ANALYTIC ZARISKI STRUCTURES AND NON-ELEMENTARY CATEGORICITY

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ABSTRACT. We study analytic Zariski structures from the point of view of non-elementary model theory. We show how to associate an abstract elementary class with a one-dimensional analytic Zariski structure and prove that the class is stable, quasi-minimal and homogeneous over models. We also demonstrate how Hrushovski's predimension arises in this general context as a natural geometric notion and use it as one of our main tools.

The notion of an analytic Zariski structure was introduced in [1] by the author and N.Peatfield in a form slightly different from the one presented here and then in [2], Ch.6 in the current form. Analytic Zariski generalises the previously known notion of a Zariski structure (see [2] for one-dimensional case and [3], [4] for the general definition) mainly by dropping the requirement of Noetherianity and weakening the assumptions on the projections of closed sets. We remark that in the broad setting, it is appropriate to consider the notion of a Zariski structure as belonging to positive model theory in the sense of [5].

In [1] we assumed that the Zariski structure is compact (or compactifiable), here we drop this assumption, which may be too restrictive in applications.

The class of analytic Zariski structures is much broader and geometrically richer than the class of Noetherian Zariski structures. The main examples come from two sources:

(i) structures which are constructed in terms of complex analytic functions and relations;

(ii) "new stable structures" introduced by Hrushovski's construction; in many cases these objects exhibit properties similar to those of class (i).

However, although there are concrete examples for both (i) and (ii), in many cases we lack the technology to prove that the structure is analytic Zariski. In particular, despite some attempts the conjecture that $\mathbb{C}_{\exp}$ is analytic Zariski, assuming it satisfies axioms of pseudo-exponentiation (see [17]), is still open.

The aim of this paper is to carry out a model-theoretic analysis of M in the appropriate language. Recall that if M is a Noetherian Zariski structure the relevant key model-theoretic result states that its first-order theory allows elimination of quantifiers and is ω-stable of finite Morley rank. In particular, it is strongly minimal (and so uncountably categorical) if $\dim M = 1$ and M is irreducible.

For analytic Zariski 1-dimensional M we carry out a model theoretic study it in the spirit of the theory of *abstract elementary classes*. We start by introducing a suitable countable fragment of the family of basic Zariski relations and a correspondent substructure of constants over which all the further analysis is carried

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out. Then we proceed to the analysis of the notion of dimension of Zariski closed sets and define more delicate notions of the *predimension* and *dimension* of a tuple in M. In fact by doing this we reinterpret dimensions which are present in every analytic structure in terms familiar to many from Hrushovski's construction, thus establishing once again conceptual links between classes (i) and (ii).

Our main results are proved under assumption that M is one-dimensional (as an analytic Zariski structure) and irreducible. No assumption on presmoothness is needed. We prove for such an M, in the terminology of [16]:

(1) M *is a quasi-minimal pregeometry structure with regards to a closure operator* cl *associated with the predimension;*

(2) M *has quantifier-elimination to the level of* $\exists$*-formulas in the following sense: every two tuples which are (first-order)* $\exists$*-equivalent over a countable submodel, are* $L_{\infty,\omega}$ *equivalent;*

(3) *The abstract elementary class associated with* M *is categorical in uncountable cardinals and is excellent.*

In fact, (3) is a corollary of (1) using the main result of [16], so the main work is in proving (1) which involves (2) as an intermediate step.

Note that the class of 1-dimensional Noetherian Zariski structures is essentially classifiable by the main result of [2], and in particular the class contains no instances of structures obtained by the proper Hrushovski construction. The class of analytic Zariski structures, in contrast, is consistent with Hrushovski's construction and at the same time, by the result above, has excellent model-theoretic properties. This gives a hope for a classification theory based on the relevant notions.

However, it must be mentioned that some natural questions in this context are widely open. In particular, we have no classification for presmooth analytic Zariski groups (with the graph of multiplication analytic). It is not known if a 1-dimensional irreducible presmooth analytic group has to be abelian. See related analysis of groups in [9].

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1. Analytic L-Zariski structures

Let M $= (M, L)$ be a structure with primitives (basic relations) L.

We introduce a topology on M^n, for all $n \geq 1$, by declaring a subset $P \subseteq M^n$ closed if there is an n-type p consisting of quantifier-free positive formulas with parameters in M such that

$$P = \{a \in M^n : \text{ M} \vDash p(a)\}.$$

In other words, the sets defined by atomic $L(M)$-formulae form a basis for the topology.

We say P is L-closed if p is over $\emptyset$.

1.1. **Remark.** Note that it follows that projections

$$\mathrm{pr}_{i_1,\dots,i_m} : M^n \to M^m, \quad \langle x_1, \dots, x_n \rangle \mapsto \langle x_{i_1}, \dots, x_{i_m} \rangle$$

are continuous in the sense that the inverse image of a closed set under a projection is closed. Indeed, $\mathrm{pr}^{-1}S = S \times M^{n-m}$.

We write $X \subseteq_{op} V$ to say that X is open in V and $X \subseteq_{cl} V$ to say it is closed.

We say that $P \subseteq M^n$ is **constructible** if P is a finite union of some sets S, such that $S \subseteq_{cl} U \subseteq_{op} M^n$.

A subset $P \subseteq M^n$ will be called **projective** if P is a union of finitely many sets of the form $\operatorname{pr} S$, for some $S \subseteq_{cl} U \subseteq_{op} M^{n+k}$, $\operatorname{pr} : M^{n+k} \to M^n$.

Note that any set S such that $S \subseteq_{cl} U \subseteq_{op} M^{n+k}$, is constructible, a projection of a constructible set is projective and that any constructible set is projective.

1.2. Dimension.
To any nonempty projective S a non-negative integer $\dim S$, called **the dimension of** S, is attached.

We assume:

(SI) (**strong irreducibility**) for an irreducible set $S \subseteq_{cl} U \subseteq_{op} M^n$ (that is S is not a union of two proper closed subsets) and its closed subset $S' \subseteq_{cl} S$,

$$\dim S' = \dim S \Rightarrow S' = S;$$

(DP) (**dimension of points**) for a nonempty projective S, $\dim S = 0$ if and only if S is at most countable.

(CU) (**countable unions**) If $S = \bigcup_{i \in \mathbb{N}} S_i$, all projective, then $\dim S = \max_{i \in \mathbb{N}} \dim S_i$;

(WP) (**weak properness**) given an irreducible $S \subseteq_{cl} U \subseteq_{op} M^n$ and $F \subseteq_{cl} V \subseteq_{op} M^{n+k}$ with the projection $\operatorname{pr} : M^{n+k} \to M^n$ such that $\operatorname{pr} F \subseteq S$ and $\dim \operatorname{pr} F = \dim S$, there exists $D \subseteq_{op} S$ such that $D \subseteq \operatorname{pr} F$.

1.3. Remark.
(CU) in the presence of (DCC) implies the *essential uncountability property* (EU) usually assumed for Noetherian Zariski structures.

We postulate further, for an irreducible $S \subseteq_{cl} U \subseteq_{op} M^{n+k}$:

(AF) $\dim \operatorname{pr} S = \dim S - \min_{u \in \operatorname{pr} S} \dim(\operatorname{pr}^{-1}(u) \cap S)$;

(FC) The set $\{a \in \operatorname{pr} S : \dim(\operatorname{pr}^{-1}(a) \cap S) \geq m\}$ is of the form $T \cap \operatorname{pr} S$ for some constructible T, and there exists an open set V such that $V \cap \operatorname{pr} S \neq \emptyset$ and
$$\min_{a \in \operatorname{pr} S} \dim(\operatorname{pr}^{-1}(a) \cap S) = \dim(\operatorname{pr}^{-1}(v) \cap S), \text{ for any } v \in V \cap \operatorname{pr}(S).$$

The following helps to understand the dimension of projective sets.

1.4. Lemma.
Let $P = \operatorname{pr} S \subseteq M^n$, for S irreducible constructible, and $U \subseteq_{op} M^n$ with $P \cap U \neq \emptyset$. Then
$$\dim P \cap U = \dim P.$$

Proof. We can write $P \cap U = \operatorname{pr} S' = P'$, where $S' = S \cap \operatorname{pr}^{-1} U$ constructible irreducible, $\dim S' = \dim S$ by (SI). By (FC), there is $V \subseteq_{op} M^n$ such that for all $c \in V \cap P$,
$$\dim \operatorname{pr}^{-1}(c) \cap S = \min_{a \in P} \dim \operatorname{pr}^{-1}(a) \cap S = \dim S - \dim P.$$

Note that $\operatorname{pr}^{-1} U \cap \operatorname{pr}^{-1} V \cap S \neq \emptyset$, since S is irreducible. Taking $s \in \operatorname{pr}^{-1} U \cap \operatorname{pr}^{-1} V \cap S$ and $c = \operatorname{pr} s$ we get, using (AF) for S',
$$\dim \operatorname{pr}^{-1}(c) \cap S' = \dim \operatorname{pr}^{-1}(c) \cap S = \min_{a \in P'} \dim \operatorname{pr}^{-1}(a) \cap S = \dim S - \dim P'.$$

So, $\dim P' = \dim P$. $\hfill \square$

1.5. **Analytic subsets.** A subset S, $S \subseteq_{\mathrm{cl}} U \subseteq_{op} M^n$, is called **analytic in** U if for every $a \in S$ there is an open $V_a \subseteq_{op} U$ such that $a \in V_a$ and $S \cap V_a$ is the union of finitely many closed in V_a irreducible subsets.

We postulate the following properties

(INT) (**Intersections**) If $S_1, S_2 \subseteq_{an} U$ are irreducible then $S_1 \cap S_2$ is analytic in U;

(CMP) (**Components**) If $S \subseteq_{an} U$ and $a \in S$, a closed point, then there is $S_a \subseteq_{an} U$, a finite union of irreducible analytic subsets of U, and some $S'_a \subseteq_{an} U$ such that $a \in S_a \setminus S'_a$ and $S = S_a \cup S'_a$;

Each of the irreducible subsets of S_a above is called an **irreducible component of S containing** a.

(CC) (**Countability of the number of components**) Any $S \subseteq_{an} U$ is a union of at most countably many irreducible components.

1.6. **Remark.** For S analytic and $a \in \mathrm{pr}\, S$, the fibre $S(a, M)$ is analytic.

1.7. **Lemma.** If $S \subseteq_{an} U$ is irreducible, V open, then $S \cap V$ is an irreducible analytic subset of V and, if non-empty, $\dim S \cap V = \dim S$.

Proof. Immediate. $\hfill \square$

1.8. **Lemma.**

(i) $\emptyset$, any singleton and U are analytic in U;
(ii) If $S_1, S_2 \subseteq_{an} U$ then $S_1 \cup S_2$ is analytic in U;
(iii) If $S_1 \subseteq_{an} U_1$ and $S_2 \subseteq_{an} U_2$, then $S_1 \times S_2$ is analytic in $U_1 \times U_2$;
(iv) If $S \subseteq_{an} U$ and $V \subseteq U$ is open then $S \cap V \subseteq_{an} V$;
(v) If $S_1, S_2 \subseteq_{an} U$ then $S_1 \cap S_2$ is analytic in U.

Proof. Immediate. $\hfill \square$

1.9. **Definition.** Given a subset $S \subseteq_{\mathrm{cl}} U \subseteq_{op} M^n$ we define the notion of the **analytic rank** of S in U, $\mathrm{ark}_U(S)$, which is a natural number satisfying

(1) $\mathrm{ark}_U(S) = 0$ iff $S = \emptyset$;
(2) $\mathrm{ark}_U(S) \le k+1$ iff there is a set $S' \subseteq_{\mathrm{cl}} S$ such that $\mathrm{ark}_U(S') \le k$ and with the set $S^0 = S \setminus S'$ being analytic in $U \setminus S'$.

Obviously, any nonempty analytic subset of U has analytic rank 1.

The next assumptions guarantees that the class of analytic subsets explicitly determines the class of closed subsets in M.

(AS) [**Analytic stratification**] For any $S \subseteq_{\mathrm{cl}} U \subseteq_{op} M^n$, $\mathrm{ark}_U S$ is defined and is finite.

We will justify this non obvious property later in 3.10 and 3.11.

1.10. **Lemma.** For any $S \subseteq_{cl} U \subseteq_{op} M^n$,

$$\dim \operatorname{pr} S + \min_{a \in \operatorname{pr} S} \dim \operatorname{pr}^{-1}(a) \cap S \geq \dim S.$$

Proof. We use (AS) and prove the statement by induction on $\operatorname{ark}_U S \geq 1$.

For $\operatorname{ark}_U S = 1$, S is analytic in U and so by (CC) is the union of countably many irreducibles S_i. By (AF)

$$\dim \operatorname{pr} S_i + \min_{a \in \operatorname{pr} S_i} \dim \operatorname{pr}^{-1}(a) \cap S_i \geq \dim S_i$$

and so by (CU) lemma follows. $\qquad\square$

1.11. **Presmoothness.** The following property (which we are not going to use in the context of the present paper) is relevant.

(PS) [**Presmoothness**] If $S_1, S_2 \subseteq_{an} U \subseteq_{op} M^n$ and S_1, S_2 and U irreducible, then for any irreducible component S_0 of $S_1 \cap S_2$

$$\dim S_0 \geq \dim S_1 + \dim S_2 - \dim U.$$

1.12. **Definition.** An L-structure M is said to be **analytic L-Zariski** if
- M satisfies (SI), (WP), (CU), (INT), (CMP),(CC), (AS);
- the expansion $M^\sharp$ of M to the language $L(M)$ (names for points in M added) satisfies all the above with the dimension extending the one for M;
- $M^\sharp$ also satisfies (AF) and (FC) with V in (FC) being L-definable whenever S is.

An analytic Zariski structure will be called **presmooth** if it has the presmoothness property (PS).

2. Model theory of analytic Zariski structures

2.1. **Lemma.** Let M be analytic L-Zariski and assume L is countable. There is a countable $M_0 \preccurlyeq M$ such that for any L-closed set S any irreducible component P of S is L-definable (L-closed); **Proof.** Use the standard Löwenheim - Skolem downward arguments. $\qquad\square$

We call M_0 a core substructure (subset) of M.

2.2. **Assumption.** By extending L to $L(M_0)$ we assume that the set of L-closed points is the core subset.

2.3. **Definition.** For finite $X \subseteq M$ we define the **predimension**

(1) $\quad \delta(X) = \min\{\dim S : \vec{X} \in S, \ S \subseteq_{an} U \subseteq_{op} M^n, \ S \text{ is } L\text{-constructible}\},$

relative predimension for finite $X, Y \subseteq M$

(2) $\quad \delta(X/Y) = \min\{\dim S : \vec{X} \in S, \ S \subseteq_{an} U \subseteq_{op} M^n, \ S \text{ is } L(Y)\text{-constructible}\},$

and **dimension of** X

$$\partial(X) = \min\{\delta(XY) : \text{ finite } Y \subset M\}.$$

We call a minimal S as in (2) an **analytic locus of X over Y**.

For $X \subseteq M$ finite, we say that X is **self-sufficient** and write $X \leq M$, if $\partial(X) = \delta(X)$.

For infinite $A \subseteq M$ we say $A \leq M$ if for any finite $X \subseteq A$ there is a finite $X \subseteq X' \subseteq A$ such that $X' \leq M$.

2.4. Assumption. $\dim M = 1$ and M is irreducible.

Note that we then have

$$0 \leq \delta(Xy) \leq \delta(X) + 1, \text{ for any } y \in M,$$

since $\vec{X}y \in S \times M$.

2.5. Lemma. Given $F \subseteq_{an} U \subseteq_{op} M^k$, $\dim F > 0$, there is $i \leq k$ such that for $\mathrm{pr}_i : (x_1, \ldots, x_k) \mapsto x_i$,

$$\dim \mathrm{pr}_i F > 0.$$

Proof. Use (AF) and induction on k. $\hspace{2cm}$ □

2.6. Proposition. Let $P = \mathrm{pr}\, S$, for some L-constructible $S \subseteq_{an} U \subseteq_{op} M^{n+k}$, $\mathrm{pr} : M^{n+k} \to M^n$. Then

$$(3) \hspace{3cm} \dim P = \max\{\partial(x) : x \in P(M)\}.$$

Moreover, this formula is true when $S \subseteq_{cl} U \subseteq_{op} M^{n+k}$.

Proof. We use induction on $\dim S$.

We first note that by induction on $\mathrm{ark}_U S$, if (3) holds for all analytic S of dimension less or equal to k then it holds for all closed S of dimension less or equal to k.

The statement is obvious for $\dim S = 0$ and so we assume that $\dim S > 0$ and for all analytic S' of lower dimension the statement is true.

By (CU) and (CMP) we may assume that S is irreducible. Then by (AF)

$$(4) \hspace{3cm} \dim P = \dim S - \dim S(c, M)$$

for any $c \in P(M) \cap V(M)$ (such that $S(c, M)$ is of minimal dimension) for some open L-constructible V.

Claim 1. It suffices to prove the statement of the proposition for the projective set $P \cap V'$, for some L-open $V' \subseteq_{op} M^n$.

Indeed,

$$P \cap V' = \mathrm{pr}(S \cap \mathrm{pr}^{-1}V'), \hspace{0.5cm} S \cap \mathrm{pr}^{-1}V' \subseteq_{cl} \mathrm{pr}^{-1}V' \cap U \subseteq_{op} M^{n+k}.$$

And $P \setminus V' = \mathrm{pr}(S \cap T)$, $T = \mathrm{pr}^{-1}(M^n \setminus V') \in L$. So, $P \setminus V'$ is the projection of a proper analytic subset, of lower dimension. By induction, for $x \in P \setminus V'$, $\partial(x) \leq \dim P \setminus V' \leq \dim P$ and hence, using 1.4,

$$\dim P \cap V' = \max\{\partial(x) : x \in P \cap V'\} \Rightarrow \dim P = \max\{\partial(x) : x \in P\}.$$

Claim 2. The statement of the proposition holds if $\dim S(c, M) = 0$ in (4).

Proof. Given $x \in P$ choose a tuple $y \in M^k$ such that $S(x^\frown y)$ holds. Then $\delta(x^\frown y) \leq \dim S$. So we have $\partial(x) \leq \delta(x^\frown y) \leq \dim S = \dim P$.

It remains to notice that there exists $x \in P$ such that $\partial(x) \geq \dim P$.

Consider the L-type

$$x \in P \,\&\, \{x \notin R : \dim R \cap P < \dim P \text{ and } R \text{ is projective}\}.$$

This is realised in M, since otherwise $P = \bigcup_R (P \cap R)$ which would contradict (CU) because (M_0, C_0) is countable.

For such an x let y be a tuple in M such that $\delta(x^\frown y) = \partial(x)$. By definition there exist $S' \subseteq_{an} U' \subseteq_{op} M^m$ such that $\dim S' = \delta(x^\frown y)$. Let $P' = \mathrm{pr}\, S'$, the projection into M^n. By our choice of x, $\dim P' \geq \dim P$. But $\dim S' \geq \dim P'$. Hence, $\partial(x) \geq \dim P$. Claim proved.

Claim 3. There is a L-constructible $R \subseteq_{an} S$ such that all the fibres $R(c, M)$ of the projection map $R \to \mathrm{pr}\, R$ are 0-dimensional and $\dim \mathrm{pr}\, R = \dim P$.

Proof. We have by construction $S(c, M) \subseteq M^k$. Assuming $\dim S(c, M) > 0$ on every open subset we show that there is a $b \in M_0$ such that (up to the order of coordinates) $\dim S(c, M) \cap \{b\} \times M^{k-1} < \dim S(c, M)$, for all $c \in P \cap V' \neq \emptyset$, for some open $V' \subseteq V$ and $\dim \mathrm{pr}\, S(c, M) \cap \{b\} \times M^{k-1} = \dim P$. By induction on $\dim S$ this will prove the claim.

To find such a b choose $a \in P \cap V$ and note that by 2.5, up to the order of coordinates, $\dim \mathrm{pr}_1 S(a, M) > 0$, where $\mathrm{pr}_1 : M^k \to M$ is the projection on the first coordinate.

Consider the projection $\mathrm{pr}_{M^n, 1} : M^{n+k} \to M^{n+1}$ and the set $\mathrm{pr}_{M^n, 1} S$. By (AF) we have

$$\dim \mathrm{pr}_{M^n, 1} S = \dim P + \dim \mathrm{pr}_1 S(a, M) = \dim P + 1.$$

Using (AF) again for the projection $\mathrm{pr}^1 : M^{n+1} \to M$ with the fibres $M^n \times \{b\}$, we get, for all b in some open subset of M,

$$1 \geq \dim \mathrm{pr}^1 \mathrm{pr}_{M^n, 1} S = \dim \mathrm{pr}_{M^n, 1} S - \dim[\mathrm{pr}_{M^n, 1} S] \cap [M^n \times \{b\}] =$$

$$= \dim P + 1 - \dim[\mathrm{pr}_{M^n, 1} S] \cap [M^n \times \{b\}].$$

Hence $\dim[\mathrm{pr}_{M^n, 1} S] \cap [M^n \times \{b\}] \geq \dim P$, for all such b, which means that the projection of the set $S_b = S \cap (M^n \times \{b\} \times M^{k-1})$ on M^n is of dimension $\dim P$, which finishes the proof if $b \in M_0$. But $\dim S_b = \dim S - 1$ for all $b \in M \cap V'$, some L-open V', so for any $b \in M_0 \cap V'$. The latter is not empty since (M_0, L) is a core substructure. This proves the claim.

Claim 4. Given R satisfying Claim 3,

$$P \setminus \mathrm{pr}\, R \subseteq \mathrm{pr}\, S', \text{ for some } S' \subseteq_{cl} S, \ \dim S' < \dim S.$$

Proof. Consider the cartesian power

$$M^{n+2k} = \{x^\frown y^\frown z : x \in M^n, \ y \in M^k, \ z \in M^k\}$$

and its L-constructible subset

$$R\&S := \{x^\frown y^\frown z : \ x^\frown z \in R \ \& \ x^\frown y \in S\}.$$

Clearly $R\&S \subseteq_{an} W \subseteq_{op} M^{n+2k}$, for an appropriate L-constructible W.

Now notice that the fibres of the projection $\mathrm{pr}_{xy} : x^\frown y^\frown z \mapsto x^\frown y$ over $\mathrm{pr}_{xy} R\&S$ are 0-dimensional and so, for some irreducible component $(R\&S)^0$ of the analytic set $R\&S$, $\dim \mathrm{pr}_{xy}(R\&S)^0 = \dim S$. Since $\mathrm{pr}_{xy} R\&S \subseteq S$ and S irreducible, we get by (WP) $D \subseteq \mathrm{pr}_{xy} R\&S$ for some $D \subseteq_{op} S$. Clearly

$$\mathrm{pr}\, R = \mathrm{pr}\, \mathrm{pr}_{xy} R\&S \supseteq \mathrm{pr}\, D$$

and $S' = S \setminus D$ satisfies the requirement of the claim.

Now we complete the proof of the proposition: By Claims 2 and 3

$$\dim P = \max_{x \in \mathrm{pr}\, R} \partial(x).$$

By induction on $\dim S$, using Claim 4, for all $x \in P \setminus \mathrm{pr}\, R$,

$$\partial(x) \leq \dim \mathrm{pr}\, S' \leq \dim P.$$

The statement of the proposition follows. $\qquad\qquad\qquad\qquad\qquad\square$

In what follows a L-substructure of M is a L-structure on a subset $N \supseteq M_0$. Recall that L is purely relational.

Recall the following well-known fact, see [10].

2.7. **Karp's characterisation of** $\equiv_{\infty,\omega}$. Given $a, a' \in M^n$ the $L_{\infty,\omega}(L)$-types of the two n-tuples in M are equal if and only if they are *back and forth equivalent* that is there is a nonempty set I of isomorphisms of L-substructures of M such that $a \in \mathrm{Dom}\, f_0$ and $a' \in \mathrm{Range}\, f_0$, for some $f_0 \in I$, and

(forth) for every $f \in I$ and $b \in M$ there is a $g \in I$ such that $f \subseteq g$ and $b \in \mathrm{Dom}\, g$;

(back) For every $f \in I$ and $b' \in M$ there is a $g \in I$ such that $f \subseteq g$ and $b' \in \mathrm{Range}\, g$.

2.8. **Definition.** For $a \in M^n$, the **projective type of** a **over** M is

$$\{P(x): \ a \in P, \ P \text{ is a projective set over } L\}\cup$$

$$\cup\{\neg P(x): \ a \notin P, \ P \text{ is a projective set over } L\}.$$

2.9. **Lemma.** Suppose $X \leq M$, $X' \leq M$ and the (first-order) quantifier-free L-type of X is equal to that of X'. Then the $L_{\infty,\omega}(L)$-types of X and X' are equal.

Proof. We are going to construct a back-and-forth system for X and X'.

Let $S_X \subseteq_{an} V \subseteq_{op} M^n$, S_X irreducible, all L-constructible, and such that $X \in S_X(M)$ and $\dim S_X = \delta(X)$.

Claim 1. The quantifier-free L-type of X (and X') is determined by formulas equivalent to $S_X \cap V'$, for V' open such that $X \in V'(M)$.

Proof. Use the stratification of closed sets (AS) to choose L-constructible $S \subseteq_{\mathrm{cl}} U \subseteq_{op} M^n$ such that $X \in S$ and $\mathrm{ark}_U S$ is minimal. Obviously then $\mathrm{ark}_U S = 0$, that is $S \subseteq_{an} U \subseteq_{op} M^n$. Now S can be decomposed into irreducible components, so we may choose S to be irreducible. Among all such S choose one which is of minimal possible dimension. Obviously $\dim S = \dim S_X$, that is we may assume that $S = S_X$. Now clearly any constructible set $S' \subseteq_{\mathrm{cl}} U' \subseteq_{op} M^n$ containing X must satisfy $\dim S' \cap S_X \geq \dim S_X$, and this condition is also sufficient for $X \in S'$.

Let y be an element of M. We want to find a finite Y containing y and an Y' such that the quantifier-free type of XY is equal to that of $X'Y'$ and both are self-sufficient in M. This, of course, extends the partial isomorphism $X \to X'$ to $XY \to X'Y'$ and will prove the lemma.

We choose Y to be a minimal set containing y and such that $\delta(XY)$ is also minimal, that is

$$1 + \delta(X) \geq \delta(Xy) \geq \delta(XY) = \partial(XY)$$

and $XY \leq M$.

We have two cases: $\delta(XY) = \partial(X) + 1$ and $\delta(XY) = \partial(X)$. In the first case $Y = \{y\}$. By Claim 1 the quantifier-free L-type r_{Xy} of Xy is determined by the formulas of the form $(S_X \times M) \setminus T$, $T \subseteq_{cl} M^{n+1}$, $T \in L$, $\dim T < \dim(S_X \times M)$.

Consider

$$r_{Xy}(X', M) = \{z \in M : X'z \in (S_X \times M) \setminus T, \ \dim T < \dim S_X, \ \text{all } T\}.$$

We claim that $r_{Xy}(X', M) \neq \emptyset$. Indeed, otherwise M is the union of countably many sets of the form $T(X', M)$. But the fibres $T(X', M)$ of T are of dimension 0 (since otherwise $\dim T = \dim S_X + 1$, contradicting the definition of the T). This is impossible, by (CU).

Now we choose $y' \in r_{Xy}(X', M)$ and this is as required.

In the second case, by definition, there is an irreducible $R \subseteq_{an} U \subseteq_{op} M^{n+k}$, $n = |X|, k = |Y|$, such that $XY \in R(M)$ and $\dim R = \delta(XY) = \partial(X)$. We may assume $U \subseteq V \times M^k$.

Let $P = \mathrm{pr}\, R$, the projection into M^n. Then $\dim P \leq \dim R$. But also $\dim P \geq \partial(X)$, by 2.6. Hence, $\dim R = \dim P$. On the other hand, $P \subseteq S_X$ and $\dim S_X = \delta(X) = \dim P$. By axiom (WP) we have $S_X \cap V' \subseteq P$ for some L-constructible open V'.

Hence $X' \in S_X \cap V' \subseteq P(M)$, for P the projection of an irreducible analytic set R in the L-type of XY. By Claim 1 the quantifier-free L-type of XY is of the form

$$r_{XY} = \{R \setminus T : T \subseteq_{cl} R, \ \dim T < \dim R\}.$$

Consider

$$r_{XY}(X', M) = \{Z \in M^k : X'Z \in R \setminus T, \ T \subseteq_{cl} R, \ \dim T < \dim R\}.$$

We claim again that $r_{XY}(X', M) \neq \emptyset$. Otherwise the set $R(X', M) = \{X'Z : R(X'Z)\}$ is the union of countably many subsets of the form $T(X', M)$. But $\dim T(X', M) < \dim R(X', M)$ as above, by (AF).

Again, an $Y' \in r_{XY}(X', M)$ is as required. $\square$

2.10. Corollary. There is at most countably many $L_{\infty,\omega}(L)$-types of tuples $X \leq M$.

Indeed, any such type is determined uniquely by the choice of a L-constructible $S_X \subseteq_{an} U \subseteq_{op} M^n$ such that $\dim S_X = \partial(X)$.

2.11. Lemma. Suppose, for finite $X, X' \subseteq M$, the projective L-types of X and X' coincide. Then the $L_{\infty,\omega}(L)$-types of the tuples are equal.

Proof. Choose finite Y such that $\partial(X) = \delta(XY)$. Then $XY \leq M$. Let $XY \in S \subseteq_{an} U \subseteq_{op} M^n$ be L-constructible and such that $\dim S$ is minimal possible, that is $\dim S = \delta(XY)$. We may assume that S is irreducible. Notice that for every proper closed L-constructible $T \subseteq_{cl} U$, $XY \notin T$ by dimension considerations.

By assumptions of the lemma $X'Y' \in S$, for some Y' in M. We also have $X'Y' \notin T$, for any T as above, since otherwise a projective formula would imply that $XY'' \in T$ for some Y'', contradicting that $\partial(X) > \dim T$.

We also have $\delta(X'Y') = \dim S$. But for no finite Z' it is possible that $\delta(X'Z') < \dim S$, for then again a projective formula will imply that $\delta(XZ) < \dim S$, for some Z.

It follows that $X'Y' \leq M$ and the quantifier-free types of XY and $X'Y'$ coincide, hence the $L_{\infty,\omega}(L)$-types are equal, by 2.9. $\square$

2.12. **Definition.** Set, for finite $X \subseteq M$,

$$\mathrm{cl}_L(X) = \{y \in M : \ \partial(Xy) = \partial(X)\}.$$

We fix L and omit the subscript below.

2.13. **Lemma.** The following two conditions are equivalent

(a) $b \in \mathrm{cl}(A)$, for $\vec{A} \in M^n$;

(b) $b \in P(\vec{A}, M)$ for some projective first-order $P \subseteq M^{n+1}$ such that $P(\vec{A}, M)$ is at most countable.

In particular, $\mathrm{cl}(A)$ is countable for any finite A.

Proof. Let $d = \partial(A) = \delta(AV)$, and $\delta(AV)$ is minimal for all possible finite $V \subseteq M$. So by definition $d = \dim S_0$, some analytic irreducible S_0 such that $\vec{AV} \in S_0$ and S_0 of minimal dimension. This corresponds to a L-definable relation $S_0(x, v)$, where x, v strings of variables of length n, m

First assume (b), thais is that b belongs to a countable $P(\vec{A}, M)$. By definition

$$P(x, y) \equiv \exists w\, S(x, y, w),$$

for some analytic $S \subseteq M^{n+1+k}$, some tuples x, y, w of variables of length $n, 1$ and k respectively, and the fibre $S(\vec{A}, b, M^k)$ is nonempty. We also assume that P and S are of minimal dimension, answering this description. By (FC), (AS) and minimality we may choose S so that $\dim S(\vec{A}, b, M^k)$ is minimal among all the fibres $S(\vec{A'}, b', M^k)$.

Consider the analytic set $S^\sharp \subseteq M^{n+m+1+k}$ given by $S_0(x, v) \,\&\, S(x, y, w)$. By (AF), considering the projection of the set on (x, v)-coordinates,

$$\dim S^\sharp \leq \dim S_0 + \dim S(\vec{A}, M, M^k),$$

since $S(\vec{A}, M, M^k)$ is a fibre of the projection. Now we note that by countability $\dim S(\vec{A}, M, M^k) = \dim S(\vec{A}, b, M^k)$, so

$$\dim S^\sharp \leq \dim S_0 + \dim S(\vec{A}, b, M^k).$$

Now the projection $\mathrm{pr}_w S^\sharp$ along w (corresponding to $\exists w\, S^\sharp$) has fibres of the form $S(\vec{X}, y, M^k)$, so by (AF)

$$\dim \mathrm{pr}_w S^\sharp \leq \dim S_0 = d.$$

Projecting further along v we get $\dim \mathrm{pr}_v \mathrm{pr}_w S^\sharp \leq d$, but $\vec{A}b \in \mathrm{pr}_v \mathrm{pr}_w S^\sharp$ so by Proposition 2.6 $\partial(\vec{A}b) \leq d$. The inverse inequality holds by definition, so the equality holds. This proves that $b \in \mathrm{cl}(A)$.

Now assume (a), that is $b \in \mathrm{cl}(A)$. So, $\partial(\vec{A}b) = \partial(\vec{A}) = d$. By definition there is a projective set P containing $\vec{A}b$, defined by the formula $\exists w\, S(x, y, w)$ for some analytic S, $\dim S = d$. Now $\vec{A}$ belongs to the projective set $\mathrm{pr}_y P$ (defined by the formula $\exists y \exists w\, S(x, y, w)$) so by Proposition 2.6 $d \leq \dim \mathrm{pr}_y P$, but $\dim \mathrm{pr}_y P \leq \dim P \leq \dim S = d$. Hence all the dimensions are equal and so, the dimension of the generic fibre is 0. We may assume, as above, without loss of generality that all fibres are of minimal dimension, so

$$\dim S(\vec{A}, M, M^k) = 0.$$

Hence, b belongs to a 0-dimensional set $\exists w\, S(\vec{A}, y, w)$, which is projective and countable. $\qquad\square$

2.14. **Lemma.** Suppose $b \in \mathrm{cl}(A)$ and the projective type of $\vec{A}b$ is equal to that of $\vec{A}'b'$. Then $b' \in \mathrm{cl}(A')$.

Proof. First note that, by (FC) and (AS), for analytic $R(u, v)$ and its fibre $R(a, v)$ of minimal dimension one has

$$\mathrm{tp}(a) = \mathrm{tp}(a') \Rightarrow \dim R(a, v) = \dim R(a', v).$$

By the second part of the proof of 2.13 the assumption of the lemma implies that for some analytic S we have $\vDash \exists w S(\vec{A}, b, w)$ and $\dim S(\vec{A}, M, M^k) = 0$. Hence $\vDash \exists w S(\vec{A}', b', w)$ and $\dim S(\vec{A}', M, M^k) = 0$. But this immediately implies $b' \in \mathrm{cl}(A')$. $\qquad\qquad\square$

2.15. **Lemma.** (i)
$$\mathrm{cl}(\emptyset) = \mathrm{cl}(M_0) = M_0.$$
(ii) Given finite $X \subseteq M$, $y, z \in M$,
$$z \in \mathrm{cl}(X, y) \setminus \mathrm{cl}(X) \Rightarrow y \in \mathrm{cl}(X, z).$$

(iii)
$$\mathrm{cl}(\mathrm{cl}(X)) = \mathrm{cl}(X).$$

Proof. (i) Clearly $M_0 \subseteq \mathrm{cl}(\emptyset)$, by definition.

We need to show the converse, that is if $\partial(y) = 0$, for $y \in M$, then $y \in M_0$. By definition $\partial(y) = \partial(\emptyset) = \min\{\delta(Y) : y \in Y \subset M\} = 0$. So, $y \in Y$, $\vec{Y} \in S \subseteq_{an} U \subseteq_{op} M^n$, $\dim S = 0$. The irreducible components of S are closed points (singletons) and $\{\vec{Y}\}$ is one of them, so must be in M_0, hence $y \in M_0$.

(ii) Assuming the left-hand side of (ii), $\partial(Xyz) = \partial(Xy) > \partial(X)$ and $\partial(Xz) > \partial(X)$. By the definition of ∂ then,

$$\partial(Xy) = \partial(X) + 1 = \partial(Xz),$$

so $\partial(Xzy) = \partial(Xz)$, $y \in \mathrm{cl}(Xz)$.

(iii) Immediate by 2.13. $\qquad\qquad\square$

Below, if not stated otherwise, we use the language $L^\exists$ the primitives of which correspond to relations $\exists$-definable in M. Also, we call a **submodel of** M any $L^\exists$-substructure closed under cl.

2.16. **Theorem.** (i) Every $L_{\infty,\omega}(L)$-type realised in M is equivalent to a projective type, that is a type consisting of existential (first-order) formulas and the negations of existential formulas.

(ii) There are only countably many $L_{\infty,\omega}(L)$-types realised in M.

(iii) $(M, L^\exists)$ is quasi minimal ω-homogeneous over countable submodels, that is the following hold:

(a) for any countable (or empty) submodel G and any n-tuples X and X', both cl-independent over G, a bijection $\phi : X \to X'$ is a G-monomorphism;

(b) given any G-monomorphism $\phi : Y \to Y'$ for finite tuples Y, Y' in M and given a $z \in M$ we can extend ϕ so that $z \in \mathrm{Dom}\,\phi$.

Proof. (i) Immediate from 2.11.

(ii) By 2.10 there are only countably many types of finite tuples $Z \leq M$. Let $N \subseteq M_0$ be a countable subset of M such that any finite $Z \leq M$ is $L_{\infty,\omega}(L)$-equivalent to some tuple in N. Every finite tuple $X \subset M$ can be extended to $XY \leq M$, so there is a $L_{\infty,\omega}(L)$-monomorphism $XY \to N$. This monomorphism

identifies the $L_{\infty,\omega}(L)$-type of X with one of a tuple in N, hence there are no more than countably many such types.

(iii) Lemma 2.15 proves that cl defines a pregeometry on M.

Consider first (a). Note that $GX \leq M$ and $GX' \leq M$ and so the types of X and X' over G are L-quantifier-free. But there is no proper L-closed subset $S \subseteq_{\text{cl}} M^n$ such that $\vec{X} \in S$ or $\vec{X}' \in S$. Hence the types are equal.

For (b) just use the fact that the G-monomorphism by our definition preserves $\exists$-formulas, so by 2.11 complete $L_{\infty,\omega}(L(G))$-types of X and X' coincide, so by 2.7 ϕ can be extended. $\square$

2.17. **Theorem.** M is a quasiminimal pregeometry structure (see [16]). In other words, the following properties of M hold:

(QM1) The pregeometry cl is determined by the language. That is, if $\text{tp}(x, Y) = \text{tp}(x', Y')$, then $x \in \text{cl}(Y)$ if and only $x' \in \text{cl}(Y')$. (Here the types are first order).

(QM2) The structure M is infinite-dimensional with respect to cl.

(QM3) (Countable closure property). If $X \subset M$ is finite, then $\text{cl}(X)$ is countable.

(QM4) (Uniqueness of the generic type). Suppose that $H, H' \subset M$ are countable closed subsets, enumerated such that $\text{tp}(H) = \text{tp}(H')$. If $y \in M \setminus H$ and $y' \in M \setminus H'$, then $\text{tp}(H, y) = \text{tp}(H', y')$.

(QM5) (ω-homogeneity over closed sets and the empty set). Let $H, H' \subset M$ be countable closed subsets or empty, enumerated such that $\text{tp}(H) = \text{tp}(H')$, and let Y, Y' be finite tuples from M such that $\text{tp}(H, Y) = \text{tp}(H', Y')$, and let $z \in \text{cl}(H, Y)$. Then there is $z' \in M$ such that $\text{tp}(H, Y, z) = \text{tp}(H', Y', z')$.

Proof. (M, cl) is a pregeometry by 2.15. (QM1) is proved in 2.14. (QM3) is 2.13 and (QM2) follows from (QM3) and (CU). (QM4)&(QM5) is 2.16(iii). $\square$

Now we define an *abstract elementary class* $\mathcal{C}$ associated with M. We follow [?], for this construction. Similar construction was used in [16].

Set

$$\mathcal{C}_0(\text{M}) = \{ \text{ countable } L^\exists\text{-structures N} : \text{ N} \cong \text{N}' \subseteq \text{M}, \ \text{cl}(N') = N'\}$$

and define embedding $\text{N}_1 \preccurlyeq \text{N}_2$ in the class as an $L^\exists$-embedding $f : \text{N}_1 \to \text{N}_2$ such that there are isomorphisms $g_i : \text{N}_i \to \text{N}_i'$, $\text{N}_1' \subseteq \text{N}_2' \subseteq \text{M}$, all embeddings commuting and $\text{cl}(N_i') = N_i'$.

Now define $\mathcal{C}(\text{M})$ to be the class of L-structures H with cl_L defined with respect to H and satisfying:

(i) $\mathcal{C}_0(\text{H}) \subseteq \mathcal{C}_0(\text{M})$ as classes with embeddings

and

(ii) for every finite $X \subseteq H$ there is $\text{N} \in \mathcal{C}_0(\text{H})$, such that $X \subseteq \text{N}$.

Given $\text{H}_1 \subseteq \text{H}_2$, $\text{H}_1, \text{H}_2 \in \mathcal{C}(\text{M})$, we define $\text{H}_1 \preccurlyeq \text{H}_2$ to hold in the class, if for every finite $X \subseteq \text{H}_1$, $\text{cl}(X)$ is the same in H_1 and H_2. More generally, for $\text{H}_1, \text{H}_2 \in \mathcal{C}(\text{M})$ we define $\text{H}_1 \preccurlyeq_f \text{H}_2$ to be an embedding f such that there are isomorphisms $\text{H}_1 \cong \text{H}_1'$, $\text{H}_2 \cong \text{H}_2'$ such that $\text{H}_1' \subseteq \text{H}_2'$, all embeddings commute, and $\text{H}_1 \preccurlyeq \text{H}_2$.

2.18. **Lemma.** $\mathcal{C}(\text{M})$ is closed under the unions of ascending $\preccurlyeq$-chains.

Proof. Immediate from the fact that for infinite $Y \subseteq M$,

$$\text{cl}(Y) = \bigcup \{\text{cl}(X) : \ X \subseteq_{\text{finite}} Y\}.$$

$\square$

2.19. Theorem. The class $\mathcal{C}(\mathrm{M})$ contains structures of any infinite cardinality and is categorical in uncountable cardinals.

Proof. This follows from 2.17 by the main result of [16].

$\square$

2.20. Proposition. Any uncountable $\mathrm{H} \in \mathcal{C}(\mathrm{M})$ is an analytic Zariski structure in the language L with parameters in H. Also H is presmooth if M is.

Proof. We define $\mathcal{C}(\mathrm{H})$ to consist of the subsets of H^n of the form $P(a, H)$ for $P \in L$ of arity $k + n$, $a \in H^k$. The assumption (L) is obviously satisfied.

Now note that the constructible and projective sets in $\mathcal{C}(\mathrm{H})$ are also of the form $P(a, H)$ for some L-constructible or L-projective P.

Define $\dim P(a, H) = d$ if $\dim P(b, M) = d$ for some $b \in M^k$ such that The $L^\exists$-quantifier-free types of a and b are equal. This is well-defined by (FC) and the fact that the any $L^\exists$-quantifier-free type realised in H is also realised in M. Moreover, we have the following.

Claim. The set of $L^\exists$-quantifier-free types realised in H is equal to that realised in M.

Indeed, this is immediate from the definition of the class $\mathcal{C}(\mathrm{M})$, stability of $\mathcal{C}(\mathrm{M})$ and the fact that the class is categorical in uncountable cardinalities.

The definition of dimension immediately implies (DP), (CU),(AF) and (FC) for H.

(SI): if $P_1(a_1, H) \subseteq_{\mathrm{cl}} P_0(a_0, H)$, $\dim P_1(a_1, H) = \dim P_0(a_0, H)$ and the two sets are not equal, then the same holds for $P_1(b_1, M)$ and $P_0(b_0, M)$ for equivalent b_0, b_1 in M. Then, $P_0(b_0, M)$ is reducible, that is for some proper $P_2(b_2, M) \subseteq_{\mathrm{cl}} P_0(b_0, M)$, $P_0(b_0, M) = P_1(b_1, M) \cup P_2(b_2, M)$. Now, by homogeneity we can choose a_2 in H such that $P_0(a_0, H) = P_1(a_1, H) \cup P_2(a_2, H)$, a reducible representation.

This also shows that the notion of irreducibility is preserved by equivalent substitution of parameters. Then the same is true for the notion of analytic subset. Hence (INT), (CMP),(CC) and (PS) follow. For the same reason (AS) holds. Next we notice that the axioms (WP) follows by the homogeneity argument. $\square$

3. Some examples

3.1. Universal covers of semiabelian varieties. Let $\mathbb{A}$ be a semiabelian variety of dimension d, e.g. $d = 1$ and $\mathbb{A}$ the algebraic torus $\mathbb{C}^\times$. Let V be the universal cover of $\mathbb{A}$, which classically can be identified as a complex manifold $\mathbb{C}^d$.

We define a structure with a (formal) topology on V and show that this is analytic Zariski.

By definition of universal cover there is a covering holomorphic map

$$\exp : V \to \mathbb{A}$$

(a generalisation of the usual exp on $\mathbb{C}$).

We will assume that $\mathcal{C}$ has no proper semiabelian subvarieties (is simple) and no complex multiplication.

We consider the two sorted structure $(V, \mathbb{A})$ in the language that has all Zariski closed subsets of $\mathbb{A}^n$, all n, the addition $+$ on V and the map exp as the primitives.

This case was first looked at model-theoretically in [14] and the special case $\mathbb{A} = \mathbb{C}^\times$ in [12], [13] and in the DPhil thesis [15] of Lucy Smith.

Our aim here is to show that the structure on the sort V with a naturally given formal topology is analytic Zariski.

The positive quantifier-free definable subsets of V^n, $n = 1, 2, \ldots$ form a base of a topology which we call the PQF-topology. In other words

3.2. **Definition.** A PQF-closed subset of V^n is defined as a finite union of sets of the form

(5) $$L \cap m \cdot \ln W$$

where $W \subseteq \mathbb{A}^n$, an algebraic subvariety, and L is a $\mathbb{Q}$-linear subspace of V^n, that is defined by a system of equations of the form $m_1 x_1 + \ldots + m_n x_n = a$, $m_i \in \mathbb{Z}$, $a \in V^n$.

The relations on V which correspond to PQF_ω-closed sets are the primitives of our language L.

PQF-closed subsets form a base for a topology on the cartesian powers of V which will underlie the analytic Zariski structure on V.

Remark. Among closed sets of the topology we have sets of the form

$$\cup_{a \in I}(S + a)$$

where S is of the form (5) and I a subset of $(\ker \exp)^n$.

Slightly rephrasing the quantifier-elimination statement proved in [14] Corollary 2 of section 3, we have the following result.

3.3. **Proposition.** (i) Projection of a PQF-closed set is PQF-constructible, that is a boolean combination of PQF-closed sets.

(ii) The image of a constructible set under exponentiation is a Zariski-constructible (algebraic) subset of $\mathbb{A}^n$. The image of the set of the form (5) is Zariski closed.

We assign **dimension** to a closed set of the form (5)

$$\dim L \cap m \cdot \ln W := \dim \exp\left(L \cap m \cdot \ln W\right).$$

using the fact that the object on the right hand side is an algebraic variety. We extend this to an arbitrary closed set assuming (CU), that is that the dimension of a countable union is the maximum dimension of its members. This immediately gives (DP). Using 3.3 we also get (WP).

The analysis of irreducibility below is more involved. Since $\exp L$ is definably and topologically isomorphic to $\mathbb{A}^k$, some $k \geq 1$, we can always reduce the analysis of a closed set of the form (5) to a one of the form $\ln W$ with $W \subseteq \mathbb{A}^k$ not contained in a coset of a proper algebraic subgroup.

For such a W consider its m-th "root"

$$W^{\frac{1}{m}} = \{\langle x_1, \ldots, x_n \rangle \in \mathbb{A}^k : \langle x_1^m, \ldots, x_k^m \rangle \in W\}.$$

Let $d = d_W(m)$ be the number of irreducible components of $W^{\frac{1}{m}}$.

It is easy to see that if $d > 1$, irreducible components $W_i^{\frac{1}{m}}$, $i = 1, \ldots, d$, of $W^{\frac{1}{m}}$ are shifts of each other by m-th roots of unity, and $m \cdot \ln W_i^{\frac{1}{m}}$ are proper closed subsets of $\ln W$ of the same dimension. It follows that $\ln W$ *is irreducible (in the sense of* (SI)*) if and only if $d_W(m) = 1$ for all $m \geq 1$.* In [18] W satisfying this condition is called **Kummer generic**. If $W \subset \exp L$ for some $\mathbb{Q}$-linear subspace $L \subset V^n$, then one uses the relative version of Kummer genericity.

We say that the sequence $W^{\frac{1}{m}}$, $m \in \mathbb{N}$, stops branching if the sequence $d_W(m)$ is eventually constant, that is if $W^{\frac{1}{m}}$ is Kummer generic for some $m \geq 1$.

The following is proved for $\mathbb{A} = \mathbb{C}^\times$ in [12], (Theorem 2, case $n = 1$ and its Corollary) and in general in [18].

3.4. Theorem. The sequence $W^{\frac{1}{m}}$ stops branching if and only if W is not contained in a coset of a proper algebraic subgroup of $\mathbb{A}^k$.

3.5. Corollary. Any irreducible closed subset of V^n is of the form $L \cap \ln W$, for W Kummer generic in $\exp L$.

Any closed subset of V^n is analytic in V^n.

It is easy now to check that (SI), (INT), (CMP), (CC), (AS) and (PS) are satisfied.

3.6. Corollary. The structure (V, L) is analytic Zariski and presmooth.

The reader may notice that the analysis above treats only *formal* notion of analyticity on the cover $\mathbb{C}$ of $\mathbb{C}^\times$ but does not address the classical one. In particular, *is the formal analytic decomposition as described by 3.5 the same as the actual complex analytic one?* In a private communication F.Campana answered this question in positive, using a cohomological argument. M.Gavrilovich proved this and much more general statement in his thesis (see [11], III.1.2) by a similar argument.

3.7. Covers in positive characteristic. Now we look into yet another version of a cover structure which is proven to be analytic Zariski, a **cover of the one-dimensional algebraic torus over an algebraically closed field of a positive characteristic.**

Let $(V, +)$ be a divisible torsion free abelian group and K an algebraically closed field of a positive characteristic p. We assume that V and K are both of the same uncountable cardinality. Under these assumptions it is easy to construct a surjective homomorphism

$$\mathrm{ex} : V \to K^\times.$$

The kernel of such a homomorphism must be a subgroup which is p-divisible but not q-divisible for each q coprime with p. One can easily construct ex so that

$$\ker \mathrm{ex} \cong \mathbb{Z}[\frac{1}{p}],$$

the additive group (which is also a ring) of rationals of the form $\frac{m}{p^n}$, $m, n \in \mathbb{Z}$, $n \geq 0$. In fact in this case it is convenient to view V and $\ker \mathrm{ex}$ as $\mathbb{Z}[\frac{1}{p}]$-modules.

In this new situation Lemma 3.3 is still true, with obvious alterations, and we can use the definition 3.2 to introduce a topology and the family L as above. The necessary version of Theorem 3.4 is proved in [18]. Hence the corresponding versions of 3.5 follows.

3.8. Remark. In all the above examples the analytic rank of any nonempty closed subset is 1, that is any closed subset is analytic.

3.9. $\mathbb{C}_{\exp}$ **and other pseudo-analytic structures.** $\mathbb{C}_{\exp}$, the structure $(\mathbb{C}; +, \cdot, \exp)$, was a prototype of **the field with pseudo-exponentiation** studied by the current author in [17]. It was proved (with later corrections, see [19]) that this structure is quasi-minimal and its (explicitly written) $L_{\omega_1,\omega}(Q)$-axioms are categorical in all uncountable cardinality. This result has been generalised to many other structures of analytic origin in [19], in particular to the the formal analogue of $\mathbb{C}_{\mathfrak{P}} = (\mathbb{C}; +, \cdot, \mathfrak{P})$, where $\mathfrak{P} = \mathfrak{P}(\tau, z)$ is the Weierstrass function of variable z with parameter τ. We call these structures **pseudo-analytic.**

It is a reasonable conjecture to assume that the pseudo-analytic structures of cardinality continuum are isomorphic to their complex prototypes. Nevertheless, even under this conjecture it is not known whether $\mathbb{C}_{\exp}$, $\mathbb{C}_{\mathfrak{P}}$ or any of the other pseudo-analytic structures (which do not satisfy 3.8) are analytic Zariski. One may start by defining the family of (formal) closed sets in the structure to coincide with the family of definable subsets which are closed in the metric topology of the complex manifold. The problem then is to conveniently classify such subsets. A suggestion for such a classification may come from the following notion.

3.10. **Generalised analytic sets.** In [6] we have discussed the following notion of generalised analytic subsets of $[\mathbf{P}^1(\mathbb{C})]^n$ and, more generally, of $[\mathbf{P}^1(K)]^n$ for K algebraically closed complete valued field.

Let $F \subseteq \mathbb{C}^2$ be a graph of an entire analytic function and $\bar{F}$ its closure in $[\mathbf{P}^1(\mathbb{C})]^2$. It follows from Picard's Theorem that $\bar{F} = F \cup \{\infty\} \times \mathbf{P}^1(\mathbb{C})$, in particular $\bar{F}$ has analytic rank 2.

Generalised analytic sets are defined as the subsets of $[\mathbf{P}^1(\mathbb{C})]^n$ for all n, obtained from classical (algebraic) Zariski closed subsets of $[\mathbf{P}^1(\mathbb{C})]^n$ and some number of sets of the form $\bar{F}$ by applying the positive operations: Cartesian products, finite intersections, unions and projections. It is clear by definition that the complex generalised analytic sets are closed (but not obvious for the case of K, algebraically closed complete non-Archimedean valued field).

3.11. **Theorem (see [6]).** *Any generalised analytic set is of finite analytic rank.*

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