MODELTHEORY AND ALGEBRAIC GEOMETRY

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It has been known for a long time that the context of the (finite) Morley rank theory involves the intuition of algebraic geometry and, more precisely, the methodology of finite-dimensional algebra over algebraically closed fields. From this point of view this context is essentially poorer than the actual Algebraic Geometry since the latter involves, in the complex case, the powerful techniques of the theory of complex manifolds and analytic methods, which are virtually applicable in the positive characteristic case too. One of the aims of the paper is to show that some classic "transcendental" structures are in fact very interesting structures of finite Morley rank. On the other hand, as is shown in last sections, model-theoretic analysis under certain assumptions leads to a theory of 'infinitesimals' or to a 'non-standard analysis' which introduces in a purely algebraic context analytic methods.
The main theorem of the paper states that any compact complex manifold considered as a structure in a natural language is ω−stable of finite Morley rank. This theorem had been obtained first directly, but immediately after the proof had been found it was realized that the properties of the compact complex manifolds used in the proof are very close to those E.Hrushovski postulated for a strongly minimal structure to get a field definable in the structure (Talk given at Europen Summer Meeting of ASL, Berlin,1989). Combining his idea with the non-trivial examples of arbitrary dimensions we jointly came to the axioms defining **a Zariski-type structure**. Since then the theory has been essentially developed. It is exposed briefly in [HZ1] and in full in [HZ2],[Z] (with some variations in the definitions). The main result of the theory states that a one-dimensional non-locally modular pre-smooth structure can be basically identified with an algebraic curve. The techniques developed in the theory also shows that to a great extent the Model Theory of Zariski-type structures is adequate to Algebraic Geometry.
The result of the present paper is more modest. Nevertheless we can point a methodologically important application of it. One of the main principles of algebraic geometry assumes that objects of algebraic geometry are supported not by sets, but by schemata or, in a simplier context, by equations

defining sets of points for every particular ground field. This is not the case for complex manifolds, since the usual analytic notions lose sense when we replace the field of complex numbers by other ones. At the same time the **compact** complex manifolds behave in many respects like smooth complete algebraic varieties. The main result of the present paper shows that indeed model theoretically compact complex manifolds and smooth complete algebraic varieties are of the same type and by going to elementary extensions one gets structures of the same type (though different ground sets) with the correct dimension theory, smoothness and in fact many other important notions (All the material is from [Z]).
The last section contains a construction of an 'analytic cover of a torus' in ultrametric analysis. The construction in fact coincides with the known Tate's construction [GvdP]. The only difference is the π-'topology', which is explained here model-theoretically and which says the torus is compact (as it should be, if we look at complex analogs). Tate's topology in algebraically closed case does not make the torus compact.

1 Axioms for Zariski-type structure M

(L) Language: all basic relations in the language are closed;
the disjunctions and conjunctions of closed relation are closed;
the universum of the structure is closed;
the equality is closed;
any singleton of the universum is closed;
cartesian products of closed relations are closed.
(P) Properness of projection: the projections $pr : M^n \to M^{n-1}$ are proper,i.e. the images and invers images of closed subsets under pr are closed.
(DCC) Descending chain condition for closed subsets: for any closed

$$S_1 \supseteq S_2 \supseteq \ldots S_i \supseteq \ldots$$

there is i such that for all $j \geq i$ $S_j = S_i$.

It foollows immediately from (DCC),that for any closed S there are closed distinct $S_1, \ldots, S_k$ such that $S = S_1 \cup \ldots \cup S_k$, where k is maximal. These S_i will be called **irreducible components of** S. They are defined up to a numeration uniquely. If $k = 1$, then S is called **irreducible**.

To any closed subset $S \subseteq M^n$ a natural number, called $dimS$ is attached, which satisfies:
(DU) $dim(S_1 \cup S_2) = max\{dimS_1, dimS_2\}$;
(DI) $dimS_1 < dimS$ for irreducible S and $S_1 \subset S$, $S_1 \neq S$;
(DP) dimension of a point is 0;
(DF) $p(S,k) = \{a : dim(S \cap pr^{-1}(a)) > k\}$ is closed for any closed $S \subseteq M^{n+m}$, where pr stands for the projection $M^{n+m} \to M^n$;
(AF) $dimS = dimpr(S) + min_{a \in pr(S)} dim(pr^{-1}(a) \cap S)$ for any closed irreducible S.
We also will use in some cases the following assumptions:
(PS) Pre-smoothness: For any irreducible $S_1, S_2 \subseteq M^n$ the dimension of any irreducible component of $S_1 \cap S_2$ is not less than $dim(S_1) + dim(S_2) - dim(M^n)$.
(EU) Essential uncountabiliy: If a closed $S \subseteq M^n$ is a union of countably many closed subsets, then there are finitely many among the subsets, the union of which is S.

Definition. For any $M' \succ M$ introduce notion of closed relation in M' by declaring closed the sets (relations) of the form

$$S(a, M'^k) = \{b \in M'^k : M' \models S(a,b)\}$$

for S a closed $k+m$-ary relation and $a \in M'^M$. Define the dimension in M' by putting $dimS(a, M') \geq k$ if a is in the corresponding set defined by dimension notion in M.
Definition. For an $M' \succ M$, $a \in M'^n$ and $A \subseteq M'$ we denote

$locus(a/A)$ the minimal A-definable closed subset of M' containing a.

$$dim(a/A) = dim(locus(a/A)).$$

Definition. Let S be an irreducible closed set. We say that a property P for points $a \in S$ is **generic** if the set

$$\{a \in S : P \text{ does not hold for } a\}$$

is contained in a proper closed subset of S.

It is convenient then to say a **is a generic point in** S, meaning "a satisfies any of the generic property considered."

Evidently, if $a \in M' \succ M$, $S = locus(a/M)$, then a satisfies any generic property definable in M. In this case we will call a **generic in** S **over** M.

Lemma 1.1 *Let* $S(x,y)$, $S^1(x,y)$ *be closed,* $a' \in M'$ *and* $S^1(a',M') \subset S(a',M')$, $S^1(a',M') \neq S(a',M')$, $dimS^1(a',M') = dimS(a',M')$. *Then there is closed* $S^2(x,y)$ *such that* $S(a',M') = S^1(a',M') \cup S^2(a',M')$ *and* $S(a',M') \neq S^2(a',M')$.

Proof. Consider $L = locus(a'/M)$, $T(x,y) = S(x,y)\&L(x)$, $T^1(x,y) = S^1(x,y)\&L(x)$. By axiom (AF) $dimT = dimT^1$. Thus by (DI) T is reducible,and using (DU) we see that T^1 must contain an irreducible component of T of dimension $dimT$, so $T = T^1 \cup T^2$, where T^2 is the union of the components of T not in T^1. Take now $S^2 = T^2$ and we are done. □

2 Compact complex manifolds as Z-type structures.

In this section,speaking on complex manifolds, we refer mainly to [GR]. As in the book we identify theorems of [GR] by a triple consisting of a Roman number, a letter and an Arabic number. Chapter V, section B, statement 20, for example, will be V.B.20 and in case the reference to a book is omitted, we mean [GR] .

First, we set some agreement on **a natural language for a compact complex manifold** M. These languages have as basic relations a collection of analytic subsets of M^n and we assume this collection is closed under disjunctions, conjunctions, cartesian products and projections. These operations preserve the assumption on analyticy (for projections it is Remmert's Theorem V.C.5, the rest is immediate from definitions).

Theorem 1 *Any compact complex manifolds M in a natural language and dimension given as complex analytic dimension is a Z-type structure and satisfies assumptions (PS) (pre-smoothness) and (EU) (essentially uncountable).*

Proof. We have to check the axioms.
(L) is given in definitions.

(P) is Remmert's Theorem, V.C.5
(DCC):to see this, first notice that any analytic S is at most a countable union of irreducible analytic S_i and the cover $S = \bigcup_i S_i$ is locally finite ([GrR],A,3,Decomposition Lemma). By compactness the number of irreducible components is finite. Now (DCC) for compact analytic sets follows from (DCC) for irreducible ones,which is a consequence of axiom (DI). The latter as well as (DU) is immediate in III.C.
The condition (AF) is the second part of Remmerts Theorem V.C.5.

(DF) is less immediate. Let U be a neighborhood of a point $b \in S$, which is locally biholomorphic to a complex disk of dimension $r_n = dim(M^n)$ and $S \cap U$ is given as the zero-set of $f_1, ... f_m$ holomorphic in U. Projection pr is given by holomorphic functions $g_1, ... g_{r_{n-1}}$. Then $p^{-1}(a) \cap S$ is the zero-set of $f_1, \dots f_m$ and $g_1 = a_1, \dots g_{r_{n-1}} = a_{r_{n-1}}$, where a_i are the coordinates of $a \in pr(S \cap U)$. And if b is a point in a-fiber, the dimension of the fiber $> k$ implies (and for b nonsingular in the set is equivalent to)
(*) the rank of Jacobian of (f, g) in b is less than $r - k$.
This condition is equivalent to vanishing of all (r-k)-minors of the Jacobian,so it is a (local) analytic condition on b, let S' be the global analytic set defined by (*) in every U. By the construction all components of dimension greater than k of fibres $pr^{-1}(a) \cap S$ lie in S'. This gives $p(S', k) = p(S, k)$. Assuming generic $a \in pr(S)$ is not in $p(S, k)$, we also deduce S' is a proper subset of S. Now the induction by $dim(S)$ finishes the proof of DF.
(EU) is is given by V.B.1.
PS(Pre-smoothness):Let S_1, S_2 be irreducible subsets of M^n. It is easy to see that $S_1 \cap S_2$ is locally biholomorphically isomorphic to $S_1 \times S_2 \cap Diag(M^n \times M^n)$. Now notice that locally M^n is represented by disks of C^d, where $d = dim(M^n)$. Now the condition S is satisfied in M by III.C.11, since the diagonal is given by d equations, each of them decreases the dimension at most by 1. □
Remark. In fact, the Theorem holds for compact analytic spaces, except for the pre-smoothness condition.

Theorem 2 *Any algebraic variety over an algebraically closed field in the language consisting of all Zariski-closed relations and the dimension notion as that of algebraic variety is a Zariski-type structure. It satisfies (PS) if the algebraic variety is smooth. It satisfies (EU) if the field is uncountable. The* *structure is always of finite Morley rank, moreover the rank coincides with the notion of dimension of algebraic varieties.*

Proof. Use any book on algebraic geometry, for example [Sh]. □

3 Zariski-type structures are of finite Morley rank.

Theorem 3 *Any Z-structure M admits elimination of quantifiers with respect to closed relations, i.e. any definable subset $Q \subseteq M^n$ is of the form*

$$(*)\ Q = \cup_{i \leq k}(S_i \setminus P_i) \text{ for some } k,$$

where S_i, P_i are closed sets, S_i irreducible, $S_i \not\subseteq S_j$ for $i \neq j$, $S_i \neq P_i$.

Proof. It is enough to prove that $pr_n(Q)$ is again of the form (*), if Q is. Without loss of generality we may assume

$$Q = S \setminus P,$$

i.e. $k = 1$. We also write pr for pr_n.
We do induction on $dim(S)$.
Let $pr(S) = F$, $d_S = min\{dim(S \cap pr^{-1}(a)) : a \in F\}$ and
$F' = \{b \in F : dim(P \cap pr^{-1}(b)) \geq d_S\}$.

Claim 1. $F \setminus pr(Q) \subseteq F'$.
Proof of the claim. Let $b \in F \setminus F'$. Then $pr^{-1}(b) \cap S \neq pr^{-1}(b) \cap P$ because of different dimensions and thus $b \in pr(S \setminus P)$. So $pr(Q) \supseteq F \setminus F'$. □ Claim

Let $P' = P \cap pr^{-1}(F')$, $S' = S \cap pr^{-1}(F')$.

Claim 2. $pr(Q) = (F \setminus F') \cup pr(S' \setminus P')$.
Proof of the claim. Evidently
$pr(Q) = pr[(S \setminus S'] \setminus P)) \cup pr(S' \setminus P)$ and $(S' \setminus P) = (S' \setminus P')$. Also
$pr(S \setminus S') = F \setminus F'$ and if $b \in F \setminus F'$ then
$dim(S \cap pr^{-1}(b)) > dim(P \cap pr^{-1}(b))$,
$b \in pr[(S \setminus S'] \setminus P))$ iff $b \in F \setminus F' \& S \cap pr^{-1}(b) \neq P \cap pr^{-1}(b)$
iff $b \in F \setminus F'$. □ Claim

Notice now that $S' \neq S$, since $F \neq F'$, the latter being a consequence of $dim(P) < dim(S)$ and axiom (AD). Thus $dim(S') < dim(S)$ and by the induction hypothesis $pr(S' \setminus P')$ is of the form (*). This finishes the proof of the theorem. □

Definition. Define **the closure** $\bar{Q}$ of a set Q to be the smallest closed set containing Q. Call Q **irreducible** if $\bar{Q}$ is irreducible.

Corollary 1 *For any definable Q*

$$\bar{Q} = \cup_{i \leq k} S_i$$

for S_i as in the statement of the Theorem.

Corollary 2 (Tarski's Theorem) *Any definable subset of K^n for algebraically closed field K is a Boolean combination of some zero-sets of polynomials.*

Proof. The projective line $P^1(K)$ is a complete algebraic variety, thus a Z-structure by Theorem 2. □

Definition. Define $dim(Q)$ to be $dim(\bar{Q})$. We will use the terms **irreducible, generic** for any definable Q, meaning it for $\bar{Q}$.

Corollary 3 *For any definable Q*

$$dim(\bar{Q} \setminus Q) < dim(Q).$$

Proof. In notations of the Theorem
$\bar{Q} \setminus Q \subseteq \cup_{i \leq k} P_i$, where $dim(P_i) < dim(S_i)$. □

Lemma 3.1 *Any Z-structure satisfying (EU) is ω_1-compact (i.e. all countable types are realized).*

Proof. We have to check that any descending chain

$$Q_0 \supseteq Q_1 \supseteq \ldots Q_i \supseteq \ldots$$

of non-empty definable subsets of M^n has a common point. We may assume that all Q_i are of the same dimension and of the form $S \setminus P_i$ for closed S and P_i. Now, apparently the intersection $\bigcap Q_i$ is non-empty iff $S \neq \bigcup P_i$, which immediately follows from (EU). □

Proposition 1 *For any essentially uncountable Z-structure M and $M' \succ M$ the structure on M' defined in Section 1 satisfies Z-axioms. It is essentially uncountable,if we choose M' to be ω_1-compact.* .

Proof. Use ω_1-compactness of M to check (DCC) and (EU) and Lemma 1.1 for (DI). All the rest are immediate from the definitions. □

Theorem 4 *Any Z-structure satisfying (EU) is ω_1-compact and of finite Morley rank. More precisely, $rk(Q) \leq dim(Q)$ for any definable set Q.*

Proof. The first statement is Lemma 3.1. Since any countable reduct of M is ω-saturated, the Morley rank in the structure can be counted as the Cantor-Bendixon rank. We prove by induction on n that $rk(Q) \geq n$ implies $dim(Q) \geq n$. $dim(Q) = 0$ iff $rk(Q) = 0$ follows from axioms. Now, suppose $rk(Q) \geq n + 1$ and Q is irreducible. Then

$$Q \supseteq Q_1 \cup \ldots Q_i \cup \ldots$$

for disjoint $Q_1, \ldots Q_i \ldots$ and $rk(Q_i) \geq n$. Hence $\bar{Q} \supseteq \bar{Q}_i$ and $\bar{Q} \neq \bar{Q}_i$ for some i. Thus $dim(\bar{Q}) > dim(\bar{Q}_i)$ by (DI). And by induction hypothesis $dim(Q_i) \geq n$, hence $dim(Q) \geq n + 1$. □

4 Examples of compact complex manifolds as structures in natural languages

We assume the natural languages contain all the analytic relations needed to carry out the constructions considered. First we remind the classic

Riemann's Existence Theorem *Let M be a 1-dimensional irreducible compact complex manifold. Then*

(i) there exist a non-constant meromorphic function on M;

(ii) there is an algebraic curve $C \subseteq \mathbf{P}^n$ and a biholomorphic isomorphism $M \to C$ or, equivalently, for any two different points $p_1, p_2 \in M$ one can find a meromorphic function f, such that $f(p_1) \neq f(p_2)$.

Proposition 2 *Suppose a compact complex manifold M contains an irreducible analytic subset D with a non-constant meromorphic function f on* *it. Then in M^{eq} there is a definable field structure K, which is isomorphic to $\mathbf{C}$ and a definable function $f' : D \to K$, which corresponds to f by the isomorphism. In particular, this holds if M contains a curve.*

If there is another analytic subset D_1 with a non-constant meromorphic function f_1, then there is a definable $f_1' : D_1 \to K$, isomorphic to $f_1 : D_1 \to \mathbf{C}$ over D_1.

Proof. The graph F of the meromorphic function f is an analytic subset of $D^0 \times P^1$, where P^1 is the projective line over $\mathbf{C}$ and D^0 is the domain of f and has the form $D \setminus B$, where B is an analytic subset of D (and so analytic in M), $dimB < dimD$. Under these conditions the topological closure $\bar{F}$ of F is analytic in $D \times P^1$. Define now a binary relation E on D by putting

$$xEy \text{ iff } \exists z \in P^1 : \langle x, z\rangle \in \bar{F} \& \langle y, z\rangle \in \bar{F}.$$

This is analytic in M and its restriction E^0 to D^0 is an equivalence relation, and both D^0 and E^0 are definable. Thus we see that the set D^0/E^0, representing $f(D^0)$ in the canonical way, is definable. Moreover, the canonical map $f' : D^0 \to D^0/E^0$ is also definable by the definition. The same arguments as above show that the (partial) field structure on
$f(D^0) = \mathbf{C}-$ finite number of points
can be translated to $f'(D^0)$ as a definable structure isomorphic to the one on $\mathbf{C}-$ finite number of points. This can be easily extended to a unique field structure, which is the K we wanted.

If D is 1-dimensional and smooth, then it is a compact complex manifold and the existence of f follows from the Riemann Theorem. In case D is not smooth the set of singular points of D is finite. Since locally any analytic curve is biholomorphically isomorphic to an algebraic one, by "blowing up" the manifold finitely many times we get a smooth compact curve D' and a biholomorphic isomorphism between
$D-$ finite number of points and $D'-$ finite number of points.
Then any meromorphic function on D' can be transferred on D.

If there are other $f_1 : D_1 \to \mathbf{C}$ then the correspondence $f^{-1} \circ f_1$ between D and D_1 again can be proved definable by the arguments as above. This correspondence provides a definable isomorphism between K and the field K_1, corresponding to D_1, f_1. This means we can identify K_1 with K. □

Corollary 4 *A non-algebraic two-dimensional complex torus A, having a one-dimensional subtorus B ([Sh], VII, 1, Example 1) is non-almost strongly* *minimal of Morley rank 2, $\aleph_1$-categorical in every reasonably large countable natural language.*

Proof. Let C be A/B. Then there is a naturally induced complex structure on C with the canonical mapping $g : A \to C$ holomorphic.
Moreover, C is a smooth (elliptic) curve. Then by the Riemann Theorem we have a meromorphic function h on C, which gives the function
$f = g \circ h : A \to \mathbf{C}$. So, we know there are a definable field K and definable mappings

$$f' : A \to K, \; e' : B \to K.$$

The first one induces $h' : C \to K$. The correspondence $e' \circ h'^{-1}$ between B and $C = A/B$ is finite-to-finite. Thus for every model A^* of $Th(A)$ the cardinality of A^* is equal to that of B^*. B is strongly minimal, since it is irreducible of dimension 1. This proves the $\aleph_1$-categoricity of $Th(A)$.
As is proved in [Sh], all irreducible curves on A are just cosets of B. This contradicts almost strong minimality, for otherwise A would be in a definable finite-to-finite correspondence with $K \times K$ and the correspondence evidently transfers any definable in K family of curves on $K \times K$ onto a definable family of curves on A of the same dimension. □

Proposition 3 *Any generic n-dimensional torus M for $n > 1$ is locally modular strongly minimal and carries an abelian group structure.*

Proof.A torus has an analytic abelian group structure from the construction. The strong minimality of the generic tori follows from the fact that they have no proper analytic subsets ([Sh].VII,1, Example 2). A.Pillay found some algebro-geometric arguments showing that the only irreducible analytic subsets of $M \times M$ are cosets of subtori of the torus $M \times M$. This is evidently eqivalent to local modularity. □

One more example can help in a better understanding of the dimension notion for Z-structures. As we saw above, the generic torus is a structure of Morley rank 1 and dimension n. To see that we can not put the dimension to be just 1, if we want to consider the structure as a natural substructure of some bigger ones, we discuss the following.

Example Let T be a generic torus of dimension n and M the compact complex manifold, which is obtained by "blowing up" T in a point $a \in T$. This means that there is a holomorphic function $f : M \to T$, which is onto and is biholomorphically one-to-one everywhere except the point a. Whereas

$$f^{-1}(a) = \mathbf{P}^{n-1}.$$

Moreover M is irreducible.
Now, if one considers M in a large natural language, T can be identified as a substructure in M^{eq}, which can be given a dimension notion in a canonical way. The Morley rank of $\mathbf{P}^{n-1}$ is $n-1$ because of its algebraic structure. Thus by axiom (DI) $dimM \geq n$. On the other hand, because of f and the axioms (DF) and (AD), $dimT \geq n$ too.

5 Pre-smoothness on algebraic curves

First we give a wider notion of pre-smoothness, which is applicable to subsets definable in Z-structures.

Definition.A will be called **pre-smooth (with** M**)** if for any irreducible $S_1, S_2 \subseteq A^k \times M^m$ any irreducible component of the intersection $S_1 \cap S_2$ is of dimension not less than

$$dim(S_1) + dim(S_2) - dim(A^k \times M^m).$$

We are going to clarify here the notion of pre-smoothness for complete algebraic curves over an algebraically closed field K. One has first to make precise the way an algebraic curve is considered a Z-structure. Of course, the best of all is to consider it in the natural language as in Theorem 2. Here for simplicity we use a different representation. It follows from the Main Theorem and results of section 5 that the present representation is equivalent to the natural one. So, speaking on an algebraic curve C we consider the algebraic variety $M = C \times \mathbf{P}^1$ over the algebraically closed field K in the language containing all the algebraic subvarieties of M^n.

Proposition 4 *Let C be complete algebraic curve over an algebraically closed field K and $\{a_1, \ldots a_m\}$ the set of all singular points of C. Then*
(i) there is a smooth projective curve A and a regular mapping $f : A \to C$,

such that f is a biregular bijection on $C \setminus \{a_1, \dots a_m\}$;
(ii) C is pre-smooth iff f is a bijection.

Proof.(i) is classic and can be found in [Ha].
(ii) One can embed any projective space into $(\mathbf{P}^1)^n$ for some n, so we consider $A \subseteq (\mathbf{P}^1)^n) \subseteq M^n$. If f is a bijection, then it is a Z-homeomorphism, i.e. it maps Z-closed subsets of C^n to that of A^n and conversely. Hence, it transfers pre-smoothness from A to C.
If f is not a bijection then $f^{-1}(c) = \{a_1, \dots a_k\}$, $k > 1$ for some $c \in C$. Take

$$F_1, F_2 \subseteq C \times M^n \times M^n,$$

defined as

$$\langle x, y, z\rangle \in F_1 \text{ iff } x = f(y),\ \langle x, y, z\rangle \in F_2 \text{ iff } x = f(z).$$

Now, $dim F_1 = dim F_2 = 1 + n \times dim M$, and the irreducible components of $F_1 \cap F_2$ are $\{\langle x, y, z\rangle : y = z \& x = f(y)\}$ and $\{\langle c, a_i, a_j\rangle\}$ for different $i, j \leq k$. Thus some of the components are of dimension 0, whereas $dim F_1 + dim F_2 - dim(C \times M^{2n}) = 1$, contradicting the pre-smoothness.□

6 Specializations

We introduce here one of the principal tools of this paper, which we call a specialization. It has analogs both in model theory and algebraic geometry. In the latter the notion under the same name has been used by A.Weil [W], namely if K is an algebraically closed field and $\bar{a}$ a tuple in an extension K' of K, then a mapping $K[\bar{a}] \to K$ is called a specialization if it preserves all equations with coefficients in K.

The model-theoretic source is the notion of **atomic compact** structures, introduced by J.Mycielski [My] and given a thorough study by B.Weglorz. A structure M is said to be atomic (positive) compact if any finitely consistent set of atomic (positive) formulas is realized in M. It can be easily seen that Z-structures are atomic compact. The main result of [We] is

Theorem 5 (B.Weglorz) *The following are equivalent for any structure M :*

(i) M is atomic compact;
(ii) M is positive compact;
(iii) M is a retract of any $M' \succ M$.

So, most of this section is essentially well known.
We assume in this section that M is a structure,satisfying (L).

Definition. Let $M^* \succ M$ be an elementary extension of M, $A \subseteq M^*$. A mapping $\pi : A \to M$ will be called a **(partial) specialization,** if $\pi(S^* \cap A^n) = S \cap A^n$ for any n-ary closed definable S, where S^* stands for the set of realizations of the relation S in M^*.

Remark. By the definition a specialization is an identity on M, since any singleton $\{s\}$ is closed.

Definition.M will be called **quasi-compact** if it satisfies (P) and
(QC) For any finitely consistent family $\{C_t : t \in T\}$ of closed subsets of M^n

$$\bigcap_{t \in T} C_t \text{ is non-empty and closed.}$$

Proposition 5 *Suppose M is a quasi-compact structure, $M^* \succ M$. Then there is a specialization $\pi : M^* \to M$, moreover, any partial specialization can be extended to a total one.*

Proof. Suppose π is defined on some $D \subseteq M^*$ and b' an element of M^*. We want to extend the definition of π to $D \cup \{b'\}$. To do this consider

$$\{S(x, \bar{d}') : S \text{ is a closed relation and } S(b', \bar{d}'), \bar{d}' \in D^n\}.$$

Now look at the corresponding family

$$\{S(M, \bar{d}) : \bar{d} = \pi(\bar{d}'), \bar{d}' \in D^n\}.$$

This is a finitely consistent family, by (P), of closed sets. Thus by (QC) the intersection of the sets of the family is non-empty. Put $\pi(b') = b$ for a b chosen in the intersection. It is immediate that the extended map is again a partial specialization.
Continuing this process one gets a total specialization. □

Corollary 5 *Suppose N is a substructure of M, M, N satisfy (L), M is quasi-compact and for every $S \subseteq M^n$ closed in M the set $S \cap N^n$ is closed in N. Then every specialization of N^* can be extended to a specialization of M^* for $(M^*, N^*) \succ (M, N)$.*

Definition. Suppose $M^* \succ M$, in the language containing all possible relations on M and $\pi : M^* \to M$ is a map identical on M. We will call a definable set $A \subseteq M^n$ π-closed if $\pi(A^*) \subseteq A$. (Evidently, $\subseteq$ can be replaced here by $=$)

Proposition 6 *For any M^* and π as in the definition above, M with the notion of closed as π-closed is quasi-compact, provided M^* is saturated over M.*

Proof. For a finitely consistent family $\{C_t : t \in T\}$ of closed subsets of M^n denote $C = \bigcap\{C_t : t \in T\}$. Then $C^* \subseteq \bigcap\{C_t^* : t \in T\}$, since $M^* \succ M$. Thus $\pi(C^*) \subseteq \bigcap\{\pi(C_t^*) : t \in T\} = C$, which means that C is closed.
C^* is non-empty, since M^* is saturated over M. Thus $C \neq \emptyset$ and (QC) is proved. □

Lemma 6.1 *If π is a specialization $N^* \to N$ and the quasi-compact topology on N, π respects, is Hausdorff (i.e. the topology is compact), then an arbitrary subset $A \subseteq N^n$ is closed iff it is π-closed.*

Proof. Closed is π-closed by definition. Suppose for the convers that A is π-closed and $a \in \bar{A}$, the closure of A, $a \ni A$. Let U_b be an open neighborhood of a, which does not intersect with an open neighborhood of $b \in N$. Choose in N^* an element a', which satisfies A and all $\bar{U}_b : b \in N$. Since $\bigcap \bar{U}_b = \{a\}$, thus $\pi(a') = (a)$, but $a' \in A^*$, which implies $\pi(a') \in A$. The contradiction. □

Corollary 6 *If there is a quasi-compact topology on a substructure N of a quasi-compact structure M, then there is a quasi-compact topology Π on M, such that N and all closed subsets of N^n are closed in Π.*

Proof. Take a specialization π as in the above corollary and Π to be the π-topology. □

Example. Let k be a locally compact field, and $K \supset k$ an algebraically closed field. It is easy to see that the projective line $P^1(k)$ is compact in the topology of k and all Zariski-closed relations on $P^1(k)$ are closed in the topology. The same is true for any finite extension of k. Consider now the language for $P^1(K)$, containing all Zariski-closed relations, a unary predicate for $P^1(k)$ and all topologically compact relations of $P^1(k)$. Then there is an extension of this language in which the structure is quasi-compact and the language induces no new closed sets on $P^1(k)$.

7 Specializations in Z-structures.

Definition. For a specialization $\pi : M^* \to M$ we say that the pair (M^*, π) is e.c. if for any $M' \succ M^* \succ M$, any finite $A \subset M'$ and a specialization $\pi' : A \cup M^* \to M$ there is an elementary isomorphism over $M \cup (A \cap M^*)$ $\alpha : A \to M^*$, such that $\pi' = \pi \circ \alpha$ on A.

Proposition 7 *For any Z-structure M there is an e.c.pair (M^*, π).*

Proof. We construct M^* and π by the following routine process:
Start with any $M_0 \succ M$ and a specialization $\pi_0 : M_0 \to M$.
We will construct a chain of length ω of elementary extensions $M_i \succ M_{i-1} \succ \ldots \succ M_0$ and specializations $\pi_i : M_i \to M$, $\pi_i \supseteq \pi_{i-1}$.
To construct M_{i+1} and π_{i+1} we first enumerate all triples of the form $\langle A, \bar{a}, p(\bar{x})\rangle$, where A is a finite subset of M_i, $\bar{a} \in M^n$ and $p(\bar{x})$ is an n-type over $M \cup A$. Let κ be the cardinality of the set of the triples. Now we construct specializations $\pi_{i,\alpha}$, $\alpha \in \kappa$, such that $\pi_{i,\alpha} \supset \pi_i$, and their domains $N_{i,\alpha}$:
$N_{i,0} = M_i$ and $\pi_{i,0} = \pi_i$.
$N_{i,\alpha+1} = N_{i,\alpha} \cup \{\bar{b}\}$, where $\bar{b}$ realizes p_α, if there is a specialization $\pi' \supset \pi_{i,\alpha}$, sending $\bar{b}$ to $\bar{a}_\alpha$. In this case $\pi_{i,\alpha+1}$ is π'.
Otherwise, $N_{i,\alpha} = N_{i,\alpha+1}$, $\pi_{i,\alpha+1} = \pi_{i,\alpha}$.
On the limit steps take unions.
Put now M_{i+1} any model containing $N_{i,\kappa}$, and $\pi_{i+1} \supset \pi_{i,\kappa}$, a specialization from M_{i+1} to M.
It follows from the construction, that for any $M' \succ M_i \succ M$, any finite $B \subset M'$ and a specialization $\pi' : B \cup M_i \to M$ there is an elementary isomorphism over M $\alpha : B \to M_{i+1}$, such that $\pi' = \pi \circ \alpha$ on B.
Now take $M^* = \bigcup_{i<\omega} M_i$, $\pi = \bigcup_{i<\omega} \pi_i$. □

Remark. Evidently, one can get M^* λ-saturated and λ-e.c. for any λ by slightly changing this construction.

8 Elements of a non-standard analysis for Z-structures

In the sequel we fix the notation (M^*, π) for an e.c. pair, constructed above.

Definition. For a point $a \in M^n$ we call **an infinitesimal neighborhood of** a the subset in M^* given as

$$\mathcal{V}_a = \pi^{-1}(a).$$

Lemma 8.1 *Suppose $a' \in \mathcal{V}_a$, $F(x,y)$ is a closed relation, such that $\models F(a,b)$. Then there is $b' \in \mathcal{V}_b \cap F(a', M^*)$ iff the following type is consistent:*

$$p(y) = \{F(a',y)\} \cup \{\neg Q(c',y) : \models \neg Q(c,b) \text{ for all closed } Q, c' \in M^{*m}, \pi(c') = c\}$$

Moreover, if the type is consistent, then for any b' realizing p in some extension and any finite $A \subset M^*$ there is $b'' \in \mathcal{V}_b \cap F(a', M^*)$, such that $tp(b'', A \cup M) = tp(b', A \cup M)$.

Proof. It follows from the definition of specialization , that p is consistent if such b' exists. For the convers just verify that if b'' realizes p, then an extension $\pi' \supseteq \pi$ to b'', defined as $\pi'(b'') = b$, is a specialization $M^* \cup \{b''\} \to M$. The rest follows from the fact that (M^*, π) is e.c. . □

Theorem 6 *Suppose $F(y)$ is a closed irreducible relation in a Z-structure M, $b \in F$ and $dimF = d$. Then there is $b' \in \mathcal{V}_b \cap F^*$, such that $dim(b'/M) = d$.*

Proof. Consider the type p defined in the preceeding Lemma, which under our assumption does not depend on a'. Now, for any $Q(c', y)$ as in the type either $dimF \& Q(c', y) < dimF$ or $F^* \subseteq Q(c', M^*)$, since F is irreducible. The latter would yield $\models Q(c,b)$, which contradicts the definition of $Q(c',y)$. The former immediately gives $dimp = dimF$, thus yielding the consistency of p and, moreover, by the Lemma the required b'. □

Definition. For a closed relation F in M and $b \in F$ define $dim_b F$ **the local dimension of F in** b to be the maximal dimension of an irreducible component of F, containing b.

Corollary 7 *$dim_b F = max\{dim(b'/M) : b' \in \mathcal{V}_b \cap F^*\}$.*

Corollary 8 *A definable set A is pre-smooth (with M) iff for any k, m and closed subsets $S_1, S_2 \subseteq A^k \times M^m$*

$$dim_x(S_1 \cap S_2) \geq dim_x(S_1) + dim_x(S_2) - dim(A^k \times M^m)$$

for any $x \in S_1 \cap S_2$.

9 Non-compact case

We assume M^* is as defined in the preceding section and also it is saturated over M. Suppose for some reasons we have chosen a structure $M^\#$, such that $M \subseteq M^\# \subseteq M^*$ and a retraction $\pi : M^\# \to M$. For $A \subseteq M^n$ define $A^\#$ (A^*) to be the subset of ${M^\#}^n$ (${M^*}^n$) which is defined by the relation naming A. This defines a $(M^\#, \pi)$-**topology** on M^n for all n by declaring closed those A which satisfy $A = \pi(A^\#)$.

Definition. We will say $N \subseteq M$, is **bounded** if $N^\# = N^*$.

Proposition 8 *If $N \subseteq M$, is bounded and closed then N is quasi-compact in $(M^\#, \pi)$-topology*

Proof. By the definitions $N \prec N^*$ in the language which names all subsets of ${N^*}^n$, π takes N^* onto N and N^* is saturated over N. Hence the required follows from 6. □

Example 2 (i) Take an a.c.field K and $P^1(K)^* \succ P^1(K)$ saturated over

$P^1(K)$, $\pi : P^1(K)^* \to P^1(K)$ a specialization with respect to Zariski topology. Considering a natural embedding $K \subset P^1(K)$, we get an induced embedding $K \prec K^*$. Define now $K^\# = K^* \setminus \pi^{-1}(\infty)$.
If π is induced on $P^1(K)^*$ =(complex projective line)*, then the $(K^\#, \pi)$-topology is the usual complex topology.
(ii) Start with the same K and define K^* and π as above. Denote $K_\circ = K \setminus \{0\}$ and put $(K_\circ)^\# = K^* \setminus \pi{-}1(\{0, \infty\})$.

Proposition 9 *Let Ω be a multiplicative cyclic subgroup of $K_\circ$ generated by a transcendental element g. Then we can choose a ${K_\circ}^\#$ and π as in Example 2(ii) such that Ω is closed and descrete in $K_\circ$.*

Proof. We first prove

Lemma 9.1 *Suppose Ω is a multiplicative subgroup of K, generated by algebraically independent elements.And suppose $t_1, \ldots, t_k \in \Omega^*$ $p(t_1, \ldots, t_k) = 0$ for $p(v_1, \ldots, v_k)$ a polynomial over Z. Then there are multiplicative semigroup words $w^i(v_1, \ldots, v_k)$ such that $w^i(t_1, \ldots, t_k) = w^j(t_1, \ldots, t_k)$ holds for some i, j and $p = 0$ is a consequence of the appropriate semigroup equations*

Proof. Let $v = \langle v_1, \ldots, v_k \rangle$, $\{v^i : i \in I(p)\}$ the monomials of p written in multiindices. Suppose,towards a contradiction,p does not satisfy the conclusion of the statement and has a minimal number of occurences of the monomials. Then $t^i \neq t^j$ for all different multiindices $i, j \in I(p)$. Then, by the fact that $\Omega \prec \Omega^*$, there are $h_1, \ldots, h_k \in \Omega$, such that $p(h_1, \ldots, h_k) = 0$ and $h^i \neq h^j$ for the same i, j. Now as h^i are elements of Ω one can write them as monomials in generators $g_1, \ldots, g_n$ of Ω : $h^i = g^{l(i)}$. The multiindices $l(i)$ are different for different i by the choice of h. Thus $\Sigma_{i \in I(p)} a_i g^{l(i)} = 0$ for a_i the coefficients of $p(v)$ is a non-trivial equation satisfied by g, which is the contradiction.□
Proof.of the Proposition. Embed Ω in $\mathbf{R}$ identifying it with $\mathbf{Z}$, which is descrete and closed in the real topology. The real topology generates a unique specialization $\sigma : \mathbf{R}^* \to \mathbf{R} \cup \{-\infty, \infty\}$, which induces a specialization $\pi : \Omega^* \to \Omega \cup \{0, \infty\})$ respecting the group structure. This specialization considered as a partial map $P^1(K)^* \to P^1(K)$ respects the full algebraic (field) structure on $P^1(K)$, by the Lemma above. Extend now π onto $P^1(K)^*$ by Proposition 5 and we are under assumptions of Example 2(ii), which construction provides $\#$. □

Remark. For π as in the Proposition one can take simply π respecting a valuation v on K as in the example in the preceding section, provided $v(g) \neq 0$.
Question. Does there exist $\#$ for $K_\circ$ which makes the quotient-space $K_\circ/\Omega$ quasi-compact for K of a positive characteristic?

Proposition 10 *Suppose we started as usual with an e.c. specialization*

$$\pi : (P^1(K), \Omega)^* \to (P^1(K), \Omega).$$

Then $K_\circ/\Omega$ is compact.

Proof. We will show that $(K_\circ/\Omega)^* = K_\circ\#/\Omega$. To do this it is enough to find for any $a' \in K_\circ{}^*$ an $a \in (K_\circ$ and $\omega \in \Omega^*$ such that $\pi(a' \cdot \omega) = a$. If $\pi(a') \in K_\circ$ then $\omega = 1$. So we assume $\pi(a') = \infty$. The case $\pi(a') = 0$ is symmetric.
Since π is e.c., we can look for ω in an extension of $\langle \pi, (P^1(K), \Omega)^* \rangle$. We construct such an extension, $\langle \pi', (P^1(K'), \Omega') \rangle$, assuming

$$\pi(a' \cdot \gamma) \in \{0, \infty\} \text{ for any } \gamma \in \Omega^* \ (1) \ .$$

Let K' be an algebraically closed field generated by a transcendental element ω over K^*. Since $a' \cdot \omega$ is transcendental over K^*, the extension $\pi' : \{a' \cdot \omega\} \cup P^1(K^*)$ of the mapping π defined as

$$\pi'(a' \cdot \omega) = 1 \ (2)$$

is a specialization with respect to the field relations. By Proposition ?? this can be extended to a total specialization $\pi' : P^1(K') \to P^1(K)$.
Now define a subgroup Ω' of the multiplicative group of K' as $\Omega \cdot \omega^Q$, where ω^Q is a minimal subgroup containing ω and such that with any δ the subgoup for each n contains some λ such that $\lambda^n = \delta$. We then denote this λ as $\delta^{\frac{1}{n}}$ and $\lambda^m = \delta^{\frac{m}{n}}$.
It follows from (2) that $\pi'(\omega^{\frac{m}{n}}) = 0$ for any positive integers m, n. By Lemma 9.1 $P^1(K') \succ P^1(K^*)$ in the language containing the field relations and Ω. Finally we show that π' preserves Ω, then π' would be the desired specialization.
Take any element of Ω' of the form $\gamma \cdot \omega$ with $\gamma \in \Omega^*$. If $\pi(\gamma) \neq \infty$, then

$\pi'(\gamma \cdot \omega) = 0$. Let $\pi(\gamma) = \infty$. By (1) we have two possibilities:
(i) $\pi'(a' \cdot \gamma^{-1}) = \infty$,
(ii) $\pi'(a' \cdot \gamma^{-1}) = 0$.
Since

$$\pi'(a' \cdot \gamma^{-1}) \cdot \pi'(\gamma \cdot \omega) = \pi'(a' \cdot (\gamma^{-1} \cdot \gamma) \cdot \omega) = 1,$$

the first case yields $\pi'(\gamma \cdot \omega) = 0$. In the second case use

$$\pi'(\gamma \cdot \omega)^{-1}) = \pi'(a' \cdot \gamma^{-1}) \cdot \pi'(a' \cdot \omega)^{-1}) = 0.$$

I.e. $\pi'(\gamma \cdot \omega) = \infty$ in this case. Thus in all cases $\pi'(\gamma^n \cdot \omega^n) \in \{\infty, 0\}$ for any non-zero integer n and $\gamma \in \Omega^*$, or, equivalently,

$$\pi'(\gamma \cdot \omega^n) \in \{\infty, 0\} \quad (3)$$

for any non-zero integer n and $\gamma \in \Omega^*$. Since $\Omega' \succ \Omega$ and the last is a group generated by a transcendental $g \in K$, for any $\gamma \in \Omega^*$ and a positive integer m there are $\delta \in \Omega^*$, an integer $0 \leq k < m$ such that $\gamma = \delta^m \cdot g^k$. It follows from (3) that

$$\pi'(\delta^m \cdot \omega^n) \in \{\infty, 0\},$$

which implies

$$\pi'(\delta \cdot \omega^{\frac{n}{m}}) \in \{\infty, 0\}$$

for any $\delta \in \Omega^*$. This proves that π' respects Ω. □

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