_W-eigenvalue of e. Indeed, this is immediate from the fact that

$$\frac{1}{\sigma} \cdot \frac{\tau}{\mu^2}$$

is infinitesimal. This proves the statement for $e' = \hat{U}e$.

The proof for $e' = \hat{V}e$ is similar. $\square$

8.16 Lemma. *Let $k, l \in {}^*\mathbb{Z}$, $\frac{k}{\mu}, \frac{l}{\mu} \in {}^*\mathbb{Q}_{fin}$. Then*

$$\mathbf{u}^{a,b}(q^{ak}) \approx \mathbf{u}(q^k), \tag{41}$$

and
$$\mathbf{v}^{b,a}(q^{bl}) \approx \mathbf{v}(q^l). \tag{42}$$

Proof. Note that for finite values of $\frac{k}{\mu}$ calculating in the $B(a,b)$-module

$$\langle \mathbf{u}^{a,b}(q^{ak})|\, \mathrm{P}\, \mathbf{u}^{a,b}(q^{ak},1)\rangle = \frac{q^{ak} - q^{-ak}}{2ia/\mu} \sim \frac{q^k - q^{-k}}{2i/\mu} \sim \frac{k\hbar}{\mu}.$$

And in the $\tilde{\mathrm{A}}$-module

$$\langle \mathbf{u}(q^k)|\mathrm{P}\mathbf{u}(q^k)\rangle = \frac{q^k - q^{-k}}{2i/\mu} \sim \frac{k\hbar}{\mu}.$$

This proves the first equivalence. The second and third equivalences follows by the same argument. $\square$

8.17 Remark. Under the above assumptions we have $a|l$ and $b|k$ by definition of a canonical basis. We then have by calculation in 7.8 in the ambient module $B(a,b)$:

$$\langle \mathbf{u}^{a,b}(q^{ak})|\mathbf{v}^{b,a}(q^{bl})\rangle_{Dir} = \mathfrak{c}\, q^{kl} \tag{43}$$

8.18 Lemma. *Let k be as in 8.16 and $d \in {}^*\mathbb{Z}$, $0 \le d \prec \mu$. Then*

$$\mathbf{u}(q^{k+d}) \approx \mathbf{u}(q^k) \tag{44}$$

and

$$\mathbf{v}(q^{k+d}, 1) \approx \mathbf{v}(q^k) \tag{45}$$

Proof. Using the same same calculations as in 8.16 we would need to prove
$$\frac{q^k - q^{-k}}{2i/\mu} \sim \frac{q^{k+d} - q^{-(k+d)}}{2i/\mu}.$$
Recall that $q^k = \exp\frac{i\hbar k}{\mu^2}$, $q^{k+d} = \exp\frac{i\hbar(k+d)}{\mu^2}$ and so

$$\frac{q^{k+d} - q^k}{2i/\mu} = q^k \mu \frac{\mathrm{e}^{i\alpha} - 1}{2i}$$

where $\alpha\mu$ is infinitesimal. The equivalence follows. $\square$

8.19 Corollary. The statements of 8.16 hold for any pair of pseudo-unitary operators S and T and their eigenvectors in place of U^a and V^b.

8.20 The space of states and the bra-ket notations.

We define the space of states as the quotient space

$$\mathbb{S} := {}^*\mathrm{E}(\mathbf{1}) / \approx$$

and the canonical quotient map as

$$\lim : {}^*\mathrm{E}(\mathbf{1}) \to \mathbb{S}. \tag{46}$$

Note that by definition ${}^*\mathrm{E}(\mathbf{1})$ is a pseudo-finite union of definable subsets of the form E^C, $C = C(S)$, $S \in \tilde{\mathrm{A}}$, $S = U^a V^b$, $a, b \in {}^*\mathbb{Q}_{fin}$. Thus

$$\lim S =: \bar{S} = U^\alpha V^\beta, \ \text{ where } \alpha = \lim a, \ \beta = \lim b,$$

According to 8.16 and 8.19

$$\lim \mathrm{E}^C = \{\lim \mathsf{s}(\mathrm{e}^{\frac{is}{\mu}}) : s \in \frac{1}{\mu}{}^*\mathbb{Z} \cap {}^*\mathbb{Q}_{fin}\} = \{|s\rangle : s \in \mathbb{R}\},$$

the image of the S-basis.

For the given $\bar{S} = U^\alpha V^\beta$ we denote the image of the basis in $\mathbb{S}$,

$$\mathbb{S}_{\alpha/\beta} := \{|s\rangle : s \in \mathbb{R}\},$$

and call it a **Lagrangian subspace of** $\mathbb{S}$.[3] Lagrangian subspaces with α, β rational will be referred to as **rational Lagrangian subspaces**.

Thus

$$\mathbb{S} = \bigcup_{\alpha, \beta \in \mathbb{R}^\times} \mathbb{S}_{\alpha/\beta},$$

and we also will use the rational part of the space of states,

$$\mathbb{S}_\mathbb{Q} := \bigcup_{\alpha, \beta \in \mathbb{Q}^\times} \mathbb{S}_{\alpha/\beta}.$$

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[3]Note that $U^\alpha V^\beta$ and $U^{m\alpha} V^{m\beta}$ commute and so generate the same Lagrangian subspace, hence the subspace is determined by the fraction α/β.

8.21 Keeping with physics notation we set

$$|x\rangle = \lim \mathbf{u}(\mathrm{e}^{\frac{ix'}{\mu}})$$

$$|p\rangle = \lim \mathbf{v}(\mathrm{e}^{\frac{ip'}{\mu}})$$

for

$$x',p' \in \frac{\hbar}{\mu} \cdot {}^*\mathbb{Z}, \ x',p' \in {}^*\mathbb{Q}_{fin}, \tag{47}$$

Note that these assumptions imply that $x',p' \in [-\frac{\mu}{2\hbar}, \frac{\mu}{2\hbar})$ and $\lim x' \in \mathbb{R}$, $\lim p' \in \mathbb{R}$, taking all possible values in $\mathbb{R}$.

These notation rely on distinguishing between the so called **position eigenstates** $|x\rangle$ and the **momentum eigenstates** $|p\rangle$ just by the letters x or p used to denote possibly the same number, the eigenvalue of the corresponding operator (to be explained below).

Often we simply right

$$|x\rangle = \lim \mathbf{u}(\mathrm{e}^{\frac{ix}{\mu}})$$

and

$$|p\rangle = \lim \mathbf{v}(\mathrm{e}^{\frac{ip}{\mu}})$$

assuming that the passage from the right-hand side to the left is clear.

8.22 We extend the domain of $\lim$ to ${}^*\mathrm{E}_{B(a,b)}(\mathbf{1})$, the eigenvector-bases of submodules, and define

$$\lim \mathbf{u}^{a,b}(\mathrm{e}^{\frac{iax}{\mu}}) = |x\rangle \tag{48}$$

$$\lim \mathbf{v}^{b,a}(\mathrm{e}^{\frac{ibp}{\mu}}) = |p\rangle \tag{49}$$

and

$$\lim \mathbf{s}^{a,b}(\mathrm{e}^{\frac{ias}{\mu}}) = |s\rangle \tag{50}$$

This agrees with $\approx$ by (41) and (42).

8.23 We say that a subset $E \subset {}^*\mathrm{E}(\mathbf{1})$ is **dense in** $\mathbb{S}_{\alpha/\beta}$ if $\lim E = \mathbb{S}_{\alpha/\beta}$.

Note that ${}^*\mathrm{E}_{B(a,b)}(\mathbf{1}) \subset \mathbf{V}_*(\mathbf{1})$ (see 3.46) and ${}^*\mathrm{E}_{B(a,b)}(\mathbf{1})$ can be embedded into ${}^*\mathrm{E}(\mathbf{1})$.

Assuming that $B(a,b)$ is a subalgebra of $\tilde{\mathrm{A}}$ of finite index, $a, b \in {}^*\mathbb{Q}$, $S \in B(a,b)$, it follows from 8.16 and 8.19 that ${}^*\mathrm{E}^S_{B(a,b)}(\mathbf{1})$ is dense in $\mathbb{S}_{\alpha/\beta}$.

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We study $\mathbb{S}$ along with the structure $\mathbf{V}_*$ and the map

$$\text{lim} : {}^*\text{E}(\mathbf{1}) \to \mathbb{S}.$$

8.24 Dirac product on $\mathbb{S}$. This is defined in accordance with 7.7 with usually fixed parametrisation in variables x or p.

We will work in the situation when we are given canonical bases $\mathbf{u}(e^{\frac{ix}{\mu}})$, $\mathbf{r}(e^{\frac{ix}{\mu}})$, and $\mathbf{s}(e^{\frac{ix}{\mu}})$ such that both $\mathbf{r}(e^{\frac{ix}{\mu}})$ and $\mathbf{s}(e^{\frac{ix}{\mu}})$ are parametrised by regular transformations

$$L_r : \mathbf{u}(e^{\frac{ix}{\mu}}) \mapsto \mathbf{r}(e^{\frac{ix}{\mu}}) \text{ and } L_s : \mathbf{u}(e^{\frac{ix}{\mu}}) \mapsto \mathbf{s}(e^{\frac{ix}{\mu}}).$$

This will define Δu, which we following above notations will write as Δx.

We will also use the notation

$$\text{lim}\, \mathbf{r}(e^{\frac{ix}{\mu}}) = |r(x)\rangle \text{ and } \text{lim}\, \mathbf{s}(e^{\frac{ix}{\mu}}) = |s(x)\rangle.$$

Now set

$$\langle r(x_1)|s(x_2)\rangle_{Dir} := \text{lim}\, \frac{1}{\Delta x} \left\{ \max_{x_1' \sim x_1,\ x_2' \sim x_2} \langle \mathbf{r}(e^{\frac{ix_1'}{\mu}})|\mathbf{s}(e^{\frac{ix_2'}{\mu}})\rangle \right\} \qquad (51)$$

Here the quantifier **max** chooses x_1' and x_2' so that the modulus of $\langle \mathbf{r}(e^{\frac{ix_1'}{\mu}})|\mathbf{s}(e^{\frac{ix_2'}{\mu}})\rangle$ reaches maximum. The argument of this expression depends continuously on x_1' and x_2' (see the definition and 3.28). Hence the right-hand side of (51) is well-defined.

On the left-hand side of (51), the use of bra-ket notation assumes that we rescale the product.

Warning. (51) only makes sense when we have chosen the corresponding canonical bases, that is the two consecutive vectors of each basis. Otherwise, we can only determine the **absolute value of** $\langle r(x_1)|s(x_2)\rangle_{Dir}$.

Nevertheless, (51) is a function of x_1, x_2 defined "up to the phase", that is the scalar of modulus 1.

Similarly one defines the parametrisation on variable p, just replacing $\mathbf{u}(e^{\frac{ix}{\mu}})$ by $\mathbf{v}(e^{\frac{ip}{\mu}})$.

We will have to use respectively Δp in the definition of the Dirac product. But note, that the application of the Fourier transform will produce the equality

$$\Delta p = \Delta x.$$

8.25 Example. Reinterpreting the calculation in 7.8 and taking into account (37) we have the classical

$$\langle x|p\rangle_{Dir} = \frac{1}{\sqrt{2\pi\hbar}}\mathrm{e}^{ixp}.$$

8.26 We also use the Dirac delta in $\mathbf{V}_*(\mathbf{1})$ as defined in 7.7. Since the dimension of the module is pseudo-finite the Δx and δ are nonstandard integers-valued. In the limit structure on $\mathbb{S}$ we introduce the Dirac $\delta(x)$ as an "infinite constant"-valued function, or symbol, along with symbol $\mathrm{d}x$ and set

$$\lim \Delta x := \mathrm{d}x,$$

$$\lim \delta(x) := \delta(\lim x).$$

8.27 Using integration notation. It is helpful to re-write the definition (51) in the following suggestive terms.

From the notations agreement above we get in particular,

$$dx\,\langle r(x_1)|s(x_2)\rangle = \langle \mathbf{r}(\mathrm{e}^{\frac{ix_1}{\mu}})|\mathbf{s}(\mathrm{e}^{\frac{ix_2}{\mu}})\rangle. \tag{52}$$

In functional Hilbert spaces elements $|r(x)\rangle$ and $|s(x)\rangle$ are understood as a family of functions $\rho(x,p)$ and $\sigma(x,p)$, with variable, say p. Now a summation formula

$$\mathbf{r}(\mathrm{e}^{\frac{ix}{\mu}}) = \sum_y \langle \mathbf{r}(\mathrm{e}^{\frac{ix}{\mu}})|\mathbf{s}(\mathrm{e}^{\frac{iy}{\mu}})\rangle \mathbf{s}(\mathrm{e}^{\frac{iy}{\mu}})$$

corresponds to the integral

$$\rho(x,p) = \int_{y\in\mathbb{R}} dy\,\langle r(x)|s(y)\rangle \cdot \sigma(y,p).$$

E.g. the summation formula for the Fourier transform (23) becomes, for $\frac{1}{\sqrt{2\pi\hbar}}\mathrm{e}^{\frac{ixp}{\hbar}}$, an eigenvector of the operator $f \mapsto -i\hbar\frac{\mathrm{d}f}{\mathrm{d}x}$, and $\delta(x-p)$, an eigenvector of the operator $f \mapsto xf$,

$$\delta(x-p) = \int_{y\in\mathbb{R}} \mathrm{d}y\,\frac{1}{\sqrt{2\pi\hbar}}\mathrm{e}^{-\frac{iyx}{\hbar}}\frac{1}{\sqrt{2\pi\hbar}}\mathrm{e}^{\frac{iyp}{\hbar}},$$

i.e.

$$\delta(0) = \frac{1}{2\pi\hbar}\int_{y\in\mathbb{R}} \mathrm{d}y$$

Conversely,

$$\frac{1}{\sqrt{2\pi\hbar}}\mathrm{e}^{ixp} = \int_{y\in\mathbb{R}} \mathrm{d}y \frac{1}{2\pi\hbar}\mathrm{e}^{iyx}\delta(p-y),$$

8.28 Note also that by definition the integral reinterpretation of definitions 7.7 yield

$$\int_{\mathbb{R}} \delta(x)\mathrm{d}x = 1.$$

8.29 Topology on the space of states. Note that the construction of the space of states $(\mathbb{S}, \mathbb{C})$ as the limit (46) of the pseudo-finitary space $\mathbf{V}_*(\mathbf{1}) = ({}^*\mathrm{E}(\mathbf{1}), {}^*\mathbb{F}_0)$ introduces a topology on the pseudo-Hibert space. Namely, $\mathbb{C}$ canonically acquires the metric topology from the standard part map $\mathsf{lim} :$ ${}^*\mathbb{F}_0 \to \mathbb{C}$. The predicates $e_1 \approx_{\frac{1}{n}} e_2$ define topology on $\mathbb{S}$. More generally, on Cartesian products of $\mathbb{S}$ and $\mathbb{C}$ the topology is given by declaring closed subsets those which are definable by positive quantifier free formulas in the language $L_{\mathbf{V}}^{Glob}$ extended by metric relation $|x| \le 1$ on $\mathbb{C}$. All such relations are preserved by map lim.

8.30 Theorem. *The object $\mathbf{V}(\mathbb{R})$ is universally attracting for the category $\mathcal{V}_{fin}$. In other words, for every $\mathbf{V}_A \in \mathcal{V}_{fin}^*$ there is a unique morphism*

$$\mathbf{p}_A : \mathbf{V}_A \to \mathbf{V}(\mathbb{R}).$$

Proof. First we consider the trivial morphism

$$\pi_A : \mathrm{Sp}_A(\mathbb{R}) \to \mathbf{1},$$

the right-hand side being the spectrum of $\mathbf{V}(\mathbb{R})$, by definition.

Now we need to define the embeddings

$$\mathbf{p}_A^\beta : \mathbf{V}_A(\beta) \to \mathbf{V}(\mathbb{R})$$

for every $\beta \in \mathrm{Sp}_A(\mathbb{R})$.

Since $A \subseteq A(\frac{1}{m}, \frac{\mathrm{h}}{m})$ for $m \in \mathbb{Z}$ large enough (in terms of divisibility), such that $\frac{m}{\mathrm{h}}$ is an integer, we may assume $A = A(\frac{1}{m}, \frac{\mathrm{h}}{m})$. Recall that $N = n_A = \frac{m^2}{\mathrm{h}}$ is the dimension of $\mathbf{V}_A(\beta)$.

By definition, $\beta = \langle u^N, v^N \rangle$ for some $u, v \in \mathbb{C}$ of modulus 1, that is

$$u = \mathrm{e}^{i\frac{x}{m}}, \ v = \mathrm{e}^{i\frac{p}{m}}, \ \text{for some unique real } x, p \in [0, 2m\pi).$$

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For $b = \frac{\mu}{m}$ consider the subalgebra $B(1,b) = \langle \hat{U}, V^{\frac{h}{m}} \rangle$ and a canonical $\hat{U}$-basis $\{ \mathbf{u}^{1,b}(q^k) : k = 0, \dots, \frac{m\mu}{h} - 1 \}$ of $\mathbf{V}_{B(1,b)}(\mathbf{1})$, $q = e^{\frac{2\pi i h}{m\mu}}$.

Since μ is an infinite integer, for $k = 0, \dots, m\mu - 1$, $\lim q^k$ takes all the complex values of modulus 1. Let k_0 be such that

$$\lim q^{k_0} = u$$

and denote

$$|uq^{bn}\rangle = \lim \mathbf{u}^{1,b}(q^{k_0 + bn}) : \quad n = 0, 1, \dots \frac{m^2}{h} - 1.$$

By definition

$$U^{\frac{1}{m}} : |uq^{bn}\rangle \mapsto uq^{bn}|uq^{bn}\rangle$$

and

$$V^{\frac{h}{m}} : |uq^{bn}\rangle \mapsto |uq^{b(n-1)}\rangle.$$

Write the $U^{\frac{1}{m}}$-basis of $\mathbf{V}_A(\beta)$ as $\{ \mathbf{u}(uq^{bn}, v) : n = 0, 1, \dots, \frac{m^2}{h} - 1 \}$ and define $\mathbf{p}_A^\beta$ on the basis as

$$\mathbf{p}_A^\beta : \mathbf{u}(uq^{bn}, v) \mapsto v^{-n}|uq^{bn}\rangle, \quad n = 0, 1, \dots, \frac{m^2}{h} - 1.$$

It is clear that $\mathbf{p}_A^\beta$ commutes with $U^{\frac{1}{m}}$. One also checks that

$$V^{\frac{h}{m}} : \mathbf{p}_A^\beta \mathbf{u}(uq^{bn}, v) = v^{-n}|uq^{b(n-1)}\rangle = v\mathbf{p}_A^\beta \mathbf{u}(uq^{b(n-1)}, v),$$

which proves that $\mathbf{p}_A^\beta$ commutes with $V^{\frac{h}{m}}$. Hence it is an isomorphism between the A-modules.

$\square$

9 Some basic calculations.

9.1 We calculate with operators Q and P defined in 8.7.

$$Q|x\rangle = \lim \frac{\hat{U} - \hat{U}^{-1}}{\frac{2i}{\mu}} \mathbf{u}(e^{\frac{ix}{\mu}}) = \\ = \lim \frac{\mu}{2i}(e^{\frac{ix}{\mu}} - e^{-\frac{ix}{\mu}}) \mathbf{u}(e^{\frac{ix}{\mu}}) = x|x\rangle. \tag{53}$$

It is clear, that this calculation is invariant under replacing x by x', with $x \sim x'$.

Hence
$$Q|x\rangle = x|x\rangle.$$

On the other hand

$$P|x\rangle = \lim \frac{\hat{V} - \hat{V}^{-1}}{\mathbf{u}}(e^{\frac{ix}{\mu}}) =$$
$$= \lim \frac{\mu}{2i} \left\{ \mathbf{u}(e^{(\frac{ix}{\mu} - \frac{i\hbar}{\mu^2})}) - \mathbf{u}(e^{(\frac{ix}{\mu} + \frac{i\hbar}{\mu^2})}) \right\} \tag{54}$$

is ill-defined since the vector in the bracket is an infinitesimal and μ is infinite.

Similarly,
$$P|p\rangle = p|p\rangle$$

but
$$Q|p\rangle \text{ is undefined.}$$

9.2 Comments. The real reason for P and Q to be only partially defined operators on E is that these are *unbounded* operators, that is

$$||P|| = \infty = ||Q||.$$

For Q this follows from the calculation (53) and similarly can be checked for P.

In mathematical physics this is the main difficulty for treating the Heisenberg algebra as a C^*-algebra.

9.3 Let us calculate the application of $QP - PQ$ to $|x\rangle$ and $|p\rangle$. To simplify the computation, we rewrite

$$Q = \frac{1}{2}(Q' + Q''), \quad P = \frac{1}{2}(P' + P'')$$

where

$$Q' = \frac{\hat{U} - 1}{i/\mu}, \quad Q'' = \frac{1 - \hat{U}^{-1}}{i/\mu},$$

$$P' = \frac{\hat{V} - 1}{i/\mu}, \quad P'' = \frac{1 - \hat{V}^{-1}}{i/\mu}.$$

First we calculate

$$(Q'P' - P'Q')\mathbf{u}(e^{\frac{ix}{\mu}}) = -4\mu^2 \left(\hat{U}\hat{V} - \hat{V}\hat{U}\right)\mathbf{u}(e^{\frac{ix}{\mu}}) =$$
$$-\mu^2 \left(1 - e^{\frac{i\hbar}{\mu^2}}\right)\hat{U}\hat{V}\mathbf{u}(e^{\frac{ix}{\mu}}) = -\mu^2 \left(1 - e^{\frac{i\hbar}{\mu^2}}\right)e^{\frac{ix}{\mu} - \frac{i\hbar}{\mu^2}} \cdot \mathbf{u}(e^{\frac{ix}{\mu} - \frac{i\hbar}{\mu^2}})$$

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Now note that by definition and (44)

$$\lim \mathbf{u}(\mathrm{e}^{\frac{ix}{\mu}}) = |x\rangle = \lim \mathbf{u}(\mathrm{e}^{\frac{ix}{\mu} - \frac{i\hbar}{\mu^2}}),$$

so, applying $\lim$ to the initial and final term of the equality and using that

$$\lim \mu^2 (\mathrm{e}^{\frac{i\hbar}{\mu^2}} - 1) \cdot \mathrm{e}^{\frac{ix}{\mu} - \frac{i\hbar}{\mu^2}} = i\hbar$$

we get

$$(\mathrm{Q}'\mathrm{P}' - \mathrm{P}'\mathrm{Q}')|x\rangle = i\hbar|x\rangle.$$

Similarly we get the same result for $(\mathrm{Q}'\mathrm{P}'' - \mathrm{P}''\mathrm{Q}')$, $(\mathrm{Q}''\mathrm{P}' - \mathrm{P}'\mathrm{Q}'')$ and $(\mathrm{Q}''\mathrm{P}'' - \mathrm{P}''\mathrm{Q}'')$. It follows

$$(\mathrm{QP} - \mathrm{PQ})|x\rangle = i\hbar|x\rangle.$$

Analogously one gets

$$(\mathrm{QP} - \mathrm{PQ})|p\rangle = i\hbar|p\rangle.$$

In fact, these are the instances of the more general fact.

9.4 Theorem. *The Canonical Commutation Relation* (2) *holds in* $\mathbb{S}$:

$$\mathrm{QP} - \mathrm{PQ} = i\hbar I.$$

In other words, given $|s\rangle \in \mathbb{S}$

$$(\mathrm{QP} - \mathrm{PQ})|s\rangle = i\hbar|s\rangle.$$

Proof. By definition $|\mathbf{s}\rangle = \lim \mathbf{s}(\mathrm{e}^{i\frac{s}{\mu}})$, for some $\mathbf{s}(\mathrm{e}^{i\frac{s}{\mu}}) \in {}^*\mathrm{E}(\mathbf{1})$, an element of a canonical S-basis.

Now we apply $(\mathrm{Q}'\mathrm{P}' - \mathrm{P}'\mathrm{Q}')$ to $\mathbf{s}(\mathrm{e}^{i\frac{s}{\mu}})$ and get by the calculation in 9.3

$$(\mathrm{Q}'\mathrm{P}' - \mathrm{P}'\mathrm{Q}')\mathbf{s}(\mathrm{e}^{i\frac{s}{\mu}}) = \mu^2 \left(\mathrm{e}^{\frac{i\hbar}{\mu^2}} - 1\right) \hat{U}\hat{V}\mathbf{s}(\mathrm{e}^{i\frac{s}{\mu}}).$$

By 8.15 $\hat{U}\hat{V}\mathbf{s}(\mathrm{e}^{i\frac{s}{\mu}}) \approx \mathbf{s}(\mathrm{e}^{i\frac{s}{\mu}})$. Also $\mu^2 \left(\mathrm{e}^{\frac{i\hbar}{\mu^2}} - 1\right) \sim i\hbar$. Hence

$$(\mathrm{Q}'\mathrm{P}' - \mathrm{P}'\mathrm{Q}')\mathbf{s}(\mathrm{e}^{i\frac{s}{\mu}}) \approx i\hbar\,\mathbf{s}(\mathrm{e}^{i\frac{s}{\mu}})$$

and hence,

$$(\mathrm{QP} - \mathrm{PQ})\mathbf{s}(\mathrm{e}^{i\frac{s}{\mu}}) \approx i\hbar\,\mathbf{s}(\mathrm{e}^{i\frac{s}{\mu}}).$$

$\square$

9.5 By the argument in 9.3 we also have for $\mathbb{S}$

$$Q' = Q \text{ and } P' = P.$$

9.6 The Fourier transform is defined on $\mathbb{S}$ by applying lim to (23):

$$\Phi|x\rangle = |p\rangle, \quad \text{for } p = x.$$

Applying Φ twice we get

$$\Phi^2 : |x\rangle \mapsto |-x\rangle.$$

9.7 The structure on $^*\mathrm{E}(\mathbf{1})$ in the global language plays a central role in the theory below. This is a pseudo-finite structure which can be seen as a model of "huge finite universe" of quantum mechanics. Its specialisation (image under lim), the "standard" structure $\mathbb{S}$, the space of states, is a close analogue of the standard Hilbert space where operators of quantum mechanics are being represented. The calculations in the next sections demonstrate that we may carry out rigorous quantitative analysis of processes in the "huge finite universe" and then, specialising these to the standard space of states, obtain well-defined formulas in complete agreement with traditional methods of quantum mechanics.

10 Regular unitary transformations on $\mathbf{V}_*$ and $\mathbb{S}$.

10.1 A **regular unitary transformation** L **of finite type on** $\mathbf{V}_*$ is defined as in 6.1(i)-(iv) for the algebra $\tilde{\mathrm{A}}$ with the extra assumption:

(v) The index $\nu_B := (\mathrm{N} : n_B)$ of the subalgebra B in $\tilde{\mathrm{A}}$ is finite.
Then we have the following refinement of Theorem 6.8.

10.2 Theorem. (i) *A regular unitary transformation L of finite type on $\mathbf{V}_*$ associated with the subalgebra $B = \langle \hat{U}^a \hat{V}^b, \hat{U}^c \hat{V}^d \rangle$, $a, b, c, d \in \mathbb{Z}$, and the isomorphism $\sigma : \langle \hat{U}^a \hat{V}^b, \hat{U}^c \hat{V}^d \rangle \to \langle \hat{U}^{a'} \hat{V}^{b'}, \hat{U}^{c'} \hat{V}^{d'} \rangle$ induces a bijection*

$$\hat{L} : \mathbb{S}_{a/b} \to \mathbb{S}_{a'/b'}.$$

(ii) *If $C = \langle \hat{U}^{a_C} \hat{V}^{b_C}, \hat{U}^{c_C} \hat{V}^{d_C} \rangle \subset B$ is another subalgebra of finite index, σ_C is the restriction of σ to C and L_C the transformation associated to C and σ_C, then $\hat{L}_C = \hat{L}$. In particular,*

$$\hat{L} : \mathbb{S}_{a_C/b_C} \to \mathbb{S}_{a'_C/b'_C},$$

and so $\hat{L}$ is defined on all rational Lagrangian subspaces

$$\hat{L} : \mathbb{S}_{\mathbb{Q}} \to \mathbb{S}_{\mathbb{Q}}.$$

(iii) *The set $\mathcal{G}_{fin}$ of all transformations $\hat{L}$, for regular unitary transformation L of finite type on $\mathbf{V}_*$, forms a group acting on $\mathbb{S}_{\mathbb{Q}}$. The group is isomorphic to $\mathrm{SL}(2, \mathbb{Q})$.*

Proof. By definition and 8.19 we have

$$\hat{L} : \{|s\rangle : s \in \mathbb{R}\} \to \{|s'\rangle : s' \in \mathbb{R}\}$$

where $S = \hat{U}^a \hat{V}^b$ and $S' = \hat{U}^{a'} \hat{V}^{b'}$. This is exactly the statement of (i).

(ii) follows by 6.4.

(iii). It follows from 6.8(iii)-(iv) that $\mathcal{G}$ is a group. Moreover, a regular transformations acts on elements of the rational Heisenberg group

$$\hat{U}^a \hat{V}^b \mapsto \hat{U}^{a'} \hat{V}^{b'}$$

as an automorphism (and so symplectomorphism). So $\mathcal{G}$ is the subgroup of all (rational) symplectomorphisms on $\mathbb{S}_{\mathbb{Q}}$, that is $\mathrm{SL}(2, \mathbb{Q})$. On the other hand 6.13, 6.19 and 6.22 prove that the generators of $\mathrm{SL}(2, \mathbb{Q})$ are in $\mathcal{G}$. So $\mathcal{G} = \mathrm{SL}(2, \mathbb{Q})$. $\square$

10.3 A **tame regular unitary transformation** L **on $\mathbf{V}_*$** is defined as in 6.1(i)-(iv) for the algebra $\tilde{\mathrm{A}}$ with the extra assumption:

(v) The index $\nu_B := (\mathrm{N} : n_B)$ of the subalgebra B in $\tilde{\mathrm{A}}$ is "small compared to μ, that is

$$\nu_B \prec \mu.$$

This condition is equivalent to $B = \langle \hat{U}^a \hat{V}^b, \hat{U}^c \hat{V}^d \rangle$, $a, b, c, d \in {}^*\mathbb{Z}$,

$$a, b, c, d \prec \mu.$$

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10.4 Theorem. (i) *A tame regular unitary transformation L on $\mathbf{V}_*$ associated with the subalgebra $B = \langle \hat{U}^a \hat{V}^b, \hat{U}^c \hat{V}^d \rangle$, $a, b, c, d \in {}^*\mathbb{Z}$, and the isomorphism $\sigma : \langle \hat{U}^a \hat{V}^b, \hat{U}^c \hat{V}^d \rangle \to \langle \hat{U}^{a'} \hat{V}^{b'}, \hat{U}^{c'} \hat{V}^{d'} \rangle$ induces a bijection*

$$\hat{L} : \mathbb{S}_{\alpha/\beta} \to \mathbb{S}_{\alpha'/\beta'}$$

where $\alpha/\beta = \lim a/b$, $\alpha'/\beta' = \lim a'/b'$,

(ii) *The set $\mathcal{G}$ of all transformations $\hat{L}$, for tame regular unitary transformation L on $\mathbf{V}_*$, forms a group acting on $\mathbb{S}$. The group is isomorphic to $\mathrm{SL}(2, \mathbb{R})$.*

Proof. The same arguments as in 10.2. $\square$

11 Time evolution operators. The case of the free particle

11.1 Let t be a rational number. We introduce a regular unitary transformation K_{fr}^t on the space of states, which corresponds to the so called *time evolution operator for the free particle*. In terms of the Heisenberg-Weyl algebra this operator could be defined as

$$\mathrm{e}^{-it \frac{\mathrm{P}^2}{2\hbar}}.$$

Equivalently, this is the operator acting on the position eigenstates as

$$\mathrm{e}^{-it \frac{\mathrm{P}^2}{2\hbar}} |p\rangle = \mathrm{e}^{-it \frac{p^2}{2\hbar}} |p\rangle. \tag{55}$$

Our K_{fr}^t satisfies the same property.

As above K_{fr}^t will first be defined on the pseudo-finitary space $\mathbf{V}_*(\mathbf{1})$ and then transferred to $\mathbb{S}$ by $\lim$.

11.2 In fact, we can only fully define K_{fr}^t on a submodule of $\mathbf{V}_*(\mathbf{1})$ the construction of which depends on t.

We assume that $t = \frac{b}{d}$, where $b, d \in {}^*\mathbb{Z}$, $|b|, |d| \prec \mu$.

We calculate over the subalgebra $B(b, d)$ of $\tilde{\mathrm{A}}$.

By 3.46 the $\tilde{\mathrm{A}}$-module $\mathbf{V}_*(\mathbf{1})$ splits into the direct sum of d of its $B(b, d)$-submodules. We define K_{fr}^t on the submodule $\mathbf{V}_{B(b,d)}(\mathbf{1})$.

Set
$$\check{q} = \mathrm{e}^{\frac{ibd\hbar}{\mu^2}} = q^{bd} \text{ and } \check{N} = \frac{\mu^2}{bd\mathrm{h}} = \frac{\mathrm{N}}{bd}.$$

Clearly,
$$\check{N} = \dim \mathbf{V}_{B(b,d)}(\mathbf{1}).$$

Choose a canonical $\hat{U}^d$-basis of $\mathbf{V}_{B(b,d)}(\mathbf{1})$ in the form
$$\{\check{\mathbf{u}}(\check{q}^m) : \ m = 0, \dots, N-1\}$$

satisfying
$$\hat{U}^d : \check{\mathbf{u}}(\check{q}^m) \mapsto \check{q}^m \check{\mathbf{u}}(\check{q}^m)$$
$$\hat{V}^b : \check{\mathbf{u}}(\check{q}^m) \mapsto \check{\mathbf{u}}(\check{q}^{m-1})$$

Set K_{fr}^t on the U-basis by:

$$K_{\mathrm{fr}}^t \check{\mathbf{u}}(\check{q}^m) = \frac{\mathfrak{a}}{\sqrt{\check{N}}} \sum_{l=0}^{\check{N}-1} \check{q}^{\frac{(l-m)^2}{2}} \check{\mathbf{u}}(\check{q}^l) =: \mathbf{s}(\check{q}^m) \qquad (56)$$

where $\mathfrak{a}$ is defined in 6.18.

One sees immediately that K_{fr}^t is the Gaussian transformation with the domain of definition and range $\mathbf{V}_{B(b,d)}(\mathbf{1})$. So we may use 29 - 6.20.

11.3 An alternative definition of K_{fr}^t. According to quantum mechanics the time evolution operator for the free particle is $K_{\mathrm{fr}}^t := \mathrm{e}^{-it\frac{\mathrm{P}^2}{2\hbar}}$ and the expression on the right makes sense in the operator algebra Then the canonical commutation relation (2) gives

$$[\mathrm{Q}, K_{\mathrm{fr}}^t] = t\mathrm{P}K_{\mathrm{fr}}^t.$$

Hence, $\mathrm{Q}K_{\mathrm{fr}}^t - K_{\mathrm{fr}}^t\mathrm{Q} = t\mathrm{P}K_{\mathrm{fr}}^t$ and

$$K_{\mathrm{fr}}^t i\mathrm{Q}K_{\mathrm{fr}}^{-t} = i(\mathrm{Q} - t\mathrm{P}).$$

Note that by the Baker-Campbell-Hausdorff formula, for any $r \in \mathbb{R}$

$$U^r \cdot V^{-r\mathrm{h}t} = \exp ir\mathrm{Q} \cdot \exp\{-irt\mathrm{P}\} = \exp(ir\mathrm{Q} - irt\mathrm{P} + \frac{1}{2}[ir\mathrm{Q}, -irt\mathrm{P}]) =$$

$$= \exp(ir\mathrm{Q} - irt\mathrm{P} + i\pi r^2 t\mathrm{h}) = \mathrm{e}^{\pi i\mathrm{h}r^2 t} K^t U^r K^{-t}.$$

Letting $t = \frac{b}{d}$ and $r = \frac{b}{\mu}$ as above, we get

$$K_{\mathrm{fr}}^t \hat{U}^d K_{\mathrm{fr}}^{-t} = S, \quad S = \check{q}^{-\frac{1}{2}} \hat{U}^d \hat{V}^{-b} \tag{57}$$

which is (27) in corresponding notation.

Another property that follows from the form of the time evolution operator, is its commutation with P and so with $\hat{V}^b$:

$$K^t \hat{V}^b K^{-t} = \hat{V}^b \tag{58}$$

(57) and (58) can be taken as the defining properties of K_{fr}^t. Note that these two algebraic properties alone do not allow to determine the scalar coefficient $\mathfrak{a}$.

11.4 Now we define K_{fr}^t on $\mathbb{S}$ *by continuity*, that is set

$$\lim K_{\mathrm{fr}}^t \mathbf{s}(\check{q}^m) = K_{\mathrm{fr}}^t \lim \mathbf{s}(\check{q}^m),$$

for all $\mathbf{s} \in {}^*\mathrm{E}(\mathbf{1})_{B(d,b)}$.

By (49), remembering that

$$\check{\mathbf{v}}(\check{q}^n) = \mathbf{v}^{\frac{b}{\mu}, \frac{d}{\mu}}(\mathrm{e}^{\frac{i\hbar n b d}{\mu^2}}) = \mathbf{v}^{b,a}(\mathrm{e}^{\frac{i b p}{\mu}}),$$

where

$$p = \frac{nd\hbar}{\mu}$$

satisfies (47). We have by definition

$$|p\rangle = \lim \hat{\mathbf{v}}^{\frac{b}{\mu}, \frac{d}{\mu}}(\mathrm{e}^{\frac{i b p}{\mu}}).$$

We also note that

$$\check{q}^{-\frac{n^2}{2}} = \mathrm{e}^{-t\frac{p^2}{2}}, \quad t = \frac{b}{d}.$$

By (28) as

$$K_{\mathrm{fr}}^t |p\rangle = e^{-it\frac{p^2}{2\hbar}} |p\rangle. \tag{59}$$

Respectively, in (56) we set

$$x_1 = \frac{lb\hbar}{\mu}, \quad x_2 = \frac{mb\hbar}{\mu},$$

we get by definition and recalling that

$$\check{\mathbf{u}}(\check{q}^l) = \mathbf{u}^{b,d}(\mathrm{e}^{\frac{lbd\hbar}{\mu^2}}) = \mathbf{u}^{b,d}(\mathrm{e}^{\frac{dx_1}{\mu}}),$$

$$\check{\mathbf{u}}(\check{q}^m) = \mathbf{u}^{b,d}(\mathrm{e}^{\frac{mbd\hbar}{\mu^2}}) = \mathbf{u}^{b,d}(\mathrm{e}^{\frac{dx_2}{\mu}}),$$

$$|x_1\rangle = \lim \check{\mathbf{u}}(\mathrm{e}^{\frac{dx_1}{\mu}})$$

and

$$K_{\mathrm{fr}}^t|x_2\rangle = \lim \mathbf{s}(\mathrm{e}^{\frac{dx_2}{\mu}}).$$

11.5 Our aim now is to determine the value of

$$\langle \check{\mathbf{u}}(\mathrm{e}^{\frac{dx_1}{\mu}})|\mathbf{s}(\mathrm{e}^{\frac{dx_2}{\mu}})\rangle_{Dir},$$

for which we need to determine Δx. This was determined in 7.7 taking into account (37) and gives for $B = D = B(d,b)$

$$\Delta x = \frac{b\hbar}{\mu}.$$

By (52) and (56)

$$\langle \check{\mathbf{u}}(\mathrm{e}^{\frac{dx_1}{\mu}})|\mathbf{s}(\mathrm{e}^{\frac{dx_x}{\mu}})\rangle_{Dir} = \frac{1}{\Delta x}\langle \check{\mathbf{u}}(\check{q}^l)|\mathbf{s}(\check{q}^m)\rangle = \frac{\sqrt{bd\mathrm{h}}}{\mu\Delta x}\mathrm{e}^{\frac{-\pi i}{4}}\check{q}^{\frac{(l-m)^2}{2}}$$

which gives us by application of $\lim$:

$$\langle x_1|K_{\mathrm{fr}}^t x_2\rangle_{Dir} = \mathrm{e}^{-\frac{\pi i}{4}}\sqrt{\frac{d\mathrm{h}}{b}}\cdot\frac{1}{\hbar}\mathrm{e}^{\frac{i(x_1-x_2)^2}{2}} = \frac{1}{\sqrt{2\pi i\hbar t}}\mathrm{e}^{\frac{i(x_1-x_2)^2}{2t\hbar}} \tag{60}$$

The left-hand side of (60) is called **the kernel of a time evolution operator** (Feynman propagator for the corresponding particle).

12 Harmonic oscillator.

12.1 In the Heisenberg Weyl algebra consider the operator

$$H = \frac{\mathrm{P}^2 + \mathrm{Q}^2}{2},$$

the *Hamiltonian for the Harmonic oscillator*, where we suppressed some parameters of physical significance).

We are interested in the time evolution operator for the Harmonic oscillator, i.e. the operator

$$K^t := K_{HO}^t := e^{-it\frac{H}{\hbar}}.$$

In order to motivate our technical definition we assuming again as in 11.3 that P and Q belong to a Banach algebra. Then well-known calculations, which we omit here, produce the following formulae

$$K^t U^a K^{-t} = e^{-a^2 \pi i h \sin t \cos t} U^{a \cos t} V^{-ah \sin t}. \tag{61}$$

and

$$K^t V^{ah} K^{-t} = e^{a^2 \pi i h \sin t \cos t} U^{a \sin t} V^{ah \cos t}. \tag{62}$$

12.2 We will assume that $t \in \mathbb{R}$ is such that $\sin t$ and $\cos t$ are rational (there is a dense subset of $t \in (0, 2\pi]$ satisfying this condition), and even allow these to be non-standard rational.

For the rest of this section we fix

$$\sin t = \frac{e}{c}, \quad \cos t = \frac{f}{c}, \quad \text{for some nonzero } e, f, c \in {}^*\mathbb{Z}, \ 0 \leq |e|, |f|, c \prec \mu.$$

We work in the principal $\tilde{A}$-module $\mathbf{V}_*(\mathbf{1})$ and in the space of states $\mathbb{S}$ and want to construct, as suggested by (61) and (62), an operator K^t so that

$$K^t \hat{U}^c (K^t)^{-1} = q^{-\frac{1}{2}ef} \hat{U}^f \hat{V}^{-e} \tag{63}$$

$$(K^t) \hat{V}^{ce} (K^t)^{-1} = q^{\frac{1}{2}e^3 f} \hat{U}^{e^2} \hat{V}^{fe} \tag{64}$$

where

$$q^{\frac{1}{2}} = e^{\frac{\pi i h}{\mu^2}}.$$

12.3 Denote

$S_t := q^{-\frac{1}{2}ef} \hat{U}^f \hat{V}^{-e}, \ R_t := q^{\frac{1}{2}ef} \hat{U}^e \hat{V}^f$ and check that $R_t^m := q^{\frac{1}{2}efm^2} \hat{U}^{em} \hat{V}^{fm}$.

Note that the algebra $\langle S_t, R_t^e \rangle$ is isomorphic to $\langle \hat{U}^c, \hat{V}^{ce} \rangle$ by the isomorphism given by conjugation by K^t, so its irreducible modules are of the same dimension $\frac{N}{c^2 e}$.

Clearly, S_t and R_t are pseudo-unitary. Recall that with our notation

$$\dim \mathbf{V}_{B(1,e)}(\mathbf{1}) = \frac{\mathrm{N}}{e}.$$

Note also that the $\hat{U}$ basis $\mathbf{u}^{1,e}(q^{el})$ can be identified with the subset $\{\mathbf{u}(q^k) : e|k\}$ of the $\hat{U}$-basis of $\mathbf{V}_*(\mathbf{1})$.

12.4 Define the operator[4]

$$K^t : \mathbf{V}_{B(c,ce)}(\mathbf{1}) \to \mathbf{V}_*(\mathbf{1})$$

by the action on an U-basis,

$$K^t : \mathbf{u}^{c,ce}(q^{c^2em}) \mapsto \mathbf{s}(q^{c^2em}) := C_0 \sqrt{\frac{e}{\mathrm{N}}} \sum_{l=0}^{\frac{\mathrm{N}}{e}-1} q^{ef\frac{l^2-e^2m^2}{2}-e^3ml}\mathbf{u}(q^{e(l+mf)}), \quad (65)$$

where $C_0 \in \mathbb{F}_0$ is a constant of modulus 1, the exact value of which we determine at the end of this section.

12.5 Lemma. *For any integer m,*

$$\begin{aligned} S_t &: \mathbf{s}(q^{c^2em}) \mapsto q^{c^2em}\mathbf{s}(q^{c^2em}) \\ R_t^e &: \mathbf{s}(q^{c^2em}) \mapsto \mathbf{s}(q^{c^2e(m-1)}). \end{aligned}$$

Proof. One checks that

$$S_t : \mathbf{u}(q^{e(l+mf)}) \mapsto q^{ef(l+\frac{1}{2})+emf^2}\mathbf{u}(q^{e(l+1+mf)})$$

and thus, using $f^2 + e^2 = c^2$,

$$S_t : q^{ef\frac{l^2-e^2m^2}{2}-e^3ml}\mathbf{u}(q^{e(l+mf)}) \mapsto q^{c^2em} \cdot q^{ef\frac{(l+1)^2-e^2m^2}{2}-e^3m(l+1)}\mathbf{u}(q^{e(l+1+mf)}).$$

This implies the first statement.

For the second statement first note that

$$\mathbf{s}(\mathbf{1}) := C_0 \sqrt{\frac{e}{\mathrm{N}}} \sum_{l=1}^{\frac{\mathrm{N}}{e}-1} q^{ef\frac{l^2}{2}}\mathbf{u}(q^{el})$$

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[4]We state without proof that K^t can be defined as a regular unitary operator on $\mathbf{V}_{\bar{A}}$.

and that

$$R_t^{-em} : \mathbf{u}(q^{el}) \mapsto q^{k(e,f,m,l)} \mathbf{u}(q^{el+emf})$$

for

$$k(e,f,m,l) = -e^3 f \frac{m^2}{2} - e^3 ml.$$

It follows

$$R_t^{-em} : \mathbf{s}(1) \mapsto C_0 \sqrt{\frac{e}{\mathrm{N}}} \sum_{l=1}^{\frac{\mathrm{N}}{e}-1} q^{ef\frac{l^2-e^2m^2}{2}-e^3ml} \mathbf{u}(q^{e(l+mf)}) = \mathbf{s}(q^{c^2em}).$$

$\square$

12.6 For $n, m \in {}^*\mathbb{Z}$ we have, expressing $\mathbf{u}^{c,ce}(q^{c^2en})$ in terms of $\mathbf{u}$ (see 3.47),

$$\langle \mathbf{u}^{c,ce}(q^{c^2en}) | \mathbf{s}(q^{c^2em}) \rangle =$$

$$= \frac{C_0}{\sqrt{c}} \cdot \sqrt{\frac{e}{\mathrm{N}}} \langle \sum_{k=0}^{c-1} \mathbf{u}(q^{cen+k\frac{\mathrm{N}}{c}}) | \sum_{l=0}^{\frac{\mathrm{N}}{e}-1} q^{ef\frac{l^2-e^2m^2}{2}-e^3ml} \mathbf{u}(q^{e(l+mf)}) \rangle$$

Set $M := \frac{\mathrm{N}}{c}$ and $l_k = kM + l_0$, for $k = 0, \ldots, c-1$.

Note that the necessary condition for the product $\langle \mathbf{u}^{c,ce}(q^{c^2en}) | \mathbf{s}(q^{c^2em}) \rangle$ to be non-zero is that

$$cen + kM = el + emf \text{ for some } l = 0, \ldots, M-1, \ k = 0, \ldots, c-1. \quad (66)$$

Let $l_0 = cn - mf$.
Claim.

$$fl_0 - e^2 m \equiv 0 \bmod c.$$

Note that $fl_0 - e^2 m = fcn - mf^2 - me^2 = fcn - mc^2$. Claim proved.
So

$$cn + kM = l_k + mf.$$

Recall that $0 < |c|, |e| \prec \mathrm{N}$ which implies that

$$M^2 \equiv 0 \bmod \mathrm{N}.$$

Now we can continue

$$\langle \mathbf{u}^{c,ce}(q^{c^2en}) | \mathbf{s}(q^{c^2em}) \rangle =$$

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$$= C_0 \sqrt{\frac{e}{Nc}} \sum_{k=0}^{c-1} \langle \mathbf{u}(q^{cen+k\frac{N}{c}}) | q^{ef\frac{l_k^2-e^2m^2}{2}-e^3ml_k} \mathbf{u}(q^{e(l_k+mf)}) \rangle =$$

$$= C_0 \sqrt{\frac{e}{Nc}} \sum_{k=0}^{c-1} q^{ef\frac{l_k^2-e^2m^2}{2}-e^3ml_k} = C_0 \sqrt{\frac{e}{Nc}} q^{ef\frac{l_0^2-e^2m^2}{2}-e^3ml_0} \sum_{k=0}^{c-1} q^{efl_0kM-e^3mkM}$$

Now note that $efl_0kM - e^3mkM = ekM(fl_0 - e^2m)$ is divisible by N by Claim above. Hence the last sum is equal to c and we have finally

$$\langle \mathbf{u}^{c,ce}(q^{c^2en}) | \mathbf{s}(q^{c^2em}) \rangle = C_0 \sqrt{\frac{ec}{N}} q^{ef\frac{l_0^2-e^2m^2}{2}-e^3ml_0}. \tag{67}$$

12.7 Note by direct substitution and using $f^2 + e^2 = c^2$ that

$$ef\frac{l_0^2 - e^2m^2}{2} - e^3ml_0 = \frac{1}{2}ec^2\{(n^2 + m^2)f - 2cnm\}. \tag{68}$$

Now rename

$$x_1 := \frac{cen\hbar}{\mu}, \quad x_2 := \frac{cem\hbar}{\mu}.$$

Then, using 8.2, 8.6 and (68), the equality (67) becomes

$$\langle \mathbf{u}^{c,ce}(q^{c^2en}) | \mathbf{s}(q^{c^2em}) \rangle = C_0 \frac{\sqrt{ec\hbar}}{\mu} \exp i \frac{(x_1^2 + x_2^2)f - 2x_1x_2c}{2e\hbar} = \tag{69}$$

$$= C_0 \frac{\sqrt{ec\hbar}}{\mu} \exp i \frac{(x_1^2 + x_2^2)\cos t - 2x_1x_2}{2\hbar \sin t}.$$

12.8 We rescale the last product according to 7.7 and (32). We have $\langle \hat{U}^c, \hat{V}^{ce} \rangle$ in place of $\langle R, R_\dagger \rangle$, and $\langle S_t, R_t^e \rangle$ in place of $\langle S, S_\dagger \rangle$. $N = \mathrm{N} = \frac{\mu^2}{h}$.

We have

$$SR = q^{ce}RS$$

and $a_R = c$, the maximal with the property

$$R^{\frac{1}{c}} \in \tilde{A},$$

while for S the corresponding number $a_S = 1$.

Hence, by 7.7 and (37)

$$\Delta x = \mathfrak{c} \frac{e\sqrt{h}}{\mu} = \frac{e\hbar}{\mu}.$$

Thus

$$\langle \mathbf{u}^{c,ce}(q^{c^2en}) | \mathbf{s}(q^{c^2em}) \rangle_{Dir} = \frac{\mu}{e\hbar} \langle \mathbf{u}^{c,ce}(q^{c^2en}) | \mathbf{s}(q^{c^2em}) \rangle =$$

$$= \frac{\mu}{e\hbar} C_0 \frac{\sqrt{ech}}{\mu} \exp i \frac{(x_1^2 + x_2^2) \cos t - 2x_1x_2}{\hbar \sin t} = \frac{C_0}{\sqrt{2\pi\hbar \sin t}} \exp i \frac{(x_1^2 + x_2^2) \cos t - 2x_1x_2}{2\hbar \sin t}.$$

12.9 For the Feynman propagator we need to apply the limit

$$\langle x_1 | K^t | x_2 \rangle = \lim \langle \mathbf{u}^{c,ce}(q^{c^2en}) | \mathbf{s}(q^{c^2em}) \rangle_{Dir}$$

which by the final formula of 12.8 above gives us

$$\langle x_2 | K_{HO}^t | x_1 \rangle = C_0 \sqrt{\frac{1}{2\pi\hbar \sin t}} \exp i \frac{(x_1^2 + x_2^2) \cos t - 2x_1x_2}{2\hbar \sin t}.$$

This is in accordance with the well-known formula [15], p552, if we set

$$C_0 = \mathfrak{a} = e^{-\frac{\pi i}{4}}. \tag{70}$$

Our definition can only determine K^t up to the coefficient C_0 of modulus 1.

12.10 In this subsection we will motivate a stronger definition for K_{HO}^t which will have (70) as a corollary.

First we need to formulate and prove several claims. Note, that in the definition (65) C_0 may depend on t. We need to have reasons to claim that it does not.

Claim 1. Given t_1 and t_2 as in 12.2 we have the following operator identity

$$K^{t_1} K^{t_2} = K^{t_1+t_2}$$

where K^{t_1}, K^{t_2} and $K^{t_1+t_2}$ are defined by (65) with the same C_0 (but with different values of e and c, which depend on t).

Proof of the Claim. TODO.

Corollary. For all $t_1 \in \mathbb{Q} \cdot t$,

$$C_0(t_1) = C_0(t).$$

Now we may use the fact used by physicists and motivated by simple algebraic (the Trotter product formula) and continuity arguments which imply that the physicists' values of the operator must satisfy

$$\lim_{t \to 0} \frac{\langle x_1 | K_{HO}^t x_2 \rangle}{\langle x_1 | K_{Fr}^t x_2 \rangle} = 1.$$

Postulating that this identity holds for the operators K_{HO}^t and K_{Fr}^t as defined above and comparing the corresponding formulas we conclude that (70) holds.

13 Trace of the time evolution operator for the Harmonic oscillator.

13.1 First note that by (66)

$$\langle \mathbf{u}^{c,ce}(q^{c^2en}) | \mathbf{s}(q^{c^2en}) \rangle \neq 0 \Leftrightarrow en(c-f) + kM = el \text{ for some } l = 0, \ldots, \frac{N}{e} - 1, \ k = 0, \ldots, c-1.$$

Hence, the n on the right-hand side can only take values

$$n = 0, 1, \ldots, \frac{N}{ec(c-f)} - 1$$

13.2 Note for $n = m$ (68) is equal to

$$-ec^2(c-f)n^2.$$

Hence, by (67) and 13.1

$$\text{Tr}(K^t) := \mathfrak{a} \sqrt{\frac{ec}{N}} \sum_{n=0}^{\frac{N}{ec(c-f)} - 1} q^{-ec^2(c-f)n^2}.$$

Denote $L = \frac{N}{ec^2(c-f)}$. We can rewrite the above as

$$\text{Tr}(K^t) = \mathfrak{a} \sqrt{\frac{1}{c(c-f)L}} \sum_{n=0}^{cL-1} e^{-\frac{2\pi i}{L} n^2}.$$

By the Gauss quadratic sums formula (29)

$$\sum_{n=0}^{L-1} e^{-\frac{2\pi i}{L} n^2} = 2e^{-\frac{\pi i}{4}} \sqrt{\frac{L}{2}} = e^{-\frac{\pi i}{4}} \sqrt{2L}; \quad \sum_{n=0}^{cL-1} e^{-\frac{2\pi i}{L} n^2} = c\sqrt{2L}.$$

Hence

$$\text{Tr}(K^t) = c\,\mathfrak{a}^2 \sqrt{2L} \sqrt{\frac{1}{c(c-f)L}} = \mathfrak{a}^2 \sqrt{\frac{2}{(1 - \frac{f}{c})}}.$$

Returning to the original notation 12.2 we have

$$\sqrt{\frac{2}{(1 - \frac{f}{c})}} = \sqrt{\frac{2}{(1 - \cos t)}} = \frac{1}{|\sin \frac{t}{2}|}.$$

Since $\mathfrak{a}^2 = -i$, we finally get,

$$\mathrm{Tr}(K_{HO}^t) = \frac{1}{i|\sin \frac{t}{2}|}.$$

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