Model theory of special subvarieties and Schanuel-type conjectures

November 15, 2013

Abstract

1 Introduction

1.1 The first part of the paper (section 2) is essentially a survey of developments around the program outlined in the talk to the European Logic Colloquium 2000 and publication [22]. It then continues with new research which aims, on the one hand side, to extend the model-theoretic picture of [22] and of section 2 to the very broad mathematical context of what we call here *special coverings of algebraic varieties*, and on the other hand, to use the language and the tools available in model theory to redefine and clarify the rather involved notion of a *special subvariety* known from the theory of Shimura varieties (mixed and pure) and some extensions of this theory.

Our definition of special coverings of algebraic varieties includes semi-abelian varieties, Shimura varieties (definitely the pure ones, and we also hope but do not know if the mixed ones in general satisfy all the requirement) and much more, for example, the Lie algebra covering a simple complex of Lie group $\mathrm{SL}(2,\mathbb{C})$.

Recall from the discussion in [22] that our specific interest in these matters arose from the connection to Hrushovski's construction of *new* stable structures (see e.g. [21]) and their relationship with generalised Schanuel conjectures. This subject is also closely related to the Trichotomy Principle and Zariski geometries. In the current paper we establish that the geometry of an arbitrary special covering of an algebraic variety is controlled by a Zariski geometry the closed subsets of which we call (weakly) special. The combinatorial type of simple (i.e. strongly minimal) weakly special subsets are classifiable by the Trichotomy Principle. Using this geometry and related dimension notions we can define a corresponding very general analogue of "Hrushovski's predimension" and formulate corresponding "generalised Schanuel's conjecture" as well as a very general forms of André-Oort, the CIT and Pink's conjectures (Zilber-Pink conjectures). Note that in this generality one can see a considerable overlap of the generalised Schanuel conjectures with the André conjecture on periods [1] (generalising the Grothendieck period conjecture) which prompt further questions on the model-theoretic nature of fundamental mathematics.

1.2 Acknowledments. The idea of approaching the most general setting of Schanuel-type conjectures crystallised during my 4-weeks visit at IHES at Bures-Sur-Yvette in February 2012, and considerable amount of work on this project was carried out during my 2-months stay at MPIM, Bonn in April-May 2012. My sincere thanks go to these two great mathematical research centres and people with whom I met and discussed the subject there. Equally crucial impact on this work comes from many conversations that I had with Jonathan Pila whose work on diophantine problems around Shimura varieties using model-theoretic tools introduced me to the subject. Moreover, the success of his work and of the Pila-Wilkie method brought together people working in different areas and made it possible for me to consult E.Ullmo, A.Yafaev, B.Klinger and others on the topic. On model-theoretic side I used Kobi Peterzil's advice on o-minimality issues. My thanks go to all these people.

2 Analytic and pseudo-analytic structures

Recall that a strongly minimal structure $\mathbf{M}$ (or its theory) can be given a coarse classification by the type of *the combinatorial geometry* that is induced by the pregeometry (M, acl) on the set $[M \setminus \mathrm{acl}(\emptyset)]/_\sim$, where $x \sim y$ iff $\mathrm{acl}(x) = \mathrm{acl}(y)$.

The Trichotomy conjecture by the author stated that for any strongly minimal structure $\mathbf{M}$ the geometry of $\mathbf{M}$ is either trivial or linear (the two united under the name *locally modular*), or the geometry of $\mathbf{M}$ is the same as of an algebraically closed field, and in this case the structure $\mathbf{M}$ is bi-interpretable with the structure of the field.

E.Hrushovski refuted this conjecture in the general setting [21]. Nevertheless the conjecture was confirmed, by Hrushovski and the author in [15], for an important subclass of structures, Zariski geometries, except for the clause stating the bi-interpretability, where the situation turned out to be more delicate.

2.1 Recall that the main suggestion of [22] was to treat an (amended version of) Hrushovski's counter-examples as *pseudo-analytic* structures, analogues of classical analytic structures. Hrushovski's predimension, and the corresponding inequality $\delta(X) \geq 0$, a key ingredient in the construction, can be seen then to directly correspond to certain type of conjectures of transcendental number theory, which we called *Generalised Schanuel* conjectures.

The ultimate goal in classifying the above mentioned pseudo-analytic structures has been to give a (non-first-order) axiomatisation and prove a categoricity theorem for the axiomatisable class.

2.2 The algebraically closed fields with pseudo-exponentiation, $\mathrm{F}_{\exp} = (\mathrm{F}; , +, \cdot, \exp)$, analogues of the classical structure $\mathbb{C}_{\exp} = (\mathbb{C}; , +, \cdot, \exp)$, was the first class studied in detail.

The axioms for $\mathrm{F}_{\exp}$ are as follows.

ACF_0 : F is an algebraically closed fields of characteristic 0;

EXP: $\exp : \mathbb{G}_a(\mathrm{F}) \to \mathbb{G}_m(\mathrm{F})$
is a sujective homomorphism from the additive group $\mathbb{G}_a(\mathrm{F})$ to the multiplicative group $\mathbb{G}_m(\mathrm{F})$ of the field F, and

$$\ker \exp = \omega\mathbb{Z}, \text{ for some } \omega \in \mathrm{F};$$

SCH: for any finite X,

$$\delta(X) := \mathrm{tr.deg}_{\mathbb{Q}}(X \cup \exp(X)) - \mathrm{ldim}_{\mathbb{Q}}(X) \geq 0.$$

Here $\mathrm{tr.deg}(X)$ and $\mathrm{ldim}_{\mathbb{Q}}(X)$ are the transcendence degree and the dimension of the $\mathbb{Q}$-linear space spanned by X, dimensions of the classical pregeometries associated with the field F, and $\delta(X)$ takes the role of *Hrushovski's predimension*, which gives rise to a new dimension notion and new pregeometry following Hrushovski's receipe. The inequality can be recognised as the Schanuel conjecture, if one also assumes $\mathrm{F}_{\exp} = \mathbb{C}_{\exp}$.

Since $\mathrm{F}_{\exp}$ results from a Hrushovski-Fräisse amalgamation, the structure is **Existentially Closed** with respect to the embeddings respecting the predimension. This takes the form of the following property.

EC: For any *rotund* and *free* system of polynomial equations

$$P(x_1, \ldots, x_n, y_1, \ldots, y_n) = 0$$

there exists a (generic) solution satisfying

$$y_i = \exp x_i \quad i = 1, \ldots, n.$$

The term *rotund* has been coined by J.Kirby, [17], rotund and free is the same as *normal* and free in [23]. We refer the reader to [17] or [23] for these technical definitions.

Finally we have the following **Countable Closure** property.

CC: For maximal rotund systems of equations the set of solutions is at most countable.

The main result of [23] is the following.

2.3 Theorem. *Given an uncountable cardinal λ, there is a unique model of axioms* ACF_0 + EXP + SCH + EC + CC *of cardinality λ.*

This is a consequence of Theorems A and B.

Theorem A. *The $L_{\omega_1,\omega}(Q)$-sentence*

$$\mathrm{ACF}_0 + \mathrm{EXP} + \mathrm{SCH} + \mathrm{EC} + \mathrm{CC}$$

is axiomatising a **quasiminimal excellent abstract elementary class** *(AEC).*

Theorem B. *A quasiminimal excellent AEC has a unique model in any uncountable cardinality.*

2.4 The original definition of quasiminimal excellence and the proof of Theorem B is in [24] based on earlier definitions and techniques of S.Shelah.

Definition. Let **M** be a structure for a countable language, equipped with a pregeometry cl. We say that the pregeometry of **M** is **quasiminimal** if the following hold:

1. The pregeometry is determined by the language. That is, if $\mathrm{tp}(a, \bar{b}) = \mathrm{tp}(a', \bar{b}')$ then $a \in \mathrm{cl}(\bar{b})$ if and only if $a' \in \mathrm{cl}(\bar{b}')$.
2. **M** is infinite-dimensional with respect to cl.
3. (Countable closure property) If $X \subseteq M$ is finite, then $\mathrm{cl}(X)$ is countable.

4. (Uniqueness of the generic type) Suppose that $H, H' \subseteq M$ are countable closed subsets, enumerated such that $\mathrm{tp}(H) = \mathrm{tp}(H')$. If $a \in M \setminus H$ and $a' \in M \setminus H'$ then $\mathrm{tp}(a, H) = \mathrm{tp}(a', H')$.

5. ($\aleph_0$-homogeneity over closed sets (submodels) and the empty set) Let $H, H' \subseteq M$ be countable closed subsets or empty, enumerated such that $\mathrm{tp}(H) = \mathrm{tp}(H')$,), let $\bar{b}, \bar{b}'$ be finite tuples from M such that $\mathrm{tp}(\bar{b}, H) = \mathrm{tp}(\bar{b}', H')$, and let $a \in \mathrm{cl}(H, \bar{b})$. Then there is $a' \in M$ such that $\mathrm{tp}(a, \bar{b}, H) = \mathrm{tp}(a', \bar{b}', H')$.

Excellence of a quasiminimal pregeometry is an extra condition on the amalgams of *independent systems of submodels* which we do not reproduce here in the general form but going to illustrate it in examples below.

The proof of Theorem A relies on essential algebraic and diophantine-geometric facts and techniques and goes through an intermediate stage which is the following.

Theorem A$_0$. *The natural $L_{\omega_1,\omega}$-axiomatisation of the two-sorted structure $(\mathbb{G}_a(\mathbb{C}), \exp, \mathbb{C}_{\mathrm{field}})$ defines a quasiminimal excellent AEC.*

Here $\mathbb{C}_{\mathrm{field}}$ is the field of complex numbers, and exp is the classical homomorphism

$$\exp : \mathbb{G}_a(\mathbb{C}) \to \mathbb{G}_m(\mathbb{C}) \tag{1}$$

of the additive group onto the multiplicative group of complex numbers.

The natural axiomatisation consists of the first-order part which consists of the theory of $\mathbb{C}_{\mathrm{field}}$, the theory of $\mathbb{G}_a(\mathbb{C})$ and the statement that exp is a surjective homomorphism. The only proper $L_{\omega_1,\omega}$-sentence which is added to this states that the kernel of exp is a cyclic group.

These proofs were not without errors. The original paper [25] of A$_0$ established quasiminimality of $(\mathbb{G}_a(\mathbb{C}), \exp, \mathbb{C}_{\mathrm{field}})$ but has an error in the part proving excellence. This was corrected in [2], where also a generalisation of this theorem to a positive characteristic analogue was given. But after [2] a related error in the proof of the main Theorem A still required an extra argument which did arrive but from an unexpected direction. M.Bays and J.Kirby in [3] (using [16]), followed by M.Bays, B.Hart, T.Hyttinen, M.Kesaala and J.Kirby [4], and further on followed by L.Haykazyan [14] found an essential strengthening of Theorem B, which made certain algebraic steps in the proofs of Theorems A$_0$ and A redundant.

The final result, see [4], is the following.

2.5 Theorem B*. *A quasi-minimal pregeometry can be axiomatised by an $L_{\omega_1,\omega}(Q)$-sentence which determines an uncountably categorical class. In particular, the class is excellent.*

With the proof of this theorem the proof of the main theorem 2.3 has been completed.

Theorem B* by itself is a significant contribution to the model theory of abstract elementary classes which, remarkably, has been found while working on applications.

The significance of this model-theoretic theorem will be further emphasised in the discussion of its implications for diophantine geometry below.

2.6 Analogues of Theorem A_0 are now established for elliptic curves, M.Bays, and Abelian varieties by M.Bays, B.Hart and A.Pillay, based on earlier contributions by M.Gavrilovich, M.Bays and the author.

Theorem $\mathbf{A}_{\mathrm{Ell}}$ (M.Bays, [5]). *Let $\mathbb{E}$ be an elliptic curve without complex multiplication over a number field $\mathrm{k}_0 \subset \mathbb{C}$. Then the natural $L_{\omega_1,\omega}$-axiomatisation of the two-sorted structure $(\mathbb{G}_a(\mathbb{C}), \exp, \mathbb{E}(\mathbb{C}))$ defines an uncountably categorical AEC. In particular, this AEC is excellent.*

Here $\exp : \mathbb{G}_a(\mathbb{C}) \to \mathbb{E}(\mathbb{C})$ is the homomorphism onto the group on the elliptic curve given by the Weierstrass function and its derivative. The *natural axiomatisation* includes an axioms Weil_m which fixes the polynomial relation between two torsion points of order m for certain choice of m. If this (first-order) axiom is dropped, the categoricity fails but not gravely – the resulting $L_{\omega_1,\omega}$-sentence still has only finitely many (fixed number of) models in each uncountable cardinality.

The following is an extension of the previous result to abelian varieties incorporated in [6] by M. Bays, B. Hart and A. Pillay

Theorem $\mathbf{A}_{\mathrm{AbV}}$. *Let $\mathbb{A}$ be an Abelian variety over a number field $\mathrm{k}_0 \subset \mathbb{C}$ and such that every endomorphism $\theta \in \mathcal{O}$ (complex multiplication) is defined over k_0. Then the natural $L_{\omega_1,\omega}$-axiomatisation of the two-sorted structure $(\mathbb{C}^g_{\mathcal{O}\cdot\mathrm{mod}}, \exp, \mathbb{A}(\mathbb{C}))$ along with the first-order type of the kernel of* exp *in the two-sorted language defines an uncountably categorical AEC. In particular, this AEC is excellent.*

Here $\mathbb{C}^g_{\mathcal{O}\cdot\mathrm{mod}}$ is the structure of the $\mathcal{O}$-module on the covering space $\mathbb{C}^g$, where $g = \dim \mathbb{A}$.

We explain the main ingredients of the proof.

2.7 Defnition. Let F be an algebraically closed field of countable transcendence degree and B a finite (possibly empty) subset of its transcendence basis. An independent system of algebraically closed fields is the collection $\mathcal{L} = \{L_s \subseteq \mathrm{F} : s \subseteq B\}$ of algebraically closed subfields, $L_s = \mathrm{acl}(s)$.

The **boundary** (or the *crown*) $\partial\mathcal{L}$ of $\mathcal{L}$ is the field generated by the L_s, $s \neq B$,

$$\partial\mathcal{L} = \langle L_s : s \subsetneq B \rangle.$$

Now let us also consider a semi-Abelian variety $\mathbb{A}$ over a number field k_0 and assume that F is of characteristic 0, $\mathrm{k}_0 \subset \mathrm{F}$.

For any subfield $\mathrm{k}_0 \subseteq \mathrm{k} \subseteq \mathrm{F}$, the set $\mathbb{A}(\mathrm{k})$ of k-rational points of $\mathbb{A}$ is well-defined. And conversely, for $D \subseteq \mathbb{A}(\mathrm{F})$ we write $\mathrm{k}_0(D)$ the extension of k_0 by (canonical) coordinates of points of D.

The $\mathbb{A}$-boundary $\partial_{\mathbb{A}}\mathcal{L}$ of the system $\mathcal{L}$ is the complex multiplication submodule

$$\partial_{\mathbb{A}}\mathcal{L} = \langle \mathbb{A}(L_s) : s \subsetneq B \rangle + \mathrm{Tors}(\mathbb{A}),$$

where $\mathrm{Tors}(\mathbb{A})$ is the torsion subgroup of $\mathbb{A}$.

The extension $\mathrm{k}_\infty = \mathrm{k}_0(\mathrm{Tors}(\mathbb{A}))$ will be of importance below.

We also need to use the Tate module $T(\mathbb{A})$ of $\mathbb{A}$ which can be defined as the limit

$$T(\mathbb{A}) = \lim_{\leftarrow} \mathbb{A}_m$$

of torsion subgroup $\mathbb{A}_m$ of $\mathbb{A}$ of order m.

Given $\sigma \in \mathrm{Gal}(\mathrm{F}:\mathrm{k}_\infty)$ and $a \in \mathbb{A}(\mathrm{k}_\infty)$ define, for $n \in \mathbb{N}$,

$$\langle \sigma, a\rangle_n = \sigma b - b \in \mathbb{A}_n,$$

for an arbitrary $b \in \mathbb{A}$ such that $nb = a$. This does not depend on the choice of b and taking limits we have the map

$$\langle\ \cdot, \cdot\rangle_\infty :\ \mathrm{Gal}(\mathrm{F}:\mathrm{k}_\infty) \times \mathbb{A}(\mathrm{k}_\infty) \to T(\mathbb{A}),\quad \langle \sigma, a\rangle_\infty = \lim_{\leftarrow} \langle \sigma, a\rangle_n.$$

This also works then for $\mathrm{k} \supseteq \mathrm{k}_\infty$ in place of k_∞ and we can consider, given $a \in \mathbb{A}(\mathrm{k})$, the submodule $\langle \mathrm{Gal}(\mathrm{F}:\mathrm{k}), a\rangle_\infty$ of the Tate module.

The main ingredient of the proof of Theorems A_0 and $\mathrm{A}_{\mathrm{AbV}}$ is the following.

Theorem ("Thumbtack Lemma"). *Let D be an $\mathbb{A}$-boundary, let k be a finitely generated extension of $\mathrm{k}_0(D)$ and let $a \in \mathbb{A}(\mathrm{k})$ be such that $\mathcal{O}\cdot\gamma \cap D = \{0\}$. Then $\langle \mathrm{Gal}(\mathrm{F}:\mathrm{k}), a\rangle_\infty$ is of finite index in $T(\mathbb{A})$.*

This splits naturally into 3 cases depending on the size $n := |B|$, namely cases $n = 0$, $n = 1$ and $n > 1$.

The case $n = 0$ essentially follows from a combination of a finiteness theorem by Faltings and the Bashmakov-Ribet Kummer theory for Abelian varieties just recently finalised by M.Larsen [20]. The similar result needed for the proof of Theorem A_0 for $\mathbb{G}_m$ is just Dedekind's theory of ideals and the classical Kummer theory.

The case $n = 1$ is the field of functions case of the Mordell-Weil theorem due to Lang and Néron, see [19], Theorem 6:2.

The case $n > 1$ is new and required a special treatment. For $\mathbb{G}_m$ it was done in [2] using the theory of specialisations (places) of fields, but [6] finds a more direct model-theoretic argument which covers also the case of Abelian varieties. Using the new model-theoretic result above one has now the following, for all reasonable forms of the Thumbtack Lemma (see, in particular, 2.12).

Corollary to Theorem B*. *The case $n > 1$ in the Thumbtack Lemma follows from the cases $n = 0, 1$.*

2.8 Recall that the statement of Theorem $\mathrm{A}_{\mathrm{AbV}}$ is weaker than that of Theorems A_0 and $\mathrm{A}_{\mathrm{Ell}}$ in including the complete description of the kernel of exp in the two-sorted language. The stronger version requires an extension of the Thumbtack Lemma which includes the case $n = 0$ & $\gamma \in \mathrm{Tors}$. In other words one needs to characterise the action of $\mathrm{Gal}(\tilde{\mathrm{k}}:\mathrm{k})$ on Tors. For $\mathbb{G}_m$ this is given by the theory of cyclotomic extensions and for elliptic curves without complex multiplication by Serre's theorem. For the general Abelian variety this is an open problem.

On the other hand Bays, Hart and Pillay in [6] prove a broad generalisation of A-style theorems at the cost of fixing more parameters in the "natural axiomatisation". Their axioms include the complete diagram of the prime model. In this setting Theorem A holds for an arbitrary commutative algebraic group over algebraically closed field of arbitrary characteristic. In fact, [6] building up on [7] by Bays, Gavrilovich and Hils shows how to generalise the statement to an arbitrary commutative finite Morley rank group and proves it in this formulation.

In terms of the Thumbtack Lemma the latter requires only $n = 1$ case. This is essentially Kummer theory over function fields in its most general, in fact model-theoretic, form.

Bays, Gavrilovich and Hils in [7] use this technique for an application in algebraic geometry.

2.9 Note a version of Theorem $\mathrm{A}_{\mathrm{AbV}}$ formulated in [6].

Theorem. *Models of the natural first order axiomatisation of* $(\mathbb{C}^g_{\mathcal{O}\cdot\mathrm{mod}}, \exp, \mathbb{A})$ *are determined up to isomorphism by the transcendence degree of the field and the isomorphism type of the kernel.*

This is a model-theoretic "decomposition" statement for the rather complex algebraic structure, similar to the Ax-Kochen-Ershov "decomposition" of henselian valued fields into residue field and value group.

2.10 The study of the above pseudo-analytic structures shed some light on classical transcendental functions, namely the complex exponentiation exp (Theorem A, 2.3), the Weierstrass function $\mathfrak{P}(\tau_0, z)$ as a function of z (Theorem $\mathrm{A}_{\mathrm{Ell}}$) and more generally Abelian integrals and the corresponding exponentiation (Theorem $\mathrm{A}_{\mathrm{AbV}}$). Although a lot of questions remain still open, especially for the latter, a natural continuation of this program leads to questions on model theory of $\mathfrak{P}(\tau, z)$ as a function of two variables z, and similar multy-variable maps related to Abelian varieties.

But before anything could be said about $\mathfrak{P}(\tau, z)$ the classical function $j(\tau)$, the modular invariant of elliptic curves $\mathbb{E}_\tau$, which can be defined in terms of $\mathfrak{P}(\tau, z)$.

2.11 The two-sorted setting for $j(\tau)$ analogous to settings in 2.6 is the structure $(\mathcal{H}, j, \mathbb{C}_{\mathrm{field}})$, where $\mathcal{H}$ is the upper half-plane as a $\mathrm{GL}^+(2, \mathbb{Q})$-set, that is with the action by individual elements $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ from the group (rational matrices with positive determinants)

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : \tau \mapsto \frac{a\tau + b}{c\tau + d}.$$

In the langiage we have names for fixed points $t_g \in \mathcal{H}$ of transformations g, which are exactly the quadratic points on the upper half-plane, and the list of statements $gt_g = t_g$ is part of the axiomatisation. The images $j(t_g) \in \mathbb{C}$ are called *special points.* They are algebraic and their values are given by the axioms of the structure.

The natural axiomatisation of this structure states that $j : \mathcal{H} \to \mathbb{C}$ is a surjection such that for every $g_1, \ldots, g_n \in \mathrm{GL}^+(2, \mathbb{Q})$ there is an irreducible algebraic curve $C_g \subset \mathbb{C}^{n+1}$ over $\mathbb{Q}(S)$, where S is the set of special points, such that

$$\langle x, y_1, \ldots, y_n \rangle \in C_g \Leftrightarrow \exists \tau \in \mathcal{H}\ x = j(\tau), y_1 = j(g_1\tau), \ldots y_n = j(g_n\tau).$$

This is given by a list of first-order sentences. Finally, the $L_{\omega_1,\omega}$-sentence

$$j(\tau) = j(\tau') \Leftrightarrow \bigvee_{g \in \mathrm{SL}(2,\mathbb{Z})} \tau' = g\tau$$

states that the fibres of j are $\mathrm{SL}(2, \mathbb{Z})$-orbits.

Adam Harris proves in [13] an analogue of Theorems A above.

2.12 Theorem Aj *The natural axiomatisation of $(\mathcal{H}, j, \mathbb{C}_{\text{field}})$, the two-sorted structure for the j-invariant, defines an uncountably categorical AEC.*

The structure of the proof is similar to the proofs discussed above. The key model-theoretic tool is Theorem B*. The appropriate thumbtack lemma takes the following form.

Theorem *Let $\mathbb{A}$ be an abelian variety defined over* k, *a finitely generated extension of $\mathbb{Q}$ or a finitely generated extension of an algebraically closed field L, such that $\mathbb{A}$ is a product of r non-isogenous elliptic curves (with j-invariants which are transcendental over* k *in the second case). Then the image of the Galois representation on the Tate module $T(\mathbb{A})$ is open in the Hodge group* $\mathrm{Hg}(\mathbb{A})(\hat{\mathbb{Z}})$.

The *Hodge group* is a subgroup of the *Mumford-Tate group*, for definitions see e.g. [11].

This theorem, essentially a version of the *adellic Mumford-Tate conjecture,* is explained in [13] as a direct consequence of a version of the hard theorem of Serre mentioned above and a further work by Ribet. Consequently Harris deduces the categoricity theorem Aj.

What is even more striking, that assuming the statement of theorem Aj does hold, [13] deduces the statement of the adellic Mumford-Tate conjecture as a consequence. This sort of equivalence was observed in [27] for categoricity statements for semi-Abelian varieties. The case considered by Harris, the simplest case of a Shimura variety, looks very different. And yet a similar tight connection between the model theory and the diophantine geometry of the j-invariant is valid.

2.13 We would like to remark that in terms of the discussion in 2.8, 2.9 the axiomatisation 2.11 assumed in Theorem Aj is not the most natural one. Its language includes constants naming special points, as a result of which the action of $\mathrm{Gal}(\tilde{\mathbb{Q}} : \mathbb{Q})$ on special points is not seen in the automorphism group of the structure. Working in a more basic language one would require a stronger version of the corresponding "Thumbtack Lemma" and thus a deeper statement of Hodge theory.

2.14 Finally we would like to discuss the first-order versus non-first-order (AEC) alternative in choosing a formalism to develop the model theory of pseudo-analytic structures as above.

In this regard there is a substantial difference between results of type A (Theorems A) and the main result about the "one-sorted" pseudo-exponentiation stated in 2.3. In the first situation, as shown in [6], one can use essentially the same techniques to classify models of the first order theory of the two-sorted structure in question.

In the one-sorted case the key property on which the whole model-theoretic study relies is Hrushovski's inequality (recall 2.1) or in more concrete form the relevant Schanuel's condition. The "decomposition" approach as in 2.9 can still be attempted but, as was noted already in [22], to formulate Schanuel's condition in the first order way one requires certain degree of diophantine uniformity, which was formulated in [28] as the Conjecture on Intersections in Tori, CIT. [22] discusses a broader formulation of this which includes semi-Abelian varieties. An equivalent to CIT conjecture was later formulated by E.Bombieri, D.Masser and U.Zannier, and a very general form, which covers the whole class of mixed

Shimura varieties, formulated by R.Pink. Without giving precise definition of *special subvarieties* (the second part of this paper is devoted to this, see section 6) we formulate what is currently referred to as the Zilber-Pink conjecture.

2.15 Conjecture Z-P. *Let $\mathbb{X}$ be an algebraic variety for which the notion of special subvarieties is well-defined. For any algebraic subvariety $V \subseteq \mathbb{X}^n$ there is a finite list of special subvarieties $S_1, \ldots, S_m \subsetneq \mathbb{X}^n$ such that, given an arbitrary special subvariety $T \subset \mathbb{X}^n$ and an irreducible component W of the intersection $V \cap T$, either* $\dim W = \dim V + \dim T - \dim \mathbb{X}^n$ *(a typical case), or $W \subseteq S_i$ for some $i = 1, \ldots, m$ (in the atypical case* $\dim W > \dim V + \dim T - \dim \mathbb{X}^n$*).*

This is a fundamental diophantine conjecture, in particular containing the Mordell-Lang and André - Oort conjecture.

The analysis of CIT in [28] specifically concentrates on $\mathbb{X} = \mathbb{G}_m$, the algebraic torus (and characteristic 0), in which case the special subvarieties S of $\mathbb{G}_m^n$ are subvarieties of the form $T \cdot t$, where $T \subseteq \mathbb{G}_m^n$ is a subtorus and $t \in \mathbb{G}_m^n$, a torsion point.

2.16 In [26] J.Kirby and the present author study the first order theory of the field with pseudo-exponentiation, $\mathrm{F}_{\exp}$, whose non-elementary theory is described in 2.3. In this case, due to the richer language the structure on the kernel of exp is highly complex, more precisely, in the standard model it is effectively the ring of integers. So the theory of the kernel is the *complete arithmetic*. Nevertheless we can aim at "decomposing" $\mathrm{Th}(\mathrm{F}_{\exp})$ into complete arithmetic and the theory *modulo the kernel*. The main result states that assuming CIT such a description is possible. Moreover, the theory modulo the kernel is quite tame and even superstable in a certain sense. The further analysis shows that the key property of the structure, Schanuel's condition, is first order axiomatisable over the kernel if and only if CIT holds.

The same must be true for other one-sorted pseudo-analytic structures, that is the study of the first order theories of such structures depend on the corresponding generalisations of CIT. On the other hand, the result of [26] shows that any progress in the studies of the first order theories should shed light on the corresponding Z-P conjecture.

2.17 Finally, we conclude this section with the remark that to study the model theory of a one-sorted pseudo-analytic structure one needs to know at the very least the statement of the corresponding Schanuel's condition. This is true both for AEC and the first order settings. A closer look at this problem quickly relates it with the problem of defining the notion of special subvarieties. The second part of the paper deals exactly with both issues.

2.18 Introduction to the second part.

Consider a complex algebraic variety $\mathbb{X}$ and a semi-algebraic[1] set $\mathbb{U}$ which is also an open subset of $\mathbb{C}^m$, some m. $\mathbb{X}$ will be treated as a structure with the universe $\mathbb{X}$ and n-ary relations given by Zariski closed subsets of $\mathbb{X}^n$. This is a classical Noetherian Zariski structure in the sense of [29]. The structure on $\mathbb{U}$ we define in a more delicate way so that eventually it is Zariski of analytic type.

[1] Semi-algebraic sets are by definition subsets of $\mathbb{R}^n$ that can be represented as Boolean combinations of subsets defined by the inequality $p(x_1, \ldots, x_n) \geq 0$ for p a polynomial over $\mathbb{R}$.

Our full structure consists of two sorts $\mathbb{U}$ and $\mathbb{X}$ with an analytic surjective mapping $\mathbf{p} : \mathbb{U} \to \mathbb{X}$ connecting the sorts in a "nice" way so that $(\mathbb{X}, \mathbb{U}, \mathbf{p})$ is model-theoretically tame (in general $\mathbf{p}$ should be a correspondence, but in this text we deal with a mapping only).

By analogy with the theory of mixed Shimura varieties call an irreducible Zariski closed $S \subset \mathbb{X}^n$ weakly special if there is an analytically irreducible $\check{S} \subset \mathbb{U}^n$ such that $\mathbf{p}(\check{S}) = S$ and $\check{S}$ is semi-algebraic (equivalently, definable in $\mathbb{U}$). Analogously but with more work one defines special subsets $S \subseteq \mathbb{X}^n$. The counterparts $\check{S} \subseteq \mathbb{U}^n$ of special sets S are called co-special. Assuming "niceness" of the definitions involved the co-special subsets form a Zariski geometry (see Theorem 4.11 below), so satisfy the Trichotomy principle: its strongly minimal subsets Y can be classified as being of one of the three types:

- trivial type: essentially a homogeneous space of a countable group;
- linear type: typically a commutative group;
- algebro-geometric type: Y is a complex algebraic curve with n-ary relations on Y given by Zariski closed subsets of Y^n.

In particular, we have a well-defined notion of a combinatorial dimension (rank) $d_{\mathrm{Spec}}(u_1, \dots, u_n)$ of a tuple $\langle u_1, \dots, u_n \rangle \in \mathbb{U}^n$. We also have a combinatorial dimension (transcendence degree) $\mathrm{tr.deg}_{\mathbb{Q}}(y_1, \dots, y_n)$ for tuples $\langle y_1, \dots, y_n \rangle \in \mathbb{X}^n$ as well as for tuples $\langle y_1, \dots, y_n \rangle \in \mathbb{U}^n$.

From model-theoretic point of view $\mathbf{p}$ can be interpreted as a "new" relation on an algebraically closed field $(\mathbb{C}, +, \cdot)$ and in order for the structure $\mathbb{C}_{\mathbf{p}} := (\mathbb{C}, +, \cdot, \mathbf{p})$ to be model-theoretically "nice" the only construction known today (after more than 20 years since [21]) is to employ the Hrushovski fusion method. In our context Hrushovski's construction suggests the following.

Introduce a relevant Hrushovski predimension. For $u_1, \dots, u_n \in \mathbb{U}$ set

$$\delta(u_1, \dots, u_n) = \mathrm{tr.deg}_{\mathbb{Q}}(u_1, \dots, u_n, \mathbf{p}(u_1), \dots, \mathbf{p}(u_n)) - d_{\mathrm{Spec}}(u_1, \dots, u_n).$$

The standard assumption (Hrushovski's inequality) then would be

$$\delta(\bar{u}) \geq 0, \text{ for all } \bar{u} \subset \mathbb{U} \tag{2}$$

which is the *generalised Schanuel conjecture* (in the sense of [22]) corresponding to our $\mathbf{p}$.

2.19 We can rewrite (2) as

$$\mathrm{tr.deg}_{\mathbb{Q}}(u_1, \dots, u_n, \mathbf{p}(u_1), \dots, \mathbf{p}(u_n)) \geq d_{\mathrm{Spec}}(u_1, \dots, u_n) \tag{3}$$

and in this form compare it with André's generalisation of Grothendieck period conjecture ([1], 23.4)

$$\mathrm{tr.deg}_{\mathbb{Q}} k(\mathrm{periods}(\mathbb{X}^n)) \geq \dim G_{\mathrm{mot}}(\mathbb{X}^n) \tag{4}$$

where $\mathrm{tr.deg}_{\mathbb{Q}} k(\mathrm{periods}(\mathbb{X}^n))$ is the transcendence degree of periods of $\mathbb{X}^n$ with parameters in k and $G_{\mathrm{mot}}(\mathbb{X}^n)$ is the motivic Galois group of $\mathbb{X}^n$.

C. Bertolin in [8] considers the "1-motives" case of André's conjecture and translates it in the form that generalises Schanuel's conjecture covering the case of elliptic functions.

This comparison suggests that motivic objects can be explained in terms of the geometry of co-special sets and co-special points. An approach to the classification of these geometries is the subject of this paper, see Theorems 4.11 and 5.17. In particular we show that any such geometry is a combination of 3 basic types of geometries of the Trichotomy principle.

3 Special coverings of algebraic varieties

3.1 The setting.
We consider $\mathbb{C}$ both in complex and in real co-ordinates, so we can define semi-algebraic subsets and relations in $\mathbb{C}$.

A. (i) We are given an open $\mathbb{U} \subseteq \mathbb{C}^m$ and with the complex structure induced from $\mathbb{C}$ the set $\mathbb{U}$ can be considered a complex manifold, and the same $\mathbb{U}$ we view in real co-ordinates.

(ii) We also are given a smooth complex algebraic variety $\mathbb{X}$ and an analytic surjection $\mathbf{p} : \mathbb{U} \to \mathbb{X}$ with discrete fibres.

(iii) We assume that a group Γ acts on $\mathbb{U}$ by biholomorphic transformations and discontinuosly. The fibres of $\mathbf{p}$ are orbits of Γ.

B. Further on we assume the existence of a *semi-algebraic fundamental domain* $\mathbb{F} \subseteq \mathbb{U}$ such that

(i) $\dot{\mathbb{F}} \subseteq \mathbb{F} \subseteq \overline{\mathbb{F}}$, where $\dot{\mathbb{F}}$ is the interior of $\mathbb{F}$ and $\overline{\mathbb{F}}$ is the closure of $\dot{\mathbb{F}}$ in the metric topology. We also assume that $\overline{\mathbb{F}}$ is semi-algebraic.

(ii) For each $\gamma \in \Gamma$ the set $\mathbb{F}_\gamma = \gamma \cdot \mathbb{F}$ and the restriction of the map $z \mapsto \gamma \cdot z$ on $\overline{\mathbb{F}}$ are semi-algebraic,

$$\dot{\mathbb{F}}_\gamma \cap \dot{\mathbb{F}} = \emptyset, \text{ for } \gamma \neq 1$$

and

$$\mathbb{U} = \bigcup_{\gamma \in \Gamma} \mathrm{F}_\gamma.$$

C. The set

$$\Delta := \{\gamma \in \Gamma : \overline{\mathbb{F}} \cap \overline{\mathbb{F}}_\gamma \neq \emptyset\} \text{ is finite.}$$

3.2 Examples. A large class of examples is the class of *arithmetic varieties* $\mathbb{X} := \Gamma \backslash \mathbb{U}$, where $\mathbb{U}$ is a Hermitian symmetric domain and Γ is an arithmetic subgroup of the real adjoint group Lie of biholomorphisms of $\mathbb{U}$. These have *Siegel sets*, which are semi-algebraic, for fundamental domains. Moreover, the condition 3.1.C is satisfied. See [9], Thm 13.1.
This class includes all *Shimura varieties* $\mathbb{X}$.

Do mixed Shimura varieties satisfy A-C?

3.3 Example. Let

$$\exp : \mathfrak{sl}(2, \mathbb{C}) \to \mathrm{SL}(2, \mathbb{C})$$

be the Lie exponentiation. The restriction of exp to the nilpotent part of $\mathfrak{sl}(2, \mathbb{C})$ is algebraic, in fact it is a map $z \mapsto 1 + z$ on a nilpotent matrix z (1 is the unit matrix).

Let $N \subset \mathrm{SL}(2,\mathbb{C})$ be the set of unipotent elements of $\mathrm{SL}(2,\mathbb{C})$ (including 1). We define

$$\mathbb{X} = \mathrm{SL}(2,\mathbb{C}) \setminus N \text{ and } \mathbb{U} = \mathfrak{sl}(2,\mathbb{C}) \setminus \mathrm{Ln}(N)$$

Clearly $\exp(\mathbb{U}) = \mathbb{X}$ and exp is unramified on $\mathbb{U}$.

We want to define Γ and a fundamental domain for $\exp : \mathbb{U} \to \mathbb{X}$. Looking at the Jordan normal form we see that an arbitrary $a \in \mathfrak{sl}(2,\mathbb{C})$ is of the form

$$a = \begin{pmatrix} x & 0 \\ 0 & -x \end{pmatrix}^g, \quad g \in \mathrm{SL}(2,\mathbb{C})$$

We write $b^{\mathrm{SL}(2,\mathbb{C})}$ for the conjugacy class $\{b^g : g \in \mathrm{SL}(2,\mathbb{C})\}$.

Define

$$\mathbb{F} = \left\{ \begin{pmatrix} x & 0 \\ 0 & -x \end{pmatrix}^{\mathrm{SL}(2,\mathbb{C})} : x \in \mathbb{C},\ x \neq 0 \text{ and } 0 \leq \mathrm{Im} x < 2\pi \right\}$$

The uniqueness, up to the order of eigenvalues, of diagonalisation implies that exp is injective on $\mathbb{F}$.

Define $\gamma_k : \mathbb{U} \to \mathbb{U}$ by setting

$$\gamma_k : \begin{pmatrix} x & 0 \\ 0 & -x \end{pmatrix}^g \mapsto \begin{pmatrix} x + 2\pi ik & 0 \\ 0 & -x - 2\pi ik \end{pmatrix}^g.$$

The maps are well-defined since the cetralisers of $\begin{pmatrix} x & 0 \\ 0 & -x \end{pmatrix}$ and $\begin{pmatrix} x + 2\pi ik & 0 \\ 0 & -x - 2\pi ik \end{pmatrix}$ coincide. The restriction of γ_k on $\mathbb{F}$ is a semi-algebraic since determining eigenvalues of an element of a matrix group is a definable operation over the field of reals.

The set Δ of 3.1C is equal to $\{\gamma_1, 1, \gamma_{-1}\}$.

Remark. In this example $\mathbb{U}$ is not semi-algebraic since the subset of diagonal matrices in $\mathbb{U}$ (which can be defined by an algeberaic formula) is in algebraic bijection with $\mathbb{C} \setminus 2\pi i\mathbb{Z}$.

3.4 We are going to use the notion of dimension $\dim_{\mathbb{R}} T$ for certain "nice" subsets of T of $\mathbb{U}$, $\mathbb{X}$ and their Cartesian powers. Obviously, this notion is applicable when T is semi-algebraic. More generally, note that given an open ball $B \subset \mathbb{U}^n$ of small enough radius, the two sorted structure $(B, \mathbf{p}(B))$ along with the map $\mathbf{p}$ restricted to B is definable in the o-minimal structure $\mathbb{R}_{an}$ (here and below we use the same notation for the map $\langle u_1, \ldots, u_n\rangle \mapsto \langle \mathbf{p}(u_1), \ldots, \mathbf{p}(u_n)\rangle$ for all $n \geq 1$)).

Now we can apply the dimension $\dim_{\mathbb{R}}$ in the sense of $\mathbb{R}_{an}$ to any subset definable in this structure. In particular, if T is a semi-algebraic subset of $\mathbb{U}^n$ (in respect to the embedding $\mathbb{U} \subseteq \mathbb{C}^m$) then $\dim_{\mathbb{R}}(T \cap B)$ and $\dim_{\mathbb{R}}(\mathbf{p}(T \cap B))$ are well defined. We can now define

$$\dim_{\mathbb{R}} \mathbf{p}(T) = \max_B \dim_{\mathbb{R}} \mathbf{p}(T \cap B).$$

A direct consequence of discreteness of fibres is the following.

Fact. *The map* $\mathbf{p}$ *preserves dimension. More presicely, for every semi-algebraic* $T \subseteq \mathbb{U}^n$, $\dim_{\mathbb{R}} \mathbf{p}(T) = \dim_{\mathbb{R}} T$.

For every semi-algebraic $S \subseteq \mathbb{X}^n$, $\dim_{\mathbb{R}} \mathbf{p}^{-1}(S) = \dim_{\mathbb{R}} S$.

3.5 Lemma. (i) *The equivalence relation* $\sim$ *on* $\overline{\mathbb{F}}$ *defined as*

$$u_1 \sim u_2 \text{ iff } \mathbf{p}(u_1) = \mathbf{p}(u_2)$$

is semi-algebraic.

(ii) *There is a (non-Hausdorff) topology* $\mathfrak{T}$ *on* $\overline{\mathbb{F}}$

- *the base open sets of* $\mathfrak{T}$ *are of the form* $\overline{\mathbb{F}} \cap \Gamma \cdot B$ *for* $B \subseteq \mathbb{U}$ *an open ball;*
- $\mathfrak{T}$ *induces a Hausdorff topology* $\mathfrak{T}/_\sim$ *on the quotient* $\overline{\mathbb{F}}/_\sim$ *with the base of open sets of the form* $(\overline{\mathbb{F}} \cap \Gamma \cdot B)/_\sim$ *and the canonical map associated with* $\sim$,
$$\tilde{\mathbf{p}} : \overline{\mathbb{F}}/_\sim \to \mathbb{X},$$
is a homeomorphism with respect to the metric topology on $\mathbb{X}$;
- *the restriction of* $\mathbf{p}$ *to* $\overline{\mathbb{F}}$,
$$\mathbf{p} : \overline{\mathbb{F}} \to \mathbb{X},$$
is an open and closed map.

(iii) *There are a semi-algebraic set* $\tilde{\mathbb{F}}$, $\dot{\mathbb{F}} \subseteq \tilde{\mathbb{F}} \subseteq \overline{\mathbb{F}}$, *and a bijective semi-algebraic correspondence* $\mathbf{i} : \overline{\mathbb{F}}/_\sim \to \tilde{\mathbb{F}}$ *such that* $\tilde{\mathbf{p}} = \mathbf{p} \circ \mathbf{i}$.

Proof. Recall $\Delta = \{\gamma \in \Gamma : \gamma\overline{\mathbb{F}} \cap \overline{\mathbb{F}} \neq \emptyset\}$.

We can redefine equivalently for $u_1, u_2 \in \overline{\mathbb{F}}$

$$u_1 \sim u_2 \text{ iff } \exists \gamma \in \Delta \ u_2 = \gamma \cdot u_1.$$

Since Δ is finite the relation is semi-algebraic. This proves (i).

(ii) It is clear that $\mathfrak{T}$ is a weakening of the metric topology on $\overline{\mathbb{F}}$ since ΓB is open in $\mathbb{U}$.

Let $\mathbf{p}(u) = x$, $\mathbf{p}(v) = y$ and V_x, V_y open non-intersecting neighborhoods in $\mathbb{X}$ around x and y correspondingly.

Then we can find balls $B_u \subseteq \mathbf{p}^{-1}(V_x)$ and $B_v \subseteq \mathbf{p}^{-1}(V_y)$ around u and v correspondingly. Since $\mathbf{p}^{-1}(V_x) \cap \mathbf{p}^{-1}(V_y) = \emptyset$ and both are invariant under the action of Γ, we have $\Gamma B_u \cap \Gamma B_v = \emptyset$. This proves that $\mathfrak{T}/_\sim$ is Hausdorff.

The argument above also shows that the inverse image of an open subset of $\mathbb{X}$ under $\mathbf{p}$ is $\mathfrak{T}$-open in $\overline{\mathbb{F}}$, and the inverse image of an open subset of $\mathbb{X}$ under $\tilde{\mathbf{p}}$ is $\mathfrak{T}/_\sim$-open in $\overline{\mathbb{F}}/_\sim$. This prove that $\mathbf{p}$ and $\tilde{\mathbf{p}}$ are continuous maps in the relevant topolgies.

We can also characterise $\mathfrak{T}$ in terms of its base of closed subsets. We can take for these the sets of the form $\overline{\mathbb{F}} \cap (\mathbb{U} \setminus \Gamma B)$ for B a ball in $\mathbb{U}$. By the above $\mathbf{p}$ sends such a closed set onto a closed set, and the same true for $\tilde{\mathbf{p}}$.

It follows that $\tilde{\mathbf{p}} : \overline{\mathbb{F}}/_\sim \to \mathbb{X}$ is a homeomorphism.

(iii) Since for any semi-algebraic surjection $f : X \to Y$ there is a semi-algebraic section $s : Y \to X$ there exists a semi-algebraic section $\mathbf{i} : \overline{\mathbb{F}}/_\sim \to \overline{\mathbb{F}}$ inverse to the quotient-map. Set $\tilde{\mathbb{F}} := \mathbf{i}(\overline{\mathbb{F}}/_\sim)$. $\square$

3.6 Remark. In particular, we have seen in the proof of (ii) that given a closed subset $C \subseteq \mathbb{U}$ invariant under the action of Γ (that is $\Gamma C = C$) the image $\mathbf{p}(\overline{\mathbb{F}} \cap C)$ is closed in $\mathbb{X}$.

3.7 Note that $\gamma \cdot \tilde{\mathbb{F}} \cap \tilde{\mathbb{F}} = \emptyset$ for $\gamma \neq 1$, since $\sim$ is trivial on $\tilde{\mathbb{F}}$ by definition. Without loss of generality **we assume from now on** that $\tilde{\mathbb{F}} = \mathbb{F}$ and thus have a stronger condition for 3.1.B(ii):

$$\gamma\mathbb{F} \cap \mathbb{F} = \emptyset \text{ for } \gamma \neq 1.$$

4 The geometry of weakly special subsets

4.1 Definition. We say $S \subseteq \mathbb{X}^n$ is weakly special, if S is Zariski closed irreducible and $\mathbf{p}^{-1}(S) \cap \mathbb{F}^n$ is semi-algebraic.

Note that if $\mathbf{p}^{-1}(S) \cap \gamma\mathbb{F}^n$ is semi-algebraic for $\gamma = 1$, it is semi-algebraic for every $\gamma \in \Gamma^n$. It is immediate by 3.1.B(ii).

4.2 Note that every point and the diagonal of $\mathbb{X}^2$ are weakly special.

4.3 We use the definition of an analytic cell $C \subseteq \mathbb{R}^m$ and the theorem of analytic cell decomposition in the context of the field of reals, $(\mathbb{R}, +, \cdot, \leq)$, see [30]. Such a C can be represented as an intersection $X \cap U$, where $U \subseteq \mathbb{R}^m$ is semi-algebraic open and X an algebraic variety. We also may assume that C is irreducible as real analytic variety.

We will refer to such cells just as 'cells'.

4.4 Lemma. *Let $C \subseteq \mathbb{U}^n$ be a cell, X a real analytic subset of $\mathbb{U}^n$ and $\dim_R(X \cap C) = \dim_R(C)$. Then $C \subseteq X$.*

Proof. Immediate from the fact that C is irreducible real analytic. □

4.5 Lemma. *Let X be a real analytic subset of an open G, both semialgebraic. Let $X = \bigcup_i X_i$ be its decomposition into irreducible components. Then there are only finitely many non-empty X_i and all of them are semi-algebraic.*

Proof. Note that the set X^{sing} of singular points is definable, i.e. semi-algebraic. Let $X' = X \setminus X^{\text{sing}}$ and $G' = G \setminus X^{\text{sing}}$, which is also open. Consider a cell decomposition

$$X' = \bigcup_{j=1}^{m} C_j$$

and let $X'_i = X_i \cap G'$.

For each C_j there must be an X_{i_j} such that $\dim_R(X_i \cap C_j) = \dim C_j$. By 4.4 $C_j \subseteq X_{i_j}$. This proves $X' = \bigcup_{j=1}^{m} X_{i_j}$.

Since non-empty intersections $X_i \cap X_k$ for $i \neq k$ are subsets of X^{sing}, the decomposition of X' into the X'_i is disjoint. Then X'_{i_j} either does not intersect C_l or contains it and is equal to X'_{i_l}. It follows, that X'_{i_j} is a union of finitely many cells, so is semi-algebraic.

It remains to note that the metric closure of $X'_i = X_i \setminus X^{\text{sing}}$ is equal to X_i and the X'_i are semi-algebraic, the X_i are semi-algebraic. □

4.6 Lemma. *Suppose $R \subseteq \mathbb{X}^n$ is Zariski closed and $\mathbf{p}^{-1}(R) \cap \mathbb{F}^n$ semi-algebraic. Let $\mathbf{p}^{-1}(R) = \bigcup_m T_m$ be the decomposition of the analytic subset $\mathbf{p}^{-1}(R)$ of $\mathbb{U}^n$ into irreducible components. Then $T_m \cap \mathbb{F}^n$ is semi-algebraic for every m and there are only finitely many of such subsets.*

Moreover, given

$$R = R_1 \cup \ldots \cup R_k,$$

the decomposition into Zariski irreducible components, the sets $\mathbf{p}^{-1}(R_i) \cap \mathbb{F}^n$ are semi-algebraic.

Proof. Let

$$\mathbf{p}^{-1}(R) \cap \overline{\mathbb{F}}^n = C_1 \cup \ldots \cup C_p$$

be a decomposition into semi-algebraic cells.

Suppose $T_m \cap \overline{\mathbb{F}}^n \neq \emptyset$. Then there is a cell, say C_1 of the decomposition such that $\dim_{\mathbb{R}}(T_m \cap C_1) = \dim_R(T_m \cap \overline{\mathbb{F}}^n)$.

Let G be an open semi-algebraic subset of $\mathbb{U}$ such that C_1 is a real analytic irreducible subset of G. Then by 4.4 $C_1 \subseteq T_m \cap G$. Since $T_m \cap \overline{\mathbb{F}}^n$ is closed the closure $\overline{C}_1 \subseteq T_m \cap G$. But since $C_1 \cup \ldots \cup C_p$ is closed in $\mathbb{U}$, we have that $\overline{C}_1$ is a union of cells in this decomposition (boundary cells), say $\overline{C}_1 = \bigcup_i \leq jC_i$ for some $j \leq p$. Now, if $T_m \cap \overline{\mathbb{F}}^n \neq \emptyset$ we consider $T'_m := T_m \setminus \overline{C}_1$ an analytic subset of the open set $\mathbb{U}' = \mathbb{U} \setminus \overline{C}_1$. Note that the cells are disjoint, so $C_{j+1}, \ldots, C_p \subseteq \mathbb{U}'^n \cap \overline{\mathbb{F}}^n$.

We can again find a cell, say C_{j+1} such that $\dim_{\mathbb{R}}(T_m \cap C_{j+1}) = \dim_R(T'_m \cap \overline{\mathbb{F}}^n)$ and repeating the argument will get $C_{j+1} \subseteq T'_m$. After finite number of steps we get that $T_m \cap \overline{\mathbb{F}}^n = \bigcup_i \leq kC_i$, for some $k \leq p$, up to renumeration of the decomposition. Claim proved.

It follows that every $T_m \cap \overline{\mathbb{F}}^n$ is semi-algebraic and there are only finitely many such.

Now note that by uniqueness of irreducible decomposition the analytic set $\mathbf{p}^{-1}(R_i)$ is a union of some subfamily of the T_m's. Thus $\mathbf{p}^{-1}(R_i) \cap \mathbb{F}^n$ are semi-algebraic. □

4.7 Lemma. *Let $R \subseteq \mathbb{X}^n$ be a constructible subset (a Boolean combination of Zariski closed sets) and $T := \mathbf{p}^{-1}(R) \cap \mathbb{F}^n$ be semi-algebraic. Then R is a Boolean combination of weakly special subsets of $\mathbb{X}^n$.*

Proof. R can be represeted as a finite union of constructible sets of the form $R_i \setminus P_i$, where R_i Zariski closed irreducible and P_i are Zariski closed subset of R_i, $\dim_{\mathbb{C}}(R_i) < \dim_{\mathbb{C}}(R_i)$.

Consider the metric closure $\overline{R}$. By general facts $\overline{R} = \bigcup_{i \leq k} R_i$.

On the other hand the $\mathfrak{T}$-closure $\overline{T}$ of T is semi-algebraic and the two are homemorphic by $\mathbf{p} : \overline{T} \to \overline{R}$. It follows by 4.6 that the components R_i are weakly special and $T_i := \mathbf{p}^{-1}(R_i) \cap \mathbb{F}^n$ are semi-algebraic.

By homeomorphism, $\bigcup_{i \leq k} T_i$ is the closure of T. Moreover, the semi-algebraic set $T_i \cap T$ corresponds to $\overline{R}_i \setminus P_i$, that is $T_i \cap T = T_i \setminus Q_i$ for some $Q_i \subset T_i$, $\mathbf{p}(Q_i) = P_i$. By construction Q_i is semi-algebraic. Hence, by 4.6, P_i is a union of weakly special sets. □

4.8 Proposition. *Let S, S_1 and S_2 be weakly special subsets of $\mathbb{X}^n$. Then*

(i) the irreducible components of $S_1 \cap S_2$ are weakly special.

(ii) given the projection $\mathrm{pr} : \mathbb{X}^n \to \mathbb{X}^{n-1}$ one can represent

$$\mathrm{pr}S = R \setminus P,$$

where R is weakly special and P is a Boolean combination of weakly special sets, and $\dim_{\mathbb{C}} P < \dim_{\mathbb{C}} R$.

(iii) Let $\langle x_1, \ldots, x_{n-1}\rangle$ be a generic point of $\mathrm{pr}S$, $S_{x_1 \ldots x_{n-1}}$ the fibre over $\langle x_1 \ldots x_{n-1}\rangle$ and $d = \dim S_{x_1 \ldots x_{n-1}}$, the dimension of the generic fibre. Then the set

$$S^{(d)} = \{\langle y_1, \ldots, y_{n-1}\rangle \in \mathrm{pr}S : \ \dim_{\mathbb{C}} S_{y_1 \cdots y_{n-1}} = d\}$$

contains a subset of the form $\mathrm{pr}S \setminus \bigcup_{i \leq k} Q$, where $Q_1, \ldots, Q_k$ are weakly special subsets of $\mathbb{X}^{n-1}$.

Proof. (i) Since $\mathbf{p}^{-1}(S_1 \cap S_2) \cap \mathbb{F}^n = \mathbf{p}^{-1}(S_1) \cap \mathbf{p}^{-1}(S_2) \cap \mathbb{F}^n$, the set $\mathbf{p}^{-1}(S_1 \cap S_2) \cap \mathbb{F}^n$ is semi-algebraic. Given $S_1 \cap S_2 = P_1 \cup \ldots \cup P_k$, the decomposition into

Zariski irreducible components we have by 4.6 that $\mathbf{p}^{-1}(P_i)$ is semi-algebraic, for every $i \le k$.

(ii) pr$S$ is Zariski constructible by Tarski's theorem. On the other hand pr$T$ is semi-algebraic, for $T = \mathbf{p}^{-1}(S) \cap \mathbb{F}^n$. The bijection $\mathbf{p}$ on $\mathbb{F}$ commutes with pr, so pr$T = \mathbf{p}^{-1}(\mathrm{pr}S) \cap \mathbb{F}^{n-1}$.

Now 4.7 proves that pr$S$ is a Boolean combination of weakly special sets, so a union of sets of the form $R_i \setminus P_i$ where R_i is weakly special and P_i is a union of weakly special. Since pr$S$ is irreducible, we get the desired.

(iii) The standard Fibre Dimension Theorem implies that the set $S^{(d)}$ is constructible and contains a Zariski open subset of the irreducible set pr$S$. On the other hand for $T = \mathbf{p}^{-1}(S) \cap \mathbb{F}^n$ we can define the set

$$T^{(d)} = \{\langle y_1, \ldots, y_{n-1}\rangle \in \mathrm{pr}T : \dim_{\mathbb{R}} T_{y_1\ldots y_{n-1}} = 2d\},$$

which is in bijective correspondence to $S^{(d)}$ via $\mathbf{p}$ since $\mathbf{p}$ preserves the real dimension. Now note that $T^{(d)}$ is semi-algebraic since dimension of fibres is definable in the field of reals too. Thus we get the conditions of 4.7 for $S^{(d)}$, which proves that $S^{(d)}$ can be presented as a Boolean combination of weakly special sets. The statement follows.□

4.9 Definition. We denote $\mathcal{S}_w$ the family of weakly special sets. $\mathbb{X}_{S_w}$ will stand for the structure $(\mathbb{X}, \mathcal{S}_w)$ with the universe $\mathbb{X}$ and n-relations given by weakly special subsets of $\mathbb{X}^n$.

We consider $\mathbb{X}_{S_w}$ a topological structure with dimension. The closed sets of the topology on $\mathbb{X}^n$, any n, are finite unions of weakly special subsets and the dimension that of algebraic geometry.

$\mathbb{F}_{S_w}$ will stand for the structure $(\mathbb{F}, \mathcal{S}_w)$ with the universe $\mathbb{F}$ and n-relations given by subsets $\mathbf{p}^{-1}(S) \cap \mathbb{F}^n$ for $S \in \mathcal{S}_w$, weakly special subsets.

4.10 Remark. $\mathbf{p}$ restricted to $\mathbb{F}$ is an isomorphism between $\mathbb{F}_{S_w}$ and $\mathbb{X}_{S_w}$.

4.11 Theorem.
(i) $\mathbb{X}_{S_w}$ is a Noetherian Zariski structure.
(ii) $\mathbb{X}_{S_w}$ has quantifier elimination and is saturated.

Proof. (i) We refer to [29] for the definition of a Zariski structure:

The properties (L), (DCC), (SI) and (AF) follow from Proposition 4.8(i) and 4.2 along with the obvious properties of Zariski topology in algebraic geometry as discussed in [29], 3.4.1.

The property (SP) of projection is checked by 4.8(ii).

The fibre condition (FC) is 4.8(iii).

(ii) Every Zariski structure has QE. Saturation follows from the fact that $\mathbb{X}_{S_w}$ is definable in the saturated structure, the field $\mathbb{C}$.

□

4.12 We call $\mathbb{X}$ **simple** if it has no proper infinite almost special subsets.

In model theoretic terminology this is equivalent, by 4.11, to say that $\mathbb{X}_{S_w}$ is strongly minimal.

4.13 Lemma. *Suppose* $\mathbb{X}$ *is* **simple**.

Set $\mathbb{X}^{reg}_{S_w} \subseteq \mathbb{X}_{S_w}$ *to be the substructure on the subset* $\mathbb{X} \setminus \mathbb{X}^{\mathrm{Sing}}$ *(by restrictions of predicates and constants of* $\mathcal{S}_w$*).*

Then $\mathbb{X}^{reg}_{S_w}$ *is a presmooth strongly minimal Noetherian Zariski structure.*

Proof. For convenience of notations we rename below $\mathbb{X}^{reg} = \mathbb{Y}$.

Claim. Given a weakly special subset S of $\mathbb{X}^n$ either $\dim_{\mathbb{C}} S \cap \mathbb{Y}^n = \dim_{\mathbb{C}} S$ or $S \cap \mathbb{Y}^n = \emptyset$. Moreover, if no projection $\mathrm{pr}_1 : \mathbb{X}^n \to \mathbb{X}$ along $n-1$ co-ordinates $\mathrm{pr}_1 S$ is a point of $\mathbb{X}^{\mathrm{Sing}}$, we get $\dim_{\mathbb{C}} S \cap \mathbb{Y}^n = \dim_{\mathbb{C}} S$.

Suppose towards a contradiction that $\dim_{\mathbb{C}} S \cap \mathbb{Y}^n < \dim_{\mathbb{C}} S$. Then, since S is irreducible, S is a subset of Zariski closed set $\mathbb{X}^n \setminus \mathbb{Y}^n$, that is $S \cap \mathbb{Y}^n = \emptyset$, This is the same as to say that, up to the order of co-ordinates, $S \subseteq \mathbb{Y}^{n-1} \times \mathbb{X}^{\mathrm{Sing}}$. Then the corresponding projection $\mathrm{pr}_1 S \subseteq \mathbb{X}^{\mathrm{Sing}}$. Thus the Zariski closure $\overline{\mathrm{pr}_1 S}$ is a subset of the Zariski closed set $\mathbb{X}^{\mathrm{Sing}}$. But $\overline{\mathrm{pr}_1 S}$ is weakly special by 4.8(ii), so by assumptions the embedding can only happen if $\mathrm{pr}_1 S$ is finite, so a point, which contradicts the assumption on S. Claim proved.

We will call weakly special in regards to $\mathbb{Y}$ the sets of the form $S \cap \mathbb{Y}^n$ for S weakly special subsets of $\mathbb{X}^n$. Now we can check that the statements (i), (ii) and (iii) of 4.8 hold for $\mathbb{Y}$. (i) and (ii) in regards to $\mathbb{Y}$ follow from (i) and (ii) for $\mathbb{X}$ by definition. (iii) follows by the Claim by the same argument as in the proof of 4.8.

Finally, $\mathbb{X}^{reg}_{S_w}$, the structure on $\mathbb{Y}$, is Noetherian Zariski by the same argument as in the proof of 4.11. The structure is presmooth since the underlying quasiprojective algebraic variety $\mathbb{X} \setminus \mathbb{X}^{\mathrm{Sing}}$ is smooth, see [29], 3.4.1. □

4.14 Theorem. *The simple weakly special subsets* $Y \subseteq \mathbb{X}^n$, *all* $n \geq 1$, *are classifiable as follows:*

the geometry of the structure $\mathbb{Y}$ *induced from* $\mathbb{X}_{S_w}$ *is either trivial, or linear (locally modular), or of algebro-geometric type, in which case*

(i) $\dim_{\mathbb{C}} Y = 1$, *that is* Y *is an algebraic curve, and*

(ii) every irreducible Zariski closed subset of Y^k, $k \in \mathbb{N}$, *is weakly special.*

Proof. By the Weak Trichotomy Theorem (see [29], Appendix, Thm B.1.43) the geometry of the strongly minimal structure $\mathbb{Y}$ is either trivial, or locally modular, or a pseudo-plane is definable in $\mathbb{Y}$. We need to analyse the latter case.

In this case by B.1.39 of [29] on a subset S of $Y \times Y$ there is a definable family L of "curves" C_l, $l \in L$, with Morley ranks $\mathrm{Mrk}\, Y = 1$, $\mathrm{Mrk}\, L = 2$ and $\mathrm{Mrk}\, C_l = 1$ for each $l \in L$. By elimination of quantifiers in $\mathbb{Y}$ each C_l is a Boolean combination of weakly special subsets, so we may assume C_l is of the form $C_l = R_l \setminus Q_l$, where R_l is weakly special and Q_l a finite union of weakly special subsets.

Note that once it is established that $\mathrm{Mrk}\, Y = 1$, we have an easy translation between the Morley rank and the (complex) dimension of definable subsets Z of $\mathbb{Y}^n$: $\dim_{\mathbb{C}} Z = \mathrm{Mrk}\, Z \cdot \dim_{\mathbb{C}} Y$.

By a standard argument for generic $l \in L$, we have $\mathrm{Mrk}\,(\mathrm{pr} C_l) = 1$, for projections of $Y \times Y \to Y$ along the both co-ordinates. This condition is definable in the structure $\mathbb{Y}$, hence we may assume $\mathrm{Mrk}\,(\mathrm{pr} C_l) = 1$ for all $l \in L$. In particular, the Zariski closure of $\mathrm{pr} C_l$ is $\mathbb{Y}$ for all $l \in L$.

Consider the substructure $\mathbb{Y}^{reg}$ of $\mathbb{Y}$ obtained by removing the singular points of Y. By 4.13 $\mathbb{Y}^{reg}$ is a presmooth strongly minimal Noetherian Zariski structure such that the trace of the family $C_l : l \in L$ on the substructure gives us a Morley rank 2 family of "curves" $\mathrm{Mrk}\,(C_l \cap \mathbb{Y}^{reg} \times \mathbb{Y}^{reg}) = 1$ This proves that $\mathbb{Y}^{reg}$ is not locally modular.

The classification of a non locally modular presmooth strongly minimal Zariski structure is given by [29], Theorem 4.4.1 (originally [15]). By the theorem a structure $\mathrm{F} = (F, +, \cdot)$ of an algebraically closed field F is definable in $\mathbb{Y}^{reg}$. But $\mathbb{Y}^{reg}$ by construction is definable in the field $\mathbb{C}$ of complex numbers. So F is definable in $(\mathbb{C}, +, \cdot)$. By standard model-theoretic fact we can definably identify F and $\mathbb{C}$.

The above implies that there is a rational function $r : \mathbb{Y}^{reg} \to \mathbb{C}$ such that any Zariski closed subset $R \subset \mathbb{C}^n$ is also definable in $\mathbb{Y}^{reg}$ By strong minimality of $\mathbb{Y}^{reg}$ the map r must be finite (that is with finite fibres $r^{-1}(x)$). This implies that $\dim_{\mathbb{C}} \mathbb{Y}^{reg} = \dim_{\mathbb{C}} \mathbb{C} = 1$, that is $\mathbb{Y}^{reg}$ is an a complex algebraic curve. Recall that $\mathbb{Y}^{reg} = \mathbb{Y} \setminus \mathbb{Y}^{\mathrm{Sing}}$ and $\mathbb{Y}$ is an algebraic variety. It follows that $\mathbb{Y}$ is a complex algebraic curve. We have proved (i).

Moreover, it follows that $\mathbb{Y}^{\mathrm{Sing}}$ is a finite subset, so in terms of definability $\mathbb{Y}$ and $\mathbb{Y}^{reg}$ are equivalent. So the statement of 4.4.1 is applicable to $\mathbb{Y}$.

The Classification Theorem 4.4.1 also states that the field is "purely definable", that is any subset of $\mathbb{C}^m$ which is definable in $\mathbb{Y}$ is definable in the field $\mathbb{C}$ alone.

Pick an arbitrary Zariski closed subset $P \subseteq \mathbb{Y}^m$. Then $r(P) \subseteq \mathbb{C}^m$ is definable in $\mathbb{C}$, so $r^{-1}(r(P))$ is definable in $\mathbb{Y}$. By elimination of quantifiers $r^{-1}(r(P))$ is a Boolean combination of weakly special subsets, hence the Zariski closure $\bar{P}$ of $r^{-1}(r(P)) \subseteq \mathbb{Y}^m$ is a finite union of weakly special subsets, that is definable in $\mathbb{Y}$.

Obviously $P \subseteq \bar{P}$, and indeed P is an irreducible component of $\bar{P}$. By Lemma 4.8(i) P is weakly special. □

5 Co-special geometry on $\mathbb{U}$

.

5.1 Given a weakly special $S \subset \mathbb{X}^n$ consider the analytic subset $\mathbf{p}^{-1}(S) \subseteq \mathbb{U}^n$ and its decomposition into analytic irreducible components

$$\mathbf{p}^{-1}(S) = \bigcup_{i \in I_S} T_i.$$

Note that for all components $\dim_{\mathbb{C}} T_i \leq \dim_{\mathbb{C}} S$ since $\dim_{\mathbb{C}} \mathbf{p}^{-1}(S) = \dim_{\mathbb{C}} S$.

We will call a component T_i **essential** if $\dim_{\mathbb{C}} T_i = \dim_{\mathbb{C}} S$.

We will call a component T_i **essential in** $\gamma\mathbb{F}^n$ if $\dim_{\mathbb{C}} T_i \cap \gamma\mathbb{F}^n = \dim_{\mathbb{C}} S$.

Note that since $T_i \subseteq \bigcup_{\gamma \in \Gamma^n} \gamma\mathbb{F}^n$, every essential T_i is essential in some $\gamma\mathbb{F}^n$, $\gamma \in \Gamma^n$.

By 4.6 for a given γ there is only finitely many T_i essential in $\gamma\mathbb{F}^n$.

5.2 Lemma. *In notation of 5.1 let $T_1, \ldots, T_k$ be the essential components of $\mathbf{p}^{-1}(S)$ intersecting $\mathbb{F}^n$. Then for any $i \leq k$ there is $\gamma_i \in \Gamma^n$ such that $T_i = \gamma_i T_1$.*

Every components of $\mathbf{p}^{-1}(S)$ intersecting $\mathbb{F}^n$ is essential.

Proof. Note that for each i the set $\Gamma \cdot T_i$ is analytic and so closed in $\mathbb{U}^n$.

Let $D_i = \mathbf{p}(\Gamma T_i \cap \overline{\mathbb{F}}^n)$, $i = 1, \ldots, k$. These are closed in the metric topology of $\mathbb{X}$, by 3.6.

By definition $\bigcup_{i=1}^k D_i \subseteq S$, and since $\mathbf{p}$ preserves the dimension $\dim_{\mathbb{R}}$ converting $\dim_{\mathbb{C}}$ into $2 \cdot \dim_{\mathbb{R}}$ and taking into account that the missing components of $\mathbf{p}^{-1}(S)$ are inessential we get

$$\dim_{\mathbb{R}}(S \setminus \bigcup_{i=1}^{k} D_i) \leq \dim_{\mathbb{R}} S - 2.$$

where the dimensions here are understood in the sense of **??**. Moreover, recalling that the structure $(\mathbb{U}, \mathbb{X}, \mathbf{p})$ is locally o-minimal we may deduce from the latter that in any small neighbourhood V of a point of S $\bigcup_{i=1}^k D_i \cap V$ contains an open subset of the analytic set $S \cap V$. But also $\bigcup_{i=1}^k D_i \cap V$ is closed in V. It follows that $\bigcup_{i=1}^k D_i \cap V = S \cap V$ and hence $\bigcup_{i=1}^k D_i = S$.

This immediately implies that every components of $\mathbf{p}^{-1}(S)$ intersecting $\mathbb{F}^n$ is essential.

Note that S is connected since it is an algebraically irreducible subvariety of a complex variety $\mathbb{X}^n$. Moreover, it will stay connected if we remove from it a subset of real dimension $\dim_{\mathbb{R}} S - 2$. It follows that $\bigcup_{i=1}^k D_i$ is connected and moreover one can get from D_1 to any D_i by a chain $D_1 = D_{i_1}, \ldots, D_{i_m} = D_i$ such that $\dim_{\mathbb{R}}(D_{i_j} \cap D_{i_{j+1}}) \geq \dim_{\mathbb{R}} S - 1$, for $1 \leq j < m$. It follows that $\dim_{\mathbb{C}}(\gamma_j T_{i_j} \cap T_{i_{j+1}}) = \dim_{\mathbb{C}} S$ for some gluing $\gamma_j \in \Gamma^n$, for $j = 1, \ldots, m-1$. The statement of Lemma follows. $\square$

5.3 Lemma. *In notation of 5.1 for any components T_1, T_2 of $\mathbf{p}^{-1}(S)$ there is a $\gamma_{12} \in \Gamma^n$ such that $T_2 = \gamma_{12} T_1$.*

Proof. We may assume that T_1 is intersecting $\mathbb{F}^n$ and T_2 intersecting $\alpha \mathbb{F}^n$ for some $\alpha \in \Gamma^n$. Now $\alpha^{-1} T_2$ intersects $\mathbb{F}^n$ and using Lemma 5.2 we get a required γ_{12}. $\square$

We will need the following.

5.4 Fact. (Special case of Theorem 12.5 of [34]) *Let $G \subseteq \mathbb{C}^n$ be a semi-agebraic open set and $X \subseteq G$ an irreducible complex analytic subset which is also semi-algebraic. Then there is a complex algebraic subset $X^{\mathrm{Zar}} \subseteq \mathbb{C}^n$ such that X is an irreducible analytic component of the set $X^{\mathrm{Zar}} \cap G$.*

Note that by 4.4 there are finitely many irreducible analytic components of the set $X^{\mathrm{Zar}} \cap G$.

5.5 Lemma. *Given a weakly special $S \subseteq \mathbb{X}^n$, for each analytic component $T \subseteq \mathbf{p}^{-1}(S)$ there is a Zariski closed Zariski irreducible subset $Z \subseteq \mathbb{C}^{mn}$ such that*

(i) $\dim_{\mathbb{C}} S = \dim_{\mathbb{C}} Z$;

(ii) T is an irreducible analytic component of the set $Z \cap U$;

(iii)

$$\mathbf{p}^{-1}(S) \cap Z = \bigcup_{\gamma \in \mathrm{St}(Z)} \gamma \cdot T,$$

for $\mathrm{St}(Z) = \{\gamma \in \Gamma^n : \gamma \cdot (Z \cap U) = Z \cap U\}$.

Proof. We may assume T is essential in $\mathbb{F}^n$. By 4.6 $T \cap \mathbb{F}^n$ is semi-algebraic and thus, for some open semi-algebraic $G \subseteq \mathbb{U}^n$, $\dim_{\mathbb{R}} T \cap G = \dim_{\mathbb{R}} T$. By 5.4 there is a Zariski closed set $Z = T^{\mathrm{Zar}}$ such that $T \cap G = Z \cap G$. Clearly, the minimal such Z must be Zariski irreducible. Since the complex algebraic dimension coinsides with the complex analytic dimension on Zariski closed sets, and the latter is local, we have

$$\dim_{\mathbb{C}} Z = \dim_{\mathbb{C}} Z \cap G = \dim_{\mathbb{C}} T \cap G = \dim_{\mathbb{C}} S.$$

This gives us (i). This also implies (ii) since by irreducibility of T we will have $T \subseteq Z \cap \mathbb{U}^n$, and by equality of dimension T has to be a component in $Z \cap \mathbb{U}^n$.

Now (iii) follows by 5.3. $\Box$

5.6 Given a Zariski closed irreducible $Z \subseteq \mathbb{C}^{mn}$ we can rearrange the analytic irreducible decomposition of $Z \cap \mathbb{U}^n$ so that

$$Z \cap \mathbb{U}^n = \bigcup_{i \in \mathbb{N}} R_i \tag{5}$$

where

$$R_i = \bigcup_{\gamma \in \mathrm{St}(Z)} \gamma \cdot T_i,$$

for some T_i, analytic irreducible component of $Z \cap \mathbb{U}^n$. Obviously the decomposition (5) is unique, up to enumeration.

Clearly, $\mathrm{St}(R_i) = \mathrm{St}(Z)$.

We call the R_i's **invariant analytic components of** $Z \cap \mathbb{U}^n$.

5.7 Definition. Given a weakly special $S \subset \mathbb{X}^n$ we call an analytic subset $\check{S} \subseteq \mathbb{U}^n$ **weakly co-special set associated with** S if $\check{S} = \mathbf{p}^{-1}(S) \cap Z$ for some Zariski closed Zariski irreducible subset $Z \subseteq \mathbb{C}^{mn}$ such that $\dim_{\mathbb{C}} \mathbf{p}^{-1}(S) \cap Z = \dim_{\mathbb{C}} \mathbf{p}^{-1}(S) = \dim_{\mathbb{C}} Z$.

By 5.5(iii) and 5.6 $\check{S}$ is an invariant analytic components of $Z \cap \mathbb{U}^n$.

We call Z the Zariski closure of $\check{S}$.

5.8 Lemma. *Given a weakly co-special $\check{S} \subseteq \mathbb{U}^n$ and its Zariski closure Z there exists an analytic set $\hat{S} \subseteq \mathbb{U}^n$ complementing $\check{S}$ to $Z \cap \mathbb{U}^n$, that is*

$$\check{S} \cup \hat{S} = Z \cap \mathbb{U}^n, \quad \dim_{\mathbb{C}}(\check{S} \cap \hat{S}) < \dim_{\mathbb{C}} \check{S}.$$

Proof. Following 5.5 take for $\hat{S}$ the union of all the analytic irreducible components of $Z \cap \mathbb{U}^n$ which are not subsets of $\check{S}$. $\Box$

5.9 Lemma. *For a weakly special $S \subset \mathbb{X}^n$ and an associated weakly co-special $\check{S}$ we have the decomposition*

$$\mathbf{p}^{-1}(S) = \bigcup_{\gamma \in \Gamma^n} \gamma \cdot \check{S}.$$

The set of (distinct) components $\gamma \cdot \check{S}$ are in bijective correspondence with the cosets $\Gamma^n / \mathrm{St}(\check{S})$.

Proof. Immediate from 5.3. $\Box$

5.10 Examples. 1. For $\mathbb{X}$ an Abelian variety of dimension g the weakly pro-special subsets of $\mathbb{U}^n = \mathbb{C}^{ng}$ are cosets of $\mathbb{C}$-linear subspaces $L \subseteq \mathbb{C}^{ng}$ such that $L + \Lambda^n$ is closed in $\mathbb{C}^{ng}$.

2. For a Shimura variety $\mathbb{C}$ (the modular curve) the weakly pro-special subsets of the upper half-plane $\mathbb{U}$ are just points and the weakly pro-special subsets of $\mathbb{U}^2$ are the graphs of the maps $x \mapsto gx$, for $g \in \mathrm{GL}^+(\mathbb{Q})$.

3. Let $\mathbb{X} = \mathbb{C}^\times \setminus \{a\}$ be the complex torus $\mathbb{C}^\times$ with a point a removed. Let $\mathbb{U} = \mathbb{C} \setminus \{\ln a + 2\pi i k : k \in \mathbb{Z}\}$ and $\mathbf{p} = \exp$ restricted to $\mathbb{U}$. This satisfies all the condition 3.1.

The weakly pro-special subsets $\check{S}$ of $\mathbb{U}^n$ are the intersection of $\mathbb{Q}$-linear subspaces of $\mathbb{C}^n$ with $\mathbb{U}^n$. Unlike Example 1 above we can have here $\mathrm{pr}\check{S}$ not constructible. For example, for

$$\check{S} = \{\langle x, y\rangle \in \mathbb{U}^2 : y = 2x\}$$

one has, for $\mathrm{pr} : \langle x, y\rangle \mapsto x$,

$$\mathrm{pr}\check{S} = \mathbb{U} \setminus \{\frac{1}{2}\ln a + \pi i k : k \in \mathbb{Z}\}.$$

5.11 Lemma. *Suppose $\check{S}_1$ and $\check{S}_2$ are weakly co-special. Then*

$$\check{S}_1 \cap \check{S}_2 = \bigcup_{i=1}^{m} \check{R}_i$$

for some weakly co-special $\check{R}_1, \ldots, \check{R}_m$.

Proof. By definition $\check{S}_1$ and $\check{S}_2$ are invariant analytic components of $\mathbf{p}^{-1}(S_1) \cap Z_1$ and $\mathbf{p}^{-1}(S_2) \cap Z_2$, correspondingly, for some Zariski irreducible Z_1 and Z_2.

Let

$$S_1 \cap S_2 = \bigcup_{i=1}^{k} P_i \tag{6}$$

be the decomposition into weakly special subsets, see 4.8.

We will have correspondingly

$$\mathbf{p}^{-1}(S_1) \cap \mathbf{p}^{-1}(S_2) = \bigcup_{i=1}^{k} \mathbf{p}^{-1}(P_i). \tag{7}$$

It follows that the irreducible components of the $\mathbf{p}^{-1}(P_i)$ are exactly the irreducible components of $\mathbf{p}^{-1}(S_1) \cap \mathbf{p}^{-1}(S_2)$.

By definition

$$\check{S}_1 \cap \check{S}_2 = \mathbf{p}^{-1}(S_1) \cap \mathbf{p}^{-1}(S_2) \cap Z_1 \cap Z_2,$$

so using the decomposition

$$Z_1 \cap Z_2 = \bigcup_{j=1}^{m} Y_j$$

into Zariski irreducible components, we get

$$\check{S}_1 \cap \check{S}_2 = \mathbf{p}^{-1}(S_1) \cap \mathbf{p}^{-1}(S_2) \cap \bigcup_{j=1}^{m} Y_j. \tag{8}$$

Combining (8) with (7)

$$\check{S}_1 \cap \check{S}_2 = \bigcup_{i=1}^{k} \mathbf{p}^{-1}(P_i) \cap \bigcup_{j=1}^{m} Y_j = \bigcup_{i,j} \mathbf{p}^{-1}(P_i) \cap Y_j. \tag{9}$$

Now note that for each j the analytic subset $\check{S}_1 \cap \check{S}_2 \cap Y_j$ of $Y_j \cap \mathbb{U}^n$ can be complemented (see 5.8) by the analytic subset $(\check{S}_1 \cap \hat{S}_2 \cup \hat{S}_1 \cap \check{S}_2 \cup \hat{S}_1 \cap \hat{S}_2) \cap Y_j$ to $Y_j \cap \mathbb{U}^n$. This implies that any irreducible component T of $\check{S}_1 \cap \check{S}_2$ is also a component of $Y_j \cap \mathbb{U}^n$. It follows that $\dim_{\mathbb{C}} T = \dim_{\mathbb{C}} Y_j$. It further implies that

$$\dim_{\mathbb{C}} \mathbf{p}^{-1}(P_i) \cap Y_j = \dim_{\mathbb{C}} \mathbf{p}^{-1}(P_i) = \dim_{\mathbb{C}} Y_j.$$

Denote

$$\check{R}_{i,j} := \bigcup_{i,j} \mathbf{p}^{-1}(P_i) \cap Y_j.$$

These are weakly co-special subsets giving the required components of $\check{S}_1 \cap \check{S}_2$. $\square$

5.12 Lemma. *Given a weakly co-special $\check{S} \subseteq \mathbb{U}^n$, $0 \leq m < n$ and a projection* $\mathrm{pr} : \mathbb{U}^n \to \mathbb{U}^m$, *we have*

$$\check{R} \supseteq \mathrm{pr}\check{S} \supseteq \check{R} \setminus \bigcup_{\gamma \in \Gamma^m} \bigcup_{i=1}^{k} \gamma \cdot \check{P}_i,$$

for some weakly co-special $\check{R}$ and $\check{P}_1, \ldots, \check{P}_k \subseteq \check{R} \subseteq \mathbb{U}^m$, $\dim_{\mathbb{C}} \check{P}_i < \dim_{\mathbb{C}} \check{R}$.

Proof. By definition $\Gamma^n \cdot \check{S} = \mathbf{p}^{-1}(S)$, $\mathbf{p}(\check{S}) = S$ for some weakly special $S \subseteq \mathbb{X}^n$. It follows from 4.8(ii) that $R \setminus \bigcup_{i=1}^{k} P_i \subseteq \mathrm{pr}S \subseteq R$ for some weakly special R and $P_1, \ldots, P_k \subsetneq R \subseteq \mathbb{X}^m$. Let $\check{R}$ be a weakly co-special set associated with R, that is $\mathbf{p}(\check{R}) = R$.

Since $\mathrm{pr}(\mathbf{p}^{-1}(A)) = \mathbf{p}^{-1}(\mathrm{pr}(A))$ for every $A \subseteq \mathbb{X}^n$, we have

$$\Gamma^m \cdot \check{R} \supseteq \mathrm{pr}(\Gamma^n \cdot \check{S}) \supseteq \Gamma^m \cdot \check{R} \setminus \mathbf{p}^{-1}(P_1 \cup \ldots \cup P_k).$$

It follows that

$$\check{R} \supseteq \mathrm{pr}\check{S} \supseteq \check{R} \setminus \mathbf{p}^{-1}(P_1 \cup \ldots \cup P_k).$$

Let $\check{P}_i$ be weakly co-special sets associated with the P_i's. Then by definition $\mathbf{p}^{-1}(P_i) = \Gamma^m \cdot \check{P}_i$ and we get the required. $\square$

5.13 Lemma. *Given a weakly co-special $\check{S} \subseteq \mathbb{U}^n$, $0 \leq m < n$, a projection* $\mathrm{pr} : \mathbb{U}^n \to \mathbb{U}$ *and a number $d = \dim_{\mathbb{C}} \check{S}(a)$, where $\check{S}(a)$ is a fibre over the point $a \in \mathrm{pr}\check{S}$ of the minimum dimension, define*

$$\check{S}^{(d)} = \{b \in \mathrm{pr}\check{S} : \dim_{\mathbb{C}} \check{S}(b) = d\}.$$

Then there are weakly co-special $\check{P}_1, \ldots, \check{P}_k \subset \mathrm{pr}\check{S}$ each of dimension less than $\dim_C \mathrm{pr}\check{S}$ *such that*

$$\check{S}^{(d)} \supseteq \mathrm{pr}\check{S} \setminus \bigcup_{\gamma \in \Gamma^m} \bigcup_{i=1}^{k} \gamma \cdot \check{P}_i.$$

Proof. As in the proof of 5.12 consider the weakly special $S \subseteq \mathbb{X}^n$, $S = \mathbf{p}(\check{S})$. By 4.8(iii) $S^{(d)} \supseteq \mathrm{pr}S \setminus \bigcup_{i=1}^k P_i$ for some weakly special $P_1, \ldots, P_k \subset \mathbb{X}^m$ of dimension less than that of $\mathrm{pr}S$.

Taking $\check{P}_i$ to be weakly co-special sets associated with the P_i's, we get the required. $\square$

5.14 Definition. $\mathbb{U}_{S_w}$ is the structure with the universe $\mathbb{U}$ and the basic n-ary relations given by the weakly co-special subsets of $\mathbb{U}^n$.

We consider $\mathbb{U}_{S_w}$ a *topological structure* in the sense of [29], with closed subsets defined as finite unions of weakly co-special sets.

By our definitions above, in particular 3.4, there is a dimension notion $\dim_{\mathbb{C}}$ defined for all projective sets (i.e. the constructible sets and their projections).

5.15 We call a weakly co-special subset $\check{S} \subseteq \mathbb{U}^n$ **simple** if it is infinite and has no proper infinite weakly co-special subsets.

In [29], Ch.6 an *analytic Zariski structure* has been defined. In the special case of irreducible one-dimensional structure, which corresponds to a simple case here, a combinatorial pregeometry has been defined and a closure operator cl introduced, see ibid. 6.3.

A more narrow but appropriate definition of an **ω_1-proper Zariski geometry M** is introduced by B.Elsner in [12]. This definition requires that **M** is ω_1-compact (that is whenever $\{A_i : i \in I\}$ is a fintely consistent countable family of constructible sets, $\bigcap_{i \in I} A_i \neq \emptyset$) and the further axioms are:

(Z0) The topology on **M** and its cartesian powers is Noetherian.

(Z1) For any closed $S \subseteq \mathbf{M}^{n+p}$ and a projection $\mathrm{pr} : \mathbf{M}^{n+p} \to \mathbf{M}^n$ there is a countable family R_j, $j \in J$, of closed subsets of $\mathbf{M}^n$ such that $\mathrm{pr}S \supseteq \overline{\mathrm{pr}S} \setminus \bigcup_{j \in J} R_j$ and $\dim R_j < \dim \mathrm{pr}S$.

(Z2) For any simple $Y \subseteq \mathbf{M}^p$ and closed $S \subseteq \mathbf{M}^n \times Y$ there is a number l such that for every $a \in \mathbf{M}^n$ either the fibre $S(a) = Y$ or $|S(a)| \leq l$.

The last axiom (Z3) requires a *presmoothness* condition for simple subsets of **M**. We do not use it below.

Before we formulate and prove the main theorem of this section we need the following result which may be of interest in its own right. The proof of it is obtained jointly with Y.Peterzil.

5.16 Proposition. *Let a be a positive real number and $D \subset \mathbb{C}$ an open domain of the form $\{z \in \mathbb{C} : -a < \mathrm{Re}(z) < a \;\&\; -a < \mathrm{Im}(z) < a\}$, $p : D \to \mathbb{C}$ a holomorphic injective function defined on D and in a small neighborhood around D, and suppose that there is a semi-algebraic 4-ary relation $T(u_1, u_2, u_3, , u_4)$ on $\mathbb{C}$ such that for any $x, y, z, w \in D$*

$$w = x \cdot y + z \Leftrightarrow T(p(w), p(x), p(y), p(z)).$$

Then p is a complex algebraic function on D, that is there is a polynomial $f \in \mathbb{C}[z_1, z_2]$ such that the graph of p is an irreducible analytic component of the analytic subset

$$\{\langle z_1, z_2 \rangle \in D \times \mathbb{C} : f(z_1, z_2) = 0\} \subset D \times \mathbb{C}.$$

Proof. First we note that the structure $(D; <, p,\ w = x \cdot y + z)$ is definable in the canonical o-minimal structure $\mathbb{R}_{an}$ (where $<$ is defined on the interval (a, b))

In the terminology of [32] the ternary relation $w = x \cdot y + z$ defines a *normal family of curves* on the interval $(-a, a)$ in $(D; <, w = x \cdot y + z)$. Hence by the main result of [32] a field $(R_1, +, \cdot)$ definably isomorphic to the field $\mathbb{R}$ of reals is definable in $(D; <, w = x \cdot y + z)$. Since $(D; <, w = x \cdot y + z)$ is semi-algebraic, $(D; <, w = x \cdot y + z)$ is bi-interpretable with $(R_1, +, \cdot)$.

On the other hand, since the map $p : D \to C =: p(D)$ induces an isomorphism,

$$(D; <, w = x \cdot y + z) \to (C; <, T(u_1, u_2, u_3, u_4))$$

(where $<$ on the right is defined on $p(-a, a)$) a field $(R_2, +, \cdot)$ definably isomorphic to the field $\mathbb{R}$ of reals is bi-interpretable with $(C; T(u_1, u_2, u_3, u_4))$. Moreover, this bi-interpretation is given by the same formulas as in the first case and it extends the isomorphism p between the two structures to an isomorphism

$$p : (R_1, +, \cdot) \to (R_2, +, \cdot).$$

Now we have two fields definable and definably isomorphic in $(D; <, p,\ w = x \cdot y + z)$. It is well-known (and easy to prove) that the only field-automorphism definable in an o-minimal expansion of $\mathbb{R}$ is the identity. Hence the isomorphism obtained by composing $i_1 : \mathbb{R} \to R_1$, $p : R_1 \to R_2$ and $i_2^{-1} : R_2 \to \mathbb{R}$ is the identity. Since i_1 and i_2 are semi-algebraic interpretations, we get that $p : R_1 \to R_2$ is semi-algebraic. But this is bi-interpretable with $p : D \to C$. Hence p is semi-algebraic.

Finally, recalling that $p : D \to \mathbb{C}$ is holomorphic, by 5.4 we get the required characterisation of p. $\square$

5.17 Theorem.

(i) $\mathbb{U}_{S_w}$ is an analytic Zariski structure. More precisely, $\mathbb{U}_{S_w}$ is an ω_1-proper Zariski geometry (not necessarily presmooth).

(ii) Let $\check{\mathbb{Y}} \subseteq \mathbb{U}^n$ be a simple weakly co-special set considered a substructure of $\mathbb{U}_{S_w}$. Then the corresponding weakly special set $\mathbb{Y} := \mathbf{p}(\check{\mathbb{Y}})$ is simple and the combinatorial geometry on $\check{\mathbb{Y}}$ is isomorphic to the combinatorial geometry on $\mathbb{Y}$.

(iii) $\check{\mathbb{Y}}$ satisfies the Trichotomy principle, i.e. the geometry on $\check{\mathbb{Y}}$ is either trivial, or linear, or of algebro-geometric type, in which case $\dim_{\mathbb{C}} Y = 1$ and every Zariski closed subset of Y^k, $k \in \mathbb{N}$, is weakly co-special. Moreover, in this case the restriction $\mathbf{p}$ on $\check{\mathbb{Y}}$, $\mathbf{p} : \check{\mathbb{Y}} \to \mathbb{Y}$, is an algebraic map from every irreducible analytic component of $\check{\mathbb{Y}}$ onto the corresponding weakly special subset $\mathbb{Y} \subseteq \mathbb{X}^n$.

Proof. As was noted above $\mathbb{U}_{S_w}$ is a topological structure with a good dimension, in the sense of [29]. The axioms on irreducible components and intersections follow from definition and 5.11. The axiom on projections is given by 5.12. The dimension of fibres condition is 5.13. The rest follow from the fact that our formally analytic sets are actually complex analytic.

The topological structure $\mathbb{U}_{S_w}$ satisfies Elsner's axioms.

First note that $\mathbb{U}_{S_w}$ is ω_1-compact. This is equivalent to the statement that if a constructible set P is a union of countably many constructible subsets P_i, $i \in \mathbb{N}$, then $P = \bigcup_{i=1}^k P_i$ for some $k \in \mathbb{N}$. The latter follows by induction on $\dim_{\mathbb{C}} P$, using the fact that an irreducible analytic set (the copies of which comprise a weakly co-special subset) can not be represented as a countable union of closed subsets of smaller dimension.

(Z0) is given by 5.11 and (Z1) by 5.12. To see (Z2) one notes that the fibre $S(a)$ can be obtained by intersecting the weakly co-special sets $\mathbb{U}^n \times Y$ with $\{a\} \times \mathbb{U}^p$. Now we can use 5.11 which tells us that $S(a)$ is a finite union of weakly co-special subsets, in fact, checking the proof, the number of subsets is bounded by $k \cdot m$, where k is the number of components in $(\mathbb{X}^n \times \mathbf{p}(Y)) \cap (\{\mathbf{p}(a)\} \times \mathbb{X}^p)$, and m is the number of components in the intersection of Zariski closures of $\mathbb{U}^n \times Y$ and $\{a\} \times \mathbb{U}^p$ in $\mathbb{C}^{m(n+p)}$. The first does not depend on a and the second is just $\mathbf{p}(a) \times \mathbb{C}^{mp}$, hence m does not depend on a. But k is also bounded from above as the number of components in a Zariski fibration.

It follows that either $S(a)$ is just $\{a\} \times Y$ or $S(a)$ is the union of at most $k \cdot m$ proper weakly co-special subsets of $\{a\} \times Y$. Since Y is simple the subsets are finite, so singletons. This proves (Z2) and finishes the proof of (i).

(ii) We define *combinatorial dimension* $\mathrm{cdim}(A)$ for finite subsets $A \subset \check{\mathbb{Y}}$ following [12].

$$\mathrm{cdim}(a_1, \ldots, a_n) = \min\{\dim \check{S} : \langle a_1, \ldots, a_n\rangle \in \check{S},\ \check{S} \subseteq_{\mathrm{cl}} \check{Y}^n\}.$$

The original definition in [29] and [12] assumes that S runs among the projective subsets of $\mathbb{U}^n$ but as Elsner notes, under the given axioms (essentially (Z1)) we may assume that S is closed.

Combinatorial dimension gives rise to a closure operator and a pregeometry on $\check{Y}$ given by

$$b \in \mathrm{cl}(a_1, \ldots, a_n) \Leftrightarrow \mathrm{cdim}(a_1, \ldots, a_n) = \mathrm{cdim}(a_1, \ldots, a_n, b).$$

Since weakly special subsets S of $\mathbb{X}^n$ and associated with them weakly co-special subsets $\check{S}$ of $\mathbb{U}^n$ are related by $\mathbf{p}(\check{S}) = S$ we have

$$b \in \mathrm{cl}(a_1, \ldots, a_n) \Leftrightarrow \mathbf{p}(b) \in \mathrm{cl}(\mathbf{p}(a_1), \ldots, \mathbf{p}(a_n)),$$

where cl on the right is defined on $\mathbb{Y}$ by the corresponding condition. This proves (ii)

(iii) The Trichotomy Theorem follows from 4.14.

It remains to prove that in the case the geometry is not locally modular, $\mathbf{p} : \check{\mathbb{Y}} \to \mathbb{Y}$ is an algebraic map.

Suppose $\mathbb{Y}$ is algebro-geometric, that is $\mathbb{Y}$ is an algebraic curve and every Zariski closed subset of $\mathbb{Y}^n$ is weakly special. We also have $\dim_{\mathbb{C}} \check{\mathbb{Y}} = 1$, since $\mathbf{p}$ preserves dimension.

Let $C \subset \check{Y} \cap \mathbb{F}^n$ be an open subset such that $\mathbf{p} : C \to \mathbf{p}(C)$ is bi-holomorphic.

Let $f : \mathbb{Y} \to \mathbb{C}$ be given by a rational map. We may choose C and f so that f has no singularities on $\mathbf{p}(C)$ and so f^{-1} exists as an algebraic function on $f(\mathbf{p}(C)) \subset \mathbb{C}$. We may assume $0 \in f(\mathbf{p}(C))$ and choose a neighbourhood $D \subset f(\mathbf{p}(C))$ of 0 of the form described in 5.16. We can even adjust C so that $D = f(\mathbf{p}(C))$.

Now consider the holomorphic isomorphism $p = \mathbf{p}^{-1} \circ f^{-1} : D \to C$. Since $\check{\mathbb{Y}}$ is an analytic subset of the complex manifold, we have an induced embedding $C \subset \mathbb{C}$, so we may assume $p : D \to \mathbb{C}$. Finally, we note that the image $T \subset \mathbb{C}^4$ under p of the Zariski closed set

$$R = \{\langle w, x, y, z\rangle \in D^4 : w = xy + z\}$$

is semi-algebraic, since the Zariski closed subset $f^{-1}(R)$ of Y^4 is weakly special.

We have now satisfied all the assumptions of 5.16. Hence p on D is algebraic. Hence $\mathbf{p}$ on C is algebraic. But then $\mathbf{p}$ is algebraic on the irreducible analytic component of $\check{Y}$ which contains C. Since all irreducible analytic components are conjugated by semi-algebraic transformations γ (see 5.6) we proved (iii) and the theorem. $\square$

6 Special and co-special sets and points

In this section we use the extra assumtions.

Assumption D. (i) the Zariski closure of $\mathbb{U}$ in $\mathbb{C}^m$ is defined over $\mathbb{Q}$.
(ii) $\mathbb{X}$ is defined over $\tilde{\mathbb{Q}}$,

6.1 Definitions.
Call an $S \subset \mathbb{X}^n$ **strongly special** if
(i) S is weakly special,
(ii) S is defined over $\tilde{\mathbb{Q}}$,
(iii) for some weakly co-special $\check{S}$ corresponding to S the Zariski closure $\check{S}^{\text{Zar}}$ is defined over $\mathbb{Q}$.

We denote $\mathcal{S}_s$ the family of strongly special sets.

Define $\mathcal{S}$, the family of **special** sets to be the minimal family of sets containing $\mathcal{S}_s$ and closed under cartesian products, intersections, Zariski closures of projections and taking irreducible components. A **special point** is a singleton set which is special.

Clearly,

$$\mathcal{S}_s \subseteq \mathcal{S} \subseteq \mathcal{S}_w.$$

We denote $\mathbb{X}_S$ the structure on $\mathbb{X}$ with n-ary relations given by special subsets of $\mathbb{X}^n$.

6.2 Example. Consider the algebraic torus $\mathbb{C}^*$ as $\mathbb{X}$ and $\exp : \mathbb{C} \to \mathbb{C}^*$ as $\mathbf{p}$.

By Lindemann the trivial special point 1 is the only strongly special point and 0 a strongly pro-special point.

Weakly special sets will be cosets of tori. The strongly special sets are exactly tori, that is 0-definable connected algebraic subgroups of $(\mathbb{C}^*)^n$.

Now the special sets by our definition are exactly Zariski closed subsets 0-definable in the multiplicative group $(\mathbb{C}^*; \cdot, 1)$. The set of points defined by the condition $x^k = 1$ (k-torsion points) is definable by $\exists y x^k = y \ \&\ y = 1$, and so any torsion point is special. It is easy to conclude that special sets are exactly the torsion cosets of tori.

6.3 Proposition. *$\mathbb{X}_S$ is a Noetherian Zariski structure, if we consider any singleton in $\mathbb{X}$ to be closed.*

(ii) The simple special sets satisfy the Trichotomy.

Proof. Immediate from 4.11 and 4.14.$\square$

6.4 Example. Let $\mathbb{X}$ be a Shimura variety. The main theorem of [31] characterises weakly special subvarieties in the sense of Shimura as weakly special in the sense above. Shimura-special points are strongly special in our sense, and Shimura-special sets are those which are weakly special and contain a special point. It follows that special sets and special points in both senses are the same.

6.5 Conjecture A. Any weakly special set is definable with parameters in $\mathbb{X}_S$. Equivalently, every weakly special subset $P \subseteq \mathbb{X}^n$ is a fibre of a special subset $S \subseteq \mathbb{X}^{n+m}$ under the projection $\mathbb{X}^{n+m} \to \mathbb{X}^n$.

6.6 Note that the reference in 6.4 confirms the conjecture A for Shimura varieties.

6.2 confirms the conjecture A in case of the algebraic tori $\mathbb{G}_m(\mathbb{C})^n$.

Special sets are also well understood for Abelian varieties and the conjecture A can be confirmed in this class as well.

For the general mixed Shimura varieties to the best of our knowledge this is open but may be well within the reach of conventional methods.

6.7 Conjecture B. Given a special subset $S \subseteq \mathbb{X}^n$, the set of special points in S is Zariski dense in S.

This important property of special points and sets in mixed Shimura varieties was pointed out to the author by E.Ullmo.

Using the fact that $\mathbb{X}_S$ has elimination of quantifiers it is easy to see that Conjectures A and B together are equivalent to the following model theoretic conjecture.

Conjecture AB. Let $\mathbb{X}_S^0$ be the substructure of $\mathbb{X}_S$ the universe of which is the set of special points. Then

$$\mathbb{X}_S^0 \preccurlyeq \mathbb{X}_S.$$

Conjecture Z-P. *For any algebraic subvariety $V \subseteq \mathbb{X}^n$ there is a finite list of special subvarieties $S_1, \ldots, S_m \subsetneq \mathbb{X}^n$ such that, given an arbitrary special subvariety $T \subset \mathbb{X}^n$ and an irreducible component W of the intersection $V \cap T$, either* $\dim W = \dim V + \dim T - \dim \mathbb{X}^n$ *(a typical case), or $W \subseteq S_i$ for some $i = 1, \ldots, m$ (in the atypical case* $\dim W > \dim V + \dim T - \dim \mathbb{X}^n$*).*

See comments in section 2, 2.15.

6.8 Remark. The validity of Conjecture AB implies that $\mathbb{X}_S^0$ is a Zariski structure (without extra requirement that the singletons to be closed).

6.9 Definition. A **co-special subset** of $\mathbb{U}^n$ is a weakly co-special subset $\check{S}$ corresponding to a special subset $S \subseteq \mathbb{X}^n$.

Correspondingly, a **period** is a point in $\mathbf{p}^{-1}(s)$ for $s \in \mathbb{X}$ a special point.

We denote $\mathbb{U}_S$ the structure on $\mathbb{U}$ with n-ary relations given by co-special subsets of $\mathbb{U}^n$.

6.10 Proposition. *$\mathbb{U}_S$ is an ω-proper Zariski structure, if we consider closed any singleton in $\mathbb{U}$.*

Proof. Immediate from 5.17(i), since by definition the family of co-special sets are closed under taking irreducible components of intersections. □

6.11 Definition. Given $u_1, \ldots, u_n \in \mathbb{U}$ we define the **special locus** of $\langle u_1, \ldots, u_n \rangle$ to be the minimal co-special subset $\check{S}$ of $\mathbb{U}^n$ containing $\langle u_1, \ldots, u_n \rangle$ and definable over $\tilde{\mathbb{Q}}$. Write in this case $\check{S} = \text{Sp.locus}(u_1, \ldots, u_n)$.

We define a (combinatorial) dimension for tuples of points in $\mathbb{U}$ by setting, for $u_1, \ldots, u_n \in \mathbb{U}$

$$d_{\text{spec}}(u_1, \ldots, u_n) = \dim_{\mathbb{C}} \text{Sp.locus}(u_1, \ldots, u_n).$$

6.12 Examples. If $\mathbf{p} = \mathrm{id}$ and $\mathbb{U} = \mathbb{X}$ then special and co-special subsets of $\mathbb{X}^n$ and $\mathbb{U}^n$ are exactly the Zariski closed subsets defined over $\tilde{\mathbb{Q}}$. In this case $d_{\mathrm{spec}}(u_1, \ldots, u_n) = \mathrm{tr.deg}_{\mathbb{Q}}(u_1, \ldots, u_n)$.

If $\mathbf{p} = \exp$, $\mathbb{U} = \mathbb{C}$ and $\mathbb{X} = \mathbb{C}^*$, then the pro-special subsets of $\mathbb{C}^n$ are given by systems of equations of the form

$$m_1 u_1 + \ldots + m_n u_n = 2\pi i k$$

for $m_1, \ldots, m_n, k \in \mathbb{Z}$. These are defined over $\tilde{\mathbb{Q}}$ if and only if $k = 0$. Hence

$$d_{\mathrm{spec}}(u_1, \ldots, u_n) = \mathrm{ldim}_{\mathbb{Q}}(u_1, \ldots, u_n),$$

the $\mathbb{Q}$-linear dimension.

Recall that Schanuel's conjecture can be stated as the following

$$\mathrm{tr.deg}_{\mathbb{Q}}(u_1, \ldots, u_n, e^{u_1}, \ldots, e^{u_n}) - \mathrm{ldim}_{\mathbb{Q}}(u_1, \ldots, u_n) \geq 0.$$

Now we have all ingredients to state the most general form of such a conjecture.

6.13 Conjecture D. Under the definitions above

$$\mathrm{tr.deg}_{\mathbb{Q}}(u_1, \ldots, u_n, \mathbf{p}(u_1), \ldots, \mathbf{p}(u_n)) - d_{\mathrm{spec}}(u_1, \ldots, u_n) \geq 0.$$

References

[1] Y. André, **Une introduction aux motifs**, Panoramas et Synthèses 17, 2004

[2] M.Bays and B.Zilber, *Covers of Multiplicative Groups of Algebraically Closed Fields of Arbitrary Characteristic.* Bull. London Math. Soc. 43 (4) (2011), 689–702

[3] M.Bays and J.Kirby, *Excellence and uncountable categoricity of Zilber's exponential fields,* arXiv:1305.0493 (2013)

[4] M.Bays, B.Hart, T.Hyttinen, M.Kesaala and J.Kirby, *Quasiminimal structures and excellence*, arXiv:1210.2008v2 (2013)

[5] M.Bays, **Categoricity Results for Exponential Maps of 1-Dimensional Algebraic Groups and Schanuel Conjectures for Powers and the CIT**. PhD thesis, Oxford University, 2009.

[6] M.Bays, B.Hart and A.Pillay, *Universal covers of commutative finite Morley rank groups* In preparation

[7] M.Bays, M.Gavrilovich and M.Hils, *Some Definability Results in Abstract Kummer Theory*, International Mathematics Research Notices,

[8] C. Bertolin. *Périodes de 1-motifs et transcendance.* J. Number Theory, 97(2):204–221, 2002

[9] , A. Borel,**Introduction aux groupes arithmétiques**, Publications de l'Institut de mathématique de l'Université de Strasbourg (15), 1969

[10] G. Boxall, D. Bradley-Williams, C. Kestner, A. Aziz, and D. Penazzi, *Weak one-basedness* Modnet preprints, 2011.

[11] P.Deligne, *Theorié de Hodge II.* Publ.Math. IHES, v.40 (1972), 5–52.

[12] B.Elsner, **Presmooth geometries**, DPhil Thesis, University of Oxford, 2013

[13] A.Harris, *Categoricity of the two sorted j-function*, arXiv:

[14] L.Haykazyan *Categoricity in Quasiminimal Pregeometry Classes*

[15] E.Hrushovski and B.Zilber, *Zariski geometries*, Journal of AMS, 9(1996), 1-56

[16] J.Kirby, *On Quasiminimal Excellent Classes*, Journal of Symbolic Logic, 75 (2010), no. 2, 551-564

[17] J.Kirby, *The theory of the exponential differential equations of semiabelian varieties*, Selecta Mathematica 15, (2009) no. 3, 445–486

[18] Kontsevich and Zagier, *Periods*, In **Mathematics unlimited 2001 and beyond**, Springer, 2001, 771-808

[19] S. Lang, **Fundamentals of Diophantine geometry**. Springer- Verlag, New York, 1983

[20] M.Larsen, *A Mordell-Weil theorem for abelian varieties over fields generated by torsion points.* 2005. arXiv:math/0503378v1

[21] E.Hrushovski, *A new strongly minimal set,* Annals of Pure and Applied Logic 62 (1993) 147-166

[22] B.Zilber, *Analytic and pseudo-analytic structures*, Logic Coll. 2000 Paris, Lect.Notes in Logic v. 19, 2005, eds. R.Cori, A.Razborov, S.Todorcevic and C.Wood, AK Peters, Wellesley, Mass. pp.392-408

[23] B.Zilber, *Pseudo-exponentiation on algebraically closed fields of characteristic zero*, Annals of Pure and Applied Logic, Vol 132 (2004) 1, pp 67-95

[24] B.Zilber, *A categoricity theorem for quasi-minimal excellent classes,* In: **Logic and its Applications,** eds. Blass and Zhang, Contemp. Maths, v.380, 2005, pp.297-306

[25] B.Zilber, *Covers of the multiplicative group of an algebraically closed field of characteristic zero,* J. London Math. Soc. (2), 74(1):41–58, 2006

[26] J.Kirby and B.Zilber, *Exponential fields and atypical intersections,* Submitted, arXiv:1108.1075

[27] B.Zilber, *Model theory, geometry and arithmetic of the universal cover of a semi-abelian variety.* In **Model Theory and Applications,** pp.427-458, Quaderni di matematica, v.11, Napoli, 2005

[28] B.Zilber, *Exponential sums equations and the Schanuel conjecture*, J.London Math.Soc.(2) 65 (2002), 27-44

[29] B.Zilber, **Zariski geometries**, CUP, 2010

[30] L. van den Dries, **Tame Topology and o-minimal Structures**, CUP, 1998

[31] E.Ullmo and A.Yafaev, *A characterisation of special subvarieties*, web

[32] Y.Peterzil and S. Starchenko, *A trichotomy theorem for o-minimal structures*, Proceedings LMS, 77 (3) 1998,

[33] Y.Peterzil and S. Starchenko, *Uniform definability of the Weierstrass P functions and generalized tori of dimension one*, Selecta Mathematica, New Series, 10 (2004) pp. 525-550.

[34] Y.Peterzil and S. Starchenko, *Complex analytic geometry in nonstandard setting*, in **Model Theory with Applications to Algebra and Analysis**, ed. Z. Chatzidakis, D. Macpherson, A. Pillay and A. Wilkie, 117-166.