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Contents

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Finite homogeneous geometries

by B. Zil'ber

The notion of a pregeometry (matroid) was introduced at the beginning of the 1930s to study a general notion of dependence. Recently it was found out that the combinatorics of homogeneous pregeometries is closely connected with important problems in stability theory. From the other hand the techniques and ideology of stability theory allow one to get serious results on homogeneous geometries. The aim of the present paper is to give a proof of the following:

Main Theorem. A finite homogeneous geometry of (projective) dimension not less than 7 with more than 2 points on its lines is an affine or projective geometry (possibly truncated).

Strictly speaking we present here only the draft of the proof omitting details. However we hope the draft is quite comprehensible, in fact, the details omitted could be reconstructed using the proof of the infinite version of the theorem in [Z1], [Z2] and a close work [Z3].

The methods of the proof are based on simple ideas of stability theory and develop those of [Z1]-[Z3].

A <u>pregeometry</u> is a set A together with a closure operator cl: $2^A \to 2^A$ satisfying the following conditions for any $X, Y \subseteq A$, $x, y \in A$:

(i) $X \subseteq cl(X)$;

(ii) $X \subseteq cl(Y) \Rightarrow cl(X) \subseteq cl(Y)$;

(iii) $x \in cl(X \cup \{y\}) \setminus cl(X) \Rightarrow y \in cl(X \cup \{x\})$.

If A is allowed to be infinite then usually the following condition is added:

(iv) $cl(X) = \bigcup\{cl(X') : X' \subseteq X, X' \text{ is finite}\}$.

Here we consider only finite A.

An automorphism of a pregeometry is any bijection $\alpha: A \longrightarrow A$ for which

$$cl(\alpha(X)) = \alpha(cl(X))$$

holds for any $X \subseteq A$. The group of all automorphisms fixing a set X pointwise is denoted $Aut(A/X)$ and $Aut(A/\emptyset) = Aut(A)$.

A pregeometry is said to be <u>homogeneous</u> if $x, y \in A \setminus cl(X)$ implies the existence of an $\alpha \in Aut(A/X)$ such that $\alpha(x) = y$.

A pregeometry is called a <u>geometry</u> if $cl(\emptyset) = \emptyset$ and $cl(\{x\}) = \{x\}$ for any $x \in A$.

For any pregeometry A one can construct the geometry $\hat{A}$ by putting

$$\hat{X} = \{cl(\{x\}) : x \in X \setminus cl(\emptyset)\}$$

for any $X \subseteq A$ and defining the closure on $\hat{A}$ to be as follows: $cl(\hat{X}) = cl(X)\hat{}$.

Another construction called localization gives a new pregeometry on the set A given a subset $C \subseteq A$. Define the new closure cl_C to be: $cl_C(X) = cl(X \cup C)$ for any $X \subseteq A$. The new pregeometry on A is denoted A_C. dim X denotes the cardinality of a maximal independent (in the sense of cl) subset of X, called a base of X. The cardinality does not depend on the choice of the base.

$dim_C X$ is the dimension of X in A_C.

Note that dim X - 1 is what is called the projective dimension of X.

1. <u>Sets over a pregeometry</u>

We shall call a subset $S \subseteq A^n$ <u>X-definable</u> for an $X \subseteq A$ if S is invariant under all automorphisms from $Aut(A/X)$. This definition defines also X-definable relations on S as subsets of A^{nk}.

An <u>X-definable set over A</u> is a set of the form S/E, where S is an

X-definable subset of A^n and E is an X-definable equivalence relation on S.

It is easy to see that Aut(A/X) acts on any X-definable set $U = S/E$. Any Aut(A/X)-invariant subset of U can be in a natural way presented as an X-definable set, so we call it X-definable too.

If E is trivial then S/E can be identified as S, so the X-definable subsets of A^n are in this sense X-definable sets over A.

If $u \in U$ and U is an X-definable set then denote by $O(u/X)$ the orbit of u under the action of Aut(A/X). This is an X-definable set (cf. tp(u/X) in model theory).

We shall call an X-definable set S/E ($S \subseteq A^n$) strictly coordinatizable over X if for any $\langle s_1,\ldots,s_n\rangle, \langle s'_1,\ldots,s'_n\rangle \in S$, $\langle s_1,\ldots,s_n\rangle \, E \, \langle s'_1,\ldots,s'_n\rangle$ implies $cl_X(s_1,\ldots,s_n) = cl_X(s'_1,\ldots,s'_n)$.

Throughout the paper <u>all X-definable sets are considered to be strictly coordinatizable over X</u>.

An example: The set L of all lines in a geometry A is a 0-definable set over A. More precisely $L = S/E$, where $S = \{\langle x,y\rangle \in A^2 : x \neq y\}$,

$$\langle x,y\rangle \, E \, \langle x',y'\rangle \text{ iff } cl(x,y) = cl(x',y').$$

If $U = S/E$ is an X-definable set, $u_1,\ldots,u_k \in U$, $u_i = \bar{s}_i E$, $\bar{s}_i = \langle s_{i1},\ldots,s_{in}\rangle \in S \subseteq A^n$ then we put

$$(u_1,\ldots,u_k,X) = cl(\{s_{11},\ldots,s_{1n},\ldots,s_{k1},\ldots,s_{kn}\} \cup X).$$

Note that for $a_1,\ldots,a_k \in A$

$$(a_1,\ldots,a_k) = cl(a_1,\ldots,a_k),$$

thus we can use the operator ( ) instead of cl.

For $u \in U$ we define

$$\mathrm{rank}(u/X) = \dim_X(u,X).$$

It follows from the definition that

1.1. $\mathrm{rank}(\langle u_1,u_2\rangle/X) =$

$$= \mathrm{rank}(u_1/(u_2,X)) + \mathrm{rank}(u_2/X)$$

$$= \mathrm{rank}(u_2/(u_1,X)) + \mathrm{rank}(u_1/X).$$

Define for sets

$$\mathrm{rank}(U/X) = \max\{\mathrm{rank}(u/X) : u \in U\}.$$

1.2. From the homogeneity it follows that $\mathrm{rank}(U/X) = \mathrm{rank}(U/Y)$ provided U is X-definable, $X \subseteq Y \subseteq A$, $\mathrm{rank}(U/X) = r$, $r < \dim_X A$, $r < \dim_Y A$. □

For any $Y \subseteq A$, define $U[Y] = \{u \in U : (u,X) \subseteq (Y)\}$.

1.3. Polynomial Theorem. For any X-definable strictly coordinatizable set U over A there is a unique polynomial $p_U(v)$ of one variable over the rationals such that

(i) for any closed $Y \subseteq A$, if $|Y| = n$, $Y \supseteq X$, then

$$|U[Y]| = p_U(n),$$

(ii) $\deg p_U = \mathrm{rank}(U/X)$,

(iii) if U' is an X'-definable set over A such that for some $\alpha \in \mathrm{Aut}(A)$, $X' = \alpha(X)$, $U' = \alpha(U)$, then $p_{U'} = p_U$.

A proof of the theorem is in fact given in [Z1], Theorem 2.2.

1.4. Let U be an X-definable set, $\mathrm{rank}(U/X) = r$. Define for any n a binary relation E_n on U:

$u_1 \, E_n \, u_2 \iff$ there are $y_1,\dots,y_n \in A$ independent over (u_1,u_2,X) and $\alpha \in \mathrm{Aut}(A/(y_1,\dots,y_n,X))$ such that $\alpha(u_1) = u_2$.

If $n + 2r \leqslant \mathrm{codim}\, X$, $(X) \neq \emptyset$ and planes in A are not projective, then E_n is an equivalence relation on U.

Proof. The only problem is transitivity. Let $u_1 \, E_n \, u_2$ and $u_2 \, E_n \, u_3$. By homogeneity to prove $u_1 \, E_n \, u_3$ it is sufficient to find $y_1,\dots,y_n$ independent over (u_1,u_2,X) as well as over (u_2,u_3,X) and over (u_1,u_2,X). If $y_1,\dots,y_i$ $(i < n)$ have been found already then

$$y_{i+1} \in A \setminus (u_1,u_2,y_1,\dots,y_i,X) \cup (u_2,u_3,y_1,\dots,y_i,X) \cup (u_1,u_3,y_1,\dots,y_i,X).$$

The sum of the three subspaces is less than A since the number of points on a line in $\hat{A}_{(X,y_1,\dots,y_i)}$ is greater than 3. □

1.5. Suppose $n + 2r \leqslant \mathrm{codim}\, X$, E_n is an equivalence relation on U, $n \geqslant r = \mathrm{rank}(U/X)$.

Under these conditions any class U_0 of the equivalence E_n is (Z)-definable, provided $X \subseteq Z \subseteq A$, $\dim_X Z \geqslant r$.

Proof. It suffices to find $u_0 \in U$ such that $(u_0,X) \subseteq (Z)$. Let $u_1 \in U_0$, $\mathrm{rank}(U_0/(X,u_1)) = r_0$. By 1.2 we can find $u_2 \in U$ with $\mathrm{rank}(u_2/(Z)) = r_0$ and $u_3 \in U$ with $\mathrm{rank}(u_3/(Z,u_2)) = r_0$. Since

$$\dim_{(X,u_2)}(X,u_3) = r_0 \leqslant \dim_{(X,u_2)} Z,$$

there is $\alpha \in \mathrm{Aut}(A/(X,u_2))$ such that $\alpha((u_3)) \subseteq (Z)$, U_0 is invariant under α. Put $u_0 = \alpha(u_2)$. □

Let U_0 be an X'-definable set $X \subseteq X' \subseteq A$, $\mathrm{rank}(U_0/X') = r$. U_0 is called almost X-definable if for any $Z \supseteq X$ with $\dim_X Z \geqslant r$, U_0 is (Z)-definable.

$$\mathrm{rank}(u/X) = \dim_X(u,X).$$

It follows from the definition that

1.1. $\mathrm{rank}(\langle u_1,u_2\rangle/X) =$

$= \mathrm{rank}(u_1/(u_2,X)) + \mathrm{rank}(u_2/X)$

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Define for sets

$$\mathrm{rank}(U/X) = \max\{\mathrm{rank}(u/X) : u \in U\}.$$

1.2. From the homogeneity it follows that $\mathrm{rank}(U/X) = \mathrm{rank}(U/Y)$ provided U is X-definable, $X \subseteq Y \subsetneq A$, $\mathrm{rank}(U/X) = r$, $r < \dim_X A$, $r < \dim_Y A$. □

For any $Y \subseteq A$, define $U[Y] = \{u \in U : (u,X) \subseteq \langle Y\rangle\}$.

1.3. Polynomial Theorem. For any X-definable strictly coordinatizable set U over A there is a unique polynomial $p_U(v)$ of one variable over the rationals such that

(i) for any closed $Y \subseteq A$, if $|Y| = n$, $Y \supseteq X$, then

$|U[Y]| = p_U(n)$,

(ii) $\deg p_U = \mathrm{rank}(U/X)$,

(iii) if U' is an X'-definable set over A such that for some $\alpha \in \mathrm{Aut}(A)$, $X' = \alpha(X)$, $U' = \alpha(U)$, then $p_{U'} = p_U$.

A proof of the theorem is in fact given in [Z1], Theorem 2.2.

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If $n + 2r \leqslant \operatorname{codim} X$, $(X) \neq \emptyset$ and planes in A are not projective, then E_n is an equivalence relation on U.

Proof. The only problem is transitivity. Let $u_1 \, E_n \, u_2$ and $u_2 \, E_n \, u_3$. By homogeneity to prove $u_1 \, E_n \, u_3$ it is sufficient to find $y_1,\dots,y_n$ independent over (u_1,u_2,X) as well as over (u_2,u_3,X) and over (u_1,u_2,X). If $y_1,\dots,y_i$ $(i < n)$ have been found already then

$$y_{i+1} \in A \setminus (u_1,u_2,y_1,\dots,y_i,X) \cup (u_2,u_3,y_1,\dots,y_i,X) \cup (u_1,u_3,y_1,\dots,y_i,X).$$

The sum of the three subspaces is less than A since the number of points on a line in $\hat{A}_{(X,y_1,\dots,y_i)}$ is greater than 3. □

1.5. Suppose $n + 2r \leqslant \operatorname{codim} X$, E_n is an equivalence relation on U, $n \geqslant r = \mathrm{rank}(U/X)$.

Under these conditions any class U_0 of the equivalence E_n is (Z)-definable, provided $X \subseteq Z \subseteq A$, $\dim_X Z \geqslant r$.

Proof. It suffices to find $u_0 \in U$ such that $(u_0,X) \subseteq (Z)$. Let $u_1 \in U_0$, $\mathrm{rank}(U_0/(X,u_1)) = r_0$. By 1.2 we can find $u_2 \in U$ with $\mathrm{rank}(u_2/(Z)) = r_0$ and $u_3 \in U$ with $\mathrm{rank}(u_3/(Z,u_2)) = r_0$. Since

$$\dim_{(X,u_2)}(X,u_3) = r_0 \leqslant \dim_{(X,u_2)} Z,$$

there is $\alpha \in \mathrm{Aut}(A/(X,u_2))$ such that $\alpha((u_3)) \subseteq (Z)$. U_0 is invariant under α. Put $u_0 = \alpha(u_2)$. □

Let U_0 be an X'-definable set $X \subseteq X' \subseteq A$, $\mathrm{rank}(U_0/X') = r$. U_0 is called <u>almost X-definable</u> if for any $Z \supseteq X$ with $\dim_X Z \geqslant r$, U_0 is (Z)-definable.

1.6. Under the conditions of 1.5, U_0 satisfies the following: for any Z with $\dim_X Z \leq n$ and any (Z)-definable set V,

$$\mathrm{rank}(U_0 \cap V/(Z)) < r_0 \quad \text{or} \quad \mathrm{rank}(U_0 \setminus V/(Z)) < r_0.$$

This follows from the definition of E_n. □

U_0 as in 1.6 will be called n-irreducible.

2. Parallelism

In what follows in this section A is a finite homogeneous geometry, L the set of all lines in A.

Two lines ℓ_1, ℓ_2 are called weakly parallel if $\ell_1 = \ell_2$ or $\dim(\ell_1,\ell_2) = 3$ and $(\ell_1) \cap (\ell_2) = \emptyset$. The fact is denoted $\ell_1 \mid \ell_2$.

We say three lines ℓ_1, ℓ_2, ℓ_3 satisfy the relation of triple parallelism if

$$\ell_1|\ell_3 \;\&\; \ell_2|\ell_3 \;\&\; \ell_1 \neq \ell_2 \;\&\; (\ell_3) \not\subseteq (\ell_1,\ell_2).$$

This fact is denoted $\ell_1 \uparrow \ell_2 \uparrow \ell_3$.

2.1. Suppose $\ell_1 \uparrow \ell_2 \uparrow \ell_3$ holds. Then:

(i) $\dim(\ell_1,\ell_2,\ell_3) = 4$;

(ii) $(\ell_1,\ell_2) \cap (\ell_3) = \emptyset$;

(iii) for any $a \in A \setminus (\ell_1,\ell_2)$ there is a unique $\ell \in L$ such that $a \in (\ell)$ and $\ell_1 \uparrow \ell_2 \uparrow \ell$;

(iv) $\ell_1 \mid \ell_2$;

(v) $(\ell_{i_1} \uparrow \ell_{i_2} \uparrow \ell_{i_3})$ for any permutation (i_1,i_2,i_3).

The proof is an exercise in elementary properties of homogeneous geometries.

Fix a pair of distinct points a, b $\in$ A and put

$$R_{ab} = \{\langle \ell_1, \ell_2 \rangle \in L^2 : a \in (\ell_1) \ \& \ b \in (\ell_2) \ \& \ (\exists \ell \in L)\ \ell_1 \uparrow \ell_2 \uparrow \ell\}.$$

For $\tau = \langle \ell_1, \ell_2 \rangle \in R_{ab}$ denote

$$\bar{\tau} = \{\ell \in L : \ell_1 \uparrow \ell_2 \uparrow \ell\}.$$

2.2. If $\tau_1, \tau_2 \in R_{ab}$, $\tau_1 \neq \tau_2$, then $\bar{\tau}_1 \cap \bar{\tau}_2$ contains at most one line.

Proof. Let $\tau_1 = \langle \ell_{11}, \ell_{12} \rangle$, $\tau_2 = \langle \ell_{21}, \ell_{22} \rangle$, $m_1, m_2 \in \bar{\tau}_1 \cap \bar{\tau}_2$, $m_1 \neq m_2$.

For some i, j $\in$ {1,2}, $(m_i) \not\subseteq (\ell_{1j}, \ell_{2j})$. Otherwise $(m_1, m_2) \subseteq (\ell_{11}, \ell_{21}) \cap (\ell_{12}, \ell_{22})$, this implies $(\ell_{11}, \ell_{21}) = (\ell_{12}, \ell_{22})$, since $\dim(m_1, m_2) \geqslant 3$. Moreover $(m_1, m_2) = (\ell_{11}, \ell_{12}) = (\ell_{21}, \ell_{22})$. This contradicts with $\ell_{11} \uparrow \ell_{12} \uparrow m_1$.

So, let $(m_1) \not\subseteq (\ell_{11}, \ell_{21})$. Together with $m_1 \in \bar{\tau}_1 \cap \bar{\tau}_2$ it implies $\ell_{11} \uparrow \ell_{21} \uparrow m_1$, provided $\ell_{11} \neq \ell_{21}$. By 2.1(iv) it contradicts $a \in (\ell_{11}) \cap (\ell_{21})$. Thus $\ell_{11} = \ell_{21}$. Now we have $\ell_{11} \uparrow \ell_{12} \uparrow m_1$ and $\ell_{11} \uparrow \ell_{22} \uparrow m_1$ and $b \in (\ell_{12}) \cap (\ell_{22})$. By 2.1(v) and (iii) we get $\ell_{12} = \ell_{22}$, thus $\tau_1 = \tau_2$. □

2.3. It is easy to see that R_{ab} is an (a,b)-definable set with $\mathrm{rank}(R_{ab}/(a,b)) = 1$. Let $R^1_{ab}, \ldots, R^m_{ab}$ be all the E_1-classes. R^i_{ab} are almost (a,b)-definable and 1-irreducible by 1.6, provided $\dim A \geqslant 6$, $R_{ab} \neq \emptyset$.

If $\tau_1 \in R^i_{ab}$, $\tau_2 \in R^j_{ab}$, $i,j \in \{1,\ldots,m\}$, $\tau_1 \neq \tau_2$, $\bar{\tau}_1 \cap \bar{\tau}_2 \neq \emptyset$ then for any distinct $\tau'_1 \in R^i_{ab}$, $\tau'_2 \in R^j_a$ it holds that $\bar{\tau}'_1 \cap \bar{\tau}'_2 \neq \emptyset$ and $(\tau'_1) \neq (\tau'_2)$.

Proof. One can assume $\tau_1 = \tau'_1$. Note that $(\tau_1) \neq (\tau_2)$, since there is $\ell \in \bar{\tau}_1 \cap \bar{\tau}_2$ and by 2.2, $(\ell) \subseteq (\tau_1, \tau_2)$, but by the definition of $\bar{\tau}_1$ $(\ell) \not\subseteq (\tau_1)$. We show that we may assume $(\tau_1) \not\subseteq (\tau_1, \tau'_2)$ and this will finish the proof by using the definition of E_1.

So, suppose $(\tau_1) \subseteq (\tau_2, \tau'_2)$, then either $(\tau_2) = (\tau'_2) = (\tau_1)$ or $\dim_{(\tau_1)}(\tau_2, \tau'_2) = 1$. The first one is impossible. If the second holds there is $\alpha \in Aut(A/(\tau_1))$ such that $(\alpha(\tau'_2)) \nsubseteq (\tau_2, \tau'_2)$. Denote $\alpha(\tau'_2) = \tau''_2$, then $(\tau_1) \nsubseteq (\tau_2, \tau''_2)$. Take τ''_2 instead of τ'_2. □

2.4. Denote

$$S^{ij} = \{\langle \tau_1, \tau_2, \ell \rangle : \tau_1 \in R^i,\ \tau_2 \in R^j,\ \ell \in \bar{\tau}_1 \cap \bar{\tau}_2,\ \tau_1 \neq \tau_2\},$$

fix $\ell_0 \in L$, such that $(\ell_0) \cap (a,b) = \emptyset$, and a plane of the form (a,b,c), $c \in A \setminus (a,b)$. Denote

$$\lambda = |\ell_0|, \quad \rho^i = |\{\tau \in R^i_{ab} : \ell_0 \in \bar{\tau}\}|, \quad \pi = |(a,b,c)|,$$
$$\mu^i = |\{\tau \in R^i : (\tau) = (a,b,c)\}|,$$

$$\delta^{ij} = \begin{cases} 0 & \text{if } i \neq j, \\ 1 & \text{if } i = j. \end{cases}$$

If we put $z = |Z|$ for any closed set $Z \subseteq A$ containing c, a, b, then the following hold:

(i) $|L[Z]| = \dfrac{z(z-1)}{\lambda(\lambda-1)}$;

(ii) $|R^i[Z]| = \dfrac{z-\pi}{\lambda-1} \cdot \rho^i + \mu^i$;

if $S^{ij} \neq \emptyset$

(iii) $|S^{ij}[Z]| = |R^i[Z]| \cdot (|R^j[Z]| - \mu^j)$;

and also

(iv) $|S^{ij}[Z]| = \dfrac{(z-\lambda)(z-\pi)}{\lambda(\lambda-1)} \cdot \rho^i \cdot (\rho^j - \delta^{ij})$.

Proof. (i) is well-known and easy. (ii) follows from computations of the number of elements in

$$T^i[Z] = \{\langle\tau,\ell\rangle : c \in (\ell), \ell \in L[Z], \ell \in \bar{\tau}, \bar{\tau} \in R^i[Z]\}.$$

For $(\tau) = (a,b,c)$ there is no $\ell \in \bar{\tau}$ with $c \in \ell$ by 2.1(ii). If $(\tau) \neq (a,b,c)$ then there is a unique ℓ such that $\langle z,\ell\rangle \in T^i$. It follows that

$$|T^i[Z]| = |R^i[Z]| - \mu^i.$$

From the other hand for any $\ell \in L$, provided $c \in (\ell)$ and $(\ell) \not\subseteq (a,b,c)$ there are exactly ρ^i elements $\tau \in R^i{}_{ab}$ such that $\langle\tau,\ell\rangle \in T^i$. Using 2.1(iii) one gets

$$|T^i[Z]| = \frac{z - \pi}{\lambda - 1}\cdot\rho^i$$

where $(z - \pi)/(\lambda - 1)$ is counted as the number of $\ell \in L[Z]$ such that $c \in (\ell) \not\subseteq (a,b,c)$.

(iii) follows from 2.3 and 2.2 if one counts $|S^{ij}[Z]|$ as the number of $\langle\tau_1,\tau_2\rangle \in R^i[Z] \times R^j[Z]$ such that $\bar{\tau}_1 \cap \bar{\tau}_2 \neq \emptyset$.

(iv) is the result of counting first the number of the lines in

$$\{\ell \in L[Z] : (\exists\tau_1 \in R^i{}_{ab})(\exists\tau_2 \in R^j{}_{ab})\langle\tau_1,\tau_2,\ell\rangle \in S^{ij}\}$$
$$= \{\ell \in L[Z] : \dim(a,b,\ell) = 4\}.$$

This number is equal to $(z - \lambda)(z - \pi)/\lambda(\lambda - 1)$. Now for each ℓ from the set there are exactly $\rho^i.(\rho^j - \delta^{ij})$ pairs of different τ_1, τ_2 such that $\langle\tau_1,\tau_2,\ell\rangle \in S^{ij}[Z]$. □

2.5. If $\dim A \geq 6$, then for any $\tau_1, \tau_2 \in R_{ab}$

$$\bar{\tau}_1 \cap \bar{\tau}_2 \neq \emptyset \quad \text{iff} \quad \tau_1 = \tau_2.$$

Proof. It suffices to show that $S^{ij} = \emptyset$ for all $i, j \in \{1,\dots,m\}$. For this use 2.4 and compare the leading coefficients of the polynomials given by (iii) and (iv) if $S^{ij} \neq \emptyset$. The coefficients are distinct though the polynomials must coincide by 1.3. □

2.6. If dim $A \geqslant 6$ then one of the following hold:

(i) every plane in A is projective;

(ii) every plane in A is affine;

(iii) there are two distinct lines ℓ_1, ℓ_2 such that $\ell_1 | \ell_2$ & $\neg(\exists \ell) \ell_1 \uparrow \ell_2 \uparrow \ell$.

Proof. Suppose (i) and (iii) do not hold. Then there are $\ell_1, \ell_2 \in L$, $\ell_1 \neq \ell_2$, and there is $\ell \in L$ such that $\ell_1 \uparrow \ell_2 \uparrow \ell$. Let $a \in (\ell)$, $a \notin (\ell_1, \ell_2)$, $\ell' \in L$, $a \in (\ell')$ and $\ell_1 | \ell'$. Then $\ell' \uparrow \ell_2 \uparrow \ell_1$ and by 2.1, $\ell' = \ell$. Thus we have proved that through any $a \notin (\ell_1, \ell_2)$ there is a unique ℓ such that $\ell | \ell_1$. By homogeneity we get the same for any ℓ_1 and any $a \notin (\ell_1)$. This is exactly (ii). □

A geometry (A,cl) is called <u>truncated projective (affine)</u> if one can define a new closure cl^* on A such that (A, cl^*) is isomorphic to a projective (affine) geometry over a field and there is $d \leqslant \dim^* A$ (dimension of A with respect to cl^*) such that $cl(X) = cl^*(X)$ if $\dim^* X \leqslant d$ and $cl(X) = A$ if $\dim^* X > d$.

2.7. If all planes in A are projective (affine), then A is a truncated projective (affine) geometry.

This is a consequence of the transitivity of Aut(A) on the set of all non-collinear triples of points from A and Theorem 1 of [CK]. □

2.8. If dim $A \geqslant 6$ then one of the following hold:

(i) A is a truncated projective geometry;

(ii) A is a truncated affine geometry;

(iii) the binary relation I on the set of lines is not empty:

$$\ell_1 I \ell_2 \iff \ell_1 | \ell_2 \ \& \ \neg(\exists \ell)\ \ell_1 \uparrow \ell_2 \uparrow \ell.$$

This is a reformulation of 2.6 taking into account 2.7. □

3. Quasi-design over A.

In this section we suppose dim A is finite, homogeneous and the relation I defined in 2.8 is not empty. We denote for $\ell \in L$

$$I\ell = \{\ell' \in L : \ell I \ell'\}.$$

The results of the section and their proofs are completely analogous to those of [Z1, section 3]. We only improved the proofs and modified them to the finite-dimensional case.

3.1. (i) $\text{rank}(I\ell/(\ell)) = 1$ for all $\ell \in L$;

(ii) if $\ell_1 \neq \ell_2$ for $\ell_1, \ell_2 \in L$, then $\text{rank}(I\ell_1 \cap I\ell_2/(\ell_1,\ell_2)) = 0$ or $I\ell_1 \cap I\ell_2 = \emptyset$.

The proof is immediate from the definitions. □

Studying L with respect to I it is convenient to treat elements of L as points and subsets of the form $I\ell$ as blocks. As in [Z1] we will call this incidence system a quasi-design.

To the end of the section we fix $X \subseteq A$ such that codim $X \geq 3$ and the partition of L

$$L = L_1 \cup \ldots \cup L_n$$

where L_i are orbits with respect to Aut(A/X). By homogeneity among $L_1, \ldots, L_n$ there is exactly one set of rank 2. Let

3.2. $\text{rank}(L_1/X) = 2$; $\text{rank}(L_i/X) \leq 1$ for $i > 1$.

3.3. If $\text{rank}(L_i/X) = 1$, $\ell \in L$, $\text{rank}(L_i \cap I\ell/(X,\ell)) = 1$ then $\ell \in L[X]$.

Proof. Under the hypotheses there is $\ell' \in L_i \cap I\ell$ such that $(\ell') \nsubseteq (\ell,X)$. Since $\text{rank}(\ell'/X) = 1$ and $\text{rank}(\ell/(\ell')) = 1$, one has

$$2 \geq \dim_X(\ell,\ell') > \dim_X(\ell).$$

Since $\mathrm{codim}(\ell,\ell',X) \geq 1$, hence supposing $(\ell) \not\subseteq (X)$ we can find $\ell'' \in L$ such that $(\ell'') \not\subseteq (\ell,\ell',X)$ and there is $\alpha \in \mathrm{Aut}(A/(\ell',X))$ such that $\alpha(\ell) = \ell''$. Then $\ell''I\ell'$, $(\ell') \not\subseteq (\ell,\ell'')$, thus it holds that $\ell''I\ell I\ell'$, which contradicts $\ell I\ell'$. □

3.4. If $\mathrm{rank}(L_i/X) = 1$ and $\mathrm{rank}(L_i \cap I\ell/(X,\ell)) < 1$ for all $\ell \in L$ then for any $q \in L_i \setminus L_i[X]$ there is $\ell_1 \in L_i$ such that $\mathrm{rank}(\ell_1/(X,q)) = 1 = \mathrm{rank}(\ell_1/(q))$.

Proof. Fix L_i. Denote for an $\ell \in L$

$$S_\ell = \{\langle \ell_1,\ell_2 \rangle \in I : \ell_2 I\ell \ \&\ \ell_1 \in L_i \ \&\ \ell_1 \neq \ell\}.$$

It is easy to compute $\mathrm{rank}(S_\ell/(X,\ell)) = 1$.

Take an arbitrary closed $Y \subseteq A$ such that $(\ell,X) \subseteq Y$. By 1.3, $|S_\ell[Y]|$ is the value of a polynomial of degree 1 depending on $|Y|$. Denote $O^\ell{}_j$, $j = 1,\dots,m$, all orbits on L under $\mathrm{Aut}(A/(\ell))$, except $\{\ell\}$. Denote

$$L^\ell{}_{ij} = L_i \cap O^\ell{}_j$$

and let $L_i \setminus \{\ell\} = L^\ell{}_{i1} \cup \dots \cup L^\ell{}_{im}$. Then

(1) $$|S_\ell[Y]| = \Sigma_{1\leq j\leq m} |L^\ell{}_{ij}[Y]| \cdot \nu^\ell{}_j$$

where $\nu^\ell{}_j$ is $|I\ell \cap I\ell_1|$ when $\ell_1 \in L^\ell{}_{ij}$.

From the other hand

(2) $$|S_\ell[Y]| = \Sigma_{1\leq k\leq n} |I\ell \cap L_k[Y]| \cdot \lambda^\ell{}_k,$$

where $\lambda^\ell{}_k = |I\ell_2 \cap L_i \setminus \{\ell\}|$ when $\ell_2 \in L_k$.

Count now the ranks of all the subsets involved and the degrees of all the polynomials and consider the leading coefficients of the polynomials (lcp).

Then from (2) we have

$$(3) \qquad \text{lcp } |S_\ell[Y]| = \text{lcp } |I\ell \cap L_1[Y]| \cdot \lambda^\ell{}_1.$$

Now we assume $(\ell) \not\subseteq (X)$. Then by 3.2 and 3.3

$$\text{lcp } |I\ell \cap L_1[Y]| = \text{lcp } |I\ell[Y]|$$

and thus

$$(4) \qquad \text{lcp } |S_\ell[Y]| = \text{lcp } |I\ell[Y]| \cdot \lambda^\ell{}_1.$$

Now we consider two possibilities for ℓ: $\ell = q \in L_i[Y]$ and $\ell = p \in L_1[Y]$. It is easy to see that $\lambda^q{}_1 = \lambda^p{}_1 - 1$, therefore

$$(5) \qquad \text{lcp } |S_q[Y]| < \text{lcp } |S_p[Y]|.$$

Looking to (1) we get

$$(6) \qquad \text{lcp } |S_p[Y]| = \text{lcp } |L_{i1}[Y]| \cdot \nu^p{}_1$$

since any two $\ell_1, \ell'_1 \in L_i \setminus L_i[p]$ are conjugated by Aut(A/(p)). And also $\nu^p{}_1 = |I\ell_1 \cap Ip|$ when $\text{rank}(\ell_1/(p)) = 2$. (5), (6) and (1) imply that $\nu^p{}_1 > \nu^q{}_j$ for any j such that $\text{rank}(L^q{}_{ij}/(X,q)) = 1$. It implies that $\langle \ell_1, p\rangle$ and $\langle \ell_1, q\rangle$ are not conjugated when $\ell_1 \in L^q{}_{ij}$, i.e. $\text{rank}(\ell_1/q) \neq 2$. □

3.5. If $z \in (y,X)$ and codim $X \geqslant 3$, then there are $x_1, x_2 \in (X)$ such that $z \in (y,x_1,x_2)$.

Proof. If (i) or (ii) of 2.8 holds then it is evident. Otherwise we use 3.3 and 3.4.

Let $y \neq z$, $z \notin (X)$, let q be the line through y and z, L_i the orbit of q under Aut(A/X). Then $\text{rank}(L_i/X) = 1$.

If there is $\ell \in L$ such that $\mathrm{rank}(L_i \cap I\ell) = 1$ then $(\ell) \subseteq (X)$ by 3.3 and let $(x_1, x_2) = (\ell)$.

If not then use 3.4. There are two possibilities for ℓ_1 from 3.4: $(\ell_1) \cap (q) \neq \emptyset$ or there is $\ell \in L$ such that $q \uparrow \ell_1 \uparrow \ell$. In the first case $(x_1) = (x_2) = (\ell_1) \cap (q) \subseteq (X)$. In the second case $(\ell_1) \subseteq (X)$ or it is possible to find ℓ such that $(\ell) \subseteq (X)$ and $q \uparrow \ell_1 \uparrow \ell$. Then $\mathrm{rank}(q/(x_1, x_2)) = 1$ for $(x_1, x_2) = (\ell_1)$ or $(x_1, x_2) = \ell$ respectively. □

4. <u>Definable transformations</u>.

Under the assumption $\dim A \geq 7$ and A is neither a projective nor an affine geometry, we construct here a definable set V over A so that there are "sufficiently many" definable transformations on V.

We begin with a broader notion. An <u>almost X-definable semitransformation</u> on A is an almost X-definable set $f \subseteq A \times A$ of rank 1 which is 2-irreducible and does not contain $\langle v, u \rangle$ with $v \in (X)$ or $u \in (X)$.

4.1. If $\mathrm{codim}\ X \geq 5$, $\langle u, v \rangle \in A^2$, $\mathrm{rank}(\langle u, v \rangle / X) = 1$, $v, u \notin (X)$, then there is an almost X-definable semitransformation f on A with $\langle u, v \rangle \in f$.

This follows from 1.6.

4.2. If f_i is an almost X_i-definable semitransformation on A for $i = 1,2$ and $\dim X_1 \cup X_2 \leq \dim X_1 + 2$, $\mathrm{rank}(f_1 \cap f_2 \ / \ X_1 \cup X_2) > 0$ then $\mathrm{rank}(f_1 \setminus f_2 \ / \ X_1 \cup X_2) = 0$.

This is a consequence of 2-irreducibility. □

4.3. If $\dim A \geq 7$, $\mathrm{codim}\ X \geq 3$, $\langle u, v \rangle$ as in 4.1, then there are $x_1, x_2 \in (X)$ and an almost (x_1, x_2)-definable semintransformation f on A with $\langle u, v \rangle \in f$.

This follows from 3.5 and 4.1. □

Denote F the set of all almost (x_1,x_2)-definable semitransformations on A for all $x_1, x_2 \in A$. It is easy to see that if $f \in F$, then $f^{-1} \in F$, where $f^{-1} = \{\langle v,u\rangle : \langle u,v\rangle \in f\}$.

For almost X-definable sets g_1, g_2 of rank 1 we denote by $g_1 \sqsubset g_2$ the fact that $\mathrm{rank}(g_2 \backslash g_1 / X) = 0$, and $g_1 \sqsubset\!\!\sqsupset g_2$ denotes $g_1 \sqsubset g_2$ & $g_2 \sqsubset g_1$.

4.4. It follows from 4.2 that $\sqsubset$ coincides with $\sqsubset\!\!\sqsupset$ on F and $\sqsubset\!\!\sqsupset$ is an equivalence relation on F. It follows from 4.3 that for any $f_1, f_2 \in F$ there are $g_1,\ldots,g_k \in F$ such that $g_1 \cup \ldots \cup g_k \sqsubset\!\!\sqsupset f_1 \circ f_2$, where

$$f_1 \circ f_2 = \{\langle u,w\rangle : (\exists v)\ \langle u,v\rangle \in f_1 \ \&\ \langle v,w\rangle \in f_2\}.$$

If f_i is almost (x_{i1},x_{i2})-definable for $i = 1,2$ then g_j are almost (y_{j1},y_{j2})-definable for some $y_{j1}, y_{j2} \in (x_{11}, x_{12}, x_{21}, x_{22})$. The set $\{g_1,\ldots,g_k\}$ is uniquely determined up to $\sqsubset\!\!\sqsupset$.

Define F_I to be the subset of F containing all almost (x_1,x_2)-definable semitransformations f such that: if $\langle u,v\rangle \in f$, $(u,v) = (q)$, $q \in L$, $u \notin (x_1,x_2)$, $(x_1,x_2) = (\ell)$, $\ell \in L$, then $qI\ell$.

4.5. $F_I \neq \emptyset$ iff A is neither a projective nor an affine truncated geometry.

This is in fact 2.8. □

4.6. If $f_i \in F_I$, $i = 1,2$, $f_1 \sqsubset\!\!\sqsupset f_2$ and f_i are almost (x_{i1},x_{i2})-definable, then $(x_{11},x_{12}) = (x_{21},x_{22})$.

This follows from 3.1(ii). □

The observation above makes it possible to treat the quotient-set $F_I/\sqsubset\!\!\sqsupset = \hat{F}_I$ as $\emptyset$-definable. An element of $\hat{F}_I$ corresponding to $f \in F_I$ will be denoted $\hat{f}$.

$(\hat{f}) = (x_1,x_2)$, $\mathrm{rank}(\hat{f}/X) = \dim_X(x_1,x_2)$ if f is almost (x_1,x_2)-definable.

4.7. (i) If $\hat{f} \in \hat{F}_I$, then $\hat{f}^{-1} \in \hat{F}_I$;

(ii) if $f_1 \in F$, $f_2 \in F_I$, $\mathrm{rank}(\hat{f}_2/(\hat{f}_1)) = 2$, $f \sqsubset f_1 o f_2$, then $f \in F_I$.

(i) is evident. (ii) is again a consequence of 2.1 and elementary geometric considerations. □

It is natural to use the following notation for $v \in A$, $f \in F$:

$$f(v) = \{u : \langle v,u \rangle \in f\}.$$

4.8. If $g, f \in F_I$, $\mathrm{rank}(\hat{g}/(\hat{f})) = 2$, $v \in A \setminus (\hat{f},\hat{g})$, $u_1,u_2 \in f(v)$ and $u_1 \neq u_2$, then $g(u_1) \cap g(u_2) = \emptyset$.

Proof. Assume the contrary, $w \in g(u_1) \cap g(u_2)$. Then $u_1, u_2 \in (\hat{f},v) \cap (\hat{g},w)$, hence $u_2 \in (\hat{f},u_1) \cap (\hat{g},u_1)$. It follows that $\mathrm{rank}(\hat{g}/(u_{11},u_2)) \leq 1$, which contradicts

$$\mathrm{rank}(\hat{g}/(u_1,u_2)) \geq \mathrm{rank}(\hat{g}/(\hat{f},u_1,u_2)) = \mathrm{rank}(\hat{g}/(\hat{f},v)) = 2. \ \square$$

4.9. Let $f, h \in F_I$, $\mathrm{rank}(\hat{f}/(\hat{h})) = 2$ and $k = |f(v)| = \max\{|s(v)| : s \in F_I, v \in A \setminus (\hat{s})\}$. Taking $g \sqsubset f^{-1} o h$ we get $h \sqsubset f o g$, $g \in F_I$ (by 4.7) and $\mathrm{rank}(\hat{g}/(\hat{f})) = 2$. Under these assumptions for any $v \in A \setminus (\hat{f},\hat{g})$, $u \in f(v)$ there exists a unique $w \in g(u) \cap h(v)$.

Proof. Let $f(v) = \{u_1,\ldots,u_k\}$, $u_i \neq u_j$ if $i \neq j$. Denote $m_i = |g(u_i) \cap f(v)|$, let $\langle v,u \rangle \in f$, $\langle u,w \rangle \in g$, $\langle v,w \rangle \in h$,

$$f' = \{\langle v',u' \rangle : (\exists w')(w' \in g(u') \cap h(v')\}.$$

Since $f' \subseteq f$ and $\langle v,u \rangle \in f'$, $\mathrm{rank}(f'/(\hat{f},\hat{h})) = 1$, hence $f' \,\square\, f$. It follows that $\langle v,u_i \rangle \in f$ iff $\langle v,u_i \rangle \in f'$, therefore $g(u_i) \cap h(v) \neq \emptyset$ and $m_i > 0$ for $i = 1,\ldots,k$.

From the other hand $\Sigma_{i \leq k} m_i \leq k$, since $\bigcup_{i \leq k} g(u_i) \cap f(v) = h(v)$. Thus

$m_i = 1$ for all $i = 1,\dots,k$. □

4.10. Fix $f \in F_I$ as a set. For any $\langle v,u\rangle \in f$, $\langle t,w\rangle \in f$ such that $\mathrm{rank}(\langle v,w\rangle/\hat{f}) = 2$ there exist $x_1, x_2 \in A$ and an $(\hat{f},x_1,x_2)$-definable mapping $\gamma: f \to f$ such that $\mathrm{rank}(\langle v,u\rangle/(\hat{f},x_1,x_2)) = 1$ and $\gamma(\langle v,u\rangle) = \langle t,w\rangle$.

Proof. For given $\langle v,u\rangle \in f$ take $g, h \in F_I$ as in 4.9 so that $w \in g(u) \cap h(v)$. Such a choice is possible by homogeneity. Note that 4.9 gives an $(\hat{f},\hat{h})$-definable mapping $\alpha: f \to h$ by the law $\alpha: \langle v,u\rangle \to \langle v,w\rangle$. Let i be the inversion i: $\langle v,w\rangle \to \langle v,w\rangle$. Let β be again an $(\hat{f},\hat{h})$-definable mapping $h^{-1} \to f^{-1}$ such that $\langle w,v\rangle \to \langle w,t\rangle$. Then $\gamma = \alpha o i o \beta o i$ is the required mapping, $(x_1,x_2) = (\hat{h})$. □

4.11. γ in 4.10 is a bijection of $f \setminus (\hat{f},x_1,x_2)^2$ onto itself.

This is easily seen from the construction. □

An $(\hat{f},x_1,x_2)$-definable bijection of $f \setminus (\hat{f},x_1,x_2)^2$ onto itself will be called a transformation of f. One constructed as in 4.10 will be called generic.

4.12. For any $v_1, v_2, t_1, t_2 \in A$ such that $\mathrm{rank}(\langle v_1,v_2,t_1,t_2\rangle/\hat{f}) = 4$, any $u_1, u_2 \in A$ such that $\langle v_1,u_1\rangle \in f$, $\langle v_2,u_2\rangle \in f$, there exists a transformation γ' and $w_1, w_2 \in A$ with $\langle t_1,w_1\rangle \in f$, $\langle t_2,w_2\rangle \in f$, $\gamma'(\langle v_1,u_1\rangle) = \langle t_1,w_1\rangle$, $\gamma'(\langle v_2,u_2\rangle) = \langle t_2,w_2\rangle$.

Proof. Let γ, h be as in the proof of 4.10, $\mathrm{rank}(\langle v_1,v_2\rangle/(\hat{f},\hat{h})) = 2$. Let $\gamma(\langle v_1,u_1\rangle) = \langle t'_1,w'_1\rangle$, $\gamma(\langle v_2,u_2\rangle) = \langle t'_2,w'_2\rangle$. It is easy to see that $(v_1,v_2,t'_1,t'_2,\hat{f}) = (v_1,v_2,w'_1,w'_2,\hat{f}) = (v_1,v_2,\hat{h},\hat{f})$ and therefore v_1, v_2, t'_1, t'_2 are independent over $(\hat{f})$. Take $\alpha \in \mathrm{Aut}(A/(\hat{f},v_1,v_2))$ such that $\alpha(t'_1) = t_1$, $\alpha(t'_2) = t_2$, and put $w_1 = \alpha(w'_1)$, $w_2 = \alpha(w'_2)$, $\gamma' = \alpha(\gamma)$. □

4.13. Let γ_1 be a $(\hat{f},x_1,x_2)$-definable transformation, γ_2 a generic $(\hat{f},\hat{h}_2)$-definable transformation and $\mathrm{rank}(\hat{h}_2/(\hat{f},x_1,x_2)) = 2$. Then there is a unique generic γ_3 which is $(\hat{f},\hat{h})$-definable for $(\hat{h}) \subseteq (\hat{f},\hat{h}_2,x_1,x_2)$ such that for any $\langle v,u\rangle \in f \setminus (\hat{f},\hat{h}_2,x_1,x_2)^2$,

$$\gamma_3(\langle v,u\rangle) = \gamma_2(\gamma_1(\langle v,u\rangle)).$$

Proof. Let $\langle v,u\rangle \in f \setminus (\hat{f},\hat{h}_2,x_1,x_2)^2$, $\gamma_1(\langle v,u\rangle) = \langle s,r\rangle$, $\gamma_2(\langle s,r\rangle) = \langle t,w\rangle$. Then by 3.5, $r \in h_1(v)$ for $h_1 \in F$, h_1 almost $(\hat{f},x_1,x_2)$-definable, $s \in f^{-1}(r)$ and $w \in h_2(s)$, i.e. $w \in h_1 o f^{-1} o h_2(v)$. By 4.7 there is $h \in F_I$ such that $(\hat{h}) \subseteq (\hat{h}_1,\hat{f}_1,\hat{h}_2)$ and $w \in h(v)$, $\mathrm{rank}(\hat{h}/(\hat{f})) = 2$. This is sufficient to construct γ_3 as in 4.10 with $\gamma_3(\langle v,u\rangle) = \langle t,w\rangle$. By 4.3, γ_3 is unique. □

4.14. If β_i is a $(\hat{f},x_{i1},x_{i2})$-definable transformation, $i = 1,2$, and $\dim(\hat{f},x_{11},x_{12},x_{21},x_{22}) \leq 5$ then there is a unique $(\hat{f},y_1,y_2)$-definable transformation with y_1, $y_2 \in (\hat{f},x_{11},x_{12},x_{21},x_{22})$ such that $\beta_3(\langle v,u\rangle) = \beta_2(\beta_1(\langle v,u\rangle))$ for any $\langle v,u\rangle \in f \setminus (\hat{f},x_{11},x_{12},x_{21},x_{22})^2$.

Proof. As in the proof of 4.13 there are h_1, $h_2 \in F$, such that $r \in h_1(v)$, $w \in h_2(s)$, $w \in h_1 o f^{-1} o h_2(v)$ and h_1, h_2 are almost $(\hat{f},x_{11},x_{12},x_{21},x_{22})$-definable. Hence $w \in h(v)$, $h \in F$, h is almost (y_1,y_2)-definable, y_1, $y_2 \in (\hat{f},x_{11},x_{12},x_{21},x_{22})$.

There are three possibilities for h:

1. $h \in F_I$, $\mathrm{rank}(\hat{h}/(\hat{f})) = 2$. In this case β_3 can be constructed as in 4.10.

2. $h \in F_I$, $\mathrm{rank}(\hat{h}/(\hat{f})) \leq 1$. Then $\dim(\hat{f},\hat{h}) = k \leq 3$ and let β_3 be an almost $(\hat{f},\hat{h})$-definable $(5-k)$-irreducible subset of

$$\{\langle v',u',t',w'\rangle : w' \in h(v') \;\&\; u' \in f(v') \;\&\; w' \in f(t')\}$$

by 1.4. Then $\beta_3 \sqsubset \beta_1 o \beta_2$.

3. $h \notin F_I$. Then $w \in (v,\hat{f},y)$ for some $y \in (\hat{f},y_1,y_2)$ and we get β_3

repeating the previous point. □

4.15. Any $(\hat{f},x_1,x_2)$-definable transformation β satisfies one of the following:

(i) β is generic;

(ii) there is $y \in (\hat{f},x_1,x_2)$ and β' such that β' is almost $(\hat{f},y)$-definable and $\beta' \sqsubset \beta$; if $\beta'' \sqsubset \beta$ and β'' is almost $(\hat{f},y')$-definable then $(\hat{f},y) = (\hat{f},y')$;

(iii) there is β' which is almost $(\hat{f})$-definable and $\beta' \sqsubset \beta$.

Proof. Let $\langle v,u\rangle \in f \setminus (\hat{f},x_1,x_2)^2$, $\beta(\langle v,u\rangle) = \langle t,w\rangle$. Since $w \in (v,\hat{f},x_1,x_2)$, there is $h \in F$ which is (y_1,y_2)-definable, $y_1, y_2 \in (\hat{f},x_1,x_2)$. There are three possibilities:

(1) $h \in F_I$, $\mathrm{rank}(\hat{h}/(\hat{f})) = 2$. This case like case 1 of 4.14 gives (i).

(2) $h \in F_I$, $\mathrm{rank}(\hat{h}/(\hat{f})) \leq 1$. Again act like in 4.14 and get $\beta' \sqsubset \beta$ which is almost $(\hat{f},\hat{h})$-definable, $(\hat{f},\hat{h}) = (\hat{f},y)$ and we get (ii) if $y \notin (\hat{f})$ or (iii) if $y \in (\hat{f})$.

(3) $h \notin F_I$. The same as (2). □

For any transformation β of 4.15, $(\hat{f},\hat{\beta})$ is defined as $(\hat{f},x_1,x_2)$ in the case (i), $(\hat{f},y)$ in (ii) and $(\hat{f})$ in (iii).

4.16. The set of all transformations forms a group Γ. The set Γ and multiplication in Γ are $(\hat{f})$-definable, as well as the partial action of Γ on f: if $\gamma \in \Gamma$, $\bar{v} \in f \setminus (\hat{f},\hat{\gamma})$ then $\gamma(\bar{v})$ is defined.

In general Γ is not strongly coordinatizable over $(\hat{f})$ but:

(i) the subset $\Gamma_0 = \{\gamma \in \Gamma : \gamma \text{ is generic}\}$ is strongly coordinatizable over $(\hat{f})$;

(ii) Γ is strongly coordinatizable over any $a_1, a_2 \in A$ which are independent over $(\hat{f})$;

(iii) $\mathrm{rank}(\Gamma/(a_1,a_2,\hat{f})) = 2$, $\mathrm{rank}(\Gamma_0/(\hat{f})) = 2$, $\mathrm{rank}(\Gamma\setminus\Gamma_0/(a_1,a_2,\hat{f})) < 2$.

5. The structure of Γ.

If Γ has a proper $(\mathbf{f})$-definable subgroup of rank 2, take a minimal such one instead of Γ. So we may assume Γ has no proper $(\mathbf{f})$-definable subgroup of rank 2.

5.1. The center C of Γ is an $(\mathbf{f})$-definable subgroup of rank 0.

Proof. For $\overline{v} \in f \setminus (\mathbf{f})^2$ there is $\overline{w} \in f \setminus (\mathbf{f},\overline{v})^2$ and a subset

$$\Gamma_{\overline{v}\overline{w}} = \{\gamma \in \Gamma_0 : \gamma(\overline{v}) = \overline{w}\}$$

with $\mathrm{rank}(\Gamma_{\overline{v}\overline{w}}/(\mathbf{f},\overline{w},\overline{v})) = 1$. Suppose $\mathrm{rank}(C/(\mathbf{f},\overline{v},\overline{w})) > 0$. Then for any $\gamma_1, \gamma_2 \in \Gamma_{\overline{v}\overline{w}}$, $\overline{u} \in f \setminus (\mathbf{f},\overline{v},\overline{w},\gamma_1,\gamma_2)^2$ one can find $\alpha \in C$ such that

$$\alpha(\overline{v}) = \overline{u}, \quad \overline{v} \notin (\mathbf{f},\gamma_1,\alpha) \cup (\mathbf{f},\gamma_2,\alpha).$$

Then $\gamma_1(\overline{u}) = \gamma_1(\alpha(\overline{v})) = \alpha(\gamma_1(\overline{v})) = \alpha(\gamma_2(\overline{v})) = \gamma_2(\alpha(\overline{v})) = \gamma_2(\overline{u})$. It follows that $\gamma_1 = \gamma_2$, contradiction. □

5.2. Γ is 2-irreducible, provided $\dim A \geqslant 8$.

Proof. Let E_2 be the equivalence relation on Γ_0 defined in 1.4, U_0 an E_2-class of rank 2. It is easy to see that if $\gamma \in \Gamma$ then $\gamma.U_0 \sqsubset U_1$ for some E_2-class U_1. It follows that the $(\mathbf{f})$-definable group

$$\{\gamma \in \Gamma : \gamma U_i \sqsubset U_i \text{ for any } E_2\text{-class } U_i \text{ of rank } 2\}$$

is of rank 2, thus it coincides with Γ. Moreover, if $\gamma \in U_0$ then $U_0.\gamma^{-1} \sqsubset \Gamma$, thus Γ is 2-irreducible. □

5.3. $\overline{\Gamma} = \Gamma/C$ is a centerless $(\mathbf{f})$-definable group.

Proof. If $\overline{\gamma}$ is a central element of $\overline{\Gamma}$ and γ the corresponding element of Γ, then $\gamma^\Gamma = \gamma.C$ is finite, therefore $C_\Gamma(\gamma)$ is of rank 2. Thus it coincides with Γ, $\gamma \in C$, $\bar{\gamma} = \bar{e}$. □

5.4. The same arguments show that $\bar{\gamma}^{\bar{\Gamma}}$ can not be of rank 0 for $\bar{\gamma} \neq \bar{e}$. □

5.5. Suppose Δ is an X-definable group over A, $\mathrm{rank}(\Delta/X) = 1$, codim $X \geqslant 3$. Then there is a unique 1-irreducible X-definable normal subgroup Δ^0 of Δ with $\mathrm{rank}(\Delta^0/X) = 1$.

The proof is analogous to 5.2. □

The subgroup Δ^0 will be called the connected component of Δ. If $\Delta = \Delta^0$, Δ is called connected.

5.6. If Δ is as in 5.5 and connected then Δ is abelian.

Proof. For $\delta \in \Delta \setminus C(\Delta)$ consider the conjugacy class $\delta^\Delta = \phi \subseteq \Delta$. ϕ or $\Delta \setminus \phi$ is of rank 0 over (X,δ), only the second is possible, since $C_\Delta(\delta)$ is of rank 0. Take now the polynomials p_ϕ and p_Δ given by 1.3. From $\phi \sqsubset \Delta$ it follows that the leading coefficients of the polynomials coincide. On the other hand $p_\Delta = |C_\Delta(\delta)|.p_\phi$, thus $|C_\Delta(\delta)| = 1$, contradiction. □

We assume now that Γ is a centerless 2-irreducible $\emptyset$-definable group over a pregeometry A', $\mathrm{rank}(\Gamma/\emptyset) = 2$, dim $A' \geqslant 6$.

5.7. There is no normal subgroup Δ of Γ which is (x_1,x_2)-definable for some $x_1, x_2 \in A'$ and $\mathrm{rank}(\Delta/(x_1,x_2)) = 1$.

Proof. Repeating the known construction [C] we can define a (x_1,x_2,x_3)-definable field structure on Δ, provided Δ is connected, which we may assume by 5.5. But such a field can not exist since the mapping definable in the field $v \mapsto v^2 - v$ maps Δ on a subset $\phi \subseteq \Delta$ and contradicts 1.3 as in 5.6. □

5.8. Let P be a maximal p-subgroup of Γ for some prime p, Y a closed subset of A', dim $Y \geqslant 3$, P[Y] a maximal p-subgroup of Γ[Y]. Then one and only one of the following holds:

(i) $|P[Y]|$ does not depend on $|Y|$;

(ii) P is an almost (γ)-definable subgroup for some $\gamma \in P$, $\mathrm{rank}(P/(\gamma)) = 1$, $|P[Y]|$ is a polynomial of $|Y|$ of degree 1, its connected component P^0 is (γ)-definable.

Proof. Choose $\gamma \in P[Y] \cap C_\Gamma(P) \setminus \{e\}$, denote $\Delta = C_\Gamma(\gamma)$. Then $P \subseteq \Delta$, $\mathrm{rank}(\Delta/(\gamma)) \leq 1$. If $\mathrm{rank}(\Delta/(\gamma)) = 0$ then $\Delta[Y]$ does not depend on Y by 1.3, the same is true for P[Y].

If $\mathrm{rank}(\Delta/(\gamma)) = 1$ and $\Delta^0 \cap P \neq \{e\}$ then $\Delta^0 \subseteq P$ and all Δ^0-cosets in P are almost (γ)-definable, so is P. This gives (ii). If $\Delta^0 \cap P = \{e\}$ then P intersects with any Δ^0-coset at most in one point. The cosets are almost (γ)-definable, therefore the number of cosets in $\Delta[Y]$ which intersect with P does not depend on $|Y|$.

5.9. There is at most one prime p for which 5.8(ii) holds.

Proof. From 5.8(ii) and 5.7 it follows that the set of all p-elements is of rank 2. Now recall 5.2. □

5.10. The polynomial $p_\Gamma(y)$ counting $\Gamma[Y]$ by 1.3 is of degree 2. On the other hand the Sylow Theorem together with 5.8 and 5.9 gives

$$|\Gamma[Y]| = p_1^{m_1} \cdot \ldots \cdot p_n^{m_n} \cdot p_P(y)$$

where $p_1, \ldots, p_n$ are all the prime divisors of $\Gamma[Y]$ for which 5.8(i) holds and $p_P(y)$ is the polynomial of degree 1 counting P[Y] satisfying 5.8(ii). This is the final contradiction. Thus Γ does not exist. □

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Contents

* Paper is a preliminary version and should not be reviewed.