Around Logical Perfection

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Abstract: In this article we present a notion of "logical perfection". We first describe through examples a notion of *logical perfection* extracted from the contemporary logical concept of categoricity. Categoricity (in power) has become in the past half century a main driver of ideas in model theory, both mathematically (stability theory may be regarded as a way of approximating categoricity) and philosophically. In the past two decades, categoricity notions have started to overlap with more classical notions of robustness and smoothness. These have been crucial in various parts of mathematics since the nineteenth century. We postulate and present the category of logical perfection. We draw on various notions of perfection from mathematics of the 19th and 20th centuries and then trace the relation to the concept of categoricity in power as a logical notion of what a "mathematically perfect" structure is.

Keywords: categoricity, logical perfection, model theoretic stability theory, philosophy of mathematics

THIS ESSAY IS AN ATTEMPT to present the idea of **logical perfection** to a philosophical audience. Although the expression is quite often used (informally) in mathematical practice and even sometimes in more formal discussion around mathematics, we construe it here for the first time as an independent philosophical notion. Informal use of the expression often happens in the form of an (implicit) aesthetic criterion; it is arguably one of the strongest drivers of mathematical activity, as one of the main tests for its relevance. Since the advent of mathematical logic as an independent discipline, it has become possible to investigate the formal notion of *categoricity* by mathematical means. We use this notion as the main base of our notion of logical perfection.

A first, very rough, description of our idea may run as follows: a mathematical object of a certain "size" is logically perfect if in a certain formal language it allows a "concise" description fully determining the object.[1]

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1   The essay has arisen from many conversations and collaboration the authors have had during the past few years and it is originally based on two talks the first author gave, one in Paris and the other one in Bogotá, about the work of the third author. The second author then brought some additional perspective. The first author wants to mention in particular the talk given at Bogotá during the workshop *Mapping traces: Representation from Categoricity to Definability* organised by the second author and María Clara Cortés at Universidad

This notion, in particular, is central in the third author's paper (Zilber, 2014) and has been implicitly present, mainly as a motivating factor, in a number of other research papers in various branches of mathematics.

The French philosopher of mathematics Albert Lautman, blending a strong Husserlian (and remotely Kantian) influence with a vivid and active dialogue with mathematicians who were his contemporaries in the 1930s, opened the way to understanding different metaphysical categories in both phenomenological and epistemological ways when doing philosophy of mathematics. Lautman's untimely death in 1944 as a prisoner of the Germans during World War II left an enormous gap in continental, non-analytic, philosophy of mathematics of the mid-twentieth century. His colleague Jean Cavaillès was also executed in 1944 for his participation in the French *résistance*; in spite of their early deaths, they set a solid ground of connections between phenomenology, epistemology and a rich dialogue with their mathematical friends, many of whom would form the basis of Bourbaki (see Pérez-Lora's thesis (Pérez-Lora, 2020)). We are indebted to that tradition here. We do not claim that our notion of logical perfection is necessarily a Lautmanian category, but we do recognise his influence in the views we present here[2] and the presence of his imprint, enhanced by Zalamea's vivid dialogue with the three of us.

We only attempt to arrive at a *definition* of logical perfection at the end of our paper, in the conclusions. Before reaching that point, we describe historical reasons for the search of logical perfection through three examples from the 19th and 20th century, in section 1. We then discuss logical perfection and the issue of uniqueness, in section 2; and the role of geometry for *logically perfect structures* in section 3. Section 4 is the first that goes away from mathematics: we describe some analogies between our notion of logical perfection and some work of recent decades in physics. Finally, section 5 offers some concluding remarks, as well as a recapitulation of the notion of logical perfection where we arrive at an attempt of a definition.

We try to gloss over many subtle mathematical details, in order to put in the forefront the cluster notions of greater philosophical relevance. It may be worth noting that many of the notions we mention are drawn from specific, more technical, discussions. In some cases, we refer the more mathematically-minded readers (or those whose professional training or interest is closer to pure mathematical notions) to some articles, mathematical or philosophical, where the issues are discussed in greater (technical) detail. There are however some definitions we do provide, either as part of the text or, more often, as footnotes.

Nacional de Colombia in 2014. This was very helpful since the audience consisting of philosophers, mathematicians and artists made the idea of writing about logical perfection for a general audience possible.

2   We thank the philosopher of mathematics Fernando Zalamea for calling our attention to this aspect of Lautman's philosophy.

Although the core application of our ideas is eminently mathematical, we do provide some analogies with philosophical problems external to mathematics, and in section 4 we discuss the relevance of our notion to contemporary fundamental physics.

We leave open, at least here, the question whether logical perfection manifests itself in other areas of human activity (such as art - the second author explores a connection between categoricity and the possibility of representation in two works by Caravaggio in Villaveces (2020a)); we may only hope it will raise the interest of some of our readers.

Finally, we thank the two referees for many insightful comments that led to serious improvements and clarifications of this paper.

1. Three Examples on the Way to Logical Perfection

The interest in looking for some kind of *perfection* in mathematical structures is not new. We may read in the history of their discipline that mathematicians have been driven in explicit or implicit ways to think about this kind of perfection, albeit for different sorts of motivations, and have tried to capture this idea by means of mathematical tools. Let us mention a few of these attempts in the work of Galois, Riemann and Grothendieck and examine three examples for our discussion of logical perfection.

Galois made a bold switch from the classical perspective of looking directly for solutions to algebraic equations to a study of the *symmetry* of possible solutions: a move toward a *completion* of the set of *possible solutions* by means of the study of the group of symmetries of all the solutions that could exist, and by filtering out the interaction between enlarging the field where these solutions could appear and the group of such symmetries. The resulting theory (aptly named Galois Theory many decades later) goes way beyond the initial quest for solutions to algebraic equations: it shifted the focus to an idea that is still essential to many ways of doing mathematics after two centuries, and whose scope goes much beyond the wildest dreams Galois could have had: the new focus for Galois was the *symmetry of possible solutions in extensions* of fields, and the *duality between those symmetries (groups) and the possible field extensions*. This is a first form of "perfection" for us: **completeness of possible solutions,** and a **register of emerging symmetry.** More importantly even, underlying these two aspects of perfection arising in Galois' work, there is a strong notion of *uniqueness* of the field with all the solutions, only reached once all possible solutions (and their symmetries) are considered. This wholeness, this uniqueness, albeit implicit in the work of Galois, is a component of the main tenets of the notion of *logical* perfection we propose.

Our second example is Riemann's work on the foundations of geometry. In a move parallel to Galois', he went beyond understanding geometry in terms of global axioms and laid the ground for a local approach, driven by a metric (a way of measuring distances) that could change, twist, curve *itself*. Instead of placing objects such as curves, planes, surfaces inside a "global space" (as had been done for aeons in mathematics), Riemann put the *twisting* itself, so to speak, at centerstage: instead of placing twisted, curved objects inside a space, the space itself became the twisting. Here, the notion of logical perfection is of a different kind from what we had in the Galois example. There, global symmetry (and the connection between symmetry and extensions where solutions live) was the expression of that perfection. Here the perfection is rather the new *flexibility* Riemann's construction offers us, when compared with earlier incarnations of the notion of space, of geometry.[3] *All possible geometries*, in a strong sense, are put to play together, thereby going beyond older notions of having to make an "a priori choice" between three geometries. Riemann's flexibility and *globality* when treating geometries ends up providing another notion of perfection—not quite "logical" perfection on the face of it, although recent research has shown categoricity of certain constructions connected to Riemann's work.

Our third example, much more contemporary and of a different kind, is Grothendieck's new foundations of algebraic geometry. Very roughly, the concept of a general notion of space (as in Riemann) is again at stake. But here Grothendieck essentially first "disassembles" the surfaces or curves (called more generally *varieties* by mathematicians) by putting all the weight of the analysis into one single aspect (localisation) of the space and then finding a system for placing these localisations in a coherent way. By doing this, Grothendieck creates a notion of spatiality called *affine scheme* that embodies two movements: first, the localisation (and the possibility of treating only *one aspect* of the space) and second, the coherence. This highlights yet a different aspect of logical perfection: the possibility of regarding space as a coherent way of pasting localised versions of itself.

These three examples reveal different phenomena, but a common feature *emerges* from them: the greater *perfection* of a notion of wholesomeness, of completeness, and ultimately of categoricity. In Galois, the specific search for solutions to algebraic equations is replaced by a much more global quest for

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3 Suffice it to say that half a century later, Einstein would base his General Relativity Theory on Riemann's work: the mathematical content of Einstein's theory is essentially present in Riemann's approach to geometry. Here, the perfection aspect has more the flavour of a way to construct *many* possible geometries, one for each Riemann metric—one for each way to "twist" space, so to speak—and a global treatment of all of these geometries (and moreover, mathematical ways of classifying and comparing them).

symmetries of all possible solutions, and a control of how these symmetries appear (through groups). The global picture that emerges is ultimately uniquely determined and gives a duality between extensions and symmetries. In Riemann, specific geometries are replaced by arbitrary metrics and the resulting objects, the Riemannian manifolds, end up capturing *all possible ways of twisting space*. Again, a global aim and in some sense, a uniqueness (especially of ways of covering some of these manifolds by others exhibiting more symmetries). In Grothendieck, there is an apparently different element (localisations, and then the construction of sheaves, that is, of ways of building spaces where the localisations may be glued in a coherent way). These sheaves are also much more *complete* mathematical objects than the corresponding objects from where the construction started.

Looking beyond the specifics of these constructions, the cluster notions that emerge are **uniqueness, completeness, wholesomeness.** It is these notions that we claim mathematical logic (and especially its branch called **model theory** and its subbranch called **stability theory**) from the last half century provides us with sharp tools to study and claim a refined, acute, keen and at the same time powerful notion of perfection.

In the next section we explore this connection in more detail.

2. Logical Perfection and the Issue of Uniqueness

In the previous section we described why we may regard a notion of uniqueness, of completeness and wholesomeness as our main aim.

One can confidently claim that the central concept of present-day model theory is that of **stability** of formal theories and one key notion of stability theory (from which it started in the 1960s) is that of **uncountably categorical** theories. Through the efforts of many people, and most prominently by contributions of S. Shelah (see Shelah, 1990), we now have a rather comprehensive **classification theory** which establishes an effective hierarchy in the universe of mathematical structures (or their theories).[4] The hierarchy is effectively based on the complexity of the system of invariants which ultimately describe a given structure, a model of a formal theory.[5] The highest level of the hierarchy corresponds to the simplest system of invariants. This corresponds, in some sense, to a highest level of perfection.

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4 An interesting interactive visualisation of a "map of the universe" can be seen online at http://www.forkinganddividing.com.

5 Shelah also uses (super-)stability as a *first* criterion of whether the given first-order theory has a *structure theorem*, that is if the isomorphism types of models of the theory can be classified in terms of a simple combinatorial structure.

The previous emphasis on the stability hierarchy, and in particular the region near what we regard here as its pinnacle (namely, uncountably categorical theories) *describes* a mathematically rigorous (and completely abstract) approach to a notion relevant to a working definition of logical perfection. We still have to address the issue of how adequate and useful this notion is, which dividing lines it draws and which important mathematical structures satisfy the criteria.

An interesting observation from the *cumulated experience* of the past half century with the study of the stability hierarchy would establish (very roughly) that *the higher a structure is in the hierarchy, the closer it seems to be to mathematical structures that have been central to the work of mathematicians throughout many centuries*. This defines a kind of focal point of the mathematical universe - in our case, algebraic geometry in the broadest sense.[6] In some (limited) sense we may define the most general form of geometry to be the structures populating the top levels of stability hierarchy.[7]

It is in this precise sense that we regard *categoricity* (in uncountable cardinals) as a pinnacle of classification theory: the observation that many important mathematical structures (those from algebraic geometry or those corresponding to linear phenomena) seem to hover close to that region[8] or, if not quite in that region, may be regarded as good approximations of categorical structures.[9]

The notion of categoricity concretises the meaning of *uniqueness*. One says that a collection of statements in a formal language *(set of axioms)* is *categorical* if it has just one model, up to isomorphism.[10]

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6   Defining exactly what algebraic geometry "in the broadest sense" means is necessarily vague and outside the aim of this paper; for the record, we mention Grothendieck's work on affine schemes as a good possibility.

7   Working on this presumption one arrives at a meaningful notion of non-classical geometric spaces (see Zilber, 2008; Cruz Morales and Zilber, 2015; Zilber, 2016 and the discussion in section 3) which in a more conventional mathematical setting are treated via the formalism of *non-commutative (or quantum) geometry*. The latter approach is essentially a syntactic algebraic analysis avoiding geometric semantics.

8   There are important exceptions to this reason. The first one is obvious: real numbers are far from being categorical yet are also clearly central mathematical structures in many senses. However, aside from the infinite order that is the reason for their non-categoricity, the exhibit a rather simple structure of *definable sets*: each one of them is really a finite union of intervals. This notion, called **o-minimality,** provides reasons to place them in a region where some of the good properties of uncountably categorical structures still work, albeit in a different way. The role of interactions between complex analysis and real analysis is mimicked by this correspondence. The second exception is subtler: classification theory provides many other regions that, while not corresponding to the "supremely perfect" uncountably categorical region, they exhibit very strong regularity and smoothness properties.

9   One theory of such approximations is the recent work of Abdolahzadi and the third author of this paper: categorical structures are obtained as limits of certain constructions dealing with structures in a slightly lower stratum in the stability hierarchy, for the area of geometry known as *anabelian geometry*; see Abdolahzadi and Zilber (2019).

10   The expression *up to isomorphism* means that we do not want to distinguish two structures if they differ only by the way their elements are presented.

The choice of the formal language is essential. Usually it is meant to be a first-order language.[11] However, as the research in the last three decades has shown, much of what will be said below about categoricity in the first-order context holds in a more general setting (see, in this volume, Väänänen's paper (Väänänen, 2020)).

The notion of categoricity has existed for as long as logic has been formalised. But in the context of first order languages one realises very quickly, from basic facts of the theory, that the above *absolute* categoricity can only hold for descriptions of finite structures. For infinite structures **M** it is possible to have uniqueness in some cases if we add to the first order description the (non-first-order) statement fixing the cardinality κ of the structure **M**. This relative categoricity is called **categoricity in cardinality (in power) κ** or just κ-categoricity.

Furthermore, there are only two kinds of cardinality really relevant when one is interested in studying categoricity in power: countable and uncountable. Here, we are interested in uncountable categorically describable structures (this entails that the structure is much bigger than the size of its description). A remarkable fact was proved by Michael Morley in 1964, namely, that categoricity in one uncountable cardinality implies the categoricity in all uncountable cardinalities: the actual value of the uncountable cardinal is irrelevant.[12]

The study of this kind of structures has been in the focus of research in model theory for at least 60 years. The amazing conclusion derived from the research is that among the huge diversity of mathematical structures there are very few which satisfy the (slightly narrower) definition of categoricity, and those can be classified. These certainly seem to corresponding to an ideal of *logical perfection*, in the following sense: categorical structures M determine a first order theory $Th(M)$ (the set of all sentences that are true in M) and then comes the reason why we call them "logically perfect": **all other structures that satisfy the theory $Th(M)$ and are of the same cardinality as M are isomorphic to M.** In other words, uncountably categorical structures are inextricably linked to their logical description; the description $T = Th(M)$ completely determines the structure M (with the usual caveat of "up to isomorphism" and because of limitations in the expressive power of first order logic[13] provided also one considers only structure of the same cardinality as M).

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11 That is, one which allows only finite length formulas and quantifiers "for all" and "there exists" which refer to elements of the structure in question (but not to relations or functions).

12 This is in sharp contrast with countable categoricity. Countably categorical structures might also in some sense be candidates to a kind of perfection, probably - but all the geometric features of uncountably categorical structures are lost in that case. This dependence on the cardinality might be regarded as non-logical in some sense, but the case of uncountable categoricity has strong logical properties as well as strong geometric properties.

13 Namely, the Löwenheim-Skolem theorem.

It is not that surprising that a remarkable example of such theory is the theory of the field of complex numbers $\mathbb{C}$ in the language based on algebraic operations $+$ and $\times$. Note that this is the language where algebraic geometry is naturally done[14] but we can not, e.g. distinguish the real part of a complex number, so we can not speak about the real numbers when working over $\mathbb{C}$. Recall that the theory of the field $\mathbb{R}$ of real numbers is not categorical.[15]

Complex numbers are present everywhere in mathematics as are the reals. However, there is a significant difference between the theory of the reals and that of the complex numbers; in fact complex geometry and the geometry of real manifolds are two different areas of specialisation within mathematics. Classification theory sharply detects and explains the difference; it places complex geometry at the top of the hierarchy, and real geometry in an interesting region sharing just some of the good properties of categorical theories.

Of course, for a mathematician the choice of an area of research is a personal matter and is usually made on either historic or aesthetic grounds. Both complex and real geometry are equally respected fields of mathematical research although, arguably, the first is fundamental while the second is auxiliary. We stress again the fact that it was Riemann who (building on Cauchy's work) first understood how real and complex geometries interact with one another and how the study of the latter introduces a whole new range of powerful methods of algebraic geometry into the field.

The mathematical model of Newtonian physics is based on real analytic geometry. This tradition continued into quantum mechanics with the model enriched by more and more uses of complex numbers, seen rather as convenient auxiliary tools. One of the first who pointed to the importance of reversing this perspective was Roger Penrose in his 1978 address at the International Congress of Mathematicians under the title *The complex geometry of the natural world* (Penrose, 1980). In more recent decades, with the arrival of string theory, the priority or at least the centrality of complex geometry is undeniable.

To summarise the *logicality* of our notion of perfection: we started with various notions of perfection as in Section 1, coming from three examples in the history of mathematics; we then narrowed our focus to a notion of uniqueness and its logical expression, (uncountable) **categoricity**. We then remarked that a whole classification theory that encompasses *all first order theories*[16] on the one hand grew up out of the attempts to prove the Morley theorem and its generalisations

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14   In algebraic geometry classical objects are solution sets of algebraic expressions, that is, polynomials written with $+$ and $\cdot$.

15   And is not even stable!

16   And in more recent decades many more kinds of theories.

and on the other hand ended up providing ways of callibrating *exactly how far* from categoricity one is, in terms of smoothness/regularity properties that slowly vanish as we go further and further away from categoricity. It is in this very sense that categoricity has been playing the role of a logical form of perfection. A posteriori we realise that a major part (although not all) of historically central mathematics actually happens to be one of the theories that are uncountably categorical.

3. Logically Perfect Structures: The Role of Geometry

Perhaps the most remarkable feature of model-theoretic classification theory is that it exposes a geometric nature of some "perfect" structures. The geometric features of those structures arise from their logical definition, albeit in a highly non-trivial and initially unforeseen way. These were discovered in the course of proving the original ground-breaking categoricity theorem of Michael Morley, described in the previous section.[17] It took a while to realise the *geometric character* of the technical definitions and to develop a new geometric intuition around the notions. In particular, the notion of a Morley rank is a very good analogue of dimension in algebraic and analytic geometry and we thus have basic tools to think of abstract versions of curves and surfaces in any mathematical theory that turns out to be uncountably categorical. This stage of the theory is summarised in the monograph (Pillay, 1996) by A. Pillay.

In the 1980s the third author formulated a Trichotomy Conjecture (see (Zil'ber, 1984)) which, based on the above intuition, suggested that any uncountably categorical structure is in some sense reducible[18] to either an object of algebraic geometry, or linear algebra, or to a simple combinatorial structure. Although in many special classes the conjecture has been confirmed, the general case was refuted by Ehud Hrushovski who found remarkable counter-examples opening fascinating new perspectives on the nature of model theory (its interactions with geometry) and its links with the world of structures important in the area of Mathematical Analysis.

Around the same time, a way to fix the original Trichotomy Conjecture was found. This required narrowing the class of structures subject to the conjecture —in some sense, this amounted to refining the notion of logical perfection. This was done by being more careful in choosing the logic involved in the description

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17 Geometric features of those structures were discovered as the key technical instruments of the proof: Morley rank, homogeneity and, added in later versions of the proof, **dimension** (Baldwin and Lachlan), and associated combinatorial geometries (Marsh, the third author).
18 Technically, bi-interpretable.

of the relevant structures. Namely, our logical language must be able to distinguish *positively formulated* statements from their negations. The axioms of a good (perfect) theory must be *equational* just like laws of physics and objects of geometry are given by equations (and not by negating equations!). But this is precisely the principle on which algebraic geometry is built! It studies curves, surfaces, shapes given as solution sets for systems of algebraic equations; treats such sets as *closed* in an important topology in algebraic geometry called "Zariski topology". The corresponding generalisation of this notion in the context of categorical and stable structures led to the notion of a *Zariski structure* (or *Zariski geometry*) introduced by Hrushovski and the third author.

This improvement in the notion led to a desired Classification Theorem (Hrushovski, Zilber 1993, see Zilber 2010). Precisely, the class of Zariski geometries satisfies the Trichotomy Principle and therefore Zariski geometries are reducible to[19] classical structures such as the field of complex numbers and vector spaces.

There was therefore a *reduction of scope* (trichotomy only valid for Zariski structures, a special subclass of our perfect structures) but a sharper understanding of the necessary tools for this trichotomy to happen.

While it may be hard to describe what exactly the subject of geometry as practised by mathematicians is, describing non-commutative geometry is a much more daunting task. It is best identified as the study of algebraic structures, in many cases called non-commutative coordinate rings, supposed to correspond to hypothetical geometric spaces which are not necessarily visualisable. Historically, it was some physicists who, starting from the famous "magic paper" of Heisenberg of 1925 (Heisenberg, 1925), had given up on the attempts to describe the physics of micro-world in classical terms and were instead using a purely formal algebraic calculus (algebraic quantum mechanics) to explain the behaviour of elementary particles. Strangely perhaps, their move to these purely formal algebraic structures met enormous success. One can say that the physics of the micro-world lives in an unusual, previously unknown, sort of *geometric space*, only describable by means of a non-commutative algebra. A curious parallelism has started to

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19   Here "reducible to" can be taken in a first reading as a technical nuisance not requiring much explanation. The typical example of Zariski geometry is a (complex) algebraic variety (glued from affine charts) with possibly a vector bundle over it, a description of which can require quite a lot of technical detail. Such a description eventually reduces to the structure of the complex field itself. However, the constructions described by the theorem can go beyond the technicalities of this example, so beyond algebraic and complex geometry. Ten years after the classification theorem, a closer analysis of what "reducible to" could mean led to the discovery that a huge source of new Zariski structures is non-commutative (or quantum) algebraic geometry, see Zilber (2008).

emerge: the top of the stability hierarchy is occupied by structures which mathematically stem from the same source.

The fusion of geometry with other branches of mathematics, such as number theory and representation theory, was one of the biggest programs in the mathematics of the 20th century.[20] We would like to believe that the fusion of logic (model theory) with other branches of mathematics is one of the biggest and most ambitious programs of the mathematical research for the 21st century. In particular, the *new geometry* arising from model theoretical considerations has the potential to become an important area of research in mathematics and beyond. And the study of logically perfect structures gives a crucial insight.

Summarising, the search of logically perfect structures leads to consider geometric/topological ingredients in logic which has as a consequence that a refinement of the idea of logical perfection is obtained. During this process the idea of Zariski structures arises from purely logical considerations but with a geometrical flavour and motivation. So far, our discussion has not left mathematics but now, based on our previous discussions, we want to go beyond mathematics, entering what some may call the "real world" for lack of a better term. A question arises: Are logically perfect structures helpful for understanding the world outside of mathematics? We next attempt a positive answer to this question and provide some possible directions of research.

4. Logical Perfection and Physics

We now focus on a different kind of problem: programs for new foundations of quantum gravity, and the issue of tackling an appropriate notion of geometric space for physics. This problem would seem a priori quite remote from our notion of logical perfection. There is however a deep link, as we will describe.

Roger Penrose said in his 1978 ICM address (Penrose, 1980, p. 189):

"Even at the most elementary level, there are still severe conceptual problems in providing a satisfactory interpretation of quantum mechanical observations in a way compatible with the tenets of special relativity. And quantum field theory, which represents the fully special-relativistic version of quantum theory, though it has had some very remarkable and significant successes, remains beset with inconsistencies and divergent integrals whose ill effects have been only partially circumvented. Moreover, the present status of the unification of general relativity with quantum mechanics remains merely a collection of hopes, ingenious ideas and massive but inconclusive calculations. In view of this situation it is perhaps not unreasonable to search for a different viewpoint concerning the role of geometry in basic physics. Broadly speaking, "geometry", after all, means any branch of mathematics in which pictorial representations provide powerful aids to one's mathematical intuition. It is by no means necessary that these "pictures" should refer just to a spatio-temporal ordering of physical events in the familiar way..."

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20   The figure of Grothendieck was essential in formulating and developing this program in the broadest generality.

Penrose continues to discuss structures of complex geometry as new geometric tools in quantum physics. However, today this seems to be far from enough.

From a similar reasoning the physicist C. Isham and the philosopher of physics J. Butterfield reached a bold program for building a new foundation of quantum gravity physics, based on Grothendieck toposes as the most general form of geometric space (see Isham and Butterfield, 2000).

Naturally, the Isham-Butterfield program is not the only one to tackle the problem (see e.g. the non-commutative geometry approach (Connes and Marcolli, 2008) by A. Connes and M. Marcolli; they, however, do not quite reveal a *geometric* space per se); however, the Isham-Butterfield program seems to be the most ambitious and general[21] attempt so far.

The third author has suggested, and started in Zilber (2016), a project in some sense comparable in spirit to Isham-Butterfield; together with the first author, they have developed a further stage of this project (Cruz Morales and Zilber, 2015). Like in other such programs, the key is the respective notion of the geometric space for physics. Our suggestion is based on the philosophy of logical perfection; after all *it is reasonable to expect that the geometric structure of the universe should be as perfect as it goes*. Correspondingly, the geometric space of quantum mechanics as suggested in Zilber (2016, 2018) emerges from a Zariski structure (see section 3) or rather, from a sheaf of Zariski structures.[22]

It is equally important to note that the logical analysis explicitly underlying the method we describe clarifies the correspondence between (possibly noncommutative) algebras as they emerge in physics and geometry and the respective geometric spaces. In essence the algebras present us with syntactic tools allowing to check in calculations what can be seen graphically and dealt with geometrically. The geometric space is therefore a semantic interpretation of the syntactically given data.[23]

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21   Maybe too general as to the best of our knowledge there is no interesting calculation produced out of it.

22   The following three facts clarify the connection between Zariski structures and the Isham-Butterfield topos:

(1) The sheaf of Zariski structures, the model of quantum mechanics, can be interpreted as a concrete realisation of an Isham-Butterfield topos.

(2) The construction essentially generalises (Zilber, 2008) building a Zariski structure corresponding to the non-commutative algebra represented by the canonical commutation relation $QP - PQ = i\hbar$.

(3) The analysis of the language and definability issues in the structure draws a clear line between notions which are *observable* (in the sense of physics) and which are not.

23   In classical cases, such as commutative finitely generated algebras, this corresponds to the well-known duality at the foundation of algebraic geometry. For commutative C^*-algebras we have the Gel'fand-Naimark duality linking those to locally compact Hausdorff spaces. In non-commutative cases the situation becomes much more complex but model theory is in the best position to deal with the challenge.

Another, different, intersection area between categoricity and physics has been explored by D. Howard and I. Toader in the past two decades (see Howard, 2018; Toader, 2018). Their take on categoricity is more akin to the original Veblen formulation (independent of cardinality) than to the role categoricity has acquired in contemporary model theory.

We finish this section with the conclusion that the principle of logical perfection, as unconventional as it may sound to some, does not disagree with other modern approaches to the mathematical foundations of physics.

5. Concluding Remarks

Our concept of logically perfect structures emerges as a side result of a 50 year classification project in logic. We may now, well-equipped with our examples and discussions, attempt a definition!

(It is worth stressing that our theory relies on technical aspects outside the scope and aim of this article; the interested reader may consult (Cruz Morales and Zilber, 2015; Zilber, 2016; Väänänen, 2020; Villaveces, 2020a)).

The defining property of logical perfection is uniqueness, or technically uncountable categoricity. This property implies certain internal harmony: homogeneity and the presence of a notion of dimension. This harmony is a manifestation of a certain kind of geometricity, which itself is a consequence of the infusion of geometric/topological ingredients in logic that brings forth the flexibility and generality of logically perfect structures. Finally, since logical structures are at the top of the classification hierarchy, they are suitable as background structures for physics and represent a good idea of geometric space in a very broad sense.

An additional feature to support our notion of logically perfect structures is the "filtration" of perfection provided by classification theory. As mentioned above, classification theory not only places all first order theories in a sort of map *with respect to* categorical theories but provides a kind of measure of going away from perfection. It provides technical ways to measure, for arbitrary theories, what features of perfection might have been lost and which ones remain. The second author's forthcoming interview with Saharon Shelah explores further several peculiarities of this connection (see the relevant excerpt of the interview in this volume (Villaveces, 2020b)).

A recent take on *incomplete* structures and their "lego-like building blocks", due to Neil Barton (Barton, 2020) is being written at the time of press. We note the connection to our notion of perfection (and his way of measuring imperfection). On the other hand, John Baldwin's analysis of the role of stability theory in

Baldwin (2018) and the paradigm shift in model theory with the arrival of stability theory has a rather different role for categoricity.

The features described above (uniqueness, geometricity, representability) have concrete mathematical formulations, as we have briefly mentioned. In addition, they help us to understand the role of those structures in the wider program of studying the syntax/semantics duality. As we have tried to show, logically perfect structures can be seen as located in the geometric/semantical side of the mentioned duality, giving a new approach to the notion of noncommutative (or quantum) geometric space, which traditionally has been treated by means of syntactic/algebraic tools. Pursuing this program of interpreting the duality between algebraic and geometric objects as an extension of the duality between syntax and semantics appears to us as one of the most interesting lines of research for the future, not only in mathematics. The idea of representing one object by another (in this case its dual) can certainly be extrapolated beyond mathematics. This idea deserves more investigation.

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