Non-elementary categoricity and projective locally o-minimal classes

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Abstract

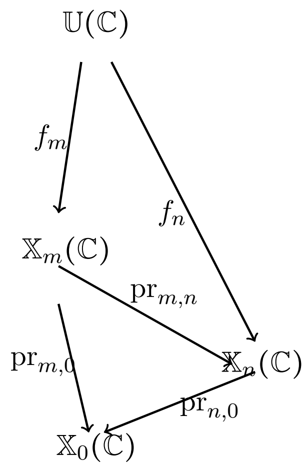
Given a cover $\mathbb{U}$ of a family of smooth complex algebraic varieties, we associate with it a class $\mathfrak{U}$, containing $\mathbb{U}$, of structures locally definable in an o-minimal expansion of the reals. We prove that the class is $\aleph_0$-homogenous over submodels and stable. It follows that $\mathfrak{U}$ is categorical in cardinality $\aleph_1$. In the case when the algebraic varieties are curves we prove that a slight modification of $\mathfrak{U}$ is an abstract elementary class categorical in all uncountable cardinals.

1 Introduction

1.1 Let $\mathrm{k}_0 \subseteq \mathbb{C}$, a countable subfield, $\{\mathbb{X}_i : i \in I\}$ a collection of non-singular irreducible complex algebraic varieties (of dim > 0) defined over k_0 and $I := (I, \geq)$ a lattice with the minimal element 0 determined by unramified k_0-rational epimorphisms $\mathrm{pr}_{i',i} : \mathbb{X}_{i'} \to \mathbb{X}_i$, for $i' \geq i$. Let $\mathbb{U}(\mathbb{C})$ be a connected complex manifold and $\{f_i : i \in I\}$ a collection of holomorphic covering maps (local bi-holomorphismss)

$$f_i : \mathbb{U}(\mathbb{C}) \twoheadrightarrow \mathbb{X}_i(\mathbb{C}), \quad \mathrm{pr}_{i',i} \circ f_{i'} = f_i.$$

as illustrated by the picture:



1.2 In a number of publications, abstract elementary classes $\mathfrak{U}$ containing structures $(\mathbb{U}, f_i, \mathbb{X}_i)$, with an abstract algebraically closed field K instead of $\mathbb{C}$ (pseudo-analytic structures) have been considered, see [1] for a survey. A typical result is a formulation of a "natural" $L_{\omega_1,\omega}$-axiom system Σ which holds for $(\mathbb{U}(\mathbb{C}), f_i, \mathbb{X}_i(\mathbb{C}))$ and defines a class $\mathfrak{U}$ categorical in all uncountable cardinals. The proofs, in each case, rely on deep results in arithmetic geometry, moreover one often is able to show that the fact of categoricity of Σ implies the arithmetic results.

The above raised the question of whether an uncountably categorical AEC $\mathfrak{U}$ containing $(\mathbb{U}(\mathbb{C}), f_i, \mathbb{X}_i(\mathbb{C}))$ exists under general enough assumptions on the data, leaving aside the question of axiomatisability and related arithmetic theory.

The current paper answers this question in positive at least in the case when the $\mathbb{X}_i$ are curves. We construct $\mathfrak{U}$ as the class of structures $\mathbb{U}(\mathrm{K})$ ($\mathrm{K} = \mathsf{R} + i\mathsf{R}$) locally definable (in the sense of M.Edmundo and others) in models R of an o-minimal expansion of the reals projected (restricted) to the language L_{glob} (global) the primitives of which are given by analytic subsets of $\mathbb{U}^m$ locally defined in the o-minimal structure. The main theorem states that, for the case when the complex dimension of $\mathbb{U}(\mathbb{C})$ is equal to 1, $\mathfrak{U}$ can be extended to a class of L_{glob}-structures which is an abstract elementary class categorical in all uncountable cardinals. For the general case we only where able to prove categoricity in $\aleph_1$.

1.3 Our main technical tool is a slightly generalised theory of K-analytic sets in o-minimal expansions of the reals developed by Y.Peterzil and S.Starchenko in [4]. We also make an essential use of the theory of quasi-minimal excellence, especially the important paper [8] by M.Bays, B.Hart, T.Hyttinen, M.Kesala and J.Kirby.

Note that our main technical results effectively prove that the structures in $\mathfrak{U}$ are *analytic Zarsiki* in a sense slightly weaker than in the paper [9], where we proved results similar to the current ones for an analytic Zariski class.

1.4 Most of our examples, see 2.3 below, have become objects of interest in the theory of o-minimality due to Pila-Wilkie-Zannier method of counting special points of Shimura varieties and more generally, see survey [2]. Effectively, one counts points of $\mathbb{U}(L) \cap D \cap S$ where D is an open subset of $\mathbb{U}(\mathbb{C})$ definable in the o-minimal structure, S an L_{glob}-definable analytic subsets of $\mathbb{U}(\mathbb{C})$ and L a number field relevant to the case at hand.

At the same time one should note that in representing an L_{glob}-structure as $\mathbb{U}(\mathrm{K})$, $\mathrm{K} = \mathsf{R} + i\mathsf{R}$, there is a remarkable degree of freedom in the choice of a model R of the underlying o-minimal theory.

This raises a lot of question on the interaction between the theory of AEC and o-minimality, the model theory - arithmetic geometry perspective of categorical classes and the o-minimal Pila-Wilkie-Zannier method.

1.5 I would like to thank Martin Bays and Andres Villaveces for some useful remarks and commentaries.

2 Preliminaries

2.1 Let $\mathbb{R}_{\text{An}}$ be an o-minimal expansion of the reals, $\mathbb{C} = \mathbb{R} + i\mathbb{R}$ in the language of rings and

$$\text{Mod}_{\text{An}} = \{\mathsf{R} : \mathsf{R} \equiv \mathbb{R}_{\text{An}}\}$$

the class of models of the complete o-minimal theory $\text{Th}(\mathbb{R}_{\text{An}})$ in the language L_{An}. To avoid unnessary complications we assume that L_{An} is a countable fragment of the full language of $\mathbb{R}_{\text{An}}$.

We write K for the algebraically closed field $\mathrm{K}(\mathsf{R}) \coloneqq \mathsf{R} + i\mathsf{R}$.

2.2 $(\mathbb{R}_{\text{An}}, \{f_i\})$**-admissible open cover of** $\mathbb{U}(\mathbb{C})$.

In addition to the data and notation spelled out in 1.1, we assume that:

(i) There is a system of connected open subsets $D_n(\mathbb{C}) \subset \mathbb{U}(\mathbb{C})$, $n \in \mathbb{N}$, definable in $\mathbb{R}_{\mathrm{An}}$ (possibly with parameters), such that

$$\text{For any } n \in \mathbb{N}, \quad D_n \subseteq D_{n+1}, \text{ and } \bigcup_n D_n(\mathbb{C}) = \mathbb{U}(\mathbb{C});$$

(ii) The restriction $f_{i,n}$ of f_i on D_n is definable in $\mathbb{R}_{\mathrm{An}}$, for each $i \in I$ and $n \in \mathbb{N}$, and for each i there is n such that $f_i(D_n) = \mathbb{X}_i$.

(iii) For all $i \in I$, there is a group Γ_i of biholomorphic transformations on $\mathbb{U}(\mathbb{C})$, so that the restrictions of the transformations to the $D_n(\mathbb{C})$ are L_{An}-definable and fibres of f_i are Γ_i-orbits, that is

$$f_i : \mathbb{U}(\mathbb{C}) \to \mathbb{X}_i(\mathbb{C}) \cong \mathbb{U}(\mathbb{C})/\Gamma_i.$$

Moreover, for $i > j$, Γ_i is a finite index subgroup of Γ_j, that is the cover $\mathrm{pr}_{i,j} : \mathbb{X}_i \to \mathbb{X}_j$ is finite.

(iv) The system of maps $f_i, i \in I$ is $\mathbb{U}$-**complete**: there is a chain $I_0 \subseteq I$ such that

$$\bigcap_{l \in I_0} \Gamma_l = \{1\}.$$

2.3 Examples of admissible $\mathbb{R}_{\mathrm{An}}$.

In all our examples $\mathbb{R}_{\mathrm{An}}$ is a L_{An}-reduct of $\mathbb{R}_{\exp,an}$, the reals with exponentiation and restricted analytic functions. What varies is $\mathbb{U}$, k_0 and the choice of the family $\{f_i, D_n : i \in I, n \in \mathbb{N}\}$ the members of which assumed to be L_{An}-definable.

1. $\mathbb{U}(\mathbb{C}) = \mathbb{C}$, $I = \mathbb{N}$, $\mathbb{X}_i = \mathbb{G}_m$, all $i \in I$, the algebraic torus, $D_n = \{z \in \mathbb{C} : -2\pi n < \mathrm{Im} z < 2\pi n\}$, $f_k(z) = \exp(\frac{z}{k})$, and $\mathrm{k}_0 = \mathbb{Q}$.

2. $\mathbb{U}(\mathbb{C}) = \mathbb{C}$, $I = \mathbb{N}$, $\mathbb{X}_i = \mathrm{E}_\tau$, all $i \in I$, an elliptic curve

$$f_k = \exp_{\tau,k} : \mathbb{C} \to \mathrm{E}_\tau \subset \mathbf{P}^2, \quad z \mapsto \exp_\tau(\frac{z}{k}),$$

the covering map for E_τ ($\exp_\tau$ is constructed from the Weierstrass $\mathfrak{P}$-function and its derivative $\mathfrak{P}'$, with period $k\Lambda_\tau = k\mathbb{Z} + \tau k\mathbb{Z}$).

D_1 is the interior of the square in $\mathbb{C}$ with vertices $(0, 1, \tau, \tau + 1)$, and $D_n = n \cdot D_1$. k_0 is the field of definition of E_τ.

3. $\mathbb{U}(\mathbb{C}) = \mathbb{H}$, the upper half-plane.

$$D_n = \{z \in \mathbb{H} : -n/2 \le \mathrm{Re}(z) < n/2 \;\&\mathrm{Im}(z) > 1/(n+1)\}.$$

For $n = 1$ this is the interior of the fundamental domain of the j-function

$$F = \{z \in \mathbb{H} : -1/2 \le \mathrm{Re}(z) < 1/2 \ \& \ \mathrm{Im}(z) > \frac{1}{2}\}$$

and the results of [5] state that the restriction of j to F is defined in $\mathbb{R}_{\mathrm{exp,an}}$. Note that, for each n, D_n can be covered by finitely many shifts of D_1 by Moebius transformations from $\Gamma := \mathrm{PSL}_2(\mathbb{Z})$. This allows one to define j on D_n in $\mathbb{R}_{\mathrm{exp,an}}$.

Moreover, we can similarly consider more general functions

$$j_N : \mathbb{H} \to \mathbb{Y}(N) \cong \mathbb{H}/\Gamma(N)$$

onto level N Shimura curves. A fundamental domain for j_N is a finite union of finitely many shifts of F and the analysis of [5] shows that the restriction of j_N on its fundamental domain is definable in $\mathbb{R}_{\mathrm{exp,an}}$. Thus we can take the family $\{j_N\}$ to be our $\{f_i\}$ $(i = N)$ and $\mathbb{Y}(N)$ to be the $\mathbb{X}_i$. It is well-known that the $\mathbb{Y}(N)$ and j_N are defined over $\mathrm{k}_0 = \mathbb{Q}^{\mathrm{ab}}$, the extension of $\mathbb{Q}$ by roots of 1.

4. $\mathbb{U}(\mathbb{C}) = \mathbb{H}$. Let Γ is a Fuchsian subgroup of $\mathrm{PGL}_2(\mathbb{R})$ and $\{\Gamma_i : i \in I\}$ the system of all finite index subgroups of Γ (see [3]). Then the $\mathbb{H}/\Gamma_i$ are bi-holomorphic to compact projective curves $\mathbb{X}_i(\mathbb{C})$ with bounded fundamental domains. Thus one can define respective D_n and f_i as in 2.2. k_0 is the union of the fields of definition of the $\mathbb{X}_i$.

5. [5] supplies us with a plethora of other examples, in particular $\mathbb{U}(\mathbb{C}) = \mathbb{H}_g$, the Siegel half-space, and $\mathbb{X}_i$ moduli spaces of polarised algebraic varieties.

3 The $\mathbb{K}$-analytic setting

3.1 Abstract structures definable in R.

Now we extend notations of 2.2 and, assuming $\mathsf{R} \in \mathrm{Mod}_{\mathrm{An}}$ be given, let $\mathbb{U}$, $\mathbb{X}_i$, $(i \in I)$, D_n, Γ_i and f_i be defined as in 2.2 in the language L_{An}. In particular, we read $\mathbb{U} := \mathbb{U}(\mathrm{K})$, $\mathbb{X}_i := \mathbb{X}_i(\mathrm{K})$, for $\mathrm{K} = \mathrm{K}(\mathsf{R})$, when the choice of the model R does not matter.

More precisely, we define

$$\mathbb{U}(\mathrm{K}) = \bigcup_n D_n(\mathrm{K}),$$

which is an $L_{\omega_1,\omega}$ interpretation of $\mathbb{U}$ in R for each $i \in I$. Now $f_i : \mathbb{U}(\mathrm{K}) \to \mathbb{X}_i(\mathrm{K})$ is defined to be the map such that it coincides with the map $f_{i,n} : D_n(\mathrm{K}) \to \mathbb{X}_i(\mathrm{K})$ for each $n \in \mathbb{N}$. Note that the latter is K-holomorphic in the sense of [4]. We will often say K-holomorphic (analytic) in an extended sense: the restriction $f_{i,n}$ of f_i to $D_n(\mathrm{K})$ is K-holomorphic.

We write $D_{\bar{n}} \subset \mathbb{U}^m$ meaning that $\bar{n} = \langle n_1, \dots, n_m \rangle \in \mathbb{N}^m$ and

$$D_{\bar{n}} = D_{n_1} \times \dots \times D_{n_m}.$$

Define f_i on $D_{\bar{n}}$ as $\langle u_1, \dots, u_m \rangle \mapsto \langle f_i(u_1), \dots, f_i(u_m) \rangle$. This obviously extends to the map f_i with the domain $\mathbb{U}^m$.

We will often restrict our analysis of K-analytic sets to open neighbourhoods, where **open** always means **definable open.**

Let k_0 be a subfield of K such that $\mathrm{k}_0 \subseteq \mathrm{dcl}(\varnothing)$, that is any point of k_0 is definable in R without parameters. Note that k_0 contains any point of the form $f_i(a)$ for $i \in I$ and a definable point $a \in D_n$.

More generally, we will work with an arbitrary k such that $\mathrm{k}_0 \subseteq \mathrm{k} \subset \mathrm{K}$.

3.2 Definition. Given $S \subset \mathbb{U}^m$ we say that S is $L_{\mathrm{glob}}(\mathrm{k})$**-primitive** if there are $I_S \subseteq I$ and Zariski closed $Z_i \subseteq \mathbb{X}_i^m$, $i \in I_S$, defined over k, such that

$$S = \bigcap_{i \in I_S} f_i^{-1}(Z_i).$$

3.3 Remark. In the definition 3.2 we may assume without loss of generality that I_S is a chain and, for $i' \geq i$ in I_S,

$$\mathrm{pr}_{i'i}(Z_{i'}) = Z_i. \tag{1}$$

Proof. First, we may assume that $I_S = I$ by setting for $i \in I \setminus I_S$, $Z_i := \mathbb{X}_i^m$.

For a finite $J \subseteq I$, take a $i_J \in I$ such that $i_J \geq J$. Set, for each $k \in J$,

$$Z_{i_J,k} := \mathrm{pr}_{i_J,k}^{-1}(Z_k) \subseteq \mathbb{X}_{i_J}^m \text{ and } Z_{i_J}^* = \bigcap_{k \in J} Z_{i_J,k}.$$

Then, since $f_k = \mathrm{pr}_{i_J,k} \circ f_{i_J}$,

$$f_{i_J}^{-1}(Z_{i_J,k}) = f_k^{-1}(Z_i) \text{ and } \bigcap_{k \in J} f_k^{-1}(Z_i) = f_{i_J}^{-1}(Z_{i_J}^*). \tag{2}$$

Since I is a countable lattice we can represent

$$I = \bigcup_{n \in \mathbb{N}} J_n$$

where $J_n \subseteq I_S$ are finite and $J_{n+1} \supseteq J_n$ for each n.

Consider (2) with $J = J_n$ and write i_{J_n} as i_n. Clearly, $i_{n+1} \geq i_n$ and

$$S = \bigcap_{n \in \mathbb{N}} f_{i_n}^{-1}(Z_{i_n}^*). \tag{3}$$

Finally, note that in (3) $\mathrm{pr}_{i_n,i_l}(Z_{i_n}^*) \subseteq Z_{i_l}^*$ for $n \geq l$, and $\mathrm{pr}_{i_n,i_l}(Z_{i_n}^*)$ is a Zariski closed subset of $\mathbb{X}_{i_l}^m$ since pr_{i_n,i_l} is unramified (and étale). Hence, we may replace $Z_{i_l}^*$ by $\bigcap_{n \geq l} \mathrm{pr}_{i_n,i_l}(Z_{i_n}^*)$ while keeping (3). Doing this consequetively for $l = 1, 2, \ldots$ delivers us (1). □

Remark. The equality relation is $L_{\mathrm{glob}}(\mathrm{k}_0)$-primitive.

3.4 K**-holomorphic maps and** K**-analytic subsets**. We use [4] for definitions and basic facts on K-analyticity in open definable subsets $D_{\bar{n}}$. By slight abuse of the terminology we call a subset $S \subseteq \mathbb{U}^m$ K-analytic if $S \cap D_{\bar{n}}$ is K-analytic for each $D_{\bar{n}} \subset \mathbb{U}^m$.

Since the complex covering maps f_i are holomorphic, the maps $f_{i,n} : D_n(\mathrm{K}) \to \mathbb{X}_i(\mathrm{K})$ are K-holomorphic and locally K-bi-holomorphic. It follows the sets $f_i^{-1}(Z_i)$ in 3.2 are K-analytic and are locally K-bi-holomorphically isomorphic to the Z_i.

The dimension dim is always the K-dimension of a K-analytic set. In case Z is an algebraic variety, $\dim Z := \dim Z(\mathrm{K})$, the dimension of the respective K-analytic set, and this coincides with the dimension in the sense of algebraic geometry.

3.5 Lemma. *Given an $L_{\mathrm{glob}}(\mathrm{k})$-primitive S, $S \cap D_{\bar{n}}$ is* K*-analytic in $D_{\bar{n}}$. S is* K*-analytic in $\mathbb{U}^m$.*

Proof. Let S be as in 3.2 with the assumption (1) and let $S_i := f_i^{-1}(Z_i)$. It follows by definition that the $S_i \cap D_{\bar{n}}$ are K-analytic. We need to prove that $\bigcap_{i \in I_S} S_i \cap D_{\bar{n}}$ is analtic.

Let $s \in S \cap D_{\bar{n}}$. For each $i \in I_S$ there is an open neighborhood $O_{s,i}$ of s such that $S_i \cap O_{s,i}$ is irreducible. We may assume that $S_{i'} \cap O_{s,i'} \subseteq S_i \cap O_{s,i}$ for $i' \geq i$. Then there exists $i_0 \in I_S$ such that for $i' \geq i \geq i_0$, $\dim S_{i'} \cap O_{s,i'} = \dim S_i \cap O_{s,i}$.

Since $S_i \cap O_{s,i}$ is irreducible, $S_{i'} \cap O_{s,i} = S_i \cap O_{s,i}$ for all $i' \geq i \geq i_0$. Thus $S \cap O_{s,i} = S_i \cap O_{s,i}$, which proves that S is K-analytic in the neighbourhood, and hence in $D_{\bar{n}}$. □

3.6 Remark. *S^{sing}, the set of singular points of $L_{\mathrm{glob}}(\mathrm{k})$-primitive S, is also an $L_{\mathrm{glob}}(\mathrm{k})$-primitive since*

$$S^{\mathrm{sing}} = \bigcap_{i \in I_S} f_i^{-1}(Z_i^{\mathrm{sing}}).$$

3.7 Proposition. *Let $S \subseteq \mathbb{U}^m$ be $L_{\text{glob}}(\mathrm{k})$-primitive and let, for some n, $S_{j,\bar{n}} \subseteq S \cap D_{\bar{n}}$ be a* K*-analytic irreducible component of $S \cap D_{\bar{n}}$. Then:*

(i) For any $D_{\bar{n}'} \supseteq D_{\bar{n}}$ there is unique $S_{j,\bar{n}'} \supseteq S_{j,\bar{n}}$ a K*-analytic irreducible component of $S \cap D_{\bar{n}'}$. The set*

$$S_j := \bigcup_{D_{\bar{n}'} \supseteq D_{\bar{n}}} S_{j,\bar{n}'}$$

is well-defined. (Call it an irreducible component of S.)

(ii) The number of K*-analytic components S_j of S is at most countable.*

(iii) The irreducible components S_j are $L_{\text{glob}}(\mathrm{k}')$-primitive for some algebraic extension k' of k.

(iv) For any i, $f_i(S_j)$ is a Zariski closed k'-definable geometrically irreducible subset of $\mathbb{X}_i^m$.

Proof. By [4], 4.12, $S_{j,\bar{n}'}$ is irreducible if and only if $S_{j,\bar{n}'} \setminus S_{j,\bar{n}'}^{\text{sing}}$ is definably connected. The union of any two irreducible extensions of $S_{j,\bar{n}} \setminus S_{j,\bar{n}}^{\text{sing}}$ will be connected, since any two points in the union can be connected by a definable path passing through $S_{j,\bar{n}} \setminus S_{j,\bar{n}}^{\text{sing}}$. Hence the extensions coinside, which gives us the first statement of Proposition.

The number of such irreducible components is at most countable since the number of components in each $D_{\bar{n}'}$ is finite. This proves (i) and (ii).

Define $\dim S_j$ to be $\dim S_{j,\bar{n}}$, which does not depend on $D_{\bar{n}}$ as long as $S_j \cap D_{\bar{n}} \neq \varnothing$, since irreducible sets are of pure dimension (the proof is the same as in the complex case, see also [4]). Define

$$\dim S := \max_j \dim S_j. \tag{4}$$

We may assume that

$$S = \bigcap_{i \in I_0} f_i^{-1}(Z_i)$$

for some chain $I_0 \subseteq I$, some Zariski closed $Z_i \subseteq \mathbb{X}_i^m$ such that $\dim Z_i = \dim S$ and $\mathrm{pr}_{i,l}(Z_i) = Z_l$ for $i > l$ in I_0.

Let $S^i := f_i^{-1}(Z_i)$ and let $S^i = \bigcup_{j \in J_i} S_j^i$ be the decomposition into irreducible analytic components with maximum dimension equal to $\dim S$. It follows that the components of S^i are also components of S^l, for $i > l$ and thus S_j is a component of $f_l^{-1}(Z_l)$.

Fix l for the time being. We can represent $Z_l = \bigcup_{p\in P} Z_{l,p}$, a finite union of geometrically irreducible algebraic subvarieties $Z_{l,p}$ defined over some algebraic extension k' of k. Respectively, S can be represnted as a finite union of $L_{\mathrm{glob}}(\mathrm{k}')$-primitives,

$$S = \bigcup_{p\in P} T_{l,p} \text{ where } T_{l,p} = S \cap f_l^{-1}(Z_{l,p})$$

and the irreducible component S_j of S is an irreducible component of one of $T_{l,p}$.

We assume without loss of generality that Z_l is geometrically irreducible, P is a singleton and, since we are only interested in S_j, assume

$$S = f_l^{-1}(Z_l).$$

We omit the subscript l in the Claim below.

Claim. $f(S_j) = Z$ and for any other component S_k of S there is $\gamma \in \Gamma$ such that $\gamma \cdot S_j = S_k$.

Proof. By 1.1 there is $\bar{n}$ such that $f(D_{\bar{n}}) = \mathbb{X}^m$.

By our assumption then

$$Z = f(\bigcup_{k\in J} S_k) = \bigcup_{k\in J} f(S_k \cap D_{\bar{n}}) = \bigcup_{k\in J_0} f(S_k \cap D_{\bar{n}})$$

where J lists all the components of S and J_0 lists the components S_k such that $S_k \cap D_{\bar{n}} \neq \emptyset$, so J_0 is finite.

Hence for the finite J_1, $J_0 \subseteq J_1 \subseteq J$,

$$Z = \bigcup_{k\in J_1} f(S_k).$$

Let Z^{sing} the singular points of Z and S^{sing} the singular points of S, which by the fact that f is a local bi-holomorphisms are related as

$$f^{-1}(Z^{\mathrm{sing}}) = S^{\mathrm{sing}}. \tag{5}$$

Note that if $s \in S_j \cap S_k$, a common point of two distinct components of S then $s \in S^{\mathrm{sing}}$. That is $S \setminus S^{\mathrm{sing}}$, the analytic subset of the open set $\mathbb{U}^m \setminus S^{\mathrm{sing}}$, splits into non-intersecting analytic components $S_k \setminus S^{\mathrm{sing}}$. We get from (5)

$$Z \setminus Z^{\mathrm{sing}} = \bigcup_{k\in J_1} f(S_k \setminus S^{\mathrm{sing}}). \tag{6}$$

The union on the right can not be disjoint, that is either J_1 is a singleton, or there are distinct $k_0, k_1 \in J_1$ such that $f(S_{k_0} \setminus S^{\rm sing}) \cap f(S_{k_1} \setminus S^{\rm sing}) \neq \varnothing$. Indeed, suppose towards a contradiction it is. Note that for a respective $D_{\bar{n}}$, $f : D_{\bar{n}} \twoheadrightarrow \mathbb{X}^m$ is a (definably) closed covering map since it is locally bi-holomorphisms. Hence $f(D_{\bar{n}} \cap S_k \setminus S^{\rm sing})$, $k \in J_1$, are disjoint definably closed subsets the union of which is the definably connected algebraic set $Z \setminus Z^{\rm sing}$, the contradiction.

Now we claim that

$$f(S_{k_0} \setminus S^{\rm sing}) = Z \setminus Z^{\rm sing}, \text{ for a } k_0 \in J_1. \tag{7}$$

Indeed, otherwise there are $k_0, k_1 \in J_1$ such that $f(S_{k_0} \setminus S^{\rm sing}) \neq f(S_{k_1} \setminus S^{\rm sing})$ but $f(S_{k_0} \setminus S^{\rm sing}) \cap f(S_{k_1} \setminus S^{\rm sing}) \neq \varnothing$. The latter means that there are $s_0 \in S_{k_0} \setminus S^{\rm sing}$ and $s_1 \in S_{k_1} \setminus S^{\rm sing}$ such that $f(s_1) = f(s_0)$, and hence $s_1 = \gamma \cdot s_0$ for some $\gamma \in \Gamma$. It follows that the K-analytic sets S_{k_1} and $\gamma \cdot S_{k_0}$ intersect in a non-singular point of $S \cap D_{\bar{n}}$ and thus $S_{k_1} \cap D_{\bar{n}} = \gamma \cdot S_{k_0} \cap D_{\bar{n}}$, and so

$$S_{k_1} = \gamma \cdot S_{k_0} \text{ and } f(S_{k_1}) = f(S_{k_0}).$$

(7) follows. This finishes the proof of the Claim and of the statement (iv).

Now, for any $i \in I$ consider

$$Z_{ij} := f_i(S_j)$$

which we proved to be Zariski closed irreducible and

$$f_i^{-1}(Z_{ij}) = \bigcup_{\gamma \in \Gamma_i} \gamma \cdot S_j.$$

Since by assumption $\bigcap_{l \in I} \Gamma_l$ is trivial, for some chain $I_1 \subseteq I$ extending I_0 we have

$$S_j = \bigcap_{l \in I_1} f_l^{-1}(Z_{lj}),$$

(iii) proved.

□

3.8 Definitions. For an m-tuple u in $\mathbb{U}$ and a subfield $\mathrm{k} \subset \mathrm{K}$ the **locus** of u over k, written $\mathrm{loc}(u/\mathrm{k})$, is the minimum $L_{\rm glob}(\mathrm{k})$-primitive containing u.

We say an $L_{\rm glob}(\mathrm{k})$-primitive S is **k-irreducible** if S can not be represented as $S_1 \cup S_2$ with $L_{\rm glob}(\mathrm{k})$-primitives S_1 and S_2 , both distinct from S.

Remark. Note that $\mathrm{loc}(u/\mathrm{k})$ is k-irreducible.

4 L_{glob}-structures

4.1 Recall, see [6], that an o-minimal structure R is a pre-geometry, i.e. has a well-behaved dependence relation, and one can define a notion of a (combinatorial) dimension $\operatorname{cdim} A$ of a subset $A \subseteq \mathsf{R}$ (not to confuse with K-dimension) as the cardinality of a maximal independent subset of A.

In particular, $\operatorname{cdim} \mathsf{R}_0 = 0$ for the prime model R_0 of the theory $\operatorname{Th}(\mathbb{R}_{\text{An}})$. And, if $\operatorname{card} \mathsf{R} = \kappa > \aleph_0$, then $\operatorname{cdim} \mathsf{R} = \kappa$.

This has the following relationship with $\dim_{\mathsf{R}} S$ (the "real" dimension in the sense of [4]) for an R-manifold $S \subseteq \mathsf{R}^m$ defined over a set C : assuming $\operatorname{cdim} \mathsf{R}/C \geq m$, for any $d \in \mathbb{N}$,

$$\dim_{\mathsf{R}} S \geq d \text{ iff exists } \langle s_1, \ldots, s_m \rangle \in S : \ \operatorname{cdim}(\{s_1, \ldots, s_m\}/C) \geq d \quad (8)$$

Recall that if S is K-analytic, then

$$\dim S = \frac{1}{2} \dim_{\mathsf{R}} S. \quad (9)$$

4.2 Definition. Given $\mathsf{R} \in \text{Mod}_{\text{An}}$, define $\mathfrak{U}(\mathsf{R})$ to be the structure with universe $\mathbb{U}(\mathrm{K})$ (K the field $\mathsf{R} + i\mathsf{R}$) in the language of $L_{\text{glob}}(\mathrm{k}_0)$-primitives.

Define $\mathfrak{U}$ to be the class of all structures of the form $\mathfrak{U}(\mathsf{R})$.

Recall the followng.

4.3 Fact. *For* K *an algebraically closed field, consider the structure* $\mathbb{X}(\mathrm{K})_{\text{Zar},\mathrm{k}_0}$ *on an infinite algebraic variety* $\mathbb{X}(\mathrm{K})$ *over* k_0 *equipped with relations* $Z \subseteq \mathbb{X}^m$, *all Zariski closed* Z *over* k_0.

The field structure K *together with its* k_0*-points is* $\varnothing$*-interpretable in* $\mathbb{X}(\mathrm{K})_{\text{Zar},\mathrm{k}_0}$.

This is well-known. A detailed proof is given in [7], Appendix A.

4.4 Proposition. $\mathfrak{U}(\mathsf{R})$ *interprets in the first order way over* $\varnothing$ *the field* K, *points of the subfield* k_0 *and all the maps* $f_i : \mathbb{U} \to \mathbb{X}_i(\mathrm{K})$.

Proof. First note that the equivalence relations on $\mathbb{U}$

$$E_i(u_1, u_2) :\equiv f_i(u_1) = f_i(u_2)$$

are $L_{\text{glob}}(\mathrm{k})$-primitives. Thus the sets $\mathbb{X}_i(\mathrm{K})$ are $\varnothing$-interpretable as $\mathbb{U}/E_i$ together with the maps $f_i : \mathbb{U} \to \mathbb{U}/E_i$.

Given a Zariski closed $Z_i \subset \mathbb{X}_i^m$ we have $Z_i^{\mathbb{U}} := f_i^{-1}(Z_i)$, a definable subset of $\mathbb{U}^m$. Thus $Z_i = f_i(Z_i^{\mathbb{U}})$ are $\varnothing$-interpretable.

Now the structure $\mathbb{X}_0(\mathrm{K})_{\mathrm{Zar},\mathrm{k}_0}$ equipped with relations $Z \subseteq \mathbb{X}_0^m$, for all Zariski closed Z over k_0, is $\varnothing$-interpretable.

It follows from 4.3, one can interpret K and k_0-points in $\mathfrak{U}(\mathsf{R})$. □

4.5 Corollary. *Any $L_{\mathrm{glob}}(\mathrm{K})$-primitive is type-definable in $\mathfrak{U}(\mathsf{R})$ using parameters.*

Below $\mathbb{U}$ is always the universe $\mathbb{U}(\mathrm{K})$ for some $\mathfrak{U}(\mathsf{R})$ in $\mathfrak{U}$.

4.6 Lemma. *If* k *is algebraically closed then* $\mathrm{loc}(u/\mathrm{k})$, *the locus of* u *over* k, *is* K-*analytically irreducible.*

If $S \subseteq \mathbb{U}^m$ is an $L_{\mathrm{glob}}(\mathrm{k})$-primitive and K-*analytically irreducible, then* $S = \mathrm{loc}(u/\mathrm{k})$, *for some* $u \in S$.

Proof. The first statement is just a corollary to 3.7(iv).

Let $\dim S = d$. By (8) and (9) there is an $u \in S$ such that $u = \langle s_1, \dots, s_m \rangle$, $\mathrm{cdim}(s_1, \dots, s_m/\mathrm{k}) = 2d$. Then $\mathrm{loc}(u/\mathrm{k}) \subseteq S$ and again by (8), and (9), $\dim \mathrm{loc}(u/\mathrm{k}) \geq d$. Since S is K-analytically irreducible, $\mathrm{loc}(u/\mathrm{k}) = S$. □

4.7 Lemma. *Let $S \subset \mathbb{U}^m$ be an $L_{\mathrm{glob}}(\mathrm{k})$-primitive,* $\dim S = d$. *Assume also* $\mathrm{cdim}(\mathsf{R}/\mathrm{k}) \geq \aleph_0$. *Then, for any family $L_{j \in J}$ of $L_{\mathrm{glob}}(\mathrm{k})$-primitives such that* $\dim L_j < d$, *all* $j \in J$,

$$S \setminus \bigcup_{j \in J} L_j \neq \varnothing. \tag{10}$$

Proof. S contains a point $u = \langle s_1, \dots, s_m \rangle$ with $\mathrm{cdim}(s_1, \dots, s_m/\mathrm{k}) = 2d$, which is not a point of any L_j. □

4.8 Proposition (The projection of an irreducible analytic set) *Let* k *be algebraically closed,* $\mathrm{cdim}(\mathsf{R}/\mathrm{k}) \geq \aleph_0$. *Let $T \subseteq \mathbb{U}^{m+1}$ be an $L_{\mathrm{glob}}(\mathrm{k})$-primitive* K-*analytically irreducible, and let* $\mathbf{p} : \mathbb{U}^{m+1} \to \mathbb{U}^m$ *be the projection onto the first m coordinates. Then there are an $L_{\mathrm{glob}}(\mathrm{k})$-primitive $S \subseteq \mathbb{U}^m$, an $i_0 \in I$ and a Zariski closed subset $R \subseteq \mathbb{X}_{i_0}^m$ defined over* k *such that* $\dim R < \dim S$ *and*

$$S \setminus f_{i_0}^{-1}(R) \subseteq \mathbf{p}(T) \subseteq S \tag{11}$$

Moreover, for any $d \leq \dim T - \dim S$, there is a Zariski closed $R_d \subset \mathbb{X}_{i_0}^m$ defined over k *such that* $R \subseteq R_d$, $\dim R_d < \dim S$ *and*

$$\mathbf{p}(T) \setminus f_{i_0}^{-1}(R_d) = \mathbf{p}_d(T) \tag{12}$$

where
$$\mathbf{p}_d(T) := \{s \in \mathbf{p}(T) : \dim(\mathbf{p}^{-1}(s) \cap T) \le d\}.$$

Proof. By 4.6
$$T = \mathrm{loc}(\bar{u}v/\mathrm{k})$$
for some $\bar{u}v \in \mathbb{U}^{m+1}$, ($\bar{u} \in \mathbb{U}^m$, $v \in \mathbb{U}$).

Let
$$S = \mathrm{loc}(\bar{u}/\mathrm{k}).$$
By definition
$$S = \bigcap_{i \in I_0} f_i^{-1}(Z_i), \quad T = \bigcap_{i \in I_0} f_i^{-1}(W_i)$$
for some Zariski closed $Z_i \subseteq \mathbb{X}_i^m$, $W_i \subseteq \mathbb{X}_i^{m+1}$ over k and we apply the same notation to the projection map $\mathbf{p} : \mathbb{X}_i^{m+1} \to \mathbb{X}_i^m$. By 3.7(iv) we may assume that all the Z_i and W_i are irreducible and of dimension equal to that of S and T respectively,
$$f_i(S) = Z_i \text{ and } f_i(T) = W_i, \text{all } i \in I_0,$$
and $f_i(\bar{u})$ is a generic point of Z_i, $f_i(\bar{u})\hat{\ } f_i(v)$ a generic point of W_i.

By basic algebraic geometry, $\mathbf{p}(W_i)$ is a constructible irreducible set and $f_i(\bar{u})$ its generic point, and thus the Zariski closure of $\mathbf{p}(W_i)$ is equal to Z_i. That is there are Zariski closed $R_i \subset Z_i$ over k such that
$$Z_i = \mathbf{p}(W_i) \cup R_i \text{ and } \dim R_i < \dim Z_i. \tag{13}$$

Since
$$\mathbf{p}(\bigcap_{i \in I} f_i^{-1}(W_i)) \subseteq \bigcap_{i \in I_0} \mathbf{p}(f_i^{-1}(W_i)) = \bigcap_{i \in I_0} f_i^{-1}(\mathbf{p}(W_i)),$$
we have
$$\mathbf{p}(T) \subseteq S.$$

Let i_0 be an element of I_0 and, for simplicity of notations, $f := f_{i_0}$, so $f(T) = W$, $f(S) = Z$ and $Z = \mathbf{p}(W) \cup R$ as in (13).

By the basic assumptions, given arbitrary $t \in T$, $s = \mathbf{p}(t)$, for some R-definable open neighbourhood $U \subset \mathbb{U}^m$ of s and open neighborhood $U \times V \subset \mathbb{U}^{m+1}$ of t, with $V \subset \mathbb{U}$, the restriction $f_U : U \to \mathbb{X}^m$ and $f_{U \times V} : U \times V \to \mathbb{X}^{m+1}$ are injective.

Thus we obtain the commuting diagram with injective horizontal arrows defined by $f_{U \times V}$ in the top line and f_U in the bottom line,
$$\begin{array}{ccc} T \cap (U \times V) & \to & W \\ \downarrow \mathbf{p} & & \downarrow \mathbf{p} \\ S \cap U & \to & \mathbf{p}(W) \supseteq Z \setminus R. \end{array} \tag{14}$$

By comparing images of down-arrows we conclude

$$S \cap U \supseteq \mathbf{p}(T \cap (U \times V)) \supseteq f_U^{-1}(Z \setminus R)$$

Note that

$$f_U^{-1}(Z \setminus R) = S \cap U \setminus f^{-1}(R)$$

and the choice of R is independent on the choice of U. Hence $\mathbf{p}(T) \supseteq S \setminus f^{-1}(R)$ and (11) is proved.

To prove the second statement recall another basic fact of algebraic geometry: there is a Zariski closed $R_d \subset \mathbb{X}^m$ such that

$$\mathbf{p}(W) \setminus R_d = \mathbf{p}_d(W) := \{z \in \mathbf{p}(W) : \dim \mathbf{p}^{-1}(z) \cap W \leq d\}.$$

Now repeat the argument with the diagram (14) with $\mathbf{p}_d(W)$ in place of $\mathbf{p}(W)$. This proves (12). □

Recall the notion of an **analytic Zariski structure**, see [9] or [10].

4.9 Corollary. *Under assumptions that* k *is algebraically closed and* $\mathrm{cdim}(\mathsf{R}/\mathrm{k}) \geq \aleph_0$, *the structure* $\mathfrak{U}(\mathsf{R})$ *in the language* $L_{\mathrm{glob}}(\mathrm{k})$ *is an analytic Zariski structure.*

Proof. The statement of Proposition 4.8 asserts that the structure on $\mathbb{U}$ determined by $L_{\mathrm{glob}}(\mathrm{k})$-primitives satisfies the key axioms (WP) and (FC) of the definition of an analytic Zariski structure. The rest of the axioms follow easily from definitions and basic algebraic geometry. □

The next statements and its proofs are similar to one of the main statements of [9] for analytic Zariski structures. More early work of M.Gavrilovich also proves this for complex analytic Zariski structures.

4.10 Proposition. $\mathfrak{U}$ *is* $\aleph_0$*-homogeneous over algebraically closed subfields:*

Suppose $\mathfrak{U}(\mathsf{R}_1), \mathfrak{U}(\mathsf{R}_2) \in \mathfrak{U}$, $\mathsf{R}_0, \mathsf{R}_1, \mathsf{R}_2 \in \mathrm{Mod}_{\mathrm{An}}$, $\mathsf{R}_0 \subseteq \mathsf{R}_1$, $\mathsf{R}_0 \subseteq \mathsf{R}_1$.

Let $\mathrm{k} \subseteq \mathrm{K}_0 = \mathrm{K}(\mathsf{R}_0)$ *be an algebraically closed subfield such that* $\mathrm{cdim}(\mathsf{R}_1/\mathrm{k}) \geq \aleph_0$ *and* $\mathrm{cdim}(\mathsf{R}_2/\mathrm{k}) \geq \aleph_0$.

Then for any $\bar{u}_1 \in \mathbb{U}^m(\mathrm{K}_1)$, $\bar{u}_2 \in \mathbb{U}^m(\mathrm{K}_2)$, *and* $w_1 \in \mathbb{U}(\mathrm{K}_1)$ *such that*

$$\mathrm{loc}(\bar{u}_1/\mathrm{k}) = \mathrm{loc}(\bar{u}_2/\mathrm{k})$$

there is $w_2 \in \mathbb{U}(\mathrm{K}_2)$ *such that*

$$\mathrm{loc}(\bar{u}_1 w_1/\mathrm{k}) = \mathrm{loc}(\bar{u}_2 w_2/\mathrm{k}).$$

Proof. Let $S = \text{loc}(\bar{u}_1/\text{k})$ and $T = \text{loc}(\bar{u}_1 w_1/\text{k})$. Note that $\bar{u}_1$ and $\bar{u}_2$ are non-singular points of S and $\bar{u}_1 w_1$ a non-singular point of T, by 3.6.

Let $d := \dim \mathbf{p}^{-1}(\bar{u}_1) \cap T$, be the dimension of the fibre over $\bar{u}_1$, and the subset $\mathbf{p}_d(T)$ as defined in 4.8. Note that by the dimension theorem of algebraic geometry $\dim \mathbf{p}_d(T) = \dim S$, since $\dim \mathbf{p}_d(W) = \dim S$ (in the notation of 4.8). Note also that

$$\dim T = \dim S + d$$

since respective equality holds for the dimensions of W and Z.

It follows that $\mathbf{p}_d(T)$ contains all generic over k points of S, $\bar{u}_2 \in \mathbf{p}_d(T)$ and thus

$$\dim \mathbf{p}^{-1}(\bar{u}_2) \cap T = d.$$

Thus there exists w_2 such that $\bar{u}_2 w_2 \in \mathbf{p}^{-1}(\bar{u}_2) \cap T$ and $\dim(w_2/\bar{u}_2\text{k}) = d$. Since T is k-irreducible,

$$T = \text{loc}(\bar{u}_2 w_2/\text{k}).$$

□

4.11 Lemma. *Let $S \subseteq \mathbb{U}^{m+n}$ be an $L_{\text{glob}}(\text{k})$-primitive and $\bar{u} \in \mathbb{U}^m$. Let*

$$S_{\bar{u}} = \{\bar{v} \in \mathbb{U}^n : \bar{u}\bar{v} \in S\}.$$

Then $S_{\bar{u}}$ is an $L_{\text{glob}}(\text{k}')$-primitive, for k', extension of k by co-ordinates of $f_i(\bar{u})$, $i \in I$.

Proof. By definition $S = \bigcap_{i \in I} f_i^{-1}(Z_i)$ for $Z_i \subseteq \mathbb{X}_i^{m+n}$.

Let, for $z_i \in \mathbb{X}_i^m(\text{K})$,

$$Z_{i,z_i} = \{x_i \in \mathbb{X}_i^n(\text{K}) : z_i x_i \in Z_i\}.$$

Thus

$$S_{\bar{u}} = \{\bar{v} \in \mathbb{U}^n : \bigwedge_{i \in I} f_i(\bar{u}) f_i(\bar{v}) \in Z_i\} =$$
$$= \bigcap_{i \in I} f_i^{-1}(Z_{i, f_i(\bar{u})}).$$

□

4.12 Corollary. *Assuming* k_0 *is algebraically closed,* $\mathfrak{U}$ *is* $\aleph_0$*-homogenous over* $\varnothing$ *and over small submodels:*

In notations of 4.10, let $V = \varnothing$ *or* $V = \mathbb{U}(\mathrm{K}_0)$ *and assume* $\mathrm{cdim}(\mathsf{R}_i/\mathrm{K}_0) \geq \aleph_0$ *for* $i = 1, 2$.

Then, for any $\bar{u}_1 \in \mathbb{U}^m(\mathrm{K}_1)$, $\bar{u}_2 \in \mathbb{U}^m(\mathrm{K}_2)$, $w_1 \in \mathbb{U}^m(\mathrm{K}_1)$ *such that*

$$\mathrm{tp}(\bar{u}_1/V) = \mathrm{tp}(\bar{u}_2/V)$$

there is $w_2 \in \mathbb{U}^m(\mathrm{K}_2)$ *such that*

$$\mathrm{tp}(\bar{u}_1 w_1/V) = \mathrm{tp}(\bar{u}_2 w_2/V),$$

where tp *is the quantifier-free type of the form (10).*

Proof. For the language without parameters use 4.10 with $\mathrm{k} = \mathrm{k}_0$. Over submodel use the statement of 4.10 with $\mathrm{k} = \mathrm{K}_0$. □

4.13 Lemma. *The structure* $\mathfrak{U}(\mathsf{R}_0)$, *for* R_0 *the prime model of the o-minimal theory* $\mathrm{Th}(\mathbb{R}_{\mathrm{An}})$, *is a prime model of* $\mathfrak{U}$, *that is there is an* L_{glob}*-embedding* $\mathfrak{U}(\mathsf{R}_0) \subseteq \mathfrak{U}(\mathsf{R})$ *for any* $\mathsf{R} \in \mathrm{Mod}_{\mathrm{An}}$.

Proof. An embedding $\mathsf{R}_0 \preccurlyeq \mathsf{R}$ induces an embedding $\mathfrak{U}(\mathsf{R}_0) \subseteq \mathfrak{U}(\mathsf{R})$. □

4.14 Theorem. *Suppose* k_0 *is algebraically closed.*

Let $\mathsf{R}_1, \mathsf{R}_2 \in \mathrm{Mod}_{\mathrm{An}}$

$$\aleph_0 \leq \mathrm{cdim}\, \mathsf{R}_1 = \mathrm{cdim}\, \mathsf{R}_2 \leq \aleph_1.$$

Then

$$\mathfrak{U}(\mathsf{R}_1) \cong \mathfrak{U}(\mathsf{R}_2).$$

In particular, $\mathfrak{U}$ *is categorical in cardinality* $\aleph_1$.

Proof. First consider the case when $\mathrm{cdim}\, \mathsf{R}_1 = \mathrm{cdim}\, \mathsf{R}_2 = \aleph_0$. Then $\mathfrak{U}(\mathsf{R}_1)$ and $\mathfrak{U}(\mathsf{R}_2)$ are countable and so we can construct an isomorphism via a countable back-and-forth process using 4.12, where $\mathrm{K}_0 = \mathrm{K}(\mathsf{R}_0)$, R_0 is the prime model of $\mathrm{Th}(\mathbb{R}_{\mathrm{An}})$.

In case $\mathrm{cdim}\, \mathsf{R}_1 = \mathrm{cdim}\, \mathsf{R}_2 = \aleph_1$ we represent

$$\mathsf{R}_1 = \bigcup_{\alpha<\aleph_1} \mathsf{R}_{1,\alpha} \text{ and } \mathsf{R}_2 = \bigcup_{\alpha<\aleph_1} \mathsf{R}_{2,\alpha}$$

the ascending chains of elementary extensions, $\mathrm{cdim}(\mathsf{R}_{i,\alpha+1}/\mathsf{R}_{i,\alpha}) = \aleph_0$, for $i = 1, 2$, and $\mathsf{R}_{1,0} = \mathsf{R}_{2,0}$ are prime models. Then the required isomorphism is constructed by induction on α :

Assume that $\mathsf{R}_{1,\alpha} \cong \mathsf{R}_{2,\alpha}$, and even that both are equal to a R_α. Now apply 4.12 with $\mathrm{K}_0 = \mathrm{K}(\mathsf{R}_\alpha)$, $\mathrm{K}_1 = \mathrm{K}(\mathsf{R}_{1,\alpha+1})$, and $\mathrm{K}_2 = \mathrm{K}(\mathsf{R}_{2,\alpha+1})$. This again produces an isomorphism $\mathsf{R}_{1,\alpha+1} \cong \mathsf{R}_{2,\alpha+1}$ by the back-and-forth procedure.

For limit indices the extension of isomorphism is obvious. □

5 The one-dimensional case

5.1 Let $P(\mathbb{U})$ stand for the power-set of $\mathbb{U}$. Define a closure operator $\mathrm{cl} : P(\mathbb{U}) \to P(\mathbb{U})$ by the condition

$$u \in \mathrm{cl}(\bar{w}) \text{ iff } \dim \mathrm{loc}(u\bar{w}/\mathrm{k}) = \dim \mathrm{loc}(\bar{w}/\mathrm{k})$$

for $\bar{w} \subset \mathbb{U}$ finite. And

$$\mathrm{cl}(W) = \bigcup \{\mathrm{cl}(\bar{w}) : w \subseteq_{\mathrm{fin}} W\}$$

for W infinite.

5.2 Lemma. *Suppose $W \in P(\mathbb{U})$ and $\mathrm{cl}(W) = W$. Then, for any $i \in I$, the subset $f_i(W) \subset \mathbb{X}_i(\mathrm{K})$ is closed under* acl, *the algebraic closure in the sense of fields.*

There is an algebraically closed subfield $L = L_W \subseteq \mathrm{K}$.

$$f_i(W) = \mathbb{X}_i(L), \text{ for all } i \in I.$$

Proof. Let $\bar{w} \in W^n$ and $f_i(\bar{w}) = \bar{x} \in \mathbb{X}_i^n(\mathrm{K})$. Let $y \in \mathbb{X}_i(\mathrm{K})$ such that $y \in \mathrm{acl}(\bar{x})$, where acl is over the base field k. Thus, for

$$X = \mathrm{loc}(\bar{x}/\mathrm{k}), \; Y = \mathrm{loc}(\bar{x}y/\mathrm{k})$$

we have $\dim X = \dim Y$. Hence, since f_i is a local bi-holomorphisms, for any $v \in f_i^{-1}(y)$

$$\dim \mathrm{loc}(\bar{w}/\mathrm{k}) = \dim \mathrm{loc}(\bar{w}v/\mathrm{k})$$

which implies $v \in \mathrm{cl}(\bar{w}) \subset W$. This proves that $f_i(W)$ is closed under acl and hence $f_i(W) = \mathbb{X}_i(L)$ for some algebraically closed field $L = L_{W,i}$.

We claim that $L_{W,i} = L_{W,j}$ for any $i, j \in I$. Indeed, consider the direct product $\mathbb{U} \times \mathbb{U}$ instead of $\mathbb{U}$ and

$$f_i \times f_j : \mathbb{U} \times \mathbb{U} \twoheadrightarrow X_i \times X_j$$

instead of f_i and f_j, which still are local bi-holomorphisms onto smooth algebraic varieties. Clearly, $\mathrm{cl}(W \times W) = W \times W$ for cl in the product structure and

$$\mathbb{X}_i(L_{W,ij}) \times \mathbb{X}_j(L_{W,ij}) = (f_i \times f_j)(W \times W) = \mathbb{X}_i(L_{W,i}) \times \mathbb{X}_j(L_{W,j})$$

that is $L_{W,ij} = L_{W,i} = L_{W,j} = L$. □

5.3 Recall (see [8]) that one calls $(\mathbb{U}, \mathrm{cl})$ a **quasiminimal pregeometry structure** if the following holds:

QM1. The pregeometry is determined by the language. That is, if $\mathrm{tp}(v\bar{w}) = \mathrm{tp}(v'\bar{w}')$ then $v \in \mathrm{cl}(\bar{w})$ if and only if $v' \in \mathrm{cl}(\bar{w}')$.

QM2. $\mathbb{U}$ is infinite-dimensional with respect to cl.

QM3. (Countable closure property) If $W \subset \mathbb{U}$ is finite then $\mathrm{cl}(W)$ is countable.

QM4. (Uniqueness of the generic type) Suppose that $W, W' \subseteq \mathbb{U}$ are countable subsets, $\mathrm{cl}(W) = W$, $\mathrm{cl}(W') = W'$ and W, W' enumerated so that $\mathrm{tp}(W) = \mathrm{tp}(W')$.

If $v \in \mathbb{U} \setminus W$ and $v' \in \mathbb{U} \setminus W'$ then $\mathrm{tp}(Wv) = \mathrm{tp}(W'v')$ (with respect to the same enumerations for W and W').

QM5. ($\aleph_0$-homogeneity over closed sets and the empty set) Let $W, W' \subseteq \mathbb{U}$ be countable closed subsets or empty, enumerated such that $\mathrm{tp}(W) = \mathrm{tp}(W')$, and let $\bar{w}, \bar{w}'$ be finite tuples from $\mathbb{U}$ such that $\mathrm{tp}(W\bar{w}) = \mathrm{tp}(W'\bar{w}')$, and let $v \in \mathrm{cl}(W\bar{w})$. Then there is $v' \in \mathbb{U}$ that $\mathrm{tp}(\bar{w}vW) = \mathrm{tp}(\bar{w}'v'W')$.

5.4 Proposition. *Assume that* k_0 *is algebraically closed,* $\dim \mathbb{U} = 1$ *and* $\mathrm{cdim}\, \mathsf{R} \geq \aleph_0$. *Then* $(\mathbb{U}(\mathsf{R}), \mathrm{cl})$ *is a quasi-minimal pregeometry.*

Proof. QM1 is by definition.

QM2 is by the assumtion on R.

QM3 follows from the fact that in the language of o-minimal structure $\mathrm{acl}(W)$ is countable and that $\mathrm{cl}(W) \subseteq \mathrm{acl}(W)$, by (8) and (9).

QM4 follows from the fact that $\mathbb{U}$ is one-dimensional irreducible and $v \notin \mathrm{cl}(W)$, $v' \notin \mathrm{cl}(W')$.

QM5. If W and W' are empty then the required follows from 4.10. when $\mathrm{k} = \mathrm{k}_0$. In the non-empty case we may assume by $\aleph_0$-homogeneity over $\varnothing$ that $W = W'$. Now 5.2 allows to replace $\mathrm{tp}(\bar{w}W)$ and $\mathrm{tp}(\bar{w}'W')$ by $\mathrm{loc}(\bar{w}/L_W)$ and $\mathrm{loc}(\bar{w}'/L_W)$ and $\mathrm{tp}(\bar{w}vW)$ and $\mathrm{tp}(\bar{w}'v'W')$ by $\mathrm{loc}(\bar{w}v/L_W)$ and $\mathrm{loc}(\bar{w}'v'/L_W)$ respectively.

The existence of v' follows from 4.10 when $\mathrm{k} = L_W$. □

Now we recall that given a quasiminimal pregeometry structure $(\mathbb{U}, \mathrm{cl})$ one can associated with it an abstract elementary class containing the structure, see [8], 2.2 - 2.3, or more general [9], 2.17 - 2.18. Call this class $\mathfrak{U}_{\mathrm{glob}}$.

By definiton, one starts with a structure $\mathbb{U} = \mathfrak{U}(\mathsf{R})$ for a R of cardinality $\aleph_1$. Define $\mathfrak{U}^-_{\mathrm{glob}}$ to be the class of all cl-closed substructures of $\mathbb{U}$ with embedding $\prec$ of structures defined as *closed* embeding, that is

$\mathbb{U}_1 \prec \mathbb{U}_2$ if and only if $\mathbb{U}_1 \subset \mathbb{U}_2$ and, for finite $W \subset \mathbb{U}_1$,

$$\mathrm{cl}_{\mathbb{U}_1}(W) = \mathrm{cl}_{\mathbb{U}_2}(W).$$

Now define $\mathfrak{U}_{\mathrm{glob}}$ to be the smallest class which contains $\mathfrak{U}^-_{\mathrm{glob}}$ and is closed under unions of $\prec$-chains.

5.5 Lemma.

$$\mathfrak{U} \subseteq \mathfrak{U}_{\mathrm{glob}}.$$

Proof. We need to show that $\mathbb{U}(\mathsf{R}) \in \mathfrak{U}_{\mathrm{glob}}$, for any $\mathsf{R} \in \mathrm{Mod}_{\mathrm{An}}$.

We prove by induction on $\kappa = \mathrm{card}\,\mathsf{R} \geq \aleph_1$ that there is a κ-chain

$$\{\mathbb{U}_\lambda \in \mathfrak{U}_{\mathrm{glob}} : \lambda \in \kappa\} \text{ such that } \bigcup_{\lambda\in\kappa} \mathbb{U}_\lambda = \mathbb{U}(\mathsf{R}).$$

Indeed, R can be represented as

$$\mathsf{R} = \bigcup_{\lambda<\kappa} \mathsf{R}_\lambda$$

for an elementary chain

$$\{\mathsf{R}_\lambda \in: \lambda \in \kappa\},\ \mathrm{card}\,\mathsf{R}_\lambda = \mathrm{card}\,\lambda + \aleph_0,\ \mathsf{R}_\lambda \prec \mathsf{R}_\mu \text{ for } \lambda < \mu.$$

Hence

$$\mathbb{U}_\lambda := \mathfrak{U}(\mathsf{R}_\lambda) \in \mathfrak{U}_{\mathrm{glob}}$$

which proves the inductive step and the lemma. □

5.6 Theorem. *Assuming* $\dim_{\mathrm{K}} \mathbb{U} = 1$, *the class* $\mathfrak{U}_{\mathrm{glob}}$ *is an abstract elementary class extending* $\mathfrak{U}$. $\mathfrak{U}_{\mathrm{glob}}$ *is categorical in uncountable cardinals and can be axiomatised by an* $L_{\omega_1,\omega}(Q)$*-sentence.*

Proof. The first part is by 5.4 - 5.5 above. The second part is the main result, Theorem 2.3, of [8]. □

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