Intersecting varieties with tori

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This paper discusses a conjecture of diophantine type that I came to in connection with an attempt to understand the complex exponentiation and other classical functions in model-theoretic context [Z1], [Z2], [Z3]. Not very surprisingly it turned out that the same conjecture could answer questions crucial for other model-theoretic works, see [P] and also an unfinished work by J.Baldwin and K.Holland. In [Z2] and [Z3] we need only the version over complex numbers with the exponentiation function, and in this form it is easier to see the links to the classical Schanuel conjecture. Nevertheless the discussion makes sense over any field $\mathbf{K}$ where an exponentiation-like function is defined (e.g. in $\mathbb{R}$) and also for much broader class of functions, e.g. elliptic and even Abelian functions, for which the 'Schanuel conjecture' can be stated. In this broader context there is an obvious generalisation of the Diophantine conjecture that can be seen also as a generalisation of both Mordell-Lang and Manin-Mumford conjectures:

Conjecture on intersection with subgroups in semi-abelian varieties. Let A be the $\mathbf{K}$-point set of a semi-abelian variety over $\mathbf{K}$ and $V \subseteq A$ an algebraic subvariety. Then there is a finite collection $\tau(V)$ of cosets of some proper algebraic subgroups of A such that, given any algebraic subgroup $B \leq A$ and a component $S \subseteq W \cap B$ of the intersection which is *atypical in dimension*, i.e.

$$\dim S > \dim W + \dim B - \dim A,$$

there is a $C \in \tau(W)$ such that $S \subseteq C$.

We discuss here mostly the case $A = (\mathbf{K}^*)^n$, a cartesian power of the multiplicative group of the field (and call it CIT, the conjecture on intersection with tori), $\mathbf{K}$ is $\mathbb{C}$ or $\mathbb{R}$, but most of the statements can be easily generalised. One of the surprising observations is the fact that the real version of CIT, due to the o-minimality, follows from the Schanuel conjecture for the real exponentiation. Thus the multiplicative version of real Mordell-Lang conjecture follows from Schanuel. Presumably this is true for Abelian functions on reals, and thus general Mordell-Lang for reals follows from the corresponding 'Schanuel'.

1 Conjecture on intersection with tori

Definition A variety $T_{m,b} \subseteq (\mathbf{K}^*)^n$ given by a set of equations of the form

$$y_1^{m_1} \cdot \ldots \cdot y_n^{m_n} = b \tag{1}$$

with $m_1, \ldots, m_n \in \mathbb{Z}$ is said to be a [**shifted**] **torus**.
Obviosly, a torus is a coset of a unique algebraic subgroup $T_m \subseteq (\mathbf{K}^*)^n$ defined by equations (1) with $b = 1$. This subgroup is a torus which we will call **a basic torus** and **the base** of $T_{m,b}$.
A torus $T \subseteq (\mathbf{K}^*)^n$ is called **proper** if $T \neq (\mathbf{K}^*)^n$.
A variety in $\mathbf{K}^n$ given by a set of equations of the form

$$m_1 \cdot x_1 + \ldots + m_n \cdot x_n = a \tag{2}$$

with $m_1, \ldots, m_n \in \mathbb{Z}$ is said to be a $\mathbb{Q}$-**affine space**.
If $a = 0$ the space is said to be $\mathbb{Q}$-**linear** and is a base of the affine variety (2).

Lemma 1.1 *For any $\mathbb{Q}$-linear space $N \subseteq \mathbf{K}^n$ $\exp N$ is a torus and*

$$\dim N = \dim \exp N.$$

Proof Proceeding by induction we may assume that N is determined by one linear equation of the form (2) with $a = 0$ and g.c.d. of $m_1, \ldots, m_n$ is 1. Then (1) with $b = 1$ holds for any $\bar{y} \in \exp N$. Conversely assume $\bar{y}$ satisfies

(1). Take $\bar{x} \in \mathbf{K}^n$ such that $\exp \bar{x} = \bar{y}$. Then $m_1 x_1 + \ldots + m_n x_n = 2k\pi i$. Let $l_1, \ldots, l_n \in \mathbb{Z}$ be such that

$$l_1 m_1 + \ldots + l_n m_n = 1$$

and define $x_i' = x_i - l_i 2k\pi i$. Then

$$m_1 x_1' + \ldots + m_n x_n' = 0 \text{ and } \exp\langle x_1', \ldots, x_n' \rangle = \langle y_1, \ldots, y_n \rangle.$$

Thus $\bar{y} \in \exp N$ and the latter is given exactly by (1).

We see also that codim $N = $ codim $\exp N$, where codim is the number of independent equations which determine the linear space N in and the torus $\exp N$ in $\mathbf{K}^n$, correspondingly. Hence $\dim N = \dim \exp N$.$\square$

Definition Let $W \subseteq \mathbf{K}^n$ be an algebraic variety defined over $\mathbb{Q}$, $T \subseteq (\mathbf{K}^*)^n$ a torus, S an irreducible component of the algebraic variety $W \cap T$.
If $\dim S > \dim W + \dim T - n$, then S is said to be **an atypical component of the intersection $W \cap T$.**
Otherwise S is said to be **typical**.

Conjecture on intersection with tori (CIT) *For any $W \subseteq \mathbf{K}^n$ algebraic variety defined over $\mathbb{Q}$ there is a finite collection*

$$\tau(W) = \{T_1, \ldots, T_k\}$$

of proper basic tori in $(\mathbf{K}^*)^n$ such that for any proper basic torus $T \subseteq (\mathbf{K}^*)^n$ and any atypical component S of $W \cap T$

$$S \subseteq T_i \text{ for some } T_i \in \tau(W).$$

Notice that CIT implies a stronger form conjecture, when W is defined over $\tilde{\mathbb{Q}}$, the algebraic closure of $\mathbb{Q}$. Indeed, such a W has finitely many conjugates W^σ (σ a field automorphism) and $W^\# = \bigcup_\sigma W^\sigma$ is defined over $\mathbb{Q}$. Applying CIT to $W^\#$ we get $\tau(W)$.

Theorem 1 (CIT with parameters) *Let $V(a) \subseteq \mathbf{K}^n$ be an algebraic variety depending on parameters $a \in \mathbf{K}^k$. Then, assuming CIT, there is a finite set $\tau_k(V)$ of basic proper tori, a natural number $t = t_k(V)$ and*

$c_1(a), \ldots, c_t(a) \in (\mathbf{K}^*)^n$ *depending on V and a such that given a proper basic torus $T \subseteq \mathbf{K}^n$ and an atypical component S_a of $V(a) \cap T$ there is $P \in \tau(V)$ and $i \le t$ for which $S_a \subseteq P \cdot c_i(a)$.*

Proof First we prove the statement with $\tau_k(V, a))$ and $t = t_k(V, a)$ depending both on V and a. For this w.l.o.g. we may assume that

$$V(a) = \{\langle x_1, \ldots, x_n \rangle : \langle x_1, \ldots, x_{n+1}, \ldots, x_{n+k} \rangle \in V \; \& \; x_{n+1} = a_1, \ldots, x_{n+k} = a_k\},$$

$V \subseteq \mathbf{K}^{n+k}$ is a $\tilde{\mathbb{Q}}$-definable irreducible variety and a a generic point in $\mathrm{pr}(V)$, where $\mathrm{pr} : \mathbf{K}^{n+k} \to \mathbf{K}^k$ is the projection onto the last k coordinates and is not contained in a proper subtorus of $(\mathbf{K}^*)^k$.
Notice that $\dim V(a) = \dim V - \dim \mathrm{pr}V$ then.
Let S_a be a connected component of the intersection $V(a) \cap T$ for a torus T. Then S_a is definable over the algebraic closure of $\mathbb{Q}(a)$ and can be represented as an equidimensional connected component of an algebraic variety of the form $S(a)$ for some $S \subseteq \mathbf{K}^{n+k}$ irreducible $\tilde{\mathbb{Q}}$-definable. Notice that $\dim S(a) = \dim S - \dim \mathrm{pr}S$ and, since $a \in \mathrm{pr}S$, a is generic in $\mathrm{pr}V$ and $S \subseteq V$, we have $\dim \mathrm{pr}V = \dim \mathrm{pr}S$. Thus

$$\dim V - \dim V(a) = \dim S - \dim S(a).$$

It follows that

$$\dim S(a) > \dim V(a) + \dim T - n \; \text{iff} \; \dim S > \dim V + \dim T \times (\mathbf{K}^*)^k - (n+k).$$

The righthandside of the above is saying that S is an atypical component of the intersection of V and torus $T \times (\mathbf{K}^*)^k$. By CIT there is a proper torus $P \in \tau(V)$ such that $S \subseteq P$. Thus $S(a) \subseteq P(a)$. But

$$P(a) = \{\langle x_1, \ldots, x_n \rangle : \langle x_1, \ldots, x_{n+1}, \ldots, x_{n+k} \rangle \in P \; \& \; x_{n+1} = a_1, \ldots, x_{n+k} = a_k\}$$

is a coset of a subtorus $P^{(k)} \subseteq (\mathbf{K}^*)^n$ and the subtorus is proper since otherwise a belongs to a proper subtorus of $(\mathbf{K}^*)^k$, which contradicts the assumptions, and thus the statement depending on a has been proved.
To prove the full statement suppose towards a contradiction that for any finite set τ of proper subtori of $\mathbf{K}^n$ and every natural number t there is an $a \in \mathbf{K}^k$ such that the pair $\langle \tau, t \rangle$ does not fit for $\langle \tau(V, a), t(V, a) \rangle$. It is a

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routine exercise to see that for every pair $\langle \tau, t \rangle$ the set $A_{\tau,t}$ of such a's is definable in the field language. Using Tarski-Seidenberg quantifier-elimination theorem for $\mathbf{K}$ we may say that $A_{\tau,t}$ is **constructible**, i.e. is obtained from algebraic subvarieties of $\mathbf{K}^k$ by finite unions, intersections and complements. Consequently, using the fact that $\mathbf{K}$ is of infinite transcendence degree, one can find an $a \in \mathbf{K}^k$ which belongs to $A_{\tau,t}$ for all $\langle \tau, t \rangle$. This contradicts the above proved statement.□

Corollary 1 *Let $V(a) \subseteq \mathbf{K}^n$ be an algebraic variety depending on parameters $a \in \mathbf{K}^k$ and $b \in \mathbf{K}^n$. Then, assuming CIT, there is a finite set $\tau_k(V)$ of basic proper tori, a natural number $t = t_k(V)$ and $c_1(a,b), \ldots, c_t(a,b) \in (\mathbf{K}^*)^n$ depending on V and a, b such that given a proper basic torus $T \subseteq \mathbf{K}^n$ and an atypical component $S_{a,b}$ of $V(a) \cap T \cdot b$ there is $P \in \tau(V)$ and $i \leq t$ for which $S_{a,b} \subseteq P \cdot c_i(a,b)$.*

We will need the following technical generalization of the notion of an atypical component of an intersection.

Definition Let $T, P \subseteq \mathbf{K}^n$ be tori, $V \subseteq \mathbf{K}^n$ an algebraic variety. Assume an irreducible component S of the intersection $V \cap T$ is a subset of P. Then S is said to be an **atypical component of $V \cap T$ with respect to P** iff

$$\dim S > \dim(V \cap P) + \dim(T \cap P) - \dim P.$$

Proposition 1 *Let $V(a) \subseteq \mathbf{K}^n$ be an algebraic variety, $b \in \mathbf{K}^n$. Then there is a finite set $\pi(V)$ of basic tori of $(\mathbf{K}^*)^n$, a number $p = p(V)$ and elements $c_1(a,b), \ldots, c_p(a,b) \in \mathbf{K}^n$ such that, given a basic torus $T \subseteq \mathbf{K}^n$, for any connected atypical component S of $V(a) \cap T \cdot b$, there is $Q \in \pi(V)$ and $c_i(a,b)$ for which $S \subseteq Q \cdot c_i(a,b)$ and S is typical in $V(a) \cap T \cdot b$ with respect to $Q \cdot c_i(a,b)$.*

Proof We shall prove a slightly generalized version of the statement:
Let $P \subseteq \mathbf{K}^n$ be a torus, $V(a) \subseteq P$ an algebraic variety. Then there is a finite set $\pi_P(V)$ of basic tori of $(\mathbf{K}^*)^n$ a number $p = p_P(V)$ and elements $c_1(a,b), \ldots, c_p(a,b) \in \mathbf{K}^n$ such that given a basic torus $T \subseteq P$ for any connected atypical component S of $V(a) \cap T \cdot b$ there is $Q \in \pi_P(V)$ and $c_i(a,b)$

for which $S \subseteq Q \cdot c_i(a, b)$ and S is typical in $V(a) \cap T \cdot b$ with respect to $Q \cdot c_i(a, b)$.

For dim $P = 1$ the statement is trivially true. Consider the general case. By CIT relativized to P there is a finite set $\tau_P(V)$ of proper subtori of P such that for any $S \subseteq V \cap P \cap T$ atypical with respect to P there is a $Q \in \tau_P(V)$ with $S \subseteq Q$. By induction on dim P we can use the finite sets $\pi_Q(V \cap Q)$ of subtori of Q. Put

$$\pi_P(V) = \bigcup_{Q \in \tau_P(V)} \pi_Q(V \cap Q).$$

□

2 CIT versus Mordell-Lang and Manin-Mumford

We discuss now the more general form of CIT as stated in the introduction. So A is a semi-abelian variety, and we write the group operation in A in the additive way. We show that in this form CIT is a conjecture of Diophantine type and is stronger than both Mordell-Lang and Manin-Mumford conjectures (now proved, see [L]). For simplicity we formulate both conjectures in a slightly restricted form.

Proposition 2 (Manin-Mumford case) *Let $W \subseteq A$ be an algebraic variety which contains no coset of an algebraic subgroup of A, and Γ the subgroup of torsion points of A. Then CIT formally implies that $W \cap \Gamma$ is finite.*

Proof Each point in Γ of order p belongs to the 0-dimensional subgroup $B_p = \{a \in A : pa = 0\}$. By assumtions dim $W < n$. Thus any point in $W \cap \Gamma$ is an atypical component of the intersection $W \cap B_p$ and thus is contained in one of P belonging to $\tau(W)$.

Since any coset P is a semi-abelian variety again, by induction on dim A it follows that $W \cap \Gamma \cap P$ is finite, hence $W \cap \Gamma$ is finite.□

Proposition 3 (Mordell-Lang case) *Let $W \subseteq A$ be an algebraic subvariety which contains no coset of an algebraic subgroup of A, Γ a finitely generated subgroup of A. Then CIT formally implies that $W \cap \Gamma$ is finite.*

Proof W.l.o.g. we assume W is not contained in any coset of a proper algebraic subgroup of A. We prove the statement by induction on $\dim A = n$. For $n = 1$ the statement is trivial.

Denote r the rank of group Γ (the size of a maximal independent subset) and $\dim W = w$. Let k be an integer such that $k(n - w) > nr$, which exists since $w < n$.

Consider any point $\langle a_1, \ldots, a_k \rangle \in \Gamma^k \cap W^k$. A maximal independent subset of $a_1, \ldots, a_k$ contains $d \le r$ elements, say $a_1, \ldots, a_d$, thus $\langle a_1, \ldots, a_k \rangle$ belongs to a subgroup $B_a \subseteq A^k$ defined by $k - d$ equations of the form

$$mx_{d+i} = m_{i,1}x_1 + \ldots + m_{i,d}x_d, \quad (i = 1, \ldots, k - d)$$

for m and $m_{i,j}$ integers. The equations are independent, hence $\dim B_a = nd$ and so $\langle a_1, \ldots, a_k \rangle$ belongs to an atypical component of the intersection $W^k \cap B_a$ since

$$\dim W^k + \dim B_a - nk = wk + nd - nk \le nr - k(n - w) < 0.$$

By Theorem 1 there is a finite set of proper subgroups $B_1, \ldots, B_l$ of A^k and points $c_1, \ldots, c_l \in A^k$ such that
(*) any point of the intersection $W^k \cap \Gamma^k$ belongs to one of the cosets $B_i + c_i$ $(i = 1, \ldots, l)$.

Suppose towards a contradiction that $W \cap \Gamma$ is infinite.

Let now k be the minimal one with the property (*). Then there are only finitely many k-tuples $\bar{w} \in W^k \cap \Gamma^k$ with $w_i = w_j$ for some $1 \le i < j \le k$. Denote $(W \cap \Gamma)^{(k)}$ the set of all k tuples with distinct coordinates from $W \cap \Gamma$. For any $U \subseteq A^k$ denote

$$U^{sym} = \left\{\bar{u}^\pi : \bar{u} \in U, \ \pi \in Sym(k) \text{ a permutation of coordinates}\right\}.$$

Then $(B_i + c_i)^{sym}$ is a finite union of cosets, and $(W \cap \Gamma)^{(k)}$ is a subset of the union of all $(B_i + c_i)^{sym}$. By the classical combinatorial fact (Ramsey Theorem) there is an infinite $X \subseteq W \cap \Gamma$ and some $i \in \{1, \ldots, l\}$ such that

$$(W \cap \Gamma)^k \cap (B_i + c_i)^{sym}.$$

It follows that there are $a_1, \ldots, a_{k-1} \in W \cap \Gamma$ such that $\langle a_1, \ldots, a_{k-1}, b \rangle \in (W \cap \Gamma)^k \cap (B_i + c_i)$ for infinitely many $b \in W \cap \Gamma$.

But the condition on b

$$\langle a_1, \ldots, a_{k-1}, b \rangle \in B_i + c_i$$

is equivalent to $b \in B + d$ for some proper subgroup $B \subseteq A$ and $d \in \Gamma$. Shifting W by d we may assume $W \cap \Gamma \cap B$ is infinite. By induction hypothesis on n the only possibility then is $W \cap B$ is not proper in B, i.e. $B \subseteq W$. The contradiction.$\square$

3   Schanuel Conjecture

We want to exhibit now some ties between CIT and the classical
Schanuel Conjecture (SchC) *For any $\mathbb{Q}$-linearly independent complex numbers $x_1, \ldots, x_n$*

$$\mathrm{tr.d.}(x_1, \ldots, x_n, \exp(x_1), \ldots, \exp(x_n)) \geq n.$$

 It easy to eliminate the assumptions on the linear independence and see that the conjecture can be stated equivalently in the following form: *For any $x_1, \ldots, x_n$ complex numbers*

$$\mathrm{tr.d.}(x_1, \ldots, x_n, \exp(x_1), \ldots, \exp(x_n)) \geq \mathrm{l.d.}(x_1, \ldots, x_n),$$

where $\mathrm{l.d.}(x_1, \ldots, x_n)$ denotes the dimension of the $\mathbb{Q}$-linear space generated by $x_1, \ldots, x_n$.

Proposition 4 *[Schanuel Conjecture with parameters]* SchC *implies:*
Let $V \subseteq \mathbf{K}^{2n+k}$ be an algebraic variety over $\tilde{\mathbb{Q}}$, $a = \langle a_1, \ldots, a_k \rangle \in \mathbf{K}^k$ and

$$V(a) = \{\langle x_1, \ldots, x_n, y_1, \ldots, y_n \rangle \in \mathbf{K}^{2n} : \langle x_1, \ldots, x_n, y_1, \ldots, y_n, a_1, \ldots, a_k \rangle \in V\}.$$

Suppose $\dim V(a) = d < n$. Then there is a finite $A \subseteq \mathbf{K}$, $|A| \leq k(n+1)$, such that either
$x_1, \ldots, x_n \in \mathbf{K}^n$ are $\mathbb{Q}$-linearly dependent over A or
$\langle x_1, \ldots, x_n, \exp(x_1), \ldots, \exp(x_n) \rangle \notin V(a)$.

Proof If there are less than $k + 1$ n-tuples $\langle x_1, \ldots, x_n \rangle$ such that $\langle x_1, \ldots, x_n, \exp(x_1), \ldots, \exp(x_n) \rangle \in V(a)$ (call such n-tuples $V(a)$-tuples) then A is just the union of all elements in the tuples. So we assume one can choose $k + 1$ of them

$$\langle x_{i,1}, \ldots, x_{i,n}, \exp(x_{i,1}), \ldots, \exp(x_{i,n}) \rangle \in V(a), \quad i = 1, \ldots, k + 1.$$

From the assumptions

$$\text{tr.d.}(\{x_{i,j}, a_l, \exp(x_{i,j}), \exp(a_l) : \ i \leq k + 1, \ j \leq n, l \leq k\}) \leq (k+1)d + 2k$$

and $(k + 1)d + 2k < (k + 1)n + k$.
By SchC then

$$\{x_{i,j} : \ i \leq k + 1, \ j \leq n\} \cup \{a_j : \ j \leq k\}$$

are linearly dependent. Thus we have proved that for any $k + 1$ $V(a)$-tuples there is a linear dependence on their coordinates over a. Let now $l \leq k$ be maximal such that there is $B = \{b_{i,j} : \ i \leq l, \ j \leq n\}$ independent over a and $\langle b_{i,1}, \ldots, b_{i,n} \rangle$ are $V(a)$-tuples. Then $A = B \cup \{a_j : \ j \leq k\}$ satisfies the requirement of the proposition.$\square$

Proposition 5 SchC + CIT *imply the following* Uniform Schanuel Conjecture (USC)*:*
for any algebraic variety $V \subseteq \mathbf{K}^{2n}$ of dimension less than n there is a finite set $\mu(V)$ of proper $\mathbb{Q}$-linear subspaces of $\mathbf{K}^n$ such that given

$$\langle x_1, \ldots, x_n, \exp(x_1), \ldots, \exp(x_n) \rangle \in V$$

there is $M \in \mu(V)$ and an integer vector $\bar{z} \in \mathbb{Z}^n$ such that

$$\langle x_1, \ldots, x_n \rangle \in M + 2\pi i \cdot \bar{z}$$

and M is of codimension at least 2 or $\bar{z} = 0$.

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Remark We can equivalently state that for any connected component S of $V \cap \{\exp(x_i) = y_i, \ i = 1, \ldots n\}$ there is M and $\bar{z}$ as above such that $\mathrm{pr}_f S \subseteq M + 2\pi i \cdot \bar{z}$ (projection onto the first n coordinates).

Proof We assume SchC+CIT and prove USC.

Denote pr_l the projection $\mathbf{K}^{2n} \to \mathbf{K}^n$ onto the last n coordinates. Let

$$\dim(V \cap \mathrm{pr}_l^{-1}(y)) = d$$

for a generic $y \in \mathrm{pr}_l(V)$. Then $\dim V = \dim \mathrm{pr}_l(V) + d$.

Under our asumptions, by SchC there is a proper $\mathbb{Q}$-linear subspace N of $\mathbf{K}^n$ which contains $\langle x_1, \ldots, x_n \rangle$, more precisely, $\dim N = \mathrm{l.d.}(x_1, \ldots, x_n)$. Hence $\langle \exp(x_1), \ldots, \exp(x_n) \rangle$ is an element of the proper subtorus $T = \exp(N)$ of $(\mathbf{K}^*)^n$.

Denote algebraic varieties

$$V_l' = \{y \in V_l : \dim(V \cap \mathrm{pr}_l^{-1}(y)) > d\}, \ \ V' = \{\langle x, y \rangle \in V : y \in V_l'\}.$$

Since V' is a proper algebraic subvariety of the irreducible variety V,

$$\dim V' < \dim V.$$

Denote also
$$d' = \min\{\dim(V \cap \mathrm{pr}_l^{-1}(y)) : y \in V_l'\}.$$

Consider two cases.

Case 1:

$\langle x_1, \ldots, x_n, \exp(x_1), \ldots, \exp(x_n) \rangle \in V'$. Then

$$\mathrm{tr.d.}(x_1, \ldots, x_n, \exp(x_1), \ldots, \exp(x_n)) \leq d' + \dim(V_l' \cap T) \leq \dim V' + \dim T - n + s' < \dim T + s',$$

where $s' = \dim V_l' + \dim T - n - \dim(V_l' \cap T)$. By SchC $s' > 0$, which means the intersection $V_l' \cap T$ is atypical. Hence $\langle \exp(x_1), \ldots, \exp(x_n) \rangle \in P_i$ for one of tori $P_i \in \tau(V_l')$.

Case 2:

$\langle x_1, \ldots, x_n, \exp(x_1), \ldots, \exp(x_n) \rangle \in V \setminus V'$. Then

$$\mathrm{tr.d.}(x_1, \ldots, x_n, \exp(x_1), \ldots, \exp(x_n)) \leq d + \dim(V_l \cap T) \leq \dim V + \dim T - n + s < \dim T + s,$$

where $s = \dim V_l + \dim T - n - \dim(V_l \cap T)$. By SchC $s > 0$, which means the intersection $V_l \cap T$ is atypical. Hence $\langle \exp(x_1), \ldots, \exp(x_n) \rangle \in Q$ for one of tori $Q \in \pi(V_l)$.

Denote now $\mu(V)$ the set of $\mathbb{Q}$-linear subspaces M of $\mathbf{K}^n$, such that $\exp(M) = P$ for $P \in \pi(V_l) \cup \pi(V_l'.)$

Suppose now $y = \langle \exp(x_1), \ldots, \exp(x_n) \rangle \in P$, $\exp(M) = P$ for $P \in \pi(V_l)$ and codim $P = 1$. Denote $V_2 = \mathrm{pr}_l(V)$. Then by definition of $\pi(V_l)$

$$\dim_y(V_2 \cap T \cap P) = \dim(T \cap P) + \dim_y(V_2 \cap P) - \dim P$$

($\dim_y$ is the dimension of a maximal component containing y.) Since $\dim(T \cap P) = \dim T + \dim P - n + s_{T,P}$ for some $s_{T,P} \geq 0$ and $\dim_y(V_2 \cap P) = \dim P + \dim V_2 - n + s_{V,P}$ for some $s_{V,P} \geq 0$, we get

$$\dim_y(V_2 \cap T \cap P) = \dim T + \dim P + \dim V_2 - 2n + s_{T,P} + s_{V,P}.$$

On the other hand

$$\dim_y(V_2 \cap T \cap P) = \dim_y(V_2 \cap T) + \dim P - n = \dim V_2 + \dim T + s + \dim P - 2n$$

for $s > 0$ defined above. Thus $s = s_{T,P} + s_{V,P}$.

If $s_{T,P} = 0$, then $s_{V,P} > 0$, which implies $V_2 \subseteq P$ and

$$\dim_y(V_2 \cap T \cap P) = \dim(T \cap P) + \dim V_2 - \dim P = \dim T + \dim V_2 - n + s_{T,P}.$$

Since

$$\dim_y(V_2 \cap T \cap P) = \dim_y(V_2 \cap T) = \dim T + \dim V_2 - n + s_{V,P},$$

we get $s_{V,P} = s_{T,P}$, a contradiction. Thus T and P intersect atypically (i.e. not transversally), which means $T \subseteq P$ in case codim $P = 1$. It follows $N \subseteq M$ and hence $\langle x_1, \ldots, x_n \rangle \in M$ in this case. Otherwise $\langle x_1, \ldots, x_n \rangle \in M + 2\pi i \cdot \bar{z}$ for some integer vector $\bar{z}$.

In case $y \in \mathrm{pr}_l(V')$ the same arguments applied to V' again give the required. $\square$

Lemma 3.1 USC *implies* SchC.

Proof Notice that

$$\text{tr.d.}(x_1, \ldots, x_n, \exp(x_1), \ldots, \exp(x_n)) < n$$

is equivalent to

$$\langle x_1, \ldots, x_n, \exp(x_1), \ldots, \exp(x_n) \rangle \in V$$

for some $\mathbb{Q}$-definable algebraic variety V of dimension less than n. By USC either $x_1, \ldots, x_n$ are $\mathbb{Z}$-linearly dependent, or there are two independent $\mathbb{Z}$-vectors $\langle z_{j,1}, \ldots, z_{j,n} \rangle$ and $k_j \in \mathbb{Z}$ $(j = 1, 2)$ such that

$$z_{j,1} \cdot x_1 + \ldots + z_{j,n} \cdot x_n = 2 k_j \pi i \ (j = 1, 2).$$

It follows there is a nonzero integer vector $\langle z_1, \ldots, z_n \rangle$ such that

$$z_1 \cdot x_1 + \ldots + z_n \cdot x_n = 0,$$

which is the statement of SchC.$\square$

Lemma 3.2 USC *implies: Let* $W(\bar{a}) \subseteq \mathbf{K}^n$ *be an algebraic variety, Then there is a finite set* $\mu(W)$ *of proper* $\mathbb{Q}$-*linear subspaces of* $\mathbf{K}^n$ *and a number* $m = m(W)$ *such that, given* $L \subseteq \mathbf{K}^n$ *a* $\mathbf{K}$-*linear subspace and* $R \subseteq L \cap \ln W(\bar{a})$ *a connected component of the analytic subset of* $\mathbf{K}^n$, *there are* $\bar{c}_1, \ldots, \bar{c}_m \in \mathbf{K}^n$ *satisfying* $R \subseteq M + c_i + 2\pi i \bar{z}$ *for some* $i \leq m$, $M \in \mu(W)$ *and* $\bar{z} \in \mathbb{Z}^n$.

Proof By assumptions $\dim R > \dim L + \dim W(\bar{a}) - n$. By intersecting R and L with generic hyperplanes we can reduce the dimensions of R and L, if needed, and come to the situation $\dim R > \dim L + \dim W(\bar{a}) - n < 0$, which we will further assume.

Let L be given by matrix equation $B\bar{x} = 0$. Consider the algebraic variety

$$V(\bar{a}, B) = \{\bar{x} \in \mathbf{K}^n : B\bar{x} = 0\} \times W(\bar{a}).$$

By the assumtion $\dim V(\bar{a}, B) < n$ and $S = \{\langle \bar{x}, \exp(\bar{x}) \rangle : \bar{x} \in R\}$ satisfies the assumptions of Proposition 5 (the remark) with parameters. Hence the statement follows.$\square$

Corollary 2 USC *implies* CIT *with parameters.*

Repeating the arguments in the proof of Proposition 1 we get from Lemma 3.1

Corollary 3 USC *implies: Let $W(\bar{a}) \subseteq \mathbf{K}^n$ be an algebraic variety, Then there is a finite set $\lambda(W)$ of proper $\mathbb{Q}$-linear subspaces of $\mathbf{K}^n$ and a number $l = l(W)$ such that, given $L \subseteq \mathbf{K}^n$ a $\mathbf{K}$-linear subspace and $R \subseteq L \cap \ln W(\bar{a})$ a connected component of the analytic subset of $\mathbf{K}^n$, there are $\bar{c}_1, \ldots, \bar{c}_l \in \mathbf{K}^n$ satisfying $R \subseteq M + c_i + 2\pi i \bar{z}$ for some $i \leq m$, $M \in \lambda(W)$ and $\bar{z} \in \mathbb{Z}^n$, and R is typical in $L \cap \ln W(\bar{a})$ with respect to $M + c_i + 2\pi i \bar{z}$.*

Lemma 3.3 *In Corollary above, denoting $M' = M + c_i + 2\pi i \bar{z}$,*

$$\dim(M' \cap L) > \dim M' + \dim L - n$$

and

$$\dim(M' \cap \ln W(\bar{a})) > \dim M' + \dim W(\bar{a}) - n.$$

Proof By the assumption that R is typical with respect to M'

$$\dim R = \dim M' \cap L + \dim_R M' \cap \ln W(\bar{a}) - \dim M', \tag{3}$$

where $\dim_R M' \cap \ln W(\bar{a})$ is the dimension of the connected component of the intersection containing R.
Let

$$\dim M' \cap L = \dim M' + \dim L - n + d_L, \tag{4}$$

$$\dim_R M' \cap \ln W(\bar{a}) = \dim M' + \dim W(\bar{a}) - n + d_W, \tag{5}$$

where d_L, d_W are some non-negative integers which are just zero iff the corresponding intersections are transversal (intersect typically). It follows from (3) and (4) that

$$\dim R = \dim L - n + d_L + \dim_R M' \cap \ln W(\bar{a}) \leq \dim L + \dim W(\bar{a}) - n + d_L$$

but $\dim R > \dim L + \dim W(\bar{a}) - n$ by the assumptions. It follows $d_L \geq 1$, hence M' intersects L not transversally.
Symmetrically, using (3) and (5), $d_W \geq 1$.$\square$

Theorem 2 *Assume* USC. *Then given a K-linear $L \subseteq \mathbf{K}^n$ not contained in any proper $\mathbb{Q}$-linear subspace of $\mathbf{K}^n$, and an algebraic family $W(\bar{a}) \subseteq \mathbf{K}^n$ of algebraic varieties there is a finite set $\eta(L, W)$ of K-linear hyperplanes of L and a natural number $l = l(W)$ such that there are $d_1, \ldots, d_l \in \mathbf{K}^n$ satisfying the property that any atypical component R of $L \cap \ln W(\bar{a})$ is contained in some $H + d_i$ for $H \in \eta(L, W)$ and $i \leq l$.*

Proof By Corollary 3 the atypical component R is contained in some $M' = M + c + 2\pi i \bar{z}$, c taking one of the $l(W)$ values depending on $\bar{a}$, M a $\mathbb{Q}$-linear subspace from $\nu(W)$ and R is typical with respect to M'. Since M' intersects with L we may assume that $c \in L$.

By $\mathbb{Q}$-linear transformation of variables we may assume that M is given by $p = \operatorname{codim} M$ equations

$$x_1 = \ldots = x_p = 0,$$

and $M + c + 2\pi i \bar{z}$ is given by

$$x_1 = 2\frac{k_1}{m}\pi i + c_1, \ldots x_l = 2\frac{k_l}{m}\pi i + c_p$$

for $k_1, \ldots, k_p, m \in \mathbb{Z}, c_1, \ldots, c_p \in \mathbf{K}$. By Lemma 3.3 M is not transversal to L, hence there is a non-trivial K-linear equation of the form $s_1 x_1 + \ldots + s_p x_p = 0$ which holds on L. In other words the projection of L onto $x_1, \ldots, x_p$ denoted $\operatorname{pr}L$ is a proper subspace of $\mathbf{K}^p$. Let N be the minimal K-linear subspace of C^p such that

$$\mathbb{Z}^p \cap N = \mathbb{Z}^p \cap \operatorname{pr}L.$$

In fact, N is $\mathbb{Q}$-linear since it has a basis consisting of integer vectors. Since L is not contained in a proper $\mathbb{Q}$-linear subspace, $N \cap \operatorname{pr}L$ is a proper subspace of $\operatorname{pr}L$.

Let $\langle x_1, \ldots, x_n \rangle \in R$. Then by the construction $\langle x_1, \ldots, x_p \rangle \in \operatorname{pr}L$ and $\langle x_1, \ldots, x_p \rangle = 2\pi i \frac{1}{m}\bar{z} + \operatorname{pr}(c)$ for some integer vector $\bar{z}$. Since $\operatorname{pr}(c) \in \operatorname{pr}L$, $\langle x_1, \ldots, x_p \rangle \in \frac{1}{m} \cdot \mathbb{Z}^p \cap \operatorname{pr}L \subseteq N$. Denote $H = \{y \in L : \operatorname{pr}(y) \in N\}$, which by the construction is a proper K-linear subspace of L and $R \subseteq H + c$. Notice that we constructed H using only M and L and c depends only on the coset M'. Define $\eta(L, W)$ to be all the H constructed from M of $\nu(W)$ and $l(W)$ the same as in Corollary 3.$\square$

4 The 'function field' case of CIT

During 1997-98 Logic Year in MSRI, Berkeley, E.Hrushovski pointed out that CIT can be treated by means of the theory of differential fields, and C.Wood and A.Pillay showed me the paper by J.Ax which contains the following basic theorem

Theorem of J.Ax [A] *Let $y_1, \ldots, y_n, z_1, \ldots, z_n$ be non-zero elements of a differential field with subfield of constants $C \supset \mathbb{Q}$ such that $Dy_i = \frac{Dz_i}{z_i}$ for $i = 1, \ldots, n$, and $Dy_1, \ldots, Dy_n$ are $\mathbb{Q}$-linearly independent. Then*

$$\text{tr.d.}_C(y_1, \ldots, y_n, z_1, \ldots, z_n) \geq n + 1.$$

In the context of the broader conjectures it is interesting to mention that W.D.Brownawell and K.K.Kubota in [BK] generalized the Ax's result to Weierstrass elliptic functions.

Proposition 6 *Given $V \subseteq \mathbf{K}^{2n+k}$ an algebraic variety there is a finite collection U of non-zero integer vectors such that for any $\bar{a} \in \mathbf{K}^k$ with $\dim V(\bar{a}) \leq n$ for any infinite component $S \subseteq V(\bar{a}) \cap \{\exp(x_i) = x_{n+i}, \ i = 1, \ldots n\}$ for one of $u \in U$ there is a constant c_S such that all $\langle s_1, \ldots, s_{2n} \rangle \in S$ satisfy $u_1 s_1 + \ldots + u_n s_n = c_S$.*

Proof Take $y_1, \ldots y_n, z_1, \ldots, z_n$ to be the coordinate functions on $V(\bar{a})$.
Consider now the restriction of functions $y_1, \ldots, y_n, z_1 \ldots, z_n$ to S and differentiation $D = \frac{d}{dt}$ along a curve in a simple point s of the component. Then

$$Dz_i = z_i Dy_i \text{ for } i = 1, \ldots, n \tag{6}$$

in the differential field $F_{S,s}$ of germs of analytic functions on S in s, and

$$\langle y_1, \ldots y_n, z_1, \ldots, z_n \rangle \in V(\bar{a})(F_{S,s}). \tag{7}$$

Now assume towards a contradiction that for any choice of finite collection U of integer vectors there is $\bar{a}$ and an infinite component $S_{\bar{a}} \subseteq V(\bar{a})$ such that the statement of the proposition fails.
Then in some formal differential field F (6) and (7) hold with F in place of $F_{S,s}$, and $Dy_1, \ldots, Dy_n$ are $\mathbb{Q}$-linearly independent. This contradicts the theorem of Ax.$\square$

It follows from the results of the preceding section

Corollary 4 *Given an algebraic variety $W(\bar{a}) \subseteq \mathbf{K}^n$ there is a finite collection $\mu(W)$ of non-zero integer vectors such that for any torus $T \subseteq \mathbf{K}^n$ and an infinite atypical component $S \subseteq W(\bar{a}) \cap T$ of the intersection there is $\bar{m} \in \mu(W)$ and a constant c such that all $(a_1, \ldots, a_n)$ in the component satisfy $a_1^{m_1} \cdot \ldots \cdot a_n^{m_n} = c$.*

Correspondingly the proof of Theorem 2 yields

Corollary 5 *Given a K-linear $L \subseteq \mathbf{K}^n$ not contained in any proper $\mathbb{Q}$-linear subspace of $\mathbf{K}^n$, and an algebraic family $W(a) \subseteq \mathbf{K}^n$ of algebraic varieties there is a finite set $\eta(L, W)$ of K-linear hyperplanes of L such that any atypical infinite component R of $L \cap \ln W(a)$ is contained in some $H + d$ for $H \in \eta(L, W)$ and $d \leq L$.*

5   The real exponent, real CIT and the Mordell conjecture

Theorem 3 *If the real exponentiation satisfies the Schanuel conjecture then it satisfies the Uniform Schanuel conjecture.*

Proof [1] Let $V \subseteq \mathbb{R}^{2n}$ be an algebraic variety, $\dim V < n$. Denote

$$V^e = \{\langle x_1, \ldots, x_n, \exp(x_1), \ldots, \exp(x_n)\rangle \in V\}.$$

Then by o-minimality of $\mathbb{R}_{\exp}$ (see [W], but in fact the Hovanski's results [Ho] suffice) V^e is an analytic subspace with finitely many connected components V_i^e, $i = 1, \ldots, k$. By the Schanuel conjecture

$$V^e \subseteq \bigcup_{M \subset \mathbb{R}^n \; \mathbb{Q}\text{-linear subspace}} \{\langle x_1, \ldots, x_n, \exp(x_1), \ldots, \exp(x_n)\rangle : \{\langle x_1, \ldots, x_n\rangle \in M\}$$

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[1] This proof is incomplete. For the full proof see the joint note with J.Kirby *The uniform Schanuel conjecture over the real numbers*. The author's web-page and to appear in the Bulletin of LMS

thus, by the connectedness, each component V_i^e lies in a set of the form $\{\langle x_1, \ldots, x_n, \exp(x_1), \ldots, \exp(x_n) \rangle : \{\langle x_1, \ldots, x_n \rangle \in M\}.\square$

Corollary 4 for reals by the same o-minimality argument takes the form

Corollary 6 [2] *Given an algebraic variety $W(\bar{a}) \subseteq \mathbb{R}^n$ there are a finite collection $\mu(W)$ of non-zero integer vectors, a number $m(W)$ independent on a, and $c_1, \ldots, c_{m(W)} \in \mathbb{R}^n$, depending on a, such that for any torus $T \subseteq \mathbb{R}^n$ and an infinite atypical component $S \subseteq W(\bar{a}) \cap T$ of the intersection there is $\bar{m} \in \mu(W)$ and $i \leq m(W)$ such that all $\langle a_1, \ldots, a_n \rangle$ in the component satisfy $a_1^{m_1} \cdot \ldots \cdot a_n^{m_n} = c_i$.*

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[2]Ignore this. The o-minimality argument does not work here. February, 2006

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