A non-desarguesian projective plane

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October 2, 2012

1 Introduction

Hrushovski's construction of "new" strongly minimal structures and more generally "new" stable structures proved very effective in providing a number of examples to classification problems in stability theory. For example, J.Baldwin used this method to construct a non-desarguesian projective plane of Morley rank 2 (see e.g. [3]). But there is still a classification problem of similar type which resists all attempt of solution, the Algebraicity (or Cherlin-Zilber) Conjecture. At present there is a growing belief that there must exists a simple group of finite Morley rank which is not isomorphic to a group of the form $G(\mathbb{F})$ for G an algebraic group and $\mathbb{F}$ an algebraically closed field (a *bad* group).

The second author developed an alternative interpretation of the "new" stable structures obtained by Hrushovski's construction, see e.g. [5]. In this interpretation the universe M of the structure is represented by a complex manifold and relation by some subsets of M^n explained in terms of the analytic structure on M. In this interpretation Hrushovski's predimension inequality corresponds to a form of (generalised) Schanuel's conjecture. We argue that looking for stable structures of analytic origin is potentially a better way of producing new stable structures.

Below we briefly explain a construction of a new non-desarguesian projective plane that originates in a complex analytic structure. The new, in comparison with previous examples of e.g. "green fields" (see [6]) is that we have to use a non-trivial collapse procedure.

2 An ω-stable analytic structure

2.1 Consider structures $K_f = (K, +, \cdot, f)$, where $(K, +, \cdot)$ is a field and $f : K \to K$ a unary function.

Let L^{alg} be a relational language for structures of the form K_f, relations of which are those of L along with all the relations corresponding to Zariski closed 0-definable subsets of K^n. We always assume that K is a field of characteristic 0. Let $C(K_f)$ be the class of all finite L^{alg}-structures that can be embedded in K_f.

Note that in the language L^{alg} we can say for an n-tuple X and a variety W over $\mathbb{Q}$ that $X \in W$. So the expression $\mathrm{tr.d.}(X) = m$ means that m is the dimension of the smallest variety W over $\mathbb{Q}$ such that $X \in W$.

2.2 Theorem. (A.Wilkie, P.Koiran, [4],[2]) The structure $\mathbb{C}_f = (\mathbb{C}, +\cdot, f)$, where f is an entire **Liouville function,** is a model of the first order ω-stable theory T_f. For every finite subset X of the structure holds the Hrushovski inequality

$$\delta(X) \geq 0, \text{ where } \delta(X) := \mathrm{tr.d.}(X \cup f(X)) - |X|.$$

2.3 Theorem. $\mathbb{C}_f$ is ω-saturated.

Proof. Let d be the dimension in $\mathbb{C}_f$ corresponding to the predimension δ. For $X \subseteq \mathbb{C}$ we denote $\langle X \rangle$ the closure of X, that is the set of those $y \in \mathbb{C}$ that satisfy $d(y/X) = 0$. We will say that $\mathbb{C}_f$ has CCP, the countable closure property, if $\langle X \rangle$ is countable for any countable X.

We need the following.

2.4 Lemma. $\mathbb{C}_f$ is the unique model of T_f of cardinality continuum which satisfies the countable closure property.

Proof. We use the general theory of "pseudo-analytic structures" developed in [7] in application to pseudo-exponentiation and in many further variations by other authors.

Claim 1. $\mathbb{C}_f$ satisfies CCP.

This can be proved following lines of the proof of the same statement for $\mathbb{C}_{\exp}$ (Lemma 5.12) in [7], only simpler. We do not need to use Ax's theorem on Schanuel's property for function fields since the "Schanuel" property for $\mathbb{C}_f$ holds (by Wilkie's theorem). Also, instead of linear dependence one uses the equality relation, since in our δ we count the size of a set rather than linear dimension.

Now we recall the **strong existential closedness axiom** of [7] (s.e.s.c) in the setting of our structure. It requires that for every algebraic variety $V \subseteq \mathbb{C}^{2n}$ over a subfield k which satisfies the property that it is normal (rotund, in more recent terminology) and free with respect to $=$, there is a generic over k point in V of the form $\langle x_1, \ldots, x_n, f(x_1), \ldots, f(x_n)\rangle$.

Claim 2. Any model of T_f satisfies the strong existential closedness axiom with respect to f.

Proof. If we omit the requirement of the point being generic, then we have the existential closedness axiom, one of the axioms of T_f. We claim that this is enough to have a generic point. Indeed, we can assume that V is defined over some finite self-sufficient $A \leq \mathbb{C}$ and that $\dim V$ is minimal, i.e. equal to n. Then either $\delta(x_1, \ldots, x_n/A) = 0$ for the point $\langle x_1, \ldots, x_n, f(x_1), \ldots, f(x_n)\rangle \in V$, or $x_i = x_j$ for some $i \neq j$, or $x_i = a \in A$ for some i. By existential closedness we still can find a point for which no of the equalities holds, so

$$\text{tr.d.}(x_1, \ldots, x_n, f(x_1), \ldots, f(x_n)/A) = n = \dim V.$$

This proves that the point is generic. Claim proved.

The main theorem of [7] states that there is unique structure of a given uncountable cardinality that satisfies the Schanuel property, strong existential closedness and CCP. Adapted to our setting, we derive using the claims that $\mathbb{C}_f$ is the unique structure of cardinality continuum satisfying the properties. $\square$ Lemma.

Moreover, by [7] and [8] the unique model is ω-homogeneous over submodels, so is ω-homogeneous. Finally, by construction of the unique model, it is universal for the class of finitely generated substructures of the form K_f. It follows that $\mathbb{C}_f$ is ω-saturated. This complets the proof of the theorem. $\square$

2.5 On class $C(\mathbb{C}_f)$ we define an equivalent predimension δ_0 as follows. For $Y \in C(\mathbb{C}_f)$

$$\delta_0(Y) := \text{tr.d.}(Y) - |Y^2 \cap F|,$$

where F is the graph of f.

Lemma. Let M be a finite L^{alg}-structure. Then $\delta_0(Y) \geq 0$ for all $Y \subseteq M$ if and only if $M \in C(\mathbb{C}_f)$.

Proof. Suppose $M \in C(\mathbb{C}_f)$ and $Y \subseteq M$. Let

$$Y^2 \cap F = \{\langle x_i, f(x_i)\rangle : i = 1, \ldots n\}.$$

We assume the x_i all distinct. Set $X = \{x_1, \dots, x_n\}$. Then by assumption

$$0 \leq \delta(X) = \text{tr.d.}(X) - n \leq \text{tr.d.}(Y) - |Y^2 \cap F| = \delta_0(Y).$$

Conversely, suppose $\delta_0(Y) \geq 0$ for all $Y \subseteq M$. Then for every $X = \{x_1, \dots, x_n\} \subseteq M$ such that $\{f(x_1), \dots, f(x_n)\} \subseteq M$, we have $\delta(X) \geq \delta_0(X \cup f(X)) \geq 0$. Extend M to M' by adding new elements of the form $f(x)$ for every $x \in M$ such that $f(x) \notin M$. We define such $f(x)$ to be mutually algebraically independent over M. It is easy to see that for every $X \subseteq M$ in regards to f on M' the inequality $\delta(X) \geq 0$ holds. It follows that the diagram of M' is consistent with the theory of $\mathbb{C}_f$. By 2.3 M' can be embedded in $\mathbb{C}_f$. □

2.6 Assumption $f(0) = 0$.

This can be achieved by setting $f(x)$ to be $f(x) - f(0)$. This does not effect the statement 2.2, except for the change in the definition of the predimension δ, we have to replace it by the predimension over 0. This does not effect our calculations with predimension below.

In particular, if $x_1, \dots, x_n$ is a generic (in the sense of Morley rank) tuple in the field $\mathbb{C}_f$, then

$$\delta(x_1, \dots, x_n) = n \tag{1}$$

3 Mild Collapse

3.1 Consider the class $C(\mathbb{C}_f)$. This is an amalgamation class with respect to strong embeddings $\leq$ determined by δ.

Let μ be a Hrushovski function satisfying $\mu(\alpha) = 1$, for any α, which is a code of a pair $(x, x_1, y_1, x_2, y_2 / a_1, a_2, b)$ in a substructure $\{x, x_1, y_1, x_2, y_2, a_1, a_2, b\}$ that satisfies relations

$$\begin{aligned} &a_1 x = x_1,\ a_2 x = x_2, \\ &f(x_1) = y_1,\ f(x_2) = y_2, \\ &y_1 - y_2 = b \end{aligned}$$

Note, that the code of type α says that $f(a_1 x) - f(a_2 x) = b$, and $\mu(\alpha) = 1$ amounts to saying that the latter has at most one solution in x.

Consider the corresponding subclass $C_\mu(\mathbb{C}_f)$. We want to porve that this class has AP with respect to $\leq$.

3.2 Proposition. Let $B, M, N \in C_\mu(\mathbb{C}_f)$, $B \leq M$, $B \leq N$, and let $M \otimes_B N \subseteq \mathbb{C}_f$ be a free amalgam over B in $C(\mathbb{C}_f)$.

Suppose a code of type α is realised in $M \otimes_B N$ by $\{x, x_1, y_1, x_2, y_2, a_1, a_2, b\}$ and $b \in M$. Then

(i) either $\{x, x_1, y_1, x_2, y_2, a_1, a_2, b\} \subseteq M$,

(ii) or $b \in B$ and $\{x, x_1, y_1, x_2, y_2, a_1, a_2, b\} \subseteq N$,

(iii) or $\{x_1, x_2\} \subseteq B$, $\{y_1, y_2, b\} \subseteq M$ and $\{x, a_1, a_2\} \subseteq N - M$,

(iv) or $\{x_1, x_2, b\} \subseteq B$, $\{y_1, y_2, \} \subseteq N$ and $\{x, a_1, a_2\} \subseteq M - N$.

Proof. Let $\mathbb{Q}(M), \mathbb{Q}(N)$ and $\mathbb{Q}(B)$ be the field generated by the corresponding subsets. Note, that since $B \leq M$, there is no new relations $f(u) = v$ on $\mathbb{Q}(M)$ and similarly with $\mathbb{Q}(N)$.

Claim. $\mathbb{Q}(M)$ and $\mathbb{Q}(N)$ are linearly dijoint over $\mathbb{Q}(B)$. $M \otimes_B N$ can be naturally embedded in the free composite of fields $\mathbb{Q}(M) \otimes_{\mathbb{Q}(B)} \mathbb{Q}(N)$.

The first follows from the freenes. The rest by strong embeddings. $\Box$ Claim.

We may assume that $1 \in B$ and $B = \mathbb{Q}(B) \cap M = \mathbb{Q}(B) \cap N$.

We continue with auxiliary lemmas.

3.3 Lemma. $\{y_1, y_2, b\} \subseteq M$, or $b \in B$ and $\{y_1, y_2, b\} \subseteq N$.

Claim. $\{y_1, y_2, b\} \subseteq \mathbb{Q}(M)$, or $b \in B$ and $\{y_1, y_2, b\} \subseteq \mathbb{Q}(N)$.

Case 1. $y_1, y_2 \in N - M$. By disjointness $\mathrm{ldim}_{\mathbb{Q}(B)}(y_1, y_2, 1) = \mathrm{ldim}_{\mathbb{Q}(Bb)}(y_1, y_2, 1)$. Since $b = y_1 - y_2$, it follows $b \in \mathbb{Q}(B)$, so $b \in B$.

Case 2. $y_1 \in N - M$, $y_2 \in M$. Since $b = y_1 - y_2$, we have $y_1 \in \mathbb{Q}(M)$. $\Box$ Claim.

Now note that $\mathbb{Q}(M) \cap N = B$, and Lemma follows by the Claim. $\Box$

Without loss of generality we will assume below that $\{y_1, y_2, b\} \subseteq M$.

3.4 Lemma. Assume $\{y_1, y_2, b\} \subseteq M$. Then $\{x_1, x_2\} \subseteq M$.

Proof. By freeness there are no relations of the form $f(u) = v$ between $N - B$ and $M - B$. $\Box$

3.5 Lemma. Assume $\{y_1, y_2, b\} \subseteq M$. Then $\{x, a_1, a_2\} \subseteq M$, or $\{x, a_1, a_2\} \subseteq N - M$ and $\{x_1, x_2\} \subseteq B$.

Proof. By definition in 3.1 $x_1 a_2 = x_2 a_1$.

Now, if $x \in M$, then $a_1 \in M$, since $a_1 = \frac{x_1}{x}$. For the same reason $a_2 \in M$. So $\{x, a_1, a_2\} \subseteq M$.

Similarly, if $a_1 \in M$, then $x = \frac{x_1}{a_1} \in \mathbb{Q}(M)$, $x \in M$, and $\{x, a_1, a_2\} \subseteq M$.

Hence, the alternative to $\{x, a_1, a_2\} \subseteq M$ is $\{x, a_1, a_2\} \subseteq N$, which implies $x_1, x_2 \in N$, since $x_1 = a_1 x$ and $x_2 = a_2 x$. $\Box$

3.6 Lemma. Assume $\{y_1, y_2, b\}$ not a subset of M. Then$\{x, x_1, y_1, x_2, y_2, a_1, a_2, b\}$ $\{y_1, y_2, b\} \subseteq N$, and either $\{x, a_1, a_2\} \subseteq N$, or $\{x, a_1, a_2\} \subseteq M - N$ and $\{x_1, x_2\} \subseteq B$.

Proof. This is just the symmetric case with proofs corresponding to that of Lemmas 3.4 and 3.5. $\square$

This lemma completes the proof of Proposition 3.2. $\square$

3.7 Lemma. In 3.2, suppose N or M is minimal over B. Then (i) or (ii) of 3.2 holds.

Proof. Suppose N is minimal. Then (iii) is not possible, since each of a_1, a_2 and x is algebraic over M.

Under the same assumption, if (iv) holds, then $y_1 \in B$ or $y_2 \in B$, because $\delta_0(y_i/B) = 0$ for $i = 1, 2$. But then both will have to be in B since $y_1 - y_2 = b$ and $b \in B$. This brings us into case (i).

Now consider the case M is minimal. Suppose (iii) holds. Then at least two of the elements of $\{y_1, y_2, b\}$ has to be in B, and again, since $y_1 - y_2 = b$, all three are in B. This brings us into the case (ii).

(iv) can not hold since $\{x, a_1, a_2\} \subseteq M - N$ is in contradiction with minimality of M. $\square$

3.8 Proposition. $C_\mu(\mathbb{C}_f)$ is an amalgamation class.

Proof. We consider $B \leq M$, $B \leq N$ with an assumption that $b \in M$ and one of the extensions is minimal. Suppose $M \otimes_B N$ is not in $C_\mu(\mathbb{C}_f)$, that is there are $\{x, x_1, y_1, x_2, y_2, a_1, a_2, b\}$ and $\{x', x_1', y_1', x_2', y_2', a_1, a_2, b\}$, substructures of code α in $M \otimes_B N$ such that $x \neq x'$.

By 3.7 we will have $\{x, x_1, y_1, x_2, y_2, a_1, a_2, b\} \subseteq M$ and $\{x', x_1', y_1', x_2', y_2', a_1, a_2, b\} \subseteq N$. Hence $\{a_1, a_2, b\} \subseteq M \cap N = B$. Now we may assume that N is minimal over B, that is $N = B \cup \{x', x_1', y_1', x_2', y_2'\}$. Since the type of $\{x', x_1', y_1', x_2', y_2'\}$ over B given by code α is complete, we can identify $\{x', x_1', y_1', x_2', y_2'\}$ with $\{x, x_1, y_1, x_2, y_2\}$ thus identifying M as an amalgam of M and N over B. $\square$

3.9 Theorem. There exists a countable rich structure K_f for class $C_\mu(\mathbb{C}_f)$.

(i) K_f is an algebraically closed field with a function f.

(ii) Given $a_1 \neq a_2$ and b in K_f, there is a unique solution to the equation

$$f(a_1 x) - f(a_2 x) = b.$$

In particular, f is a bijection on K.

(iii) K_f is embeddable in $\mathbb{C}_f$.

(iv) depending on μ, the theory of K_f is ω-stable of rank ω or strongly minimal.

Proof. (i) and (iv) follows from general theory.

(iii) is by 2.3.

(ii) follows from the definition of μ. $\square$

4 Plane

4.1 Recall [1] that a **ternary ring** R is a set R with two distinguished elements $0, 1$ and a ternary operation $T : R^3 \to R$ satisfying the following conditions:

(T1) $T(1, a, 0) = T(a, 1, 0) = a$ for all $a \in R$;

(T2) $T(a, 0, c) = T(0, a, c) = c$ for all $a, c \in R$;

(T3) If $a, b, c \in R$, the equation $T(a, b, y) = c$ has a unique solution y;

(T4) If $a, a', b, b' \in R$ and $a \neq a'$, the equations $T(x, a, b) = T(x, a', b')$ have a unique solution x in R;

(T5) If $a, a', b, b' \in R$ and $a \neq a'$, the equations

$T(a, x, y) = b,\ T(a', x, y) = b'$ have a unique solution x, y in R.

Consider the ternary operation

$$T(a, x, b) = f^{-1}(f(ax) + b)$$

on K_f.

4.2 Lemma. The ternary operation $T(a, x, b)$ on K_f determine a ternary ring with 0 and 1 of the field K..

Proof. Check using 3.9 and 2.6. $\square$

4.3 Theorem (see [1]) Every projective plane $\mathbf{P}$ is bi-interpretable with a ternary ring R (associated ternary ring).

Every Desarguesian plane has a unique associated ternary ring, which is an associative division ring.

4.4 Corollary. The projective plane associated with the ternary ring (K_f, T) is not desarguesian.

Proof. Suppose the projective plane is Desarguesian. Then by 4.3 the ternary ring (K_f, T) is an associative division ring. In particular, we will have the identity

$$T(a, x, b) = a * x \dot{+} b$$

for $a * x := T(a, x, 0) = ax$ and $x \dot{+} b := T(1, x, b) = f^{-1}(f(x) + b)$.

The associativity law will give us the identity

$$a(x_1 + x_2) = ax_1 + ax_2, \text{ equivalently } af^{-1}(f(x_1) + x_2) = f^{-1}(f(ax_1) + ax_2).$$

The latter identity implies $\delta(a, x_1, x_2) < 3$ for any elements $a, x_1, x_2 \in K_f$, in contradiction with (1). $\square$

References

[1] A.Albert and R.Sandler, **An Introduction to Finite Projective Planes**, Holt, Rinehart and Winston, 1968.

[2] P.Koiran, *The theory of Liouville functions*, J. Symbolic Logic, **68**, (2), (2003), 353– 365

[3] K.Tent and M.Ziegler, **A Course in Model Theory**, CUP, 2012

[4] A.Wilkie, *Liouville functions*, Lect. Notes in Logic **19**, Logic Colloquium 2000, Eds R. Cori, A. Razborov, S. Tudorcevic, C. Wood (2005), 383-391

[5] B.Zilber, *Analytic and pseudo-analytic structures* Lect.Notes in Logic **19** (2005),Logic Colloquium Paris 2000, eds. R.Cori, A.Razborov, S.Todorcevic and C.Wood, AK Peters, Wellesley, Mass. 392–408

[6] B.Zilber, *Bi-coloured fields on the complex numbers*, J.Symbolic Logic, **69** (4), (2004), 1171–1186

[7] B.Zilber, *Pseudo-exponentiation on algebraically closed fields of characteristic zero*, Annals of Pure and Applied Logic, Vol 132 (2004) 1, pp 67-95

[8] B.Zilber, *A categoricity theorem for quasi-minimal excellent classes*, In: Logic and its Applications eds. Blass and Zhang, Cont.Maths, v.380, 2005, pp.297-306