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ALGEBRAIC GEOMETRY VIA MODEL THEORY

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The subject of the lecture is to present a model-theoretic point of view at some more positive parts of mathematics and to demonstrate some minor applications of this approach to such respected field of mathematics as algebraic geometry also expressing hopes that more applications can be obtained along these lines.

The starting point for the model - theoretic subject under discussion is the study of uncountable structures (algebraic systems) which can be described uniquely up to isomorphism in the first order language, provided the cardinality of the structure is given. Such structures are called uncountably categorical (u.c.). By the definition u.c. structures are ones which fit ideally into the scheme of axiomatic mathematics,so not surprisingly many classical structures satisfy the definition:

(a) algebraically closed fields $\mathbb{K}$; the groups $GL(n,\mathbb{K})$; simple algebraic groups over $\mathbb{K}$;

(b) vector spaces $\mathbb{V}$ over any countable division ring $\mathbb{R}$; Grassmannian algebra of a bounded grade over $\mathbb{V}$, provided $\mathbb{R}$ is finite;

(c) ultra-homogeneous irreflexive graphs of finite valency (here ultra-homogeneous means that for any two finite isomorphic subgraphs there is an antomorphism of the whole graph sending the

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first onto the second).

In fact the condition on the cardinality of the examples can be omitted since any infinite structure of the type is first-order equivalent to a structure of any given infinite cardinality of the same type.

A fundamental step in the study of u.c. structures was made by M. Morley in 1963. He discovered that for any u.c. structure $\mathbb{M}$ (with universum $\mathbf{M}$) there is a natural way to introduce a notion of a rank for any set definable in $\mathbb{M}$. The term "definable" is basic in this paper. By a definable set S we mean a quotient-set of the form $S = \mathbf{U}/\mathbf{E}$ where $\mathbf{U} \subseteq \mathbf{M}^n$, $\mathbf{E} \subseteq \mathbf{U}^2$ are subsets of all n-tuples and 2n-tuples in $\mathbf{M}$ satisfying first-order conditions and $\mathbf{E}$ is also an equivalence relation on $\mathbf{U}$. If in the conditions parameters $\mathbf{X} \subseteq \mathbf{M}$ are used we say S is $\mathbf{X}$-definable. We say that a structure $\mathbb{N}$ is $\mathbf{X}$-definable in $\mathbb{M}$ if the universum $\mathbf{N}$ of $\mathbb{N}$ and all basic relations and operations (interpreted as subsets of $\mathbf{N}^m$) are $\mathbf{X}$-definable sets in $\mathbb{M}$.

In the examples (a) definable sets in $\mathbb{K}$ are exactly constructible sets (by a theorem of Tarski and Seidenberg) and the Morley rank coincides with the dimension in the sense of algebraic geometry. In the example (b) the Morley rank of the set of Grassmannians of grade r is exactly r. Later it was proved that Morley rank is finite for u.c. structures and enjoys all properties of a good notion of dimension. J. Baldwin and A. Lachlan showed that in many respects the study of u.c. structures can be reduced to the study of their substructures of rank 1, which cannot be divided on two parts of rank 1 (strongly minimal sets).

For example, algebraic groups in example (a) can be reduced to the field $\mathbb{K}$; the Grassmannian algebra in example (b) can be reduced to the vector space $\mathbb{V}$, and the graph (c) is reducible to a trivial structure, i.e. a set with no relations and operations except the equality.

It was also introduced very nice and useful notion of a closure of any set $\mathbf{X} \subset \mathbf{M}$ (denoted $cl(\mathbf{X})$) which in the case (a) coincides with algebraic closure of $\mathbf{X}$, in the case (b) the linear closure and in (c) $cl(\mathbf{X})$ is the set of all points connected to a point in $\mathbf{X}$.

In 1980 the present author noticed that the types in examples (a),(b),(c) are characteristic in general and proved the following

TRICHOTOMY THEOREM. *Any u.c. structure* $\mathbb{M}$ *is either*

(i) *reducible to a trivial structure*

or

(ii) *reducible to a vector space over a division ring*

or

(iii) *there is in* $\mathbb{M}$ *a definable set* $\mathbf{P}$ *of rank* 2 *and a definable family of rank* ≥ 2 *of subsets of* $\mathbf{P}$ *each having rank* 1 *and any two intersecting in a finite number of points.*

Very essential was an associated result stating that case (iii) does not hold under the condition that $cl(\mathbf{X})$ is finite for all finite $\mathbf{X} \subset \mathbf{M}$. It was also conjectured that structures in (iii) are reducible to algebraically closed fields.

One should add also that the above trichotomy can be considered in much more general context. All this, as we think, has a universal mathematical meaning. First of all,the theoretical trichotomy confirms the natural division in mathematics: algebraic geometry, linear algebra, combinatorics of finite graphs. From the other hand we see explicitly common notions, ideas and techniques in these three parts. This general approach gave serious applications at least in mathematics associated with cases (i) and (ii). We mention here two results which gave solutions to open problems.

The first one concerns combinatorial geometries and in the final analysis fits case (ii). Combinatorial geometry is a set S with a closure operator cl satisfying the replacement axiom: $z \in cl(X \cup \{y\}) - cl(X)$ implies $y \in cl(X \cup \{z\})$. It is also assumed that the closure of any point is a point. The closure of two distinct points is called a line. A subset $\{x_1, \ldots, x_n\}$ is said to be independent if $x_{i+1} \notin cl(x_1, \ldots, x_i)$ for any $i<n$. So we have a notion of dimension. A geometry is called homogeneous if any two independent n-tuples of elements are conjugated by an automorphism of the geometry.

THEOREM. *Any finite homogeneous geometry of dimension greater than 7 with at least three points on any line is a projective or affine geometry over a finite field (possibly truncated).*

The proof does not depend on classification of finite simple groups. The infinite version of the theorem is in [Z1], [Z2] and in [E]. Then D. Evans proved it for $dim \geq 23$ (unpublished) the final version belongs to Zilber [Z4].

The second result can not be stated so directly. In short, it was found that the techniques used in the study of case (i) is crucial in understanding finite ultra-homogeneous structures, in particular, graphs. This is shown by A.Lachlan, G.Cherlin and others, see survey [L].

So the applications given above show that the model-theoretic approach is rather powerful at least in cases (i) and (ii). One may hope that some applications would also be possible in case (iii), more precisely, in algebraic geometry. This is exactly what we are concerned with in the rest of the paper. But before going to discuss the applications in details we state precisely the conjectures concerning case (iii) and comment it.

The **MAIN CONJECTURE** in a precise form consisted of two parts:

(A) *In any u.c. structure* $\mathbb{M}$ *satisfying* (iii) *an algebraically closed field* $\mathbb{K}$ *is definable;*

(B) *If* $\mathbb{M}$ *in* (A) *is of rank* 1 *then* $\mathbb{M}$ *is definable in* $\mathbb{K}$.

Unfortunately the real world is not so nice. E.Hrushovski constructed series of very interesting counterexamples to (A). So the condition (iii) is too weak to be adequate to algebraic geometry. There is a hope that under the additional assumption that $\mathbb{M}$ can be equipped with a good Zariski-kind topology (A)and (B) will prove correct. Hrushovski announced he has a proof of this recently.

However a restricted form of the conjecture above is of some interest.

RESTRICTED CONJECTURE: *Suppose* $\mathbb{M}$ *is a structure definable in an algebraically closed field* $\mathbb{K}$ *and satisfies* (iii) *then a field-structure* $\mathbb{K}'$ *isomorphic to* $\mathbb{K}$ *is definable in* **M**.

Notice that part (B) of **MAIN CONJECTURE** is already in the hypothesis of the **RESTRICTED CONJECTURE**. We also need not assume $\mathbb{M}$ to be uncountably categorical. In fact, we have a slighter assertion: the Morley rank of $\mathbb{M}$ is finite. This follows from the hypothesis that $\mathbb{M}$ is definable in $\mathbb{K}$, which is a finite Morley rank structure.

It is important to notice also that it is generally an easy exercise for a given structure $\mathbb{M}$ to verify whether it satisfies (iii). In fact, structures not satisfying (iii) are described by (i) and (ii) almost explicitly (though we have not presented the exact meaning of "reducible" here). On the other hand to pass from (iii) to defining a field in $\mathbb{M}$ is a difficult problem which, we guess, is related to synthetic algebraic geometry. A very special case of the problem is the celebrated theorem of projective geometry stating that in a projective desarguesian plane given the collineation relation one can define the coordinatization field.

1. Rich structures

From now on by a structure we mean a structure definable in a fixed algebraically closed field $\mathbb{K}$.

We will call a structure $\mathbb{M}$ rich if $\mathbb{M}$ satisfies the conclusion of **RESTRICTED CONJECTURE**.

We present now some results concerning special cases of the conjecture which also give series of examples of rich structures.

The first published result was by G. Martin [M], which for simplicity we cite assuming $char\mathbb{K}=0$.

THEOREM. *Suppose* $\mathbb{M}=\langle K,+,g\rangle$ *or* $\mathbb{M}=\langle K,\cdot,f\rangle$ *where* K *is the universum of the field* $\mathbb{K}$; $+,\cdot$ *its operations,* g,f *rational one* *variable functions on* K, g *is nonlinear,* f *is not of the form* ax^m. *Then* $\mathbb{M}$ *is rich.*

The proof of the theorem was obtained by witty manipulations with algebraic terms constructing the multiplication and addition in the two cases correspondently. It is hardly plausible these arguments could work in the general situation.

In a joint work of E.D.Rabinowich and the author [RZ] another approach was proposed.

THEOREM. *Suppose* $\mathbb{M}=\langle K,+,R\rangle$, *where* K *is as above and* $R(x,y)$ *is a binary relation on* K *meaning* $\langle x,y\rangle \in C$ *for an irreducible curve* C *of degree at least* 2. *Then* $\mathbb{M}$ *is rich.*

To prove the theorem we considered curves on the affine plane K^2 as binary relations on K. Then giving two curves S_1, S_2 obtained by shifting C using + one may construct a new curve $S_1 \cdot S_2$ as the composition $\exists y(S_1(x,y) \ \& \ S_2(y,z))$. From assumptions one can deduce that if (x_0,y_0) is a generic point of S_1 and S_2 then (x_0,x_0) is a nonsingular point on the curve $S_1 \cdot S_2^{-1}$. What is more the set F of curves obtained in this way and going through (x_0,x_0) behaves well with respect to the composition: "almost always" the composition of two such curves gives a curve with (x_0,x_0) nonsingular on it.

Under these conditions intersections of these curves is studied. Combining the Bezout Theorem with properties of the family F we showed that the relation E between curves: "S_1 is tangent to S_2 in (x_0,x_0)" is expressible in terms of the numbers of points in set-theoretic intersections of S_1, S_2 and some other curves from the family, and so is definable in the language of $\mathbb{M}$ (using some parameters).

Since the relation E is an equivalence relation on F one gets the definable set F/E which is birationally ejuivalent to K. The composition on F induces a multiplication on F/E which is isomorphic to the field-multiplication. In the final analysis we define a field - structure in $\mathbb{M}$, which is isomorphic to $\mathbb{K}$.

A. Pillay and D. Marker showed that these considerations could be essentially simplified by using a combinatorial criterion, so called field-configuration discovered by E. Hrushovski in connection with **MAIN CONJECTURE**. Hrushovski's very nontrivial construction makes it possible to find a field under very weak assumptions.

The strongest result to the moment covering the theorems above is obtained by E. D. Rabinovich.

THEOREM. *Suppose* $\mathbb{M}$ *is a u.c. structure satisfying* (iii) *and a rational curve is definable in* $\mathbb{M}$. *Then* $\mathbb{M}$ *is rich.*

Since the rational curve can be identified with K the ideas from the previous proof can be used. Dealing with singularities can not be avoided this time. Though the intersection theory on projective plane is clear, it takes rather complicated combinatorial arguments to see that the relation: "there is a branch of S_1 and a branch of S_2 with the multiplicity of intersection in (x_0, x_0) at least n" is definable for every n. Then the field can be reconstructed from the group of n-jets of all definable curves for some n.

Let us illustrate the result with the following application. Let C be a smooth projective curve of genus $g \geq 2$. Consider $2g$-ary relation $E_g(x_1, \ldots x_g, y_1 \ldots y_g)$ on the set C meaning that divisors $x_1+\dots+x_g$ and $y_1+\dots+y_g$ are linearly equivalent. Then the structure $\mathbb{C}=\langle C,E_g\rangle$ is rich. To see this from the Rabinovich theorem first note that $\mathbb{C}$ is uncountably categorical and satisfies (iii). The first fact follows from a general statement that any structure whose universum is strongly minimal is u.c., C satisfies this condition. The second fact follows from **TRICHOTOMY THEOREM** after we note that in $\mathbb{C}$ the Jacobian group $J(C)$ is definable together with an embedding $C \subset J(C)$. In structures of type (i) no infinite group is definable at all and in structures of type (ii) any definable subset of a group almost contains a coset of an infinite subgroup. This is not so with C. At last,to satisfy the third assumption of the Rabinovich theorem, note that we can define any fixed class of divisors in $\mathbb{C}$. Fixing a very ample class we get all cuts by hyperplanes in the corresponding embedding of C into a projective space. Then it is easy to single out a set of cuts isomorphic to a protective line.

2. Very rich structures

The fact that a structure $\mathbb{M}$ is rich gives you the ground-field $\mathbb{K}$ reconstructed by means of the structure $\mathbb{M}$ but in general it does not let one to reconstruct the way $\mathbb{M}$ is defined in $\mathbb{K}$ and so to reconstruct $\mathbb{M}$ as an algebro-geometric object. So we introduce a stronger notion of a very rich structure. To do this one should know a model-theoretic notion of a definable closure. let $a_1,\dots a_n,b$ be elements of a set $\mathbf{N}$ which is $\emptyset$-definable in $\mathbb{M}$. We say b is in the definable closure of $\{a_1,\dots a_n\}$ if there is a first order formula $\varphi(x_1,\dots x_n,y)$ in the language of $\mathbb{M}$ and free variables $x_1,\dots x_n,y$

divisors $x_1+...+x_g$ and $y_1+...+y_g$ are linearly equivalent. Then the structure $\mathbb{C}=\langle C,E_g\rangle$ is rich. To see this from the Rabinovich theorem first note that $\mathbb{C}$ is uncountably categorical and satisfies (iii). The first fact follows from a general statement that any structure whose universum is strongly minimal is u.c., C satisfies this condition. The second fact follows from **TRICHOTOMY THEOREM** after we note that in $\mathbb{C}$ the Jacobian group $J(C)$ is definable together with an embedding $C \subset J(C)$. In structures of type (i) no infinite group is definable at all and in structures of type (ii) any definable subset of a group almost contains a coset of an infinite subgroup. This is not so with C. At last,to satisfy the third assumption of the Rabinovich theorem, note that we can define any fixed class of divisors in $\mathbb{C}$. Fixing a very ample class we get all cuts by hyperplanes in the corresponding embedding of C into a projective space. Then it is easy to single out a set of cuts isomorphic to a protective line.

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and it is true in $\mathbb{M}$ $\varphi(a_1,\dots a_n,b)$ & $\exists! y\varphi(a_1,\dots a_n,y)$.

A structure $\mathbb{M}$ is very rich if an infinite field $\mathbb{K}'$ is definable in $\mathbb{M}$ and there are $c_1,\dots c_k \in \mathbf{M}$ such that any element of $\mathbf{M}$ is in the definable closure of $\{c_1,\dots,c_k\}\cup K'$.

Proposition 1. *For any structure* $\mathbb{M}$ *the following are equivalent:*

(a) $\mathbb{M}$ *is very rich;*

(b) *there is a unique way of reconstructing the family of all constructible subsets of* $\mathbf{M}^n$ *assuming that all the relations of* $\mathbb{M}$ *are constructible;*

(c) *any abstract automorphism* α *of* $\mathbb{M}$ *is a composition of an automorphism* υ *induced by an automorphism of the field* $\mathbb{K}$ *and a constructible automorphism* ρ *of* $\mathbb{M}$.

Recall that constructible set is by definition a union of differences of Zariski-closed subsets. Constructible automorphism is one whose graph is a constructible subset of $\mathbf{M}\times\mathbf{M}$.

The proof of Proposition 1 is rather easy modulo known model-theoretic facts and constructions exposed in [P].

(b) follows from (a) since under the assumption $\mathbb{M}$ can be presented as the image of a constructible mapping of a constructible subset of K'^m. Note also that $\mathbb{K}'$ is definable in $\mathbb{K}$ and hence there is a constructible isomorphism $i:\mathbb{K}\to\mathbb{K}'$, which is $\emptyset$-definable. (We will call i canonical; it is unique up to Frobenius automorphisms. Then definable subsets of $\mathbf{M}^n$ are exactly all the constructible subsets of $\mathbf{M}^n$.

To see (c) from (a) we proved first two **LEMMAS**:

1. *There exists the minimal subfield* $Def(\mathbb{M})$ *of* $\mathbb{K}$ *such that* $\mathbb{M}$ *is* $Def(\mathbb{M})$*-definable in* $\mathbb{K}$ *and if* $\mathbb{M}$ *is also*

X-definable then Def(𝕄) is a subfield of one generated by X. Def(𝕄) is the set of all elements of K fixed under all automorphisms of 𝕂 inducing automorphisms of 𝕄. Any relation on M which is definable in 𝕄 and is Def(𝕄)-definable in 𝕂 is ∅-definable in 𝕄.

2. *In the definition of a very rich structure we may assume 𝕂' is ∅-definable.*

Now to get (c) consider the automorphism α' induced by α on 𝕂'. Then $\alpha' i = i\upsilon$ for some automorphism υ of 𝕂. For any $a \in Def(𝕄)$ by Lemma 1 $i(a)$ is fixed by α', so $\upsilon(a)=a$. Thus υ induces an automorphism (denoted by the same letter) of 𝕄. If we considered from the very start automorphism $\beta=\upsilon\alpha^{-1}$ then the corresponding automorphism β' of 𝕂' would become the identity. Such a mapping β', as is easily seen, provided **M** is in definable closure of $\{c_1,\dots c_k\}\cup K'$, can be defined by a first order formula in 𝕄 and so in 𝕂. Thus $\rho=\beta^{-1}$ is constructible.

It is important to find verifiable conditions under which a rich structure 𝕄 is always very rich. It is easy to point out a necessary model-theoretical condition: 𝕄 is "almost strongly minimal". We have found also an useful condition which is sufficient in the presence of the previous one:

There is a definable group structure on **M** *and this group has no proper subgroups of finite index.*

EXAMPLE. The structure ℂ from section 1 is very rich. To see this we use the fact that ℂ is equivalent to the group-structure J(C) with the embedding C ⊂ J(C) in respect of the problem. Any J(C) is a divisible group.

Groups have an advantage in respect to another problem, too.

PROPOSITION 2. *Suppose* $\mathbb{M}$ *is very rich and there is a definable in* $\mathbb{M}$ *operation on* $\mathbf{M}$ *which makes* $\mathbf{M}$ *an algebraic group. Then there is a unique way of reconstruction of the Zariski topology on* $\mathbf{M}^m$ *for all* m.

So this is the case were really "algebra defines the geometry". We conjecture that in the proposition the assumption of the existence of the group-structure is superfluous, it could be replaced by: all basic relations of $\mathbb{M}$ are Zariski-closed.

The proposition follows from the following easy considerations. Suppose there are two ways of defining variety structure on $\mathbf{M}$. Than we have two algebraic groups and the identity mapping $\mathbf{M} \to \mathbf{M}$ is a constructible isomorphism ρ between them. On some open subset of $\mathbf{M}$ ρ coincides with a rational bijection spoiled by Frobenins automorphisms on some coordinates. Then ρ should coincide with this bijection on the whole $\mathbf{M}$. This bijection preserves Zariski topology on $\mathbf{M}^m$.

COROLLARY. *There is a unique way of defining the Zariski topology on* C^m *for every* m *agreeing with the relation* E_g *on a smooth projective curve* C *of genus* ≥ 2 *i.e. with the structure* $\mathbb{C}$ *from section* 1.

3. Moduli of very rich structures

Now we will present a more detailed analysis of point (c) in Proposition 1 and it will led to a construction giving something which is similar to moduli.

Suppose for simplicity that $\{c_1, \ldots c_k\}$ in the definition of a very rich structure $\mathbb{M}$ is empty. Then it is easy to see from the definition $\mathbb{M}$ is definable also in $\mathbb{K}'$.

So we may consider the subfield $Def(\mathbb{M})$ of $\mathbb{K}'$ as defined in Lemma 1. In this special case it is easy to see that $Def(\mathbb{M}) = dcl(\emptyset)\cap\mathbb{K}'$. Let m′ denote any l-tuple of generators (as a subfield) of $dcl(\emptyset)\cap\mathbb{K}'$, $l=length(m')$. Since m′ is a tuple of constants it can be defined by a formula φ of $Th(\mathbb{M})$ which has m′ as the only solution. The canonical isomorphism $i:\mathbb{K}'\longrightarrow\mathbb{K}$ sends m′ to some l-tuple in $\mathbb{K}$ and this tuple we denote $m(\mathbb{M},\varphi)$.

For characteristic 0 $m(\mathbb{M},\varphi)$ does not depend on i, we restrict oursetves to this case and call $m(\mathbb{M},\varphi)$ an invariant. Observe that if $\mathbb{M}$ and $\mathbb{N}$ are two very rich structures in $\mathbb{K}$ and they are isomorphic by a definable in $\mathbb{K}$ isomorphism f then $m(\mathbb{M},\varphi)=m(\mathbb{N},\varphi)$. Indeed, f takes $m'=\varphi(\mathbb{M})$ to some $n'=\varphi(\mathbb{N})$ which is a tuple from a field 0-definable in $\mathbb{N}$ in the same way as $\mathbb{K}'$ is definable in $\mathbb{M}$. The canonical isomorphisms together with f compose a definable automorphism of $\mathbb{K}$ which sends $m(\mathbb{M},\varphi)$ to $m(\mathbb{N},\varphi)$. Since for characteristic 0 there is only one constructible isomorphism of $\mathbb{K}$ hence $m(\mathbb{M},\varphi)=m(\mathbb{N},\varphi)$.

From the other hand, under some conditions $m(\mathbb{M},\varphi)=m(\mathbb{N},\varphi)$ implies $\mathbb{M}$ is definably isomorphic to $\mathbb{N}$. These conditions can be expressed by elementary sentences. Suppose the language L of $\mathbb{M}$ is finite. Let T_φ be a finite set of sentences in L valid in $\mathbb{M}$ which fix a formula interpreting a field $\mathbb{K}'$ in any model $\mathbb{N}$, state that the field is infinite and that φ is a formula with an only solution n in $\mathbb{K}'$ for some n and the structure $\mathbb{N}$ is n-definable in $\mathbb{K}'$ by given formulas. The considerations above show that for any two models $\mathbb{N}$ and $\mathbb{N}'$ of T_φ, provided they are structures definable in $\mathbb{K}$ and are constructibly isomorphic,

$m(\mathbb{N},\varphi)=m(\mathbb{N}',\varphi)$ holds. On the contrary if $m(\mathbb{N},\varphi)=m(\mathbb{N}',\varphi)$ then the canonical isomorphism between the corresponding fields sends n to n' where n, n' are the corresponding solutions of φ in the fields. This isomorphism can be extended to an isomorphism from $\mathbb{N}$ to $\mathbb{N}'$ since the both are interpretable in the corresponding fields by the same formulas fixed in T_φ. So we have proved

THEOREM 2. *For any structure* $\mathbb{M}$ *of finite language* L *definable in an algebraically closed field* $\mathbb{K}$ *of characteristic* 0 *there are a formula* φ *in* L, *a finite set of sentences* T_φ *in* L *depending also on the way* $\mathbb{M}$ *is interpreted in* $\mathbb{K}$ *and a set* $\mathfrak{M}_\varphi \subseteq \mathbb{K}^l$ *definable in* $\mathbb{K}$ *such that*

(1) *for any structure* $\mathbb{N}$ *in language* L *definable in* $\mathbb{K}$ *which is a model of* T_φ *there is an element* $m(\mathbb{N},\varphi) \in \mathfrak{M}_\varphi$, *any* $n \in \mathfrak{M}_\varphi$ *is of the form* $m(\mathbb{N},\varphi)$ *for some model* $\mathbb{N}$ *of* T_φ;

(2) $m(\mathbb{N}_1,\varphi)=m(\mathbb{N}_2,\varphi)$ *iff* $\mathbb{N}_1$ *is constructibly isomorphic to* $\mathbb{N}_2$;

(3) *if* $\mathbb{N}$ *is* **X**-*definable in* $\mathbb{K}$ *then* $m(\mathbb{N},\varphi)$ *is an* **X**-*definable element, what is more if* $\mathbb{N}(V)$ *is a definablefamily of structures which for any* **X** *from a definable set gives* $\mathbb{N}(X)$ *a model of* T_φ *then the mapping* $m(\mathbb{N}(V),\varphi)$ *is definable.*

The only thing we didn't explain before is $\mathfrak{M}_\varphi$. To define it recall that in any model $\mathbb{N}$ of T_φ is 0-definable field-structure $\mathbb{K}'$ and $\mathbb{N}$ is n-definable in $\mathbb{K}'$ by some fixed formulas. Let $\mathfrak{M}_\varphi$ be the set of all l-tuples n in K for which there is a structure $\mathbb{N}$ satisfying T_φ and which is n-definable in $\mathbb{K}$ by the formulas mentioned above. Obviously $\mathfrak{M}_\varphi$ is 0-definable in $\mathbb{K}$ and $\mathfrak{M}_\varphi$ satisfies (1).

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