On systems of exponential sums with real exponents

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In [Z2] we studied the theory of formal exponentiation (raising to powers) and proved that it is very nice (superstable) provided a certain diophantine conjecture CIT is true (see [Z1] for a discussion of CIT). We believe that in fact the theory describes the genuin complex exponentiation. To show this we need two statements, one of them being the Schanuel conjecture [L] and the other is *the exponential-algebraic closedness* of $\mathbf{C}$, i.e. the solvability of any non-obviously-inconsistent system of equations. Here we give a proof of the latter in case of real powers modulo the Schanuel conjecture and CIT. The proof is based on the theory of exponential sums developed by D.Bernstein, A.Kushnirenko, B.Kazarnovski and A.Khovanski, see [Kh] (the Russian edition).

1 Exponential sums

Let $V \subseteq \mathbf{C}^n$ be a zero-set of p polynomials

$$f_i(y) = \sum_{m \in \Gamma_i} a_{i,m} y^m \quad (i = 1, \ldots, p) \tag{1}$$

where $y = \langle y_1, \ldots, y_n \rangle$ are variables, $m = \langle m_1, \ldots, m_n \rangle \in \mathbf{Z}^n$, $y^m = y_1^{m_1} \cdot \ldots \cdot y_n^{m_n}$, $a_m \in \mathbf{C}$ and Γ_i some finite subsets of $\mathbf{Z}^n$.

Let $L \subseteq \mathbf{C}^n$ be a K-linear subspace of $\mathbf{C}^n$ for some subfield $K \subseteq \mathbf{C}$. Up to the numeration of coordinate functions $x_1, \ldots, x_n$ we may assume L is given by $n - k$ linear equations of the form $x_j = (s_j, z)$, $j = k+1, \ldots, n$, where $z = \langle x_1, \ldots, x_k \rangle$, $s_j = \langle s_{j,1}, \ldots, s_{j,k} \rangle \in K^k$ and $(s_j, z) = \sum_{l \le k} s_{j,l} x_l$. We assume also $s_j = \langle 0, \ldots, 0, 1, \ldots 0 \rangle$ with 1 on jth place for $j = 1, \ldots, k$, thus

$$(s_j, z) = x_j \text{ for all } j = 1, \ldots, n. \tag{2}$$

We also identify the points $t \in K^n$ with the K-linear mapping $(t, x) = t_1 \cdot x_1 + \ldots + t_n \cdot x_n$ from $\mathbf{C}^n$ to $\mathbf{C}$. The restriction of such a mapping to L denote t_L.

Denote λ the linear mapping $t \to t_L$ from the space K^n of K-linear forms on $\mathbf{C}^n$ onto the space of K-linear forms L^* on L.

With this notation we associate to the system (1) the system of **exponential sums** in coordinates z ranging in L

$$\sum_{m \in \Gamma_i} a_{i,m} \exp(m_L, z) \quad (i = 1, \ldots, p) \tag{3}$$

or, equivalently, after gathering summands with common exponent, we get from (1) the system of equations

$$x \in L \ \& \ \bigwedge_{i=1,\ldots,p} \sum_{d \in \Delta_i} b_{i,d} \exp(d,x) = 0 \tag{4}$$

with $\Delta_i = \{m_L : m \in \Gamma_i\} \subseteq L^*$ and

$$b_{i,d} = \sum_{\{m \in \Gamma_i: \ d = \lambda(m)\}} a_{i,m}.$$

The set of solutions of the system (4) can be identified as $L \cap \ln V$, where

$$\ln V = \{\langle x_1, \ldots, x_n \rangle : \langle \exp(x_1), \ldots, \exp(x_n) \rangle \in V\}.$$

Thus the correspondence between pairs (L, V) and exponential systems, depending only on the choice of coordinate functions, is established.

Definition Let L be a linear subspace of $\mathbf{C}^n$ and V an algebraic variety in $\mathbf{C}^n$. Given a pair (L, V), a $\mathbf{Q}$-affine subspace M of $\mathbf{C}^n$ and a point $a \in M \cap L \cap \ln V$ it is said that (L, V) **restricted to M in a is free** if M is the minimal $\mathbf{Q}$-affine subspace of $\mathbf{C}^n$ containing $M \cap L$ and $\exp(M)$ is the minimal (shifted) torus containing the connected component of $V \cap \exp(M)$ passing through $\exp(a)$.
If $M = \mathbf{C}^n$ in the definition and $\exp(\mathbf{C}^n)$ is the minimal torus containing a connected component of V then (L, V) is said to be free.

Notice that if L is free of additive dependencies, in particular, if the pair if free, then for any $d \in K^k$ there is no more than one $m \in \mathbf{Z}^n$ such that $m_L = d$, in other words the mapping $t \mapsto t_L$ is injective on $\mathbf{Z}^n$. In this case also $b_{i,d} = a_{i,m}$ for $d = m_L$ in equations (4).

The following notions are basic in [Z].

Definition A pair (L, V) with V defined over $A \subseteq \mathbf{C}$ is said to be **normal** if there are $\langle a_1, \ldots, a_n \rangle \in L$ and $\langle b_1, \ldots, b_n \rangle \in V$ such that for any $l \leq n$ independent integer vectors $m_i = \langle m_{i,1}, \ldots m_{i,n} \rangle$, $i = 1, \ldots, l$, and

$$a_i' = m_{i,1} a_1 + \ldots + m_{i,n} a_n, \quad b_i' = b_1^{m_{i,1}} \cdot \ldots \cdot b_n^{m_{i,n}}$$

it holds

$$\text{lin.d.}_K(a_1', \ldots, a_l') + \text{tr.d.}(b_1', \ldots, b_l'/A) \geq l.$$

Equivalently, for the K-linear subspaces $L_{1,\ldots,l}' = \text{linlocus}\langle a_1', \ldots, a_l' \rangle$, the minimal K-linear subspace of $\mathbf{C}^l$ containing $\langle a_1', \ldots, a_l' \rangle$, and varieties $W_{1,\ldots,l}' = \text{alglocus}_A\langle b_1', \ldots, b_l' \rangle$, the minimal algebraic subvariety of $\mathbf{C}^l$ defined over A and which contains $\langle b_1', \ldots, a_l' \rangle$, it must hold

$$\dim L_{1,\ldots,l}' + \dim W_{1,\ldots,l}' \geq l.$$

The following connects the normality and freeness conditions to the basic invariant of the theory of exponential sums, the **mixed Minkovski volume of Newton polytopes of the system**, see [Kh] and [GW].

Lemma 1.1 *Assume V is given by a system $f = 0$ of $p = k$ independent polynomial equations. If (L, V) is normal and free and $\dim L + \dim V = n$, $(\dim L = k)$ then for the corresponding exponential system of k equations in k variables the mixed Minkovski volume of convex envelopes $\bar{\Delta}_1, \ldots \bar{\Delta}_k$*

$$\mathrm{Vol}(\bar{\Delta}_1, \ldots \bar{\Delta}_k) \neq 0.$$

Proof By a suitable replacement of variables we may assume that in each equation non-trivial constant term occurs. I.e. each Γ_i contains the zero-point.

We use the fact that, assuming each $\bar{\Delta}_i$ contains the zero point, $\mathrm{Vol}(\bar{\Delta}_1, \ldots \bar{\Delta}_k) \neq 0$ provided that for any distinct $i_1, \ldots, i_l \leq k$

$$\mathrm{lin.d.span}_K(\Delta_{i_1} \cup \ldots \cup \Delta_{i_l}) \geq l.$$

Notice that, since V is free of multiplicative dependencies and V is non-empty, $V \cap (\mathbf{C}^*)^n \neq \emptyset$.

Notice that since L has no additive dependencies the mapping $m \mapsto m_L$ is injective on $\mathbf{Z}^n$ and $(\Gamma_i)_L = \Delta_i$.

Suppose towards a contradiction that $\mathrm{span}_K(\Delta_1, \ldots \Delta_l) = H_\Delta \subseteq L^*$ is of dimension less than l. Since the mapping $\lambda : K^n \to L^*$ is linear, for $H_\Gamma = \mathrm{span}_K(\Gamma_1, \ldots \Gamma_l)$,

$$\lambda(H_\Gamma) = H_\Delta$$

and

$$\dim \ker_\Gamma \lambda = \dim H_\Gamma - \dim H_\Delta,$$

for $\ker_\Gamma \lambda$, the kernel of the mapping restricted to H_Γ. Denote $s = \dim H_\Gamma$. Then

$$\dim \ker_\Gamma \lambda > s - l.$$

Since H_Γ is generated by integer points, there is a $\mathbf{Z}$-linear transformation of variables on $\mathbf{C}^n$ of the form $x_i' = (q_i, x)$, for some $q_i \in \mathbf{Q}^n$, such that $q_1, \ldots, q_s$ is a basis of H_Γ which also generates the abelian subgroup containing $\Gamma_1, \ldots, \Gamma_l$. Letting $y_i' = \exp(x_i')$ we can equivalently rewrite equations (1) so that in the first l equations only $y_1', \ldots, y_s'$ occur. Denote V_s the projection of V onto the subspace of the first variables $y_1', \ldots, y_s'$ and L_s, correspondingly, the projection of L onto $(x_1', \ldots, x_s')$-subspace $\mathbf{C}^s$. Since the equations $f = 0$ were independent, $\dim V_s \leq s - l$. The dimension of L_s we deduce from that of $\ker_\Gamma \lambda$. Notice that $\ker_\Gamma \lambda$ is the set of all K-linear $t : \mathbf{C}^s \to \mathbf{C}$ which are zero on L_s. Thus the quotient $K^s / \ker_\Gamma \lambda$ can be identified with the space L_s^* of all K-linear forms on L_s, the dual space to L_s. Since $\dim L_s^* = \dim L_s$, we get

$$\dim L_s = s - \dim \ker_\Gamma \lambda.$$

Thus

$$\dim V_s + \dim L_s \leq (s - l) + (s - \dim \ker_\Gamma \lambda) < s,$$

which contradicts the normality of (L, V).$\square$

We use below the widely known Schanuel Conjecture SchC and a conjecture of diopantine type, *the conjecture on intersection varieties with tori in atypical components* CIT, discussed in [Z]. It is proved in [Z]

Theorem 1 *Assume* SchC+CIT. *Then given a K-linear $L \subseteq \mathbf{C}^n$ not contained in any proper $\mathbf{Q}$-linear subspace of $\mathbf{C}^n$, and an algebraic family $W(\bar{a}) \subseteq \mathbf{C}^n$ of algebraic varieties there is a finite set $\eta(L, W)$ of K-linear hyperplanes of L and a natural number $l = l(W)$ such that there are $d_1, \ldots, d_l \in \mathbf{C}^n$ satisfying the property that any atypical component R of $L \cap \ln W(\bar{a})$ is contained in some $H + d_i$ for $H \in \eta(L, W)$ and $i \leq l$.*

Notation Fix an exponential system given by $p = k$ polynomial equations (1) in $\mathbf{C}^n$ and a K-linear subspace $L \subseteq \mathbf{C}^n$ in the final form (4). Denote V the family of all systems $V(a)$ of $p = k$ **independent** equations (1). We assume all $a_{i,m} \neq 0$ in the initial system and also that L **is free,** i.e. not contained in any proper $\mathbf{Q}$-linear subspace of $\mathbf{C}^n$. We also assume that the projective completion $\bar{V}(a) \subseteq \mathbf{P}^n(\mathbf{C})$ of $V(a)$ has no component lying in the subvariety $y_i = 0$ or $y_i^{-1} = 0$ for $i \leq n$.
It is well known that the family V can be coordinatized by points of $\mathbf{C}^d$ for some d so, that the coordinatization identifies family V as an algebraic set $C(V)$, i.e. a set of the form $S_1 \setminus S_2$ with S_1 and S_2 algebraic varieties, and each point of $C(V)$ corresponds to an algebraic variety $V(a)$ of the family. W.l.o.g. we assume $C(V)$ is irreducible (i.e. S_1 is).

We give now several key definitions and results from [Kh], see also earlier publication [K]. We assume from now on $K = \mathbf{R}$ ([Kh] considers only this case).

Definition Let $\Sigma_i \subseteq \Gamma_i$ be faces of corresponding convex polytopes. The faces are said to agree if there is a common $\mathbf{R}$-linear function φ on $\mathbf{R}^n$ that takes its minimum on each convex polytope exactly on Σ_i. To each agreed set $\Sigma_1, \ldots, \Sigma_k$ of faces one associates a **shortening** of the initial system $f = 0$ of equations defined as

$$f_i^{\Sigma_i} = \sum_{m \in \Sigma_i} a_m y^m = 0 \ \ (i = 1, \ldots, k).$$

If $f = 0$ determines variety $V \subseteq \mathbf{C}^n$ we denote V^Σ the variety corresponding to the shortened system.

Correspondingly the definition is applicable to any exponential system with real powers.

One can easily see that under assumption that an exponential system is associated to a pair (L, V) any shortening of the exponential system has the form $L \cap \ln V^\Sigma$ for a correspondent shortening V^Σ of V.

Definition Given G, a domain in $\mathbf{R}^k$, the exponential system associated to a pair (L, V) is said to be non-degenerate on infinity in domain $\mathbf{R}^k \times \imath G$ $(\imath^2 = 1)$ if all points of $L \cap \ln V$ lying in the domain are isolated and all shortenings $L \cap \ln V^\Sigma$ do not have points in the domain.

4

Definition An ultrafilter D on $\mathbf{N}$ is a family of subsets of $\mathbf{N}$ satisfying for any $X, Y \subseteq \mathbf{N}$ the conditions:
(i) $X, Y \in D$ implies $X \cap Y \in D$;
(ii) $X \in D$ and $X \subseteq Y$ implies $Y \in D$;
(iii) $X \notin D$ iff $\mathbf{N} \setminus X \in D$.

In a τ_1-topological space a point ξ is said to be the limit point of a sequence $\zeta = \{\zeta_i : i \in \mathbf{N}\}$ along an ultrafilter D if for any neighborhood V of ξ

$$\{i \in \mathbf{N} : \zeta_i \in V\} \in D.$$

Notation We denote the limit point $\xi = \zeta/D$.
Remark In a compact topological space ζ/D always exists.

Lemma 1.2 *For L and V as above there is a finite set $\rho(L, V)$ of algebraic families of algebraic varieties W such that for any $W \in \rho(L, V)$ there are $l = l(W) < n$ and a $\mathbf{Q}$-linear transformation of basis of $\mathbf{C}^n$ to a new basis $x_1, \ldots, x_n$, which induces the basis $y_1, \ldots, y_n$ ($y_i = \exp(x_i)$) of $\mathbf{C}^n$ such that*
(i) the projection $\mathrm{pr}_l(L)$ of L into the $(x_1, \ldots, x_l)$-subspace $S \subseteq \mathbf{C}^n$ along $(x_{l+1}, \ldots, x_n)$ is of dimension less than $\dim L$;
(ii) for any $a \in C(V)$ $W(a)$ is a subvariety of the space $\exp(S)$ and $l - \dim W(a) > n - \dim V$;
(iii) given sequences $\{a_i \in C(V) : i \in \mathbf{N}\}$, $\zeta(i) \in \exp(L) \cap V(a_i)$ and an ultrafilter D on $\mathbf{N}$, the limit point ζ/D of the sequence along D belongs to the projective closure $\bar{V}(a)$ for some a. Suppose $a \in C(V)$ (i.e. $\dim V(a) = n - k$) but $\zeta/D \notin (\mathbf{C}^*)^n$. Then there is $W \in \rho(L, V)$ such that $\mathrm{pr}_l(\zeta/D) \in W(a) \cap \exp \mathrm{pr}_l(L)$.

Proof Let $\zeta = \langle \zeta_1, \ldots, \zeta_n \rangle$ be as in (iii). Then up to numeration of variables there is an $l < n$ such that $\zeta_{l+1}/D, \ldots, \zeta_n/D$ takes values 0 or ∞ and $\zeta_1/D, \ldots, \zeta_l/D$ are in $\mathbf{C}^*$.
By transformations of the form $y_i \to y_i^{-1}$ (corresponds to the sign-changing rational transformation $x_i \to -x_i$) we may assume

$$\zeta_{l+1}/D = \ldots = \zeta_n/D = 0.$$

Notice that there are only finitely many choices of subsets of variables and sign-changing transformations for all choices of ζ.
From that point on we begin an algorithm constructing a new basis.
For any non-negative integer d such that $d > \dim V(a) + l - n$ for any distinct $i_1, \ldots, i_d \in \{1, \ldots, l\}$ (if $d = 0$ then the set is empty) the projection $\mathrm{pr}_{i_1, \ldots, i_d, l+1, \ldots, n} V$ onto the space of coordinates $y_{i_1}, \ldots, y_{i_d}, y_{l+1}, \ldots, y_n$ is of dimension less than the dimension $n - l + d$ of the ambient space. Hence there is a non-zero polynomial

$$h_{i_1, \ldots, i_d, l+1, \ldots, n}(a, y_{i_1}, \ldots, y_{i_d}, y_{l+1}, \ldots, y_n)$$

whose construction depends on $i_1, \ldots, i_d, l+1, \ldots, n$ but not on ζ, which is zero on $V(a)$. Write the polynomial in the form

$$h_{i_1,\ldots,i_d,l+1,\ldots,n} = \sum_{m \in M_{i_1,\ldots,i_d}} g_{m,i_1,\ldots,i_d,l+1,\ldots,n}(a, y_{i_1}, \ldots, y_{i_d}) \cdot \langle y_{l+1}, \ldots, y_n \rangle^m,$$

where $M_{i_1,\ldots,i_d}$ is a set of $(n-l)$-tuples $\langle m_{l+1}, \ldots, m_n \rangle$ of integers, $g_{m,i_1,\ldots,i_d,l+1,\ldots,n}$ non-zero polynomials, and $\langle y_{l+1}, \ldots, y_n \rangle^m$ denotes $y_{l+1}^{m_{l+1}} \cdot \ldots \cdot y_n^{m_n}$. Notice that $M_{i_1,\ldots,i_d}$ contains at least two distinct elements for otherwise there is a component of projective completion $\bar{V}(a)$ of $V(a)$ in the subvariety $y_i = 0$ or $y_i^{-1} = 0$ for some i of $\mathbf{P}^n(\mathbf{C})$.
The next step in the algorithm depends on the cases.
Case 1: $\langle \zeta_{l+1}, \ldots, \zeta_n \rangle^{m-m'}/D$ is either 0 or ∞ for all distinct $m, m' \in M_{i_1,\ldots,i_d}$
Case 2: $\langle \zeta_{l+1}, \ldots, \zeta_n \rangle^{m-m'}/D$ is in $\mathbf{C}^*$ for some distinct $m, m' \in M_{i_1,\ldots,i_d}$.
In the second case, assuming $m_{l+1} - m'_{l+1} \neq 0$ we change the variables $x_i(new) = x_i$ for all $i \neq l+1$ and $x_{l+1}(new) = (m_{l+1} - m'_{l+1}) \cdot x_{l+1} + \ldots + (m_n - m'_n) \cdot x_n$. Correspondingly, $y_i(new) = \exp(x_i(new))$. Then in the new variables $\zeta_1/D, \ldots, \zeta_{l+1}/D$ are in $\mathbf{C}^*$, $\zeta_{l+2}/D = \ldots = \zeta_n/D = 0$, and we come to the first step of the algorithm with $l+1$ instead of l.
In case 1 find $m_{0,i_1,\ldots,i_d,l+1,\ldots,n} \in M_{i_1,\ldots,i_d}$ which satisfies the "minimality" condition:

$$\langle \zeta_{l+1}, \ldots, \zeta_n \rangle^{m-m_{0,i_1,\ldots,i_d,l+1,\ldots,n}}/D = 0 \text{ for all } m \in M_{i_1,\ldots,i_d} \text{ distinct from } m_{0,i_1,\ldots,i_d,l+1,\ldots,n}.$$

Since

$$h_{i_1,\ldots,i_d,l+1,\ldots,n}(a, \zeta/D) = 0$$

we get necessarily

$$g_{m_{0,i_1,\ldots,i_d,l+1,\ldots,n}}(a, \zeta_{i_1}/D, \ldots, \zeta_{i_d}/D) = 0.$$

Notice that this case is not possible for $d = 0$ since the polynomial has not been identically zero.
We terminate the algorithm and define $W(a)$ to be the algebraic subvariety of $S = \mathbf{C}^l$ determined by the set of equations

$$\{ \prod_{m_{0,i_1,\ldots,i_d,l+1,\ldots,n} \in M_{i_1,\ldots,i_d}} g_{m_{0,i_1,\ldots,i_d,l+1,\ldots,n}}(a, y_{i_1}, \ldots, y_{i_d}) = 0 : i_1 < \ldots < i_d \leq l \}$$

in variables $y_1, \ldots, y_l$. Notice that the length of the algorithm is restricted by n and on every step we choose between finitely many possibilities. Thus there are only finitely many outcomes of the algorithm.

Also, $\dim W(a) < d$ since any d coordinates on $W(a)$ are algebraically dependent. Hence, by the choice of d, $\quad l - \dim W(a) > n - \dim V(a)$.
Consider now the projection $\mathrm{pr}_l(L) = \mathrm{pr}_{1,\ldots l}(L)$ of L into C^l in coordinates $x_1, \ldots, x_l$ along $x_{l+1}, \ldots, x_n$. Notice that the mapping

$$\mathrm{pr}_1 : L \to \mathrm{pr}_l(L)$$

has a non-trivial kernel. Otherwise the mapping is the linear isomorphism and hence for $l < i \le n$ there are real $s_{i,1}, \ldots, s_{i,l} \in K$ such that for any $\langle x_1, \ldots, x_n \rangle \in L$

$$x_i = s_{i,1} \cdot x_1 + \ldots + s_{i,l} \cdot x_l.$$

It follows
$$\zeta_i = \exp(s_{i,1} \cdot \ln \zeta_1) \cdot \ldots \cdot \exp(s_{i,l} \cdot \ln \zeta_l),$$

and going to limit we obtain the contradiction with the fact that, for $l < i \le n$, $j \le l$, by the construction $\zeta_i / D = 0$, while $\exp(s_{i,j} \cdot \ln \zeta_j / D)$ are non-zero.

Since the kernel is non-trivial, $\dim \operatorname{pr}_l(L) < \dim L.\square$

Remark The varieties $W(a)$ constructed in Lemma above are just shortenings of $V(a)$ in the terminology of [Kh].

Notation Denote for a subset $X \subseteq \mathbf{C}^k$

$$\operatorname{Re}(X) = \{\operatorname{Re}(x) : \ x \in X\}.$$

Notice that $\operatorname{Re}(L)$ under the assumption that $K \subseteq \mathbf{R}$ is an $\mathbf{R}$-linear subspace of $\mathbf{R}^k$.

Lemma 1.3 *There are a real number $R = R(L, V)$ and a positive integer $m = m(L, V)$ satisfying the following: Given $a \in C(V)$, there is a set $\mu(L, V, a)$ of m $\mathbf{R}$-affine hyperplanes of $\operatorname{Re}(L)$, such that for any ball $B \subseteq \operatorname{Re}(L)$ of radius R which intersect no hyperplane from $\mu(L, V, a)$ there is a point in $(\operatorname{Re}(L) + \imath B) \cap L \cap \ln V(a)$.*

Proof By [Kh, section 6] taking into account Lemma 1.1, for any closed ball $B \subseteq \operatorname{Re}(L)$ of a large enough radius the set

$$C_B(L, V) = \{a \in C(V) : L \cap \ln V(a) \text{ is non-degenerate in } \operatorname{Re}(L) + \imath B\}$$

is dense in $C(V)$ (in the real topology) and any exponential system in the family $C_B(L, V)$ has a solution in $\operatorname{Re}(L) + \imath B$.

Suppose $a \in C(L, V)$. Then $a = a_i / D$ the limit point of a sequence $\{a_i \in C_B(V) : i \in \mathbf{N}\}$ along an ultrafilter D on $\mathbf{N}$. By the above for each i one can find a solution

$$\xi(i) \in (\operatorname{Re}(L) + \imath B) \cap \ln V(a_i)$$

(notice that $\operatorname{Re}(L) + \imath B \subseteq L$). Correspondingly, $\exp(\xi(i)) = \zeta(i) \in L \cap V(a_i)$. The limit ξ / D of the sequence $\xi(i)$ is either a point in $(\operatorname{Re}(L) + \imath B) \cap \ln V(a)$ or $\zeta / D = \exp(\xi / D)$ satisfies the conditions of Lemma 1.2 (iii). Then $\operatorname{pr}_l(\zeta / D) \in \exp(\operatorname{pr}_l(L)) \cap W(a)$ for one of finitely many $W \in \rho(L, V)$. Thus

$$\operatorname{pr}_l(\xi / D) \in \operatorname{pr}_l(\operatorname{Re}(L) + \imath B) \cap \ln W(a).$$

It follows from (ii) of Lemma 1.2 that any point in the intersection is atypical. Thus by Theorem 1 $\operatorname{pr}_l(\xi / D)$ must lie in one of finetely many affine hyperplanes of $\operatorname{pr}_l(L)$

of the form $H + c_i$ for $H \in \eta(\mathrm{pr}_l(L), W)$ and $i \leq p(\mathrm{pr}_l(L), W)$. Thus B must intersect one of the hyperplanes of $\mathrm{Re}(L)$ of the form $\mathrm{Re}[\mathrm{pr}_l^{-1}(H + c_i)]$.

Take all such hyperplanes to be $\mu(L, V, a)$. □

Lemma 1.4 *Given a natural number m, there is a real constant $R^*(m)$ such that given a ball B^* in $\mathrm{Re}(L)$ of radius $R^*(m)$ and a set $\mathcal{H}$ of m affine hyperplanes in the space there is a ball $B \subseteq B^*$ of radius R which does not intersect with any hyperplane of $\mathcal{H}$.*

Proof By induction on m one easily sees that $R^*(m) = 2^m \cdot R$ fits. □

Theorem 2 *Assume* SchC+CIT. *Let $L \subseteq \mathbf{C}^n$ be an $\mathbf{R}$-linear subspace and V a family of algebraic varieties given by equations (1) such that $(L, V(a))$ is normal and free for any $a \in C(V)$. Then there is a constant $R(L, V)$ such that, given a ball B^* of radius $R(L, V)$, there is always a solution x of the exponential system with*

$$x \in (\mathrm{Re}(L) + \imath B^*).$$

Proof Take $R = R(V, L) = R^*(m(L, V))$ of Lemma 1.4. Then for any B^* of radius R there is $B \subseteq B^*$ such that B does not intersect any hyperplane from $\mu(L, V, a)$. Apply now Lemma 1.3. □

Remark The theorem can be strengthened to fulfill the requirement that, given any extra l $\mathbf{R}$-affine hyperplanes, we can choose $R(V, L, l)$ so that one can find a solution in $\mathrm{Re}(L) + \imath B^*$ outside the l hyperplanes.

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