Analytic and pseudo-analytic structures

B. Zilber
University of Oxford

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One of the questions frequently asked nowadays about model theory is whether it is still logic. The reason for asking the question is mainly that more and more of model theoretic research focuses on concrete mathematical fields, uses extensively their tools and attacks their inner problems. Nevertheless the logical roots in the case of model theoretic geometric stability theory are not only clear but also remain very important in all its applications.
This line of research started with the notion of a κ-categorical first order theory, which quite soon mutated into the more algebraic and less logical notion of a κ-categorical structure.

A structure $\mathbf{M}$ in a first order language L is said to be **categorical in cardinality** κ if there is exactly one, up to isomorphism, structure of cardinality κ satisfying the L-theory of $\mathbf{M}$.
In other words, if we add to $\mathrm{Th}(\mathbf{M})$ the (non first-order) statement that the cardinality of the domain of the structure is κ, the description becomes categorical.

The principal breakthrough, in the mid-sixties, from which stability theory started was the answer to J.Los' problem

The Morley Theorem *A countable theory which is categorical in one uncountable cardinality is categorical in all uncountable cardinalities.*

The basic examples of uncountably categorical structures in a countable language are:

(1) Trivial structures (the language allows only equality);

(2) Abelian divisible torsion-free groups; Abelian groups of prime exponent (the language allows $+, =$); Vector spaces over a (countable) division ring

(3) Algebraically closed fields in language $(+, \cdot, =)$.

Also, any structure definable in one of the above is uncountably categorical in the language which witnesses the interpretation.

The structures definable in algebraically closed fields, for example, are effectively objects of algebraic geometry.

As a matter of fact the main logical problem after answering the question of J.Los was *what properties of* $\mathbf{M}$ *make it* κ*-categorical for uncountable* κ?

The answer is now reasonably clear:
The key factor is measurability by a dimension and high homogeneity of the structure.

This gave rise to (Geometric) Stability Theory, the theory studying structures with good dimensional and geometric properties (see [Bu] and [P]). When applied to fields, the stability theoretic approach in many respects is very close to Algebraic Geometry.

The abstract dimension notion for finite $X \subset \mathbf{M}$ mentioned above could be best understood by examples:

(1a) Trivial structures: **size of** X;

(2a) Abelian divisible torsion-free groups; Abelian groups of prime exponent; Vector spaces over a division ring: **linear dimension of** X;

(3a) Algebraically closed fields: **transcendence degree** tr.d.(X).
Dually, one can classically define another type of dimension using the initial one:

$$\dim V = \max\{\text{tr.d.}(\bar{x}) \mid \bar{x} \in V\}$$

for $V \subseteq \mathbf{M}^n$ an algebraic variety. The latter type of dimension notion is called in model theory the **Morley rank**.

The last example can serve also as a good illustration of the significance of homogeneity of the structures. So, in general, the transcendence degree makes good sense in any field, and there is quite a reasonable dimension theory for algebraic varieties over a field. But the dimension theory in arbitrary fields fails if we want to consider it for wider classes of definable subsets, e.g. the images of varieties under algebraic mappings. In algebraically closed fields any definable subset is a boolean combination of varieties, by elimination of quantifiers, which eventually is the consequence of the fact that algebraically closed fields are existentially closed in the class of fields. The latter effectively means high homogeneity, as an existentially closed structure absorbs any amalgam with another member of the class.

One of the achievements of stability theory is the establishing of some hierarchy of types of structures that allows to say which ones are more 'analysable' (see [Sh]).

The next natural question to ask is *whether there are 'very good' stable structures which are not reducible to (1) - (3) above?*

The initial hope of the present author in [Z1], that any uncountably categorical structure comes from the classical context (the trichotomy conjecture), was based on the general belief that logically perfect structures could not be overlooked in the natural progression of mathematics. Allowing some philosophical licence here, this was also a belief in a strong logical predetermination of basic mathematical structures.
As a matter of fact this turned out to be true in many cases. Specifically for *Zariski geometries*, which are defined as the structures with a good dimension theory and nice topological properties, similar to the Zariski topology on algebraic varieties (see [HZ]).
Another situation where this principle works, is the context of o-minimal structures (see [PS]).

Powerful applications of the result on Zariski geometries and of the underlying methodology were found by Hrushovski [H3],[H4]. This not only lead to new and independent solutions to some Diophantine problems, Manin-Mumford and Mordell-Lang (the functional case) conjectures, but also a new geometric vision of these.

Yet the trichotomy conjecture proved to be false in general as Hrushovski found a source of a great variety of counterexamples.
We analyse below the Hrushovski construction, purporting to answer the question of whether the counterexamples it provides dramatically overhaul the trichotomy conjecture or if there is a way to save at least the spirit of it. As the reader will find below the author is inclined to stick to the second alternative.

1 Hrushovski construction of new structures

The main steps:

Suppose we have a, usually elementary, class of structures $\mathcal{H}$ with a good dimension notion $d(X)$ for finite subsets of the structures. We want to introduce a new function or relation on $\mathbf{M} \in \mathcal{H}$ so that the new structure gets a good dimension notion.
The main principle, which Hrushovski found will allow us to do this, is that of the free fusion. That is, the new function should be related to the old structure in as a free way as possible. At the same time we want the structure to be homogeneous. He then found an effective way of writing down the condition: *the number of explicit dependencies in X in the new structure must not be greater than the size (the cardinality) of X.*
The explicit L-dependencies on X can be counted as L-codimension, size$(X) - d(X)$. The explicit dependencies coming with a new relation or function are the ones given by simplest 'equations', basic formulas.
So, for example, if we want a *new unary function f on a field* (implicit in [H2]), the condition should be

$$\text{tr.d.}(X \cup f(X)) - \text{size}(X) \geq 0, \tag{1}$$

since in the set $Y = X \cup f(X)$ the number of explicit field dependencies is $\text{size}(Y) - \text{tr.d.}(Y)$, and the number of explicit dependencies in terms of f is $\text{size}(X)$.
If we want, e.g., to put a *new ternary relation R on a field*, then the condition would be

$$\text{tr.d.}(X) - r(x) \geq 0, \tag{2}$$

where $r(X)$ is the number of triples in X satisfying R.
The very first of Hrushovski's examples (see [H1]) introduces just a *new structure of a ternary relation*, which effectively means putting new relation on the trivial structure. So then we have

$$\text{size}(X) - r(X) \geq 0. \tag{3}$$

If we similarly introduce an automorphism σ on the field (*difference fields,* [CH]), then we have to count

$$\text{tr.d.}(X \cup \sigma(X)) - \text{tr.d.}(X) \geq 0, \tag{4}$$

and the inequality here always holds.
Similarly for *differential fields* with the differentiation operator D (see [Ma]), where we always have

$$\text{tr.d.}(X \cup D(X)) - \text{tr.d.}(X) \geq 0. \tag{5}$$

The left hand side in each of the inequalities (1) - (5), denote it $\delta(X)$, is a counting function, which is called **predimension**, as it satisfies some of the basic properties of the dimension notion.

At this point we have carried out the first step of the Hrushovski construction, that is:

(Dim) we introduced the class $\mathcal{H}_\delta$ of the structures with a new function or relation, and the extra condition

$$\text{(GS)} \qquad \delta(X) \geq 0 \quad \text{for all finite } X.$$

(GS) here stands for 'Generalised Schanuel', the reason for which will be given below.
The condition (GS) allows us to introduce another counting function with respect to a given structure $\mathbf{M} \in \mathcal{H}_\delta$

$$\partial_M(X) = \min\{\delta(Y) : X \subseteq Y \subseteq_{fin} M\}.$$

We also need to adjust the notion of embedding in the class for further purposes. This is the **strong embedding**, $\mathbf{M} \leq \mathbf{L}$, meaning that $\partial_M(X) = \partial_L(X)$ for every $X \subseteq_{fin} \mathbf{M}$.

The next step is

(EC) Using the inductiveness of the class construct an existentially closed structure in $(\mathcal{H}_\delta, \leq)$.
If the class has the amalgamation property, then the existentially closed structures are sufficiently homogeneous. Also $\partial_M(X)$ *for existentially closed* $\mathbf{M}$ *becomes a dimension notion.*
So, if also the class $\mathcal{EC}$ of existentially closed structures is axiomatisable, one can rather easily check that the existentially closed structures are ω-stable. This is the case for examples (1) - (3) and (5) above.
In more general situations the e.c. structures may be unstable, but still with a reasonably good model-theoretic properties.
Notice that though condition (GS) is trivial in examples (4) - (5), the derived dimension notion ∂ is non-trivial. In both examples $\partial(x) > 0$ iff the corresponding rank of x is infinite (which is the SU-rank in algebraically closed difference fields and the Morley rank, in differentially closed fields).

Notice that the dimension notion ∂ for finite subsets, similarly to the example (3a), gives rise to a dual dimension notion for definable subsets $S \subseteq \mathbf{M}^n$ over a finite set of parameters C :

$$\dim(S) = \max\{\partial(\{x_1, \dots, x_n\}/C) : \langle x_1, \dots, x_n\rangle \in S\}.$$

(Mu) This stage originally had been considered prior to (EC), but as one easily sees, it can be equivalently introduced after.
We want to find now a finite Morley rank structure as a substructure (may be non-elementary) of a structure $\mathbf{M} \in \mathcal{EC}$. In fact an existentially closed $\mathbf{M}$ would be of finite Morley rank, if '$\dim(S) = 0$' is equivalent to 'S is finite'. But in general $\dim(S)$ may be zero for some infinite definable subsets S, e.g. the set $S = \{x \in \mathbf{M} : f(x) = 0\}$ is one such in example (1) : 'some equations have too many solutions'.
To eliminate the redundant solutions Hrushovski introduces a counting function μ for the maximal allowed size of potentially Morley rank 0 subsets. Then $\mathcal{H}_{\delta,\mu}$ is the subclass of structures of $\mathcal{H}_\delta$ satisfying the bounds given by μ. Equivalently, since existentially closed structures are universal for structures of $\mathcal{H}_\delta$, so $\mathcal{H}_{\delta,\mu}$ is the class of substructures of existentially closed structures $\mathbf{M}$, satisfying the bound by μ.
Inside this class we can just as well carry out the construction of existentially closed structures $\mathbf{M}_\mu$. Again, if the subclass has the amalgamation property and is first order definable, then an existentially closed substructure M_μ of this subclass is of finite Morley rank, in fact strongly minimal in cases (1) - (3). It is also important for the further discussion that $\mathbf{M}_\mu \subseteq \mathbf{M}$.

The infinite dimensional structures emerging after step (EC) in natural classes we call *natural Hrushovski structures*. Some but not all of them lead after step (Mu) to finite Morley rank structures.
It follows immediately from the construction, that the class of natural Hrushovski structures is singled out in $\mathcal{H}$ by three properties: the generalised Schanuel property (GS), the property of existentially closedness (EC) and the property (ID), stating the existence of n-dimensional subsets for all n.

It takes a bit more model theoretic analysis, as is done in [H1], to prove that in examples (1)-(3), and in many others, (GS), (EC) and (ID) form a complete set of axioms.

Since Hrushovski found the counterexamples, the main question that has arisen is whether the pathological structures demonstrate the failure of the general principle or if there is a classical context which explains the counterexamples.
We now want to try and find grounds for the latter.

We start with one more example of Hrushovski construction.

2 Pseudo-exponentiation

Suppose we want to put a new function ex on a field K of characteristic zero, so that ex is a homomorphism between the additive and the multiplicative groups of the field:

$$\mathrm{ex}(x_1 + x_2) = \mathrm{ex}(x_1) \cdot \mathrm{ex}(x_2).$$

Then the corresponding predimension on new structures $\mathbf{K}_{\mathrm{ex}} = (K, +, \cdot, \mathrm{ex})$ must be

$$\delta(X) = \mathrm{tr.d.}(X \cup \mathrm{ex}(X)) - \mathrm{lin.d.}(X) \geq 0, \qquad \text{(GS)}$$

where lin.d.(X) is the linear dimension of X over $\mathbb{Q}$.
Equivalently (GS) can be stated

$$\text{assuming that } X \text{ is linearly independent over } \mathbb{Q}, \quad \mathrm{tr.d.}(X \cup \mathrm{ex}(X)) \geq \mathrm{size}(X),$$

which in case $\mathbf{K}$ is the field of complex numbers and ex = exp is known as the *Schanuel conjecture* (see [La]).

Start now with the class $\mathcal{H}(\mathrm{ex/st})$ consisting of structures $\mathbf{K}_{\mathrm{ex}}$ satisfying (GS) and with the additional property that the kernel $\ker = \{x \in K : \mathrm{ex}(x) = 1\}$ is a cyclic subgroup of the additive group of the field K, which we call a **standard kernel** (see [Z2]). This class is non-empty and can be described as a subclass of an elementary class defined by omitting countably many types.
The subclass $\mathcal{EC}(\mathrm{ex/st})$ of existentially closed substructures of $\mathcal{H}(\mathrm{ex/st})$ is first order axiomatisable inside $\mathcal{H}(\mathrm{ex/st})$ (see [Z3] for the proof about a similar structure).
By the obvious analogy with the structure $\mathbb{C}_{\exp} = (\mathbb{C}, +, \cdot, \exp)$ on the complex numbers we conjecture that $\mathbb{C}_{\exp}$ is one of the structures in $\mathcal{EC}(\mathrm{ex/st})$. And we want to find the condition that might single out the isomorphism type of $\mathbb{C}_{\exp}$ among the other structures in the class. We do this by introducing an extra step to the Hrushovski construction, when it is possible, which comes after (EC). This is applicable in a wide variety of classes:

(P) Consider a ∂-independent set C of cardinality κ (embeddable in an existentially closed structure) and let $E(C)$ be a structure **prime over C in class $\mathcal{EC}$.**

When a prime structure over ∂-independent C, card $C = \kappa$, exists and is unique over C, we call $E(C)$ **the κ-canonical structure.**

The nice thing about having κ-canonical structures is that we get a link with the logical background again: the notion is a good analogue of κ-categoricity, or rather the strong minimality concepts (see also [Le] for recent developments in this direction).

We prove
If the κ-canonical structure $E(C)$ exists and the language is countable, we have card $E(C) = \kappa$ *and $E(C)$ is $\aleph_0$-quasi-minimal, i.e. any definable subset of $E(C)$ is either countable, or the complement of a countable. Moreover, when $\kappa > \aleph_0$, for a definable $S \subseteq E(C)^n$ S is countable iff* $\dim(S) = 0$.

It follows from a well-known theorem of Shelah (see [Sh] and [Bu]), that canonical structures exist if the class $\mathcal{EC}$ is first order axiomatisable, complete and ω-stable, e.g. in examples (1) - (3) and (5). There is no prime structure in the class of algebraically closed difference fields (example (4)) due to the fact that the theory is not complete. In spite of the fact that $\mathcal{EC}(\mathrm{ex/st})$ is not axiomatisable and even interprets the ring of integers we managed to prove in [Z2] that
There exists a weaker (non unique) version of κ-canonical structure for any infinite cardinal κ in $\mathcal{H}(\mathrm{ex/st})$ which is $\aleph_0$-quasi-minimal. [1]
This finally brings us to

Conjecture $\mathbb{C}_{\exp}$ *is the canonical structure of cardinality $2^{\aleph_0}$ in $\mathcal{H}(\mathrm{ex/st})$ if we put* ex *to be* exp.

The conjecture (and even the one stating that $\mathbb{C}_{\exp} \in \mathcal{EC}(\mathrm{ex/st})$) is obviously stronger than the Schanuel conjecture. Another consequence of the conjecture is the fact that $\mathbb{C}_{\exp}$ is existentially closed. We show in [Z2] that this is equivalent to the statement

EC(exp): *Any non obviously contradictory system of equations over $\mathbb{C}$ in terms of $+$, $\cdot$ and* exp *has a solution in $\mathbb{C}$.*

The definition of **non obviously contradictory system** is quite similar to ones (implicitly) formulated for other classes in the form of axiom schemes, e.g. ACFA(iii) in [CH] for algebraically closed difference fields, and see also examples in section 3. We are not going to give the definition here, but a good example of such a system is an equation of the form $t(x) = 0$, where $t(x)$ is a term in $+, \cdot, \exp$ over $\mathbb{C}$ which is not

[1] *Added in proof.* We now proved that there is an $L_{\omega_1,\omega}(Q)$-sentence axiomatizing a subclass of $\mathcal{H}(\mathrm{ex/st})$ with unique model in every infinite cardinal, and the models are $\aleph_0$-quasi-minimal. We don't know whether the models are canonical in the above sense.

of the form $\exp(s(x))$ for some other term $s(x)$. Such an equation by the conjecture should have a solution in $\mathbb{C}$. I have learned, while writing this paper, that such was exactly a conjecture by S.Schanuel which was proved by W.Henson and L.Rubel using Nevanlinna theory (see [HR]). A.Wilkie gave a rather simple proof of the solvability of an equation of the form $\sum_{i\leq N} q_i(x)e^{p_i(x)} = 0$, for q_i and p_i polynomials in one variable, $N > 1$, based on the rate of growth argument. The general case is open, but some research suggesting the truthfulness of the conjecture can be traced in the literature. See also the discussion below.
Notice also that one can easily replace the exponentiation by other classical functions and observe similar effects, including corresponding versions of the Schanuel conjecture.
Based on the analysis of pseudo-exponentiation one would like to conclude hypothetically that

1. Basic Hrushovski structures have analytic prototypes.

2. The statement of the Schanuel conjecture along with its analogs is an intrinsic property of classical analytic functions, probably responsible for a good dimension theory of the corresponding structures on the complex numbers.

3. Another basic property of classical analytic structures is the EC-property: *Any non-obviously contradictory system of equations over the structure has a solution.*

Of course, we don't possess technical means to check the truthfulness of the conjectures. But the general picture drawn above in view of the conjectures can be tested in simpliest examples.

3 Analytic interpretations

New ternary relation

Let $g : \mathbb{C}^3 \to \mathbb{C}$ be an entire function with the properties:

(GS[R]) If a system of $n+1$ equations of the form $g(v_{i_k}, v_{j_k}, v_{l_k}) = 0$ $(k = 1, \ldots, n+1)$, with $v_{i_k}, v_{j_k}, v_{l_k} \in \{v_1, \ldots, v_n\}$, has a solution $\langle a_1, \ldots, a_n\rangle$ in $\mathbb{C}$, then at least two of the triples $\langle a_{i_k}, a_{j_k}, a_{l_k}\rangle$ coincide.

(EC[R]) Let $\{\langle x_j, y_j, z_j\rangle : j \leq n + m\}$ be distinct triples, where each of x_j, y_j, z_j is either a complex constant or one of variables $v_1, \ldots, v_n$, but not all three of them constants. Then the system

$$\{g(x_i, y_i, z_i) = 0 : i \leq n\} \cup \{g(x_i, y_i, z_i) \neq 0 : n < i \leq n + m\}$$

of n equations and m inequalities has a solution in $\mathbb{C}$, provided no k of the n equations have less than k explicit variables.

It is easy to see that if we interpret the ternary relation $R(v_1, v_2, v_3)$ as $g(v_1, v_2, v_3) = 0$, by GS[R] we get condition (3) for δ, and by EC[R] the existential closedness is satisfied. Then ID[R] follows from the fact that ∂-closure of a finite set is countable, since n independent analytic equations in the complex n-space have only countably many non-singular solutions.

Problem 1. (i) Construct an entire function g such that GS[R] and EC[R] hold.
(ii) Prove that $(\mathbb{C}, R)$ is canonical in this case, i.e. the structure is prime over its basis (of cardinality $2^{\aleph_0}$).

Remark One can construct g satisfying (GS[R]), using an argument of A.Wilkie, as

$$g(v_1, v_2, v_3) = \sum_{i_1, i_2, i_3 \in \mathbb{N}} a_{i_1, i_2, i_3} v_1^{i_1} v_2^{i_2} v_3^{i_3}$$

with complex coefficients a_{i_1,i_2,i_3} algebraically independent and very rapidly decreasing.

New functions on a field.

This is another class mentioned above. One can easily write down in a first order way the following two schemes of axioms:

(GS[f]) Let $V \subseteq K^{2n}$ be a variety over $\mathbb{Q}$ in variables $x_1, \ldots, x_n, y_1, \ldots, y_n$. If $\dim V < n$, then there is no point $\langle x_1, \ldots, x_n, y_1, \ldots, y_n \rangle$ in V with $y_i = \mathrm{f}(x_i)$ and $x_i \neq x_j$ for all distinct $i, j \in \{1, \ldots, n\}$.
(EC[f]) Let $V \subseteq K^{2n}$ be an irreducible variety over K in variables $x_1, \ldots, x_n, y_1, \ldots, y_n$ such that
(i) V is not contained in a hyperplane given by an equation of the form $x_i = x_j$ for $i < j \leq n$, or $x_i = c$, for $c \in K$;
(ii) for any $0 < i_1 < \ldots < i_k \leq n$ the dimension of $V_{i_1,\ldots,i_k}$, the projection of V onto $(x_{i_1}, \ldots, x_{i_k}, y_{i_1}, \ldots, y_{i_k})$-space, is not less than k.
Then there is a point $\langle x_1, \ldots, x_n, y_1, \ldots, y_n \rangle$ in V with $y_i = \mathrm{f}(x_i)$, for all $i \in \{1, \ldots, n\}$.

It has been proved in [Z6] that

GS[f] *and* EC[f] *along with* ID[f], *the axiom of infinite ∂-dimensionality, determine a complete ω-stable theory (of Morley rank ω). The theory has canonical model $\mathbf{K}_\mathrm{f}$ in every cardinality κ.*

Problem 2 (i) Construct an entire holomorphic function $\mathrm{f} : \mathbb{C} \to \mathbb{C}$ satisfying GS[f] and EC[f] for $K = \mathbb{C}$.

(ii) Prove that $\mathbb{C}_{\mathrm{f}}$ is isomorphic to canonical $\mathbf{K}_{\mathrm{f}}$ of cardinality $2^{\aleph_0}$.

Notice that we get basically the same theory, with minor changes, if we change GS[f] and, correspondingly, the counting function δ in (1) to:

(GS'[f]) $\mathrm{tr.d.}(X \cup \mathrm{f}(X)) - \mathrm{size}(X) \geq 0$, provided X does not contain certain elements, say, 0.

A.Wilkie in [W] proves that an entire analytic function given as

$$\mathrm{f}(x) = \sum_{i \geq 0} \frac{x^i}{a_i}$$

with a_i very rapidly increasing integers, satisfies (GS'[f]) and (EC[f]), if in the latter V is defined over $\mathbb{Q}$.

New differentiable functions on a field.

To get better understanding of the 'analytic' nature of previous examples we consider the class $\mathcal{H}(F)$ of fields K of characteristic zero with a collection $F = \{\mathrm{f}^{(i)} : i \in \mathbb{Z}\}$ of unary functions on K.
We first introduce, for a finite $I \subseteq \mathbb{Z}$

$$\delta_I(X) = \mathrm{tr.d.}(X \cup \bigcup_{i \in I} f^{(i)}(X)) - \mathrm{size}(X) \cdot \mathrm{size}(I),$$

the I-predimension of $X \subseteq_{fin} K$.

Then the predimension of $X \subseteq_{fin} K$ is

$$\delta(X) = \min\{\delta_I(X) : I \subseteq_{fin} \mathbb{Z}\}.$$

We then, as usual, introduce the subclass satisfying first order definable condition

$$(\mathrm{GS}[F]) \qquad\qquad \delta(X) \geq 0,$$

and after going through the construction stage (EC) find out that the resulting structures satisfy first order axiom scheme EC[F], similar to EC[f].

If we add the corresponding ID[F], *the first order theory defined by*

$$\mathrm{GS}[F] \cup \mathrm{EC}[F] \cup \mathrm{ID}[F]$$

is ω-stable of rank ω, and the reduct of the theory to $(+, \cdot, \mathrm{f})$, where $\mathrm{f} = \mathrm{f}^{(0)}$, is just the theory in the previous example.

We then consider for each $i \in \mathbb{Z}$ a definable function

$$g^{(i)}(x_1, x_2) = \begin{cases} \frac{\mathrm{f}^{(i)}(x_1) - \mathrm{f}^{(i)}(x_2)}{x_1 - x_2}, & \text{if } x_1 \neq x_2 \\ \\ \mathrm{f}^{(i+1)}(x_1), & \text{if } x_1 = x_2 \end{cases}$$

We want to introduce on $\mathbf{K}_F$, a model of the theory, a Zariski type topology τ. Consider a formal 'completion' $\bar{K} = K \cup \{\infty\}$, which can be viewed as the projective line over the field K. Define **basic τ-closed subsets of $\bar{K}^n$** to be
(i) all Zariski-closed subsets of $\bar{K}^n$ for every $n \in \mathbb{N}$,
(ii) for $n = 2$ the 'closures' of graphs of $\mathrm{f}^{(i)}$:

$$\{\langle x, y\rangle \in \bar{K}^2 : (x \in K \wedge y = \mathrm{f}^{(i)}(x)) \vee x = \infty\}$$

and
(iii) for $n = 3$ the 'closures' of graphs of $g^{(i)}$:

$$\{\langle x_1, x_2, y\rangle \in \bar{K}^3 : (x_1, x_2 \in K \wedge y = g^{(i)}(x_1, x_2)) \vee x_1 = \infty \ \vee x_2 = \infty\}.$$

Notice that (ii) and (iii) would actually be the closures of the graphs in the complex topology, if $K = \mathbb{C}$ and $f^{(i)}$ were holomorphic functions.
Define now the family of **τ-closed sets** to be the minimal family of subsets of $\bar{K}^n$, all n, containing the τ-closed subsets and closed under intersections, finite unions and projections $\bar{K}^{n+1} \to \bar{K}^n$. We prove in [Z6] that

The topology τ is compact in the sense that any family with the finite intersection property of τ-closed sets has a non-empty intersection, and the projection of a closed set is closed (but not Hausdorff).
Notice that there is no DCC for closed sets.

We can then view K as a locally compact space. By definition, $f^{(i)}$ and $g^{(i)}$ are continuous functions on K, in the sense of the topology, and $f^{(i+1)}$ satisfies the definition of the derivative of $f^{(i)}$. Moreover, one can then carry out complex style analysis and 'non-standard analysis' on K, as shown in [Z4].
In particular, using the notion of *infinitesimal elements* in $K^* \succ K$, in the case $\mathrm{char}K = 0$, one gets the Taylor formula for any definable function h: *given $x \in K$, $n \in \mathbb{N}$ and an infinitesimal α, there is an infinitesimal β such that*

$$h(x + \alpha) = \sum_{0 \leq k \leq n} \frac{h^{(k)}(x)}{k!} \cdot \alpha^k + \alpha^n \beta,$$

where $h^{(k)}$ are the derivatives of h. In case of positive characteristic p it holds only for $n < p$.

Remark It is worth mentioning that in the example of *new function on a field* one could consider f to be a one-to-one function with the same predimension formula

(1), which is in fact the original Hrushovski example in [H2] in the case of an equi-characteristic pair of fields. In this case *we can not* expand the structure to include derivatives under a reasonable topology as above, and this agrees very well with the fact that there is no transcendental analytic one-to-one function. But any one-to-one Hrushovski structure, by the way of construction, is a substructure of a natural Hrushovski structure above.

4 Mixed characteristics structures

There are examples of Hrushovski structures which have no analytic proptotype. Such an example is *the fusion of two fields*:
Let $\mathcal{H}$ be the class of two-sorted structures (L, R) with both sorts fields, $\mathrm{char}L = p$, $\mathrm{char}R = q$, in the language of fields for both sorts and an extra binary relation between the two sorts, interpreted as a bijective function $\mathrm{f} : L \to R$.
One then introduces the predimension function for $X \subseteq_{fin} L$

$$\delta(X) = \mathrm{tr.d.}(X) + \mathrm{tr.d.}(\mathrm{f}(X)) - \mathrm{size}(X) \geq 0 \tag{6}$$

After going through steps (Dim) and (EC) one gets an ω-stable theory of rank ω which is called the **ω-stable fusion of two fields.** Notice that the pull-back of the field structure of R to L gives a second field strcuture on L, which was the original Hrushovski example. More importantly, Hrushovski showed that after carrying out step (Mu) one gets a strongly minimal fusion of two fields.

Let $\mathcal{H}$ again be the class of two-sorted structures with both sorts fields, $\mathrm{char}L = 0$, $\mathrm{char}R = p$, with the condition, that L is a valued field with the residue field R and the valuation group $\mathbb{Z}$. So there are also definable mappings $v : L \to \mathbb{Z}$ and $\rho : L_0 \to R$, where v is the valuation, $L_0 = \{x \in L : v(x) \geq 0\}$, the valuation ring, and ρ the residue ring homomorphism.
Following the freeness principle one comes to the following predimension function, analogous to (4) and (5), for the class:
Let $v(p) = 1$, and for $X \subseteq_{fin} L \quad [X] = \{p^z x : x \in X, z \in \mathbb{Z} \text{ and } v(x) = -z\}$. Then

$$\delta(X) = \mathrm{tr.d.}(X) - \mathrm{tr.d.}(\rho[X]).$$

Like in (4) and (5), $\delta(X) \geq 0$ holds automatically in this class, so $\mathcal{H}_\delta = \mathcal{H}$ and every embedding is strong.
Then, completing step (EC), one gets the subclass of existentially closed structures, which are elementary equivalent to *maximal unramified extensions of the p-adic field* $\mathbb{Q}_p$. A more complex structure, with a definable automorphism σ, is studied in [BM] and is identified there as *the field of Witt vectors with an automorphism.*

Another class of mixed charcteristics structures, *algebraically closed valued fields*, with the stability flavor is studied in [HHM], and there is a strong evidence that this example is of the pattern under consideration.

We present here a series of examples of mixed characteristics of a different kind.

Let A be a 1-dimensional torus or an elliptic curve defined over a field F_0 of characteristic p, by which we mean just the group scheme, so for every field F containing F_0 there is an algebraic group $\mathcal{A}(F)$ (written multiplicatively) of F-points of $\mathcal{A}$. Let $\mathcal{H}(\mathcal{A})$ be a class of two sorted structures (D, A), where $A = \mathcal{A}(F)$ is as above, and with the language of Zariski closed relations on A, D is a field in characteristic zero considered with family of relations and operations Σ, all definable by polynomial equations (with parameters) in the field and, at least, containing the additive operation, so D is a group with an extra structure.
Notice, that both sorts are then strongly minimal structures and so have corresponding dimension notions $\dim_D$ and $\dim_A$.
Let also the language of the class to contain a function symbol e, which is interpreted as a surjective homomorphism

$$e : D \to A.$$

We can do the Hrushovski construction for this class putting for $X \subseteq_{fin} D$

$$\delta(X) = \dim_D(X) + \dim_A(e(X)) - \text{lin.d.}(X),$$

where lin.d.(X) is the dimension in the additive group, i.e. the linear dimension over $\mathbb{Q}$.
It is easy to see that the resulting structure will have D and A existentially closed, so F is an algebraically closed field. Also, the kernel $\ker = \{x \in D : e(x) = 1\}$ is an additive subgroup of D, and by algebraic reasons ker is elementarily equivalent to:

$\mathbb{Z}[1/p]$, in case A is the multiplicative group of the field;

or to

$\mathbb{Z}[1/p]^{2-i} \times \mathbb{Z}^i$, in case A is an elliptic curve with Hasse invariant $i \in \{0, 1\}$.

Here $\mathbb{Z}[1/p]$ is the additive subgroup of rationals with denominators p^n, $n \in \mathbb{N}$, if $p > 0$, and is just the group of integers, if $p = 0$.

(See also about elliptic curves in [Ha] and the characterisation of elementary equivalence for abelian groups in [EF]).

It is well known that in the given language $A(F)$ is biinterpretable with the algebraically closed field F, perhaps expanded by finitely many constants. Thus for $Y \subseteq_{fin} A$ we have $\dim_A(Y) = \text{tr.d.}(Y/C)$, with the transcendence degree on the right calculated by means of the biinterpretation, and C is the set of constants.
Because of trichotomy results for definability in algebraically closed fields (see [R]), following also from later work [HZ], there are essentially two possibilities to consider for D :

(i) D is an algebraically closed field;

(ii) D is a vector space over a field K of characteristic zero.

Thus in the case (i) the inequality for δ takes form

$$\text{tr.d.}(X) + \text{tr.d.}(e(X)) - \text{lin.d.}(X) + d \geq 0, \tag{7}$$

where d is a non-negative integer depending on the constants needed for the interpretation.
In case (ii) we have

$$\text{lin.d.}_K(X) + \text{tr.d.}(e(X)) - \text{lin.d.}(X) + d \geq 0, \tag{8}$$

where $\text{lin.d.}_K(X)$ is the dimension in the sense of the K-vector space.

The existentially closed structures in the classes bear rather close similarity to *universal coverings of corresponding algebraic varieties* A, and the corresponding kernel ker plays the role of *the fundamental group of the variety*, denoted usually $\pi_1(A)$. Both model theoretically and algebraic geometrically the following two types of the kernel are especially interesting:

the case of minimal group satisfying the condition above, i.e. precisely the group

$$\mathbb{Z}[1/p] \quad \text{or} \quad \mathbb{Z}[1/p]^{2-i} \times \mathbb{Z}^i,$$

depending on A, which we call **standard kernel**;

the case of minimal *algebraically compact group* (see [EF]) satisfying the condition above, i.e. precisely the group

$$\prod_{l \neq p \text{ prime}} \mathbb{Z}_l \quad \text{or} \quad \prod_{l \neq p \text{ prime}} \mathbb{Z}_l^{2-i} \times \prod_{l \text{ prime}} \mathbb{Z}_l^i,$$

depending on A, which we call **compact kernel**. (Here $\mathbb{Z}_l$ stands for the additive group of l-adic numbers).
The standard kernel corresponds to the, so-called, *topological* $\pi_1(A)$, and the compact kernel corresponds to the, so-called, *algebraic* $\pi_1(A)$ [M].
So even in the mixed characteristic case the construction results in something which, even if not analytic, has a definite analytic flavour. But anyway, we hardly know what is the right generalisation of analyticity in many cases and there is hardly a satisfactory theory in the case of non-Archimedean valued fields.

The case corresponding to formula (7) with $p = 0$ is quite similar to the case of the field with pseudoexponentiation.
The case corresponding to formula (8) is studied in [Z3] mainly under assumption that the characteristic of F is zero. Notice that the corresponding structure is quite rich in expressive power, in particular, in sort A we can 'raise to powers $k \in K$' by putting for $x \in A$

$$x^k = e(k \ln(x)),$$

where $\ln(x)$ is an e-pull-back of x in D, so this is a 'multivalued operation'. We call $\mathcal{EC}$-structures corresponding to this case *the group A with 'raising to K-powers'*. It is also interesting that, given $k_1, \ldots, k_n \in K$,

$$H = k_1\text{ker} + \ldots + k_n\text{ker} \quad \text{and} \quad e(H) = e(k_1\text{ker}) \cdot \ldots \cdot e(k_n\text{ker}) \tag{9}$$

are definable subgroups of D and A, correspondingly, and when the kernel is standard these are finite group-rank subgroups. If $n = 1$ and $k_1 = N^{-1}$, where $N \in \mathcal{Z}$ is a non-standard integer, the group

$$e(\frac{1}{N}\mathrm{ker})$$

is a torsion subgroup of A.
We can prove the existence of the canonical structure in all the classes, including pseudo-exponentiation, with fixed compact kernel (see [Z2]). But it remains open

Problem 3 Prove the existence of κ-canonical structures for classes above with standard kernel.

Notice that when $K = \mathbb{Q}$, formula (8) with $d = 0$ holds for any structure in the class, and the class $\mathcal{EC}$ (with all possible kernels) is axiomatisable, complete and superstable. We call the structures in the class **group covers of $\mathcal{A}$**. Even for this class Problem 3 is non trivial and has been proved for the case $\mathcal{A}$ is the multiplicative group of a field of characteristic zero very recently in [Z7] using some rather subtle field arithmetic results and some model theoretic techniques of Shelah's.
We believe that the answer for group covers with standard kernel is crucial for understanding the general case.

Further model theoretic properties and a Diophantine conjecture.

Any deeper model theory of most of the above mentioned classes depends on the following conjecture about intersections in semi-Abelian varieties (charcteristic zero case):

Given a semi-Abelian variety B over $\mathbb{Q}$ and an algebraic subvariety $W \subseteq B$ over $\mathbb{Q}$ there is a finite collection $\tau(W, B)$ of proper semi-Abelian subvarieties of B such that: for any algebraic subgroup $S \leq B$ and any irreducible component U of $S \cap W$ either $\dim U = \dim W + \dim S - \dim B$, *or* $U \subseteq T$ *for some* $T \in \tau(W, B)$.

The conjecture, as a matter of fact, is of Diophantine type:
The conjecture on intersections in semi-Abelian varieties implies the Mordell-Lang (and Manin-Mumford) conjecture for number fields.
This can be proved *ab initio* as in [Z5], but a nicer, model theoretical way is to see directly as in [Z3] that

under the assumption that the conjecture is true, the structure A with raising to K-powers is superstable, and the definable finitely generated or torsion subgroups of A are locally modular.

In fact, the proof of this theorem goes via elimination of quantifiers to the level of existential formulas (*near model compactness,* as typical for Hrushovski structures), which yields local modularity of the kernel. Hence one gets local modularity of subgroups in (9), which by the definition of local modularity implies the Mordell-Lang statement. It remains to notice that any finitely generated or torsion subgroup of A is embeddable in a group $e(H)$ of (9) for a right choice of K and $k_1, \ldots, k_n$.

The conjecture effects the fields with pseudo-exponentiation:

Assume the conjecture on intersection in semi-Abelian varieties, the case $B = (\mathbb{C}^*)^n$*. Then on any canonical model* $\mathbf{K}_{\mathrm{ex}}$ *in* $\mathcal{EC}(\mathrm{ex/st})$ *there is a (non-Hausdorff) locally compact topology under which* ex *is continuous and, moreover, in infinitesimal neighborhoods of any point* $a \in \mathbf{K}$ ex *can be represented by the Taylor expansion*

$$\mathrm{ex}(a+x) = \mathrm{ex}(a) \cdot \sum_n \frac{x^n}{n!}.$$

Also under the conjecture the structure of genuine exponentiation (or rather raising to powers) on the complex numbers becomes very clear.

(See [Z5]) *Assume the Schanuel conjecture and the conjecture on intersections in semi-Abelian varieties (the case* $B = (\mathbb{C}^*)^n$*). Then the structure of complex numbers with the (multivalued) operations* $y = \exp(r \ln x) = x^r$ *of raising to real powers satisfies* [EC]*, i.e. any non-obviously contradictory system of equations of the form*

$$\sum_{r_1, \ldots r_n} a_{r_1,\ldots r_n} x_1^{r_1} \cdot \ldots \cdot x_n^{r_n} = 0,$$

with $a_{r_1,\ldots r_n} \in \mathbb{C}$ *and* $r_1, \ldots, r_n \in \mathbb{R}$, *has a solution in* $\mathbb{C}$.

The proof is based on a theory developed by D.Bernstein, A.Kushnirenko, B.Kazarnovski and A.Khovanski (see [Kh]).

It follows then from the results of preceeding sections that

(See [Z3]) *Assume the Schanuel conjecture and the conjecture on intersections in semi-Abelian varieties (the case* $B = (\mathbb{C}^*)^n$*). Then the structure of complex numbers with the (multivalued) operations of raising to real powers is superstable.*

It is also interesting to notice that the Diophantine conjecture is equivalent to Hrushovski-Schanuel type inequality (8) with $d = 0$ and $K = \mathcal{Q}$, a non-standard model of the field of rationals. In this form we can suggest the corresponding conjecture for case $p > 0$:

$$\mathrm{lin.d.}_{\mathcal{Q}}(X) + \mathrm{tr.d.}(e(X)) - \mathrm{lin.d.}_{Q(\mathcal{P})}(X) \geq 0, \tag{10}$$

where

$$\mathcal{P} = \{p^N : N \in \mathcal{Z}\}$$

is the subgroup of the multiplicative group of $\mathcal{Q}$ inducing automorphisms on A, and $Q(\mathcal{P})$ is the subfield of $\mathcal{Q}$ generated by $\mathcal{P}$. Thus the corresponding class $\mathcal{EC}$ generalises the class of algebraically closed difference fields, with commuting generic automorphisms. So, most probably the theory of the class is *simple*.

One can easily rewrite the conjecture in (10) in a more standard form as follows

Given a semi-Abelian variety A over an algebraically closed field F of characteristic p and an algebraic subvariety $W \subseteq A^n$ there are finite collections $\lambda(W, A^n)$ of constants and $\tau(W, A^n)$ of n-tuples $\langle f_1(v_1, \ldots, v_m), \ldots, f_n(v_1, \ldots, v_m)\rangle$ of polynomials with integer coefficients with the property:
for any algebraic subgroup $S \leq A^n$ and any irreducible component U of $S \cap W$ there are a constant $c \in \lambda(W, A^n)$, an n-tuple $\langle f_1, \ldots, f_n\rangle \in \tau(W, A^n)$ and non-negative integers $t_1, \ldots, t_m$ such that

$$S \subseteq \{\langle x_1, \ldots, x_n\rangle \in A^n : x_1^{f_1(p^{t_1}, \ldots, p^{t_m})} \cdot \ldots \cdot x_n^{f_n(p^{t_1}, \ldots, p^{t_m})} = c\}.$$

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