Ringel–Hall style vertex algebra and Lie algebra structures on the homology of moduli spaces

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Contents

1 Introduction

This is an incomplete first draft. I hope the eventual finished document will have three parts:

- I Construction of vertex algebras and Lie algebras in Algebraic Geometry: the theory, written in terms of homology of Artin stacks.
- II Application of Part I in examples, in both Geometric Representation Theory (categories of quiver representations), and Algebraic Geometry (categories of coherent sheaves). Discussion of virtual classes, enumerative invariants, and wall-crossing formulae.
- III Construction of vertex algebras and Lie algebras in Differential Geometry, on the homology of moduli spaces of connections. Application to enumerative invariants counting instanton-type moduli spaces, including $\mathrm{SU}(n)$-Donaldson theory of 4-manifolds for all $n\geqslant 2$, Seiberg–Witten invariants, Hermitian–Einstein connections on Kähler manifolds, G_2-instantons, and Spin(7)-instantons.

Despite its length, the present document is only a first draft of Part I and the first half of Part II.

To explain the basic idea, we first review Ringel–Hall algebras. There are several versions, beginning with Ringel [135–138]; we discuss the constructible functions version, following the author [74–79].

Let $\mathbb{K}$ be an algebraically closed field of characteristic zero, and write $\mathbf{Art}_{\mathbb{K}}^{\mathrm{lft}}$ for the 2-category of Artin $\mathbb{K}$-stacks locally of finite type. For each object S in $\mathbf{Art}_{\mathbb{K}}^{\mathrm{lft}}$ we can consider the $\mathbb{Q}$-vector space $\mathrm{CF}(S)$ of *constructible functions* $\alpha : S(\mathbb{K}) \to \mathbb{Q}$, as in [74]. If $f : S \to T$ is a representable morphism in $\mathbf{Art}_{\mathbb{K}}^{\mathrm{lft}}$ we can define the *pushforward* $f_* : \mathrm{CF}(S) \to \mathrm{CF}(T)$, and if $g : S \to T$ is a finite type morphism in $\mathbf{Art}_{\mathbb{K}}^{\mathrm{lft}}$ we can define the *pullback* $g^* : \mathrm{CF}(T) \to \mathrm{CF}(S)$. Pushforwards and pullbacks are functorial, and have a commutative property for 2-Cartesian squares in $\mathbf{Art}_{\mathbb{K}}^{\mathrm{lft}}$.

Let $\mathcal{A}$ be a $\mathbb{K}$-linear abelian category, satisfying some conditions. Write $\mathcal{M}$ for the moduli stack of objects E in $\mathcal{A}$, and $\mathfrak{Exact}$ for the moduli stack of short exact sequences $0 \to E_1 \to E_2 \to E_3 \to 0$ in $\mathcal{A}$. Then $\mathcal{M}, \mathfrak{Exact}$ are Artin $\mathbb{K}$-stacks, with morphisms $\Pi_1, \Pi_2, \Pi_3 : \mathfrak{Exact} \to \mathcal{M}$ such that Π_i maps $[0 \to E_1 \to E_2, \to E_3 \to 0] \mapsto [E_i]$. The conditions on $\mathcal{A}$ imply that $\mathcal{M}, \mathfrak{Exact}$ are locally of finite type, and Π_2 is representable, and $(\Pi_1, \Pi_3) : \mathfrak{Exact} \to \mathcal{M} \times \mathcal{M}$ is of finite type. We define a $\mathbb{Q}$-bilinear operation $* : \mathrm{CF}(\mathcal{M}) \times \mathrm{CF}(\mathcal{M}) \to \mathrm{CF}(\mathcal{M})$ by

$$f * g = (\Pi_2)_* \circ (\Pi_1, \Pi_3)^*(f \boxtimes g), \tag{1.1}$$

as in [77]. Then $*$ is associative, and makes $\mathrm{CF}(\mathcal{M})$ into a $\mathbb{Q}$-algebra with unit $\delta_{[0]}$, which we call a *Ringel–Hall algebra*. Hence $\mathrm{CF}(\mathcal{M})$ is also a *Lie algebra* over $\mathbb{Q}$, with Lie bracket $[f, g] = f * g - g * f$.

Now let $K(\mathcal{A})$ be a quotient of the Grothendieck group $K_0(\mathcal{A})$ of $\mathcal{A}$ such that $\mathcal{M}$ splits as $\mathcal{M} = \coprod_{\alpha \in K(\mathcal{A})} \mathcal{M}_\alpha$ with $\mathcal{M}_\alpha \subset \mathcal{M}$ the open and closed $\mathbb{K}$-substack of objects E in $\mathcal{A}$ with class $[\![E]\!] = \alpha$ in $K(\mathcal{A})$, let τ be a stability condition on $\mathcal{A}$ in the sense of [78] factoring via $K(\mathcal{A})$, and suppose that the open substack $\mathcal{M}_\alpha^{\mathrm{ss}}(\tau) \subseteq \mathcal{M}_\alpha$ of τ-semistable objects in class α is of finite type for all $\alpha \in K(\mathcal{A})$. Then the characteristic function $\delta_{\mathcal{M}_\alpha^{\mathrm{ss}}(\tau)}$ is an element of the Ringel–Hall algebra $(\mathrm{CF}(\mathcal{M}), *)$.

Let $\tilde\tau$ be another stability condition on $\mathcal{A}$, satisfying the same conditions. In [79] the author proved a universal wall-crossing formula, which writes $\delta_{\mathcal{M}_\alpha^{\mathrm{ss}}(\tilde\tau)}$ as a sum of products of $\delta_{\mathcal{M}_\beta^{\mathrm{ss}}(\tau)}$ in the Ringel–Hall algebra $(\mathrm{CF}(\mathcal{M}), *)$. By [79, Th. 5.4], an alternative version of the wall-crossing formula, for elements $\epsilon_\alpha(\tau) \in \mathrm{CF}(\mathcal{M})$ defined using the $\delta_{\mathcal{M}_\alpha^{\mathrm{ss}}(\tau)}$, makes sense solely in the Lie algebra $(\mathrm{CF}(\mathcal{M}), [\,,\,])$. All this was applied to wall-crossing for Donaldson–Thomas invariants of Calabi–Yau 3-folds in [81].

In this book, given a suitable $\mathbb{K}$-linear abelian category $\mathcal{A}$ (or $\mathbb{K}$-linear triangulated category $\mathcal{T}$), writing $\mathcal{M}$ for the moduli stack of objects in $\mathcal{A}$ (or $\mathcal{T}$), we first define the structure of a *graded vertex algebra* on the homology $H_*(\mathcal{M})$ of $\mathcal{M}$ over a commutative ring R, in the sense of Kac [85] and Frenkel and Ben-Zvi [46]. Vertex algebras are complicated algebraic structures arising in Conformal Field Theory in Mathematical Physics.

Roughly speaking, we will also define a (graded) (super) Lie bracket $[\,,\,]$ on $H_*(\mathcal{M})$, making $H_*(\mathcal{M})$ into a (graded) Lie (super)algebra over R. (Actually, we define $[\,,\,]$ either on a modification $H_*(\mathcal{M})^{t=0}$ of $H_*(\mathcal{M})$, or on the homology $H_*(\mathcal{M}^{\rm pl})$ of a modification $\mathcal{M}^{\rm pl}$ of $\mathcal{M}$.) So we replace $\mathrm{CF}(\mathcal{M})$ by $H_*(\mathcal{M})$ in the usual Ringel–Hall (Lie) algebra construction. These Lie algebras $(H_*(\mathcal{M}),[\,,\,])$ have interesting applications, including wall-crossing formulae under change of stability condition for virtual cycles in enumerative invariant problems.

In the analogue of (1.1) defining the Lie bracket $[\,,\,]$ on $H_*(\mathcal{M})$, the pushforward $(\Pi_2)_*$ is natural on homology, but the pullback $(\Pi_1,\Pi_3)^*$ is more complex, and we need extra data to define it, a perfect complex $\Theta^\bullet$ on $\mathcal{M}\times\mathcal{M}$ satisfying some conditions, similar in spirit to an obstruction theory in the sense of Behrend and Fantechi [15]. In our examples there are natural choices for $\Theta^\bullet$.

As above we write $\mathcal{M}=\coprod_{\alpha\in K(\mathcal{A})}\mathcal{M}_\alpha$. Part of the data is a symmetric biadditive map $\chi:K(\mathcal{A})\times K(\mathcal{A})\to\mathbb{Z}$ with $\operatorname{rank}\Theta^\bullet\vert_{\mathcal{M}_\alpha\times\mathcal{M}_\beta}=\chi(\alpha,\beta)$, called the *Euler form*. We define a shifted grading $\tilde H_*(\mathcal{M}_\alpha)$ on $H_*(\mathcal{M}_\alpha)$ by

$$\tilde H_i(\mathcal{M}_\alpha)=H_{i+2-\chi(\alpha,\alpha)}(\mathcal{M}_\alpha).$$

We can interpret $2-\chi(\alpha,\alpha)$ as the virtual dimension of the 'projective linear' moduli stack $\mathcal{M}_\alpha^{\rm pl}$ below. This induces a shifted grading $\tilde H_*(\mathcal{M})$ on $H_*(\mathcal{M})$. Then our Lie brackets $[\,,\,]$ on $H_*(\mathcal{M})$ are graded with respect to the shifted grading $\tilde H_*(\mathcal{M})$, that is, $[\,,\,]$ maps $\tilde H_k(\mathcal{M}_\alpha)\times\tilde H_l(\mathcal{M}_\beta)\to\tilde H_{k+l}(\mathcal{M}_{\alpha+\beta})$.

As for Ringel–Hall algebras, there are actually *many versions* of this Lie algebra construction, all variations on the same basic theme. In §3.3–§3.7 we explain five versions, which admit further modifications as in §3.8:

(i) For the '$t=0$' version in §3.3, we define a graded representation $\diamond$ of $R[t]$ on $H_*(\mathcal{M})$, where t has degree 2, and we define

$$H_*(\mathcal{M})^{t=0}=H_*(\mathcal{M})/\bigl(\langle t,t^2,\ldots\rangle_R\diamond H_*(\mathcal{M})\bigr).$$

Then we define a graded Lie bracket $[\,,\,]^{t=0}$ on $\tilde H_*(\mathcal{M})^{t=0}$.

(ii) For the 'projective linear' version in §3.4, we work with a modified moduli stack $\mathcal{M}^{\rm pl}$ parametrizing nonzero objects E in $\mathcal{A}$ or $\mathcal{T}$ up to 'projective linear' isomorphisms, meaning that we quotient out by isomorphisms $\lambda\,\mathrm{id}_E:E\to E$ for $\lambda\in\mathbb{G}_m$. The $\mathbb{K}$-points $[E]$ of $\mathcal{M}^{\rm pl}$ are the same as those of $\mathcal{M}'=\mathcal{M}\setminus\{[0]\}$, but while $\mathcal{M}'$ has isotropy groups $\mathrm{Iso}_{\mathcal{M}'}([E])=\mathrm{Aut}(E)$, $\mathcal{M}^{\rm pl}$ has isotropy groups $\mathrm{Iso}_{\mathcal{M}^{\rm pl}}([E])=\mathrm{Aut}(E)/\mathbb{G}_m\cdot\mathrm{id}_E$. There is a morphism $\Pi^{\rm pl}:\mathcal{M}'\to\mathcal{M}^{\rm pl}$ which is a fibration with fibre $[*/\mathbb{G}_m]$.

Under some assumptions (including R a $\mathbb{Q}$-algebra) we can show that $H_*(\mathcal{M}')^{t=0}\cong H_*(\mathcal{M}^{\rm pl})$, giving a geometric interpretation of $H_*(\mathcal{M}')^{t=0}$. In the general case, depending on some partially conjectural assumptions, we can define a graded Lie bracket $[\,,\,]^{\rm pl}$ on $\tilde H_*(\mathcal{M}^{\rm pl})$, such that $(\Pi^{\rm pl})_*:\tilde H_*(\mathcal{M}')^{t=0}\to\tilde H_*(\mathcal{M}^{\rm pl})$ is a Lie algebra morphism.

(iii) For the 'positive rank' version in §3.5, we choose a 'rank' morphism $\mathrm{rk}:K(\mathcal{A})\to\mathbb{Z}$, we set $\mathcal{M}^{\mathrm{rk}>0}=\coprod_{\alpha\in K(\mathcal{A}):\mathrm{rk}\,\alpha>0}\mathcal{M}_\alpha\subset\mathcal{M}$, and we define a graded Lie bracket $[\,,\,]^{\mathrm{rk}>0}$ on homology $\tilde H_*(\mathcal{M}^{\mathrm{rk}>0})$ over a $\mathbb{Q}$-algebra R.

(iv) For the 'mixed' version in §3.6, we again choose $\mathrm{rk}: K(\mathcal{A}) \to \mathbb{Z}$, we set

$$\begin{aligned}\tilde{H}_*(\mathcal{M})^{\mathrm{mix}} = \bigl(&\textstyle\bigoplus_{\alpha\in K(\mathcal{A}):\mathrm{rk}\,\alpha=0} R[s]\otimes_R \tilde{H}_*(\mathcal{M}_\alpha)^{t=0}\bigr)\\ &\oplus \bigl(\textstyle\bigoplus_{\alpha\in K(\mathcal{A}):\mathrm{rk}\,\alpha\neq 0} \tilde{H}_*(\mathcal{M}_\alpha)\bigr),\end{aligned}$$

where s is graded of degree 2, and we define a graded Lie bracket $[\,,\,]^{\mathrm{mix}}$ on $\tilde{H}_*(\mathcal{M})^{\mathrm{mix}}$. This combines the '$t=0$' version when $\mathrm{rk}\,\alpha=0$ with the 'positive rank' version when $\mathrm{rk}\,\alpha\neq 0$.

(v) For the 'fixed determinant' versions in §3.7, we assume we are given a group stack $\mathcal{P}$ called the *Picard stack* with isotropy groups $\mathbb{G}_m$, and a 'determinant' morphism $\det: \mathcal{M}\to\mathcal{P}$ satisfying some assumptions. We show that all of (i)–(iv) still work when we replace $\mathcal{M}$ by the substack $\mathcal{M}^{\mathrm{fpd}} = \mathcal{M}\times_{\mathcal{P}^{\mathrm{pl}}} *$ of objects in $\mathcal{M}$ with 'fixed projective determinant', and also discuss $\mathcal{M}^{\mathrm{fd}} = \mathcal{M}\times_{\mathcal{P}} *$ and $\mathcal{M}^{\mathrm{pfd}} = \mathcal{M}^{\mathrm{pl}}\times_{\mathcal{P}^{\mathrm{pl}}} *$.

The author regards the 'projective linear' version as the primary one.

Our definitions of Lie algebras work in great generality. For example, if X is *any* smooth projective $\mathbb{K}$-scheme and $\mathcal{A}=\mathrm{coh}(X)$, or $\mathcal{T}=D^b\mathrm{coh}(X)$, the Lie algebra constructions above apply, though for applications involving enumerative invariants and virtual classes we need to restrict X to be a curve, or surface, or Calabi–Yau 3- or 4-fold, or Fano 3-fold. Similarly, the Lie algebras on moduli spaces of connections we define in part III work on *any* compact spin manifold, though in applications we restrict to 4-manifolds, G_2-manifolds, etc.

So far as the author can tell, the core idea behind our vertex algebra and Lie algebra constructions is new. But it is related to, or similar to, work by other authors. Probably the closest to ours is work by Grojnowski [57] and Nakajima [118–122] discussed in §6, which defines representations of interesting Lie algebras on the homology of Hilbert schemes of points on surfaces. We can explain their results by realizing their Lie algebras in the homology of complexes of dimension 0 sheaves.

The author envisages four main areas of application of our theory:

(a) In Geometric Representation Theory, as a way of producing examples of interesting Lie algebras and their representations, starting from abelian or derived categories such as categories of quiver representations.

(b) In the study of (co)homology of moduli spaces in Algebraic Geometry, for example by using facts from representation theory of infinite-dimensional Lie algebras to explain modular properties of generating functions of Betti numbers of Hilbert schemes, moduli spaces of vector bundles, etc.

(c) In the study of enumerative invariants 'counting' coherent sheaves on projective varieties: Mochizuki's invariants counting sheaves on surfaces [115], Donaldson–Thomas invariants of Calabi–Yau 3-folds and Fano 3-folds [81, 91, 92, 146], and Donaldson–Thomas type invariants counting sheaves on Calabi–Yau 4-folds [20, 30, 31].

As we explain in §7, we can write the virtual classes defining invariants as elements of $H_*(\mathcal{M}^{\mathrm{pl}})$, and then (conjecturally) we can write wall-crossing formulae for these virtual classes under change of stability condition using the Lie bracket $[\,,\,]^{\mathrm{pl}}$ and the universal wall-crossing formula of [79].

(d) There should be a Differential-Geometric version of (c) for enumerative invariants in gauge theory, such as Donaldson invariants of 4-manifolds. For example, we hope to define $\mathrm{U}(n)$ and $\mathrm{SU}(n)$ Donaldson invariants of oriented Riemannian 4-manifolds (X,g) with $b^2_+(X)\geqslant 1$, and give wall-crossing formulae for them when $b^2_+(X)=1$.

In Part I, we begin in §2 with background on Artin stacks and higher stacks, and discuss homology and cohomology of stacks. Section 3 constructs the vertex algebra structures and Lie brackets on the homology of moduli spaces that are our main concern. Proofs of the main results of §3 are postponed to §4. In §2–§3 we take an axiomatic approach, stating Assumptions on both (co)homology theories $H_*(-),H^*(-)$ of stacks, and on the abelian category $\mathcal{A}$ or triangulated category $\mathcal{T}$, and then proving theorems about them.

Part II studies examples of the constructions in Part I in detail. The current incomplete version only covers abelian categories $\mathcal{A}=\text{mod-}\mathbb{C}Q$ and derived categories $\mathcal{T}=D^b\,\text{mod-}\mathbb{C}Q$ of representations of a quiver Q. In the triangulated category case, the graded vertex algebras are lattice vertex algebras, and the graded Lie algebras naturally contain Kac–Moody algebras. Future versions will discuss quivers with relations, dg-quivers, and (derived) categories $\mathcal{A}=\mathrm{coh}(X)$, $\mathcal{T}=D^b\mathrm{coh}(X)$ of coherent sheaves on a smooth projective $\mathbb{K}$-scheme X.

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Part I
Vertex and Lie algebras in Algebraic Geometry

2 Background material

2.1 Lie algebras and graded Lie algebras

We discuss Lie algebras and graded Lie algebras. Some good references are Humphreys [66] and Kac [84]. Note that our graded Lie algebras are examples of Lie *superalgebras*; for brevity we usually avoid using the prefix 'super-'.

2.1.1 Basic definitions

Definition 2.1. Let R be a commutative ring. A *Lie algebra* over R is a pair $(V,[\,,\,])$, where V is an R-module, and $[\,,\,]:V\times V\rightarrow V$ is an R-bilinear map called the *Lie bracket*, which satisfies the identities for all $u,v,w\in V$:

$$[v,u]=-[u,v], \tag{2.1}$$

$$[[u,v],w]+[[v,w],u]+[[w,u],v]=0, \tag{2.2}$$

Here (2.1) is *antisymmetry*, and (2.2) is the *Jacobi identity.* If we assume (2.1) then (2.2) is equivalent to

$$[[u,v],w]-[u,[v,w]]-[v,[u,w]]=0, \tag{2.3}$$

which we will also call the Jacobi identity.

A *representation* of a Lie algebra $(V,[\,,\,])$ is an R-module W and an R-bilinear map $[\,,\,]:V\times W\rightarrow W$ satisfying (2.3) for all $u,v\in V$ and $w\in W$. Note that (2.1)–(2.2) do not make sense for representations.

Here is the graded version of Definition 2.1:

Definition 2.2. Let R be a commutative ring. A *graded Lie algebra* over R (also called a *graded Lie superalgebra* over R) is a pair $(V_*,[\,,\,])$, where $V_*=\bigoplus_{a\in\mathbb{Z}}V_a$ is a graded R-module, and $[\,,\,]:V_*\times V_*\rightarrow V_*$ is an R-bilinear map called the *Lie bracket*, which is graded (that is, $[\,,\,]$ maps $V_a\times V_b\rightarrow V_{a+b}$ for all $a,b\in\mathbb{Z}$), and satisfies the identities for all $a,b,c\in\mathbb{Z}$ and $u\in V_a$, $v\in V_b$ and $w\in V_c$:

$$[v,u]=(-1)^{ab+1}[u,v], \tag{2.4}$$

$$(-1)^{ca}[[u,v],w]+(-1)^{ab}[[v,w],u]+(-1)^{bc}[[w,u],v]=0. \tag{2.5}$$

Here (2.4) is *graded antisymmetry*, and (2.5) is the *graded Jacobi identity.*

If we assume (2.4) then (2.5) is equivalent to

$$[[u,v],w]-[u,[v,w]]+(-1)^{ab}[v,[u,w]]=0, \tag{2.6}$$

which we will also call the graded Jacobi identity. Note that because of the signs $(-1)^{ab},(-1)^{bc},(-1)^{ca}$ in (2.4)–(2.6), graded Lie algebras are examples of

Lie superalgebras. The subspaces V_0 and $V_{\mathrm{even}} = \bigoplus_{k\in\mathbb{Z}} V_{2k}$ are ordinary Lie algebras. Any Lie algebra is a graded Lie algebra concentrated in degree 0.

A *representation* of a graded Lie algebra $(V_*, [\,,\,])$ is a graded R-module $W_* = \bigoplus_{c\in\mathbb{Z}} W_c$ and an R-bilinear map $[\,,\,] : V_* \times W_* \to W_*$ which is graded (that is, $[\,,\,]$ maps $V_a \times W_b \to W_{a+b}$ for all $a, b \in \mathbb{Z}$), satisfying (2.6) for all $a, b, c \in \mathbb{Z}$ and $u \in V_a$, $v \in V_b$ and $w \in W_c$.

2.1.2 Kac–Moody algebras

Kac–Moody algebras are an important class of Lie algebras, as in Kac [84], that will occur in our examples in §5. We summarize parts of their theory.

Definition 2.3. Let Q_0 be a finite set, and $A = (a_{vw})_{v,w\in Q_0}$ be an integer matrix on Q_0. We call A a *symmetric generalized Cartan matrix* if $a_{vv} = 2$ for all $v \in Q_0$, and $a_{vw} = a_{wv} \leqslant 0$ for all $v \neq w$ in Q_0. Let R be a field of characteristic zero. As in Kac [84, §0.3], the associated (*derived*) *Kac–Moody algebra* $\mathfrak{g}'(A)$ over R is the Lie algebra over R with generators e_v, f_v, h_v for $v \in Q_0$, and the relations

$$\begin{gathered}
[h_v, h_w] = 0, \quad [e_v, f_w] = 0 \quad \text{if } v \neq w, \quad [e_v, f_v] = h_v, \\
[h_v, e_w] = a_{vw} e_w, \quad [h_v, f_w] = -a_{vw} f_w, \\
(\operatorname{ad} e_v)^{1-a_{vw}}(e_w) = 0 \quad \text{if } v \neq w, \quad (\operatorname{ad} f_v)^{1-a_{vw}}(f_w) = 0 \quad \text{if } v \neq w.
\end{gathered} \tag{2.7}$$

For elements $\boldsymbol{d} \in \mathbb{Z}^{Q_0}$, we will use the notation

$$\begin{aligned}
\boldsymbol{d} \geqslant 0 \quad &\text{if } \boldsymbol{d} \in \mathbb{N}^{Q_0}, & \boldsymbol{d} > 0 \quad &\text{if } \boldsymbol{d} \in \mathbb{N}^{Q_0} \setminus \{0\}, \\
\boldsymbol{d} \leqslant 0 \quad &\text{if } -\boldsymbol{d} \in \mathbb{N}^{Q_0}, & \boldsymbol{d} < 0 \quad &\text{if } -\boldsymbol{d} \in \mathbb{N}^{Q_0} \setminus \{0\}.
\end{aligned} \tag{2.8}$$

Note that if $\boldsymbol{d}(v) > 0$ and $\boldsymbol{d}(w) < 0$ for some $v, w \in Q_0$ then none of $\boldsymbol{d} \geqslant 0$, $\boldsymbol{d} > 0$, $\boldsymbol{d} \leqslant 0$, $\boldsymbol{d} < 0$, or $\boldsymbol{d} = 0$ hold.

The next theorem is taken from Kac [84, §1 & §5], with (e) in [84, Rem. 1.5].

Theorem 2.4. *Let $A = (a_{vw})_{v,w\in Q_0}$ and $\mathfrak{g}'(A)$ be as in Definition 2.3. Then:*

- **(a)** *There is a unique grading $\mathfrak{g}'(A) = \bigoplus_{\boldsymbol{d}\in\mathbb{Z}^{Q_0}} \mathfrak{g}_{\boldsymbol{d}}$, such that $e_v \in \mathfrak{g}_{\delta_v}$, $f_v \in \mathfrak{g}_{-\delta_v}$ and $h_v \in \mathfrak{g}_0$ for all $v \in Q_0$ and $[\mathfrak{g}_{\boldsymbol{d}}, \mathfrak{g}_{\boldsymbol{e}}] \subseteq \mathfrak{g}_{\boldsymbol{d}+\boldsymbol{e}}$ for all $\boldsymbol{d}, \boldsymbol{e} \in \mathbb{Z}^{Q_0}$, where $\delta_v \in \mathbb{Z}^{Q_0}$ is given by $\delta_v(w) = 1$ if $v = w$ and $\delta_v(w) = 0$ otherwise.*
- **(b)** *Each graded subspace $\mathfrak{g}_{\boldsymbol{d}}$ for $\boldsymbol{d} \in \mathbb{Z}^{Q_0}$ is finite-dimensional, and $\mathfrak{g}_{\boldsymbol{d}} = 0$ unless either $\boldsymbol{d} = 0$, or $\boldsymbol{d} > 0$, or $\boldsymbol{d} < 0$, in the notation of* (2.8)*. Write $\mathfrak{h} = \mathfrak{g}_0$, $\mathfrak{n}_+ = \bigoplus_{\boldsymbol{d}>0} \mathfrak{g}_{\boldsymbol{d}}$, and $\mathfrak{n}_- = \bigoplus_{\boldsymbol{d}<0} \mathfrak{g}_{\boldsymbol{d}}$. Then $\mathfrak{h}, \mathfrak{n}_+, \mathfrak{n}_-$ are Lie subalgebras of $\mathfrak{g}'(A)$, with $\mathfrak{g}'(A) = \mathfrak{h} \oplus \mathfrak{n}_+ \oplus \mathfrak{n}_-$. We call $\mathfrak{h}$ the* ***Cartan subalgebra****, $\mathfrak{n}_+$ the* ***positive part****, and $\mathfrak{n}_-$ the* ***negative part*** *of $\mathfrak{g}'(A)$.*

  Then $\mathfrak{h}$ is abelian, and is the R-vector space with basis h_v for $v \in Q_0$. Also $\mathfrak{n}_+$ is the Lie subalgebra of $\mathfrak{g}'(A)$ generated by the e_v for $v \in Q_0$, and $\mathfrak{n}_-$ is the Lie subalgebra of $\mathfrak{g}'(A)$ generated by the f_v for $v \in Q_0$.

(c) *We call* $\boldsymbol{d} \in \mathbb{Z}^{Q_0}$ *a* ***root*** *of* $\mathfrak{g}'(A)$ *if* $\boldsymbol{d} \neq 0$ *and* $\mathfrak{g}_{\boldsymbol{d}} \neq 0$. *Then either* $\boldsymbol{d} > 0$, *when* $\boldsymbol{d}$ *is a* ***positive root****, or* $\boldsymbol{d} < 0$, *when* $\boldsymbol{d}$ *is a* ***negative root****. Write* $\Delta, \Delta_+, \Delta_-$ *for the sets of roots, positive roots, and negative roots. Then* $\mathfrak{n}_+ = \bigoplus_{\boldsymbol{d} \in \Delta_+} \mathfrak{g}_{\boldsymbol{d}}$ *and* $\mathfrak{n}_- = \bigoplus_{\boldsymbol{d} \in \Delta_-} \mathfrak{g}_{\boldsymbol{d}}$.

We also divide roots $\boldsymbol{d} \in \Delta$ *into* ***real roots****, with* $A(\boldsymbol{d}, \boldsymbol{d}) > 0$, *and* ***imaginary roots****, with* $A(\boldsymbol{d}, \boldsymbol{d}) \leqslant 0$. *If* $\boldsymbol{d}$ *is a real root then* $A(\boldsymbol{d}, \boldsymbol{d}) = 2$.

(d) *There is a unique isomorphism* $\omega : \mathfrak{g}'(A) \to \mathfrak{g}'(A)$ *satisfying* $\omega^2 = \mathrm{id}$, *and* $\omega(\mathfrak{g}_{\boldsymbol{d}}) = \mathfrak{g}_{-\boldsymbol{d}}$ *for all* $\boldsymbol{d} \in \mathbb{Z}^{Q_0}$, *and* $\omega(e_v) = -f_v$, $\omega(f_v) = -e_v$, $\omega(h_v) = -h_v$ *for all* $v \in Q_0$.

(e) *Suppose* $\mathfrak{t}$ *is an ideal in* $\mathfrak{g}'(A)$ *which is graded with respect to the grading in* **(a)**, *with* $\mathfrak{h} \cap \mathfrak{t} = 0$. *Then* $\mathfrak{t} = 0$.

Remark 2.5. **(i)** Kac [84, §1] constructs another Lie algebra $\mathfrak{g}(A)$, also called a Kac–Moody algebra, such that $\mathfrak{g}'(A) = [\mathfrak{g}(A), \mathfrak{g}(A)] \subseteq \mathfrak{g}(A)$ is the derived Lie subalgebra. We call $\mathfrak{g}'(A)$ the *derived* Kac–Moody algebra if we want to stress the difference with $\mathfrak{g}(A)$. We have $\mathfrak{g}(A) = \hat{\mathfrak{h}} \oplus \mathfrak{n}_+ \oplus \mathfrak{n}_-$, where the Cartan subalgebra $\hat{\mathfrak{h}}$ of $\mathfrak{g}(A)$ has $\mathfrak{h} \subseteq \hat{\mathfrak{h}}$, and $\dim \hat{\mathfrak{h}} - \dim \mathfrak{h} = \mathrm{null}(A)$ is the number of zero eigenvalues of A, so that $\mathfrak{g}(A) = \mathfrak{g}'(A)$ if and only if $\det A \neq 0$. We chose only to define the $\mathfrak{g}'(A)$, as they occur in our Ringel–Hall Lie algebra examples.

(ii) The definition of Kac–Moody algebra [84, §1] does not require the matrix $A = (a_{vw})_{v,w \in Q_0}$ to be symmetric, only that $a_{vw} = 0$ implies $a_{wv} = 0$. One can also relax the condition that $a_{vv} = 2$, giving a *generalized Kac–Moody algebra*, or *Borcherds–Kac–Moody algebra*.

(iii) The Lie algebra $\mathfrak{g}'(A)$ is finite-dimensional if and only if the matrix A is positive definite, and then $\mathfrak{g}'(A)$ is semisimple, a sum of Lie algebras of types A, D and E. As in Kac [84], much of the representation theory of finite-dimensional semisimple Lie algebras extends to Kac–Moody algebras in a nice way.

(iv) The theory of quantum groups [71, 106] extends to Kac–Moody algebras.

2.1.3 The Virasoro algebra

Definition 2.6. The *Virasoro algebra* Vir_R over a $\mathbb{Q}$-algebra R is the Lie algebra with basis elements L_n, $n \in \mathbb{Z}$ and c (the *central charge*), and Lie bracket

$$[c, L_n] = 0, \;\; [L_m, L_n] = (m-n)L_{m+n} + \tfrac{1}{12}(m^3 - m)\delta_{m,n}c, \;\; m, n \in \mathbb{Z}.$$

The factor $\frac{1}{12}$ is a convention, and can be omitted when defining the Virasoro algebra over a general commutative ring R.

The quotient $\mathrm{Vir}_R / \langle c \rangle$ is called the *Witt algebra*, and may be regarded as the Lie algebra of vector fields on the circle $\mathcal{S}^1$. The Virasoro algebra is the unique central extension of the Witt algebra. It is very important in Conformal Field Theory and String Theory.

2.2 Vertex algebras

Next we discuss vertex algebras. These were introduced by Borcherds [18], and some books on them are Kac [85], Frenkel and Ben-Zvi [46], Frenkel, Lepowsky and Meurman [48], Frenkel, Huang and Lepowsky [47], and Lepowsky and Li [99].

2.2.1 (Graded) vertex algebras and vertex operator algebras

Here is Borcherds' original definition of vertex algebra [18]:

Definition 2.7. Let R be a commutative ring. A *vertex algebra* over R is an R-module V equipped with morphisms $D^{(n)} : V \to V$ for $n = 0, 1, 2, \dots$ with $D^{(0)} = \mathrm{id}_V$ and $v_n : V \to V$ for all $v \in V$ and $n \in \mathbb{Z}$, with v_n R-linear in v, and a distinguished element $\mathbb{1} \in V$ called the *identity* or *vacuum vector*, satisfying:

(i) For all $u, v \in V$ we have $u_n(v) = 0$ for $n \gg 0$.

(ii) If $v \in V$ then $\mathbb{1}_{-1}(v) = v$ and $\mathbb{1}_n(v) = 0$ for $-1 \neq n \in \mathbb{Z}$.

(iii) If $v \in V$ then $v_n(\mathbb{1}) = D^{(-n-1)}(v)$ for $n < 0$ and $v_n(\mathbb{1}) = 0$ for $n \geqslant 0$.

(iv) $u_n(v) = \sum_{k\geqslant 0}(-1)^{k+n+1}D^{(k)}(v_{n+k}(u))$ for all $u, v \in V$ and $n \in \mathbb{Z}$, where the sum makes sense by (i), as it has only finitely many nonzero terms.

(v) $(u_l(v))_m(w) = \sum_{n\geqslant 0}(-1)^n\binom{l}{n}\bigl(u_{l-n}(v_{m+n}(w)) - (-1)^l v_{l+m-n}(u_n(w))\bigr)$ for all $u, v, w \in V$ and $l, m \in \mathbb{Z}$, where the sum makes sense by (i).

One can show from these axioms that $D^{(m)} \circ D^{(n)} = \binom{m}{n}D^{(m+n)}$, and so by induction $n!D^{(n)} = (D^{(1)})^n$. If R is a $\mathbb{Q}$-algebra then we write $T = D^{(1)}$ and call T the *translation operator*, and $D^{(n)} = \frac{1}{n!}T^n$ for $n = 0, 1, 2, \dots$.

As in [85, §1.3], [46, §1.3], it is very common to encode the maps $u_n : V \to V$ for $n \in \mathbb{Z}$ in generating function form as R-linear maps for each $u \in V$

$$Y(u, z) : V \longrightarrow V[[z, z^{-1}]], \quad Y(u, z) : v \longmapsto \textstyle\sum_{n\in\mathbb{Z}} u_n(v)z^{-n-1}, \tag{2.9}$$

where z is a formal variable. The $Y(u, z)$ are called *fields*, and have a meaning in Physics. Parts (i)–(v) may be rewritten as properties of the $Y(u, z)$.

There are several alternative definitions of vertex algebra in use in the literature [46, 48, 85, 99], which generally require R to be a $\mathbb{Q}$-algebra or field of characteristic zero, and are known to be equivalent to Definition 2.7 when R is a $\mathbb{Q}$-algebra, [85, §4.8].

Vertex algebras are part of the basic language of Conformal Field Theories in Mathematical Physics and String Theory, and have important mathematical applications in the representation theory of infinite-dimensional Lie algebras, the study of the Monster group, and other areas.

Next we define *graded vertex algebras*, which are examples of *vertex superalgebras*. Inconveniently, the dominant grading convention in the literature for graded vertex algebras of the type we want is different to that for graded Lie algebras in §2.1.1, as they are graded over $\frac{1}{2}\mathbb{Z}$ rather than $\mathbb{Z}$.

Definition 2.8. Let R be a commutative ring. A *graded vertex algebra* (or *graded vertex superalgebra*) over R is an R-module $V_* = \bigoplus_{a\in\frac{1}{2}\mathbb{Z}} V_a$ graded over $\frac{1}{2}\mathbb{Z}$, equipped with morphisms $D^{(n)} : V_* \to V_*$ for $n = 0,1,2,\ldots$ which are graded of degree n (i.e. $D^{(n)}$ maps $V_a \to V_{a+n}$ for $a \in \frac{1}{2}\mathbb{Z}$) with $D^{(0)} = \mathrm{id}_{V_*}$ and morphisms $v_n : V_* \to V_*$ for all $v \in V_*$ and $n \in \mathbb{Z}$ which are R-linear in v and graded of degree $a-n-1$ for $v \in V_a$ (i.e. v_n maps $V_b \to V_{a+b-n-1}$ for $b \in \frac{1}{2}\mathbb{Z}$), and an element $\mathbb{1} \in V_0$ called the *identity* or *vacuum vector*, satisfying:

(i) For all $u, v \in V_*$ we have $u_n(v) = 0$ for $n \gg 0$.

(ii) If $v \in V_*$ then $\mathbb{1}_{-1}(v) = v$ and $\mathbb{1}_n(v) = 0$ for $-1 \neq n \in \mathbb{Z}$.

(iii) If $v \in V_*$ then $v_n(\mathbb{1}) = D^{(-n-1)}(v)$ for $n < 0$ and $v_n(\mathbb{1}) = 0$ for $n \geqslant 0$.

(iv) $u_n(v) = \sum_{k\geqslant 0}(-1)^{4ab+k+n+1}D^{(k)}(v_{n+k}(u))$ for all $a, b \in \frac{1}{2}\mathbb{Z}$, $n \in \mathbb{Z}$ and $u \in V_a$, $v \in V_b$, where the sum makes sense by (i).

(v) $(u_l(v))_m(w) = \sum_{n\geqslant 0}(-1)^n\binom{l}{n}\bigl(u_{l-n}(v_{m+n}(w)) - (-1)^{4ab+l}v_{l+m-n}(u_n(w))\bigr)$ for all $a, b, c \in \frac{1}{2}\mathbb{Z}$, $l, m \in \mathbb{Z}$ and $u \in V_a$, $v \in V_b$, $w \in V_c$, where the sum makes sense by (i).

Then $V_{\mathbb{Z}} = \bigoplus_{a\in\mathbb{Z}} V_a$ is the 'even' part of V_*, which is an ordinary vertex algebra, and $V_{\frac{1}{2}+\mathbb{Z}} = \bigoplus_{a\in\frac{1}{2}+\mathbb{Z}} V_a$ is the 'odd' or 'super' part of V_*.

As in [18, 48, 85, 99], *vertex operator algebras* are a class of vertex algebras important in Conformal Field Theory:

Definition 2.9. Let R be a field of characteristic zero. A *vertex operator algebra* (or *vertex operator superalgebra*, or *conformal vertex algebra* [85, §4.10]) over R is a graded vertex algebra $V_* = \bigoplus_{a\in\frac{1}{2}\mathbb{Z}} V_a$ over R as in Definition 2.8, with a distinguished *conformal element* $\omega \in V_2$ and a *central charge* $c_V \in R$, such that writing $L_n = \omega_{n+1} : V_* \to V_*$, we have

(i) $[L_m, L_n] = (m-n)L_{m+n} + \frac{1}{12}c_V(m^3 - m)\delta_{m,n}\mathrm{id}_{V_*}$ for $m, n \in \mathbb{Z}$. That is, the L_n define an action of the *Viraso algebra* on V_*, with central charge c_V, as in §2.1.3.

(ii) $L_{-1} = D^{(1)} = T$ is the translation operator.

(iii) $L_0|_{V_a} = a \cdot \mathrm{id}_{V_a}$ for $a \in \frac{1}{2}\mathbb{Z}$.

A graded vertex algebra V_* need not admit a conformal element ω, and if ω exists it may not be unique. The term *vertex operator algebra* (rather than *superalgebra*) is often reserved for the case with odd part $V_{\frac{1}{2}+\mathbb{Z}} = 0$, but we do not do this. Some authors also impose additional conditions such as $\dim V_n < \infty$ and $V_n = 0$ for $n \ll 0$, but again we do not do this.

2.2.2 Some basic constructions for vertex algebras

Many concepts for ordinary algebras generalize to vertex algebras, graded vertex algebras, and vertex operator algebras, in an obvious way. We mention some of these, details can be found in [46, 48, 85, 99].

- *Morphisms* of (graded) vertex algebras and vertex operator algebras are defined in the obvious way, making them into categories.
- A (*graded*) *vertex subalgebra* W_* of a (graded) vertex algebra V_* is a (graded) subspace $W_* \subseteq V_*$ such that $\mathbb{1} \in W_*$ and $w_{(n)}(W_*) \subseteq W_*$ for all $w \in W_*$ and $n \in \mathbb{Z}$.
- An *ideal* in a (graded) vertex algebra, or vertex operator algebra V_*, is a (graded) R-submodule $I_* \subset V_*$ with $\mathbb{1} \neq I_*$ closed under all operations $D^{(n)} : V_* \to V_*$ and $u_n : V_* \to V_*$, and such that $u_n(V_*) \subseteq I_*$ for $u \in I_*$. Then the quotient V_*/I_* is a (graded) vertex algebra, or vertex operator algebra. We call V_* *simple* if it has no nonzero ideals.
- *Direct sums* $V_* \oplus W_*$ of (graded) vertex algebras V_*, W_* are (graded) vertex algebras.
- *Tensor products* $V_* \otimes_R W_*$ of (graded) vertex algebras or vertex operator algebras V_*, W_* are (graded) vertex algebras or vertex operator algebras.
- Suppose a group G acts on a (graded) vertex (operator) algebra V_* preserving the structures. Then the G-invariant subspace V_*^G is also a (graded) vertex (operator) algebra, which is called an *orbifold vertex algebra.*

We define representations of graded vertex algebras, following Lepowsky and Li [99] and Frenkel and Ben-Zvi [46, §5.1].

Definition 2.10. Let R be a commutative ring, and V_* be a graded vertex algebra over R. A *representation* of V_*, or V_*-*module*, is an R-module $W_* = \bigoplus_{a\in\frac{1}{2}\mathbb{Z}} W_a$ graded over $\frac{1}{2}\mathbb{Z}$, equipped with morphisms $v_n : W_* \to W_*$ for all $v \in V_*$ and $n \in \mathbb{Z}$ which are R-linear in v and graded of degree $a-n-1$ for $v \in V_a$, satisfying:

(i) For all $v \in V_*$ and $w \in W_*$ we have $v_n(w) = 0$ for $n \gg 0$.

(ii) If $w \in W_*$ then $\mathbb{1}_{-1}(w) = w$ and $\mathbb{1}_n(w) = 0$ for $-1 \neq n \in \mathbb{Z}$.

(ii) $(u_l(v))_m(w) = \sum_{n\geqslant 0}(-1)^n \binom{l}{n}\bigl(u_{l-n}(v_{m+n}(w)) - (-1)^{4ab+l} v_{l+m-n}(u_n(w))\bigr)$ for all $a,b,c \in \frac{1}{2}\mathbb{Z}$, $l,m \in \mathbb{Z}$ and $u \in V_a$, $v \in V_b$, $w \in W_c$, where the sum makes sense by (i).

One can also take $W_* = \bigoplus_{a\in\mathbb{Q}} W_a$ to be graded over $\mathbb{Q}$ rather than $\frac{1}{2}\mathbb{Z}$.

As in Borcherds [18], we can construct a (graded) Lie algebra from a (graded) vertex algebra. This will be used in §3.3 to define the '$t = 0$' Lie algebras.

Definition 2.11. Let $V_* = \bigoplus_{a\in\frac{1}{2}\mathbb{Z}} V_a$ be a graded vertex algebra over R as in Definition 2.8. For each $n \in \mathbb{Z}$ define an R-module

$$V_n^{\mathrm{Lie}} = V_{\frac{1}{2}n+1} \big/ \textstyle\sum_{k\geqslant 1} D^{(k)}(V_{\frac{1}{2}n-k+1}).$$

For all $m,n \in \mathbb{Z}$ define an R-bilinear map $[\,,\,] : V_m^{\mathrm{Lie}} \times V_n^{\mathrm{Lie}} \to V_{m+n}^{\mathrm{Lie}}$ by

$$\bigl[u + \textstyle\sum_{k\geqslant 1} D^{(k)}(V_*), v + \sum_{k\geqslant 1} D^{(k)}(V_*)\bigr] = u_0(v) + \sum_{k\geqslant 1} D^{(k)}(V_*).$$

One can show this is well defined, and makes $V_*^{\mathrm{Lie}} = \bigoplus_{n\in\mathbb{Z}} V_n^{\mathrm{Lie}}$ into a graded Lie algebra over R in the sense of Definition 2.2, so V_0^{Lie} is a Lie algebra.

As in Borcherds [18] and Prevost [131, Prop.s 4.5.3–4.5.4], if V_* is a vertex operator algebra then

$$\dot{V}_0^{\mathrm{Lie}} = \bigl\{u + \textstyle\sum_{k\geqslant 1} D^{(k)}(V_*) \in V_0^{\mathrm{Lie}} : u \in V_1,\ L_n(u) = 0,\ n \geqslant 1\bigr\}$$

is a Lie subalgebra of V_0^{Lie}, which is generally rather smaller, and may be closer to the Lie algebras one wants to construct.

The next definition follows Frenkel and Ben-Zvi [46, §1.4], and provides a first class of examples of vertex algebras.

Definition 2.12. A (graded) vertex algebra V_* is called *commutative* if $u_n(v) = 0$ for all $u, v \in V_*$ and $n \geqslant 0$.

This implies that $u_m : V_* \to V_*$ and $v_n : V_* \to V_*$ (super)commute for all $u, v \in V_*$ and $m, n \in \mathbb{Z}$. The R-bilinear product $u \cdot v := u_{-1}(v)$ is (graded), associative and (super)commutative, making V_* into a ($\frac{1}{2}\mathbb{Z}$-graded) commutative R-algebra with identity $\mathbb{1}$. The translation operator $T : V_* \to V_*$ is an (even) derivation of this algebra (graded of degree 1).

Conversely, if R is a $\mathbb{Q}$-algebra, given a ($\frac{1}{2}\mathbb{Z}$-graded) commutative R-algebra V_* with a (degree 1, even) derivation $T : V_* \to V_*$, we can reconstruct the (graded) vertex algebra structure on V_*, and this gives a 1-1 correspondence between commutative (graded) vertex algebras over R and commutative ($\frac{1}{2}\mathbb{Z}$-graded) R-algebras with (degree 1) derivation.

If V_* is a commutative (graded) vertex algebra then the (graded) Lie algebra V_*^{Lie} in Definition 2.11 is abelian (and so boring).

2.2.3 Vertex Lie algebras

We define (*graded*) *vertex Lie algebras*, following Kac [85, Def. 2.7b], Primc [133], and Frenkel and Ben-Zvi [46, §16.1].

Definition 2.13. Let R be a commutative ring. A *vertex Lie algebra* (or *conformal algebra* [85]) over R is an R-module V equipped with morphisms $D^{(n)} : V \to V$ for $n = 0, 1, 2, \dots$ with $D^{(0)} = \mathrm{id}_V$ and $u_n : V \to V$ for all $u \in V$ and $n \in \mathbb{N}$, with u_n R-linear in u, satisfying:

(i) For all $u, v \in V$ we have $u_n(v) = 0$ for $n \gg 0$.

(ii) If $u, v \in V$ then $(D^{(k)}(u))_n(v) = (-1)^k \binom{n}{k} u_{n-k}(v)$ for $0 \leqslant k \leqslant n$, and $(D^{(k)}(u))_n(v) = 0$ for $0 \leqslant n < k$.

(iii) $u_n(v) = \sum_{k\geqslant 0} (-1)^{k+n+1} D^{(k)}(v_{n+k}(u))$ for all $u, v \in V$ and $n \in \mathbb{N}$, where the sum makes sense by (i).

(iv) $(u_l(v))_m(w) = \sum_{n=0}^{l} (-1)^n \binom{l}{n} \bigl(u_{l-n}(v_{m+n}(w)) - (-1)^l v_{l+m-n}(u_n(w))\bigr)$ for all $u, v, w \in V$ and $l, m \in \mathbb{N}$.

Definition 2.14. Let R be a commutative ring. A *graded vertex Lie algebra* over R is an R-module $V_* = \bigoplus_{a\in\frac{1}{2}\mathbb{Z}} V_a$ graded over $\frac{1}{2}\mathbb{Z}$, equipped with morphisms $D^{(n)} : V_* \to V_*$ for $n = 0, 1, 2, \ldots$ which are graded of degree n (i.e. $D^{(n)}$ maps $V_a \to V_{a+n}$ for $a \in \frac{1}{2}\mathbb{Z}$) with $D^{(0)} = \mathrm{id}_{V_*}$ and morphisms $u_n : V_* \to V_*$ for all $u \in V_*$ and $n \in \mathbb{N}$ which are R-linear in u and graded of degree $a - n - 1$ for $u \in V_a$ (i.e. u_n maps $V_b \to V_{a+b-n-1}$ for $b \in \frac{1}{2}\mathbb{Z}$), satisfying:

(i) For all $u, v \in V_*$ we have $u_n(v) = 0$ for $n \gg 0$.

(ii) If $u, v \in V_*$ then $(D^{(k)}(u))_n(v) = (-1)^k \binom{n}{k} u_{n-k}(v)$ for $0 \leqslant k \leqslant n$, and $(D^{(k)}(u))_n(v) = 0$ for $0 \leqslant n < k$.

(iii) $u_n(v) = \sum_{k\geqslant 0} (-1)^{4ab+k+n+1} D^{(k)}(v_{n+k}(u))$ for all $a, b \in \frac{1}{2}\mathbb{Z}$, $n \in \mathbb{N}$ and $u \in V_a$, $v \in V_b$, where the sum makes sense by (i).

(iv) $(u_l(v))_m(w) = \sum_{n=0}^{l} (-1)^n \binom{l}{n} \bigl(u_{l-n}(v_{m+n}(w)) - (-1)^{4ab+l} v_{l+m-n}(u_n(w))\bigr)$ for all $a, b, c \in \frac{1}{2}\mathbb{Z}$, $l, m \in \mathbb{N}$ and $u \in V_a$, $v \in V_b$, $w \in V_c$.

A (graded) vertex Lie algebra has some of the structures of a (graded) vertex algebra: it has the operators $D^{(n)}$ and u_n for $n \geqslant 0$, but omits u_n for $n < 0$ and $\mathbb{1}$. Any (graded) vertex algebra is a (graded) vertex Lie algebra.

If V_* is a (graded) vertex algebra, a (*graded*) *vertex Lie subalgebra* W_* of V_* is a (graded) subspace $W_* \subseteq V_*$ such that $D^{(n)}(W_*) \subseteq W_*$ and $w_n(W_*) \subseteq W_*$ for all $w \in W_*$ and $n \geqslant 0$. Then W_* is a (graded) vertex Lie algebra.

As in Primc [133], any (graded) vertex Lie algebra W_* has a *universal enveloping vertex algebra* $\mathscr{V}(W_*)$, which is a (graded) vertex algebra with an inclusion $W_* \subseteq \mathscr{V}(W_*)$ as a vertex Lie subalgebra, with the universal property that if V_* is a (graded) vertex algebra and $\phi : W_* \to V_*$ is a morphism of (graded) vertex Lie algebras, then ϕ extends to a unique morphism $\mathscr{V}(\phi) : \mathscr{V}(W_*) \to V_*$ of (graded) vertex algebras.

Often a (graded) vertex Lie algebra W_* may be much smaller, and easier to write down, than its associated (graded) vertex algebra $\mathscr{V}(W_*)$.

The definition of (graded) Lie algebras V_*^{Lie} in Definition 2.11 also works for (graded) vertex Lie algebras V_*.

2.2.4 Lattice vertex algebras

We discuss *lattice vertex algebras*, an important class of examples of graded vertex algebras, following Borcherds [18], Kac [85, §5.4], Frenkel and Ben-Zvi [46, §5.4], Frenkel et al. [48, Th. 8.10.2], and Lepowsky and Li [99, §6.4–§6.5].

Definition 2.15. Let Q_0 be a finite set, so that $\mathbb{Z}^{Q_0}$ is a finite rank free abelian group. Let $\chi : \mathbb{Z}^{Q_0} \times \mathbb{Z}^{Q_0} \to \mathbb{Z}$ be a $\mathbb{Z}$-bilinear symmetric form, with matrix $A = (a_{vw})_{v,w\in Q_0}$. Let R be a field of characteristic zero.

As in [85, (5.14)], [46, (5.4)], suppose we are given signs $\epsilon_{\alpha,\beta} \in \{\pm 1\}$ for all

$\alpha, \beta \in \mathbb{Z}^{Q_0}$ satisfying for all $\alpha, \beta, \gamma \in \mathbb{Z}^{Q_0}$

$$\epsilon_{\alpha,\beta} \cdot \epsilon_{\beta,\alpha} = (-1)^{\chi(\alpha,\beta)+\chi(\alpha,\alpha)\chi(\beta,\beta)}, \tag{2.10}$$

$$\epsilon_{\alpha,\beta} \cdot \epsilon_{\alpha+\beta,\gamma} = \epsilon_{\alpha,\beta+\gamma} \cdot \epsilon_{\beta,\gamma}, \tag{2.11}$$

$$\epsilon_{\alpha,0} = \epsilon_{0,\alpha} = 1. \tag{2.12}$$

Note that these coincide with (3.1)–(3.3) in Assumption 3.1 below. As in Kac [85, Cor. 5.5], there exist solutions $(\epsilon_{\alpha,\beta})_{\alpha,\beta\in\mathbb{Z}^{Q_0}}$ to (2.10)–(2.12), which are unique up to an appropriate notion of equivalence (i.e. all solutions yield isomorphic lattice vertex algebras).

Define a commutative R-algebra

$$V = R[\mathbb{Z}^{Q_0}] \otimes_R R[b_{v,i} : v \in Q_0, \;\; i = 1, 2, \ldots].$$

Here $R[\mathbb{Z}^{Q_0}]$ is the group algebra over R of the abelian group $\mathbb{Z}^{Q_0}$, which has basis formal symbols e^α for $\alpha \in \mathbb{Z}^{Q_0}$ with multiplication $e^\alpha \cdot e^\beta = e^{\alpha+\beta}$, and $R[b_{v,i} : v \in Q_0,\, i \geqslant 1]$ is the polynomial algebra over R in formal variables $b_{v,i}$. Thus, V is the free R-module with basis

$$\bigl\{e^\alpha \otimes \textstyle\prod_{v\in Q_0,\, i\geqslant 1} b_{v,i}^{n_{v,i}} : \alpha \in Q_0,\, n_{v,i} \in \mathbb{N}, \text{ only finitely many } n_{v,i} \neq 0\bigr\}. \tag{2.13}$$

Define a grading $V = \bigoplus_{a\in\frac{1}{2}\mathbb{Z}} V_a$ of V over $\frac{1}{2}\mathbb{Z}$ such that

$$e^\alpha \otimes \textstyle\prod_{v\in Q_0,\, i\geqslant 1} b_{v,i}^{n_{v,i}} \in V_{-\frac{1}{2}\chi(\alpha,\alpha)+\sum_{v,i} n_{v,i}} \tag{2.14}$$

for each basis element in (2.13). (Note that although V is an R-algebra, because of the $-\frac{1}{2}\chi(\alpha, \alpha)$ term the multiplication in V does not respect the grading.) We write $V = V_*$ as a graded vector space, with even part $V_{\mathbb{Z}}$ and odd part $V_{\frac{1}{2}+\mathbb{Z}}$.

For each $\alpha \in \mathbb{Z}^{Q_0}$ and $n \in \mathbb{Z}$ we define R-linear maps $\alpha_n : V_* \to V_*$ which are graded of degree $-n$ and $\mathbb{Z}$-linear in α, by

(i) If $n > 0$ then $\alpha_n : V \to V$ is the derivation of the R-algebra V generated by $\alpha_n(b_{v,i}) = 0$ for $v \in Q_0$, $i \neq n$, and $\alpha_n(b_{v,n}) = n\alpha(v)$, and $\alpha_n(e^\beta) = 0$ for $\beta \in \mathbb{Z}^{Q_0}$. In effect, α_n acts as $n \sum_{v\in Q_0} \alpha(v) \frac{\mathrm{d}}{\mathrm{d}b_{v,n}}$.

(ii) α_0 acts by $\alpha_0 : e^\beta \otimes \prod_{v\in Q_0,\, i\geqslant 1} b_{v,i}^{n_{v,i}} \mapsto \chi(\alpha, \beta) \cdot e^\beta \otimes \prod_{v\in Q_0,\, i\geqslant 1} b_{v,i}^{n_{v,i}}$.

(iii) If $n < 0$ then $\alpha_n : V \to V$ is multiplication by $\sum_{v\in Q_0} \alpha(v) b_{v,-n}$ in the R-algebra V.

Write $\mathbb{1} = e^0 \otimes 1 \in V_0$ for the identity 1 in the R-algebra V.

The next theorem follows from Kac [85, Prop.s 5.4 & 5.5 & Th. 5.5], see also Lepowsky and Li [99, Th. 6.5.3].

Theorem 2.16. *In the situation of Definition* 2.15, *there is a unique graded vertex algebra structure over R on $V_* = \bigoplus_{a\in\frac{1}{2}\mathbb{Z}} V_a$ such that $\mathbb{1}$ is the identity vector, and for all $\alpha \in \mathbb{Z}^{Q_0}$ and $n \in \mathbb{Z}$ we have*

$$\bigl(e^0 \otimes \textstyle\sum_{v\in Q_0} \alpha(v) b_{v,1}\bigr)_n = \alpha_n : V_* \longrightarrow V_*, \tag{2.15}$$

and for all $\alpha,\beta\in\mathbb{Z}^{Q_0}$, in the notation of (2.9) we have

$$\begin{aligned}&Y(e^\alpha\otimes 1,z)|_{e^\beta\otimes R[b_{v,i}:v\in Q_0,\,i\geqslant 1]}\\&=\epsilon_{\alpha,\beta}z^{\chi(\alpha,\beta)}e^\alpha\exp\bigl[-\textstyle\sum_{n<0}\frac{1}{n}z^{-n}\alpha_n\bigr]\circ\exp\bigl[-\textstyle\sum_{n>0}\frac{1}{n}z^{-n}\alpha_n\bigr]:\\&\qquad e^\beta\otimes R[b_{v,i}:v\in Q_0,\ i\geqslant 1]\longrightarrow e^{\alpha+\beta}\otimes R[b_{v,i}:v\in Q_0,\ i\geqslant 1].\end{aligned}\tag{2.16}$$

We call V_* a ***lattice vertex algebra****.*

Now suppose that the inner product $\chi:\mathbb{Z}^{Q_0}\times\mathbb{Z}^{Q_0}\to\mathbb{Z}$ is nondegenerate, so that $A=(a_{vw})_{v,w\in Q_0}$ is invertible over $\mathbb{Q}$, and write $A^{-1}=C=(c_{vw})_{v,w\in Q_0}$ with $c_{v,w}\in\mathbb{Q}$. Then V_* is a simple graded vertex algebra (it has no nonzero ideals), and it is a graded vertex operator algebra, with conformal vector

$$\omega=\tfrac{1}{2}\sum_{v,w\in Q_0}c_{vw}\,e^0\otimes b_{v,1}b_{w,1},\tag{2.17}$$

and central charge $c_V=\operatorname{rank}\mathbb{Z}^{Q_0}=|Q_0|$.

Remark 2.17. (i) We have changed notation compared to [18,46,48,85,99], for compatibility with §5. In particular, lattice vertex algebras are usually defined using a finite rank free abelian group Λ with symmetric form $\chi:\Lambda\times\Lambda\to\mathbb{Z}$. We have taken $\Lambda=\mathbb{Z}^{Q_0}$, and written $\mathrm{Sym}_R(\Lambda\otimes_{\mathbb{Z}}t^{-1}R[t^{-1}])$ in [85, §5.4] as the polynomial algebra $R[b_{v,i}:v\in Q_0,\ i\geqslant 1]$.

(ii) As in Kac [85, Ex. 5.5b], the conformal vector ω in (2.17) may not be unique, lattice vertex algebras can admit nontrivial families of conformal vectors.

(iii) Parts of Definition 2.15 and Theorem 2.16 also work over more general commutative rings R. Borcherds [18, §2] initially defines $V_*^{\mathbb{Q}}$ over $R=\mathbb{Q}$, and then defines an integral form $V_*^{\mathbb{Z}}$ to be the smallest subring of $V_*^{\mathbb{Q}}$ as a $\mathbb{Q}$-algebra containing $e^\alpha\otimes 1$ for all $\alpha\in\mathbb{Z}^{Q_0}$ and closed under all operations $D^{(n)}$. We can then define the lattice vertex algebra over any commutative ring R to be $V_*^R=V_*^{\mathbb{Z}}\otimes_{\mathbb{Z}}R$. Note that ω in (2.17) is not defined over $\mathbb{Z}$.

2.3 Background on stacks

Stacks are a large and difficult subject. Although we give a little introduction here, it will not be enough to enable readers unfamiliar with stacks to properly understand our paper. Our discussion is intended for readers who already have good background knowledge of stacks. On core material we aim to establish notation, give references, and remind readers of the basic ideas. We also go into detail on some technical points which will be important later.

For general references on stacks we recommend Gómez [55], Olsson [127], Laumon and Moret-Bailly [98], and the online Stacks Project [34].

2.3.1 Classes of spaces in algebraic geometry

Throughout §2–§7 we fix a field $\mathbb{K}$, such as $\mathbb{K}=\mathbb{C}$, which we sometimes require to be algebraically closed. Then we can consider the following classes of algebro-geometric spaces over $\mathbb{K}$:

(i) $\mathbb{K}$-*schemes*, as in Hartshorne [62]. These form a category $\mathbf{Sch}_{\mathbb{K}}$.

We generally restrict attention to schemes S which are *locally of finite type*. We write $\mathbf{Sch}_{\mathbb{K}}^{\mathrm{lft}} \subset \mathbf{Sch}_{\mathbb{K}}$ for the full subcategory of such schemes.

(ii) *Algebraic* $\mathbb{K}$-*spaces*, as in Knutson [89] and Olsson [127]. These form a category $\mathbf{AlgSp}_{\mathbb{K}}$.

(iii) *Artin* $\mathbb{K}$-*stacks*, as in Gómez [55], Olsson [127], Laumon and Moret-Bailly [98], and the 'Stacks Project' [34]. These form a 2-category $\mathbf{Art}_{\mathbb{K}}$.

We generally restrict attention to stacks S which are *locally of finite type*, and we make the convention that Artin $\mathbb{K}$-stacks in this paper are assumed to be locally of finite type unless we explicitly say otherwise. We write $\mathbf{Art}_{\mathbb{K}}^{\mathrm{lft}} \subset \mathbf{Art}_{\mathbb{K}}$ for the full 2-subcategory of such stacks.

The typical examples of Artin stacks we will be interested in are moduli stacks $\mathcal{M}$ of objects in a $\mathbb{K}$-linear abelian category $\mathcal{A}$, such as coherent sheaves $\mathcal{A} = \mathrm{coh}(X)$ on a smooth projective $\mathbb{K}$-scheme X.

(iv) *Higher (Artin)* $\mathbb{K}$-*stacks*, as in Simpson [143], Toën [149], and Pridham [132]. These are a generalization of Artin stacks which form an ∞-category $\mathbf{HSt}_{\mathbb{K}}$, including 'geometric n-stacks' for $n = 1, 2, \ldots$, where geometric 1-stacks are Artin stacks. We generally restrict attention to higher stacks S which are *locally of finite type*, and higher stacks in this paper are assumed to be locally of finite type unless we explicitly say otherwise. We write $\mathbf{HSt}_{\mathbb{K}}^{\mathrm{lft}} \subset \mathbf{HSt}_{\mathbb{K}}$ for the full ∞-subcategory of such stacks.

The typical examples of higher stacks we will be interested in are moduli stacks $\mathcal{M}$ of objects in a $\mathbb{K}$-linear triangulated category $\mathcal{T}$, such as the bounded derived category $\mathcal{T} = D^b\mathrm{coh}(X)$ of complexes of coherent sheaves on a smooth projective $\mathbb{K}$-scheme X.

The reason we need higher stacks to study moduli spaces of complexes is that if $\mathcal{E}^\bullet$ is an object in $D^b\mathrm{coh}(X)$ with $\mathrm{Ext}^i(\mathcal{E}^\bullet, \mathcal{E}^\bullet) \neq 0$ for some $i < 0$ then the moduli space $\mathcal{M}$ of objects in $D^b\mathrm{coh}(X)$ is generally not represented by an Artin stack near $\mathcal{E}^\bullet$, but it is a higher Artin stack.

(v) *Derived* $\mathbb{K}$-*stacks*, as in Toën and Vezzosi [149–152]. These form an ∞-category $\mathbf{dSt}_{\mathbb{K}}$, including a full ∞-subcategory $\mathbf{dArt}_{\mathbb{K}}$ of *derived Artin stacks*. We generally restrict attention to derived stacks $\boldsymbol{S}$ which are *locally of finite presentation*. We write $\mathbf{dSt}_{\mathbb{K}}^{\mathrm{lfp}} \subset \mathbf{dSt}_{\mathbb{K}}$ for the full ∞-subcategory of such derived stacks. We usually write derived stacks in bold type.

The typical examples of derived stacks we will be interested in are derived moduli stacks $\boldsymbol{\mathcal{M}}$ of objects in a $\mathbb{K}$-linear abelian category $\mathcal{A}$, such as $\mathcal{A} = \mathrm{coh}(X)$ (in this case $\boldsymbol{\mathcal{M}}$ is a derived Artin stack) or in a $\mathbb{K}$-linear triangulated category $\mathcal{T}$, such as $\mathcal{T} = D^b\mathrm{coh}(X)$.

These (higher) categories have the following inclusion relations:

$$\begin{tikzcd}
\mathbf{Sch}_{\mathbb{K}} \arrow[r, hook, "\subset"'] & \mathbf{AlgSp}_{\mathbb{K}} \arrow[r, hook, "\subset"'] & \mathbf{Art}_{\mathbb{K}} \arrow[r, hook, "\subset"'] \arrow[d, hook, "\subset"', shift right] & \mathbf{HSt}_{\mathbb{K}} \arrow[d, hook, "\subset"', shift right] \\
& & \mathbf{dArt}_{\mathbb{K}} \arrow[r, hook, "\subset"] \arrow[u, dotted, "t_0"', shift right] & \mathbf{dSt}_{\mathbb{K}} . \arrow[u, dotted, "t_0"', shift right]
\end{tikzcd}$$

There is also a *classical truncation functor* $t_0 : \mathbf{dSt}_{\mathbb{K}} \to \mathbf{HSt}_{\mathbb{K}}$, which maps $t_0 : \mathbf{dArt}_{\mathbb{K}} \to \mathbf{Art}_{\mathbb{K}}$, is left inverse to the inclusion $\mathbf{HSt}_{\mathbb{K}} \hookrightarrow \mathbf{dSt}_{\mathbb{K}}$, and takes derived moduli stacks to the corresponding classical moduli stacks.

It will often not be important to us that $\mathbf{Art}_{\mathbb{K}}, \mathbf{HSt}_{\mathbb{K}}, \mathbf{dArt}_{\mathbb{K}}, \mathbf{dSt}_{\mathbb{K}}$ are 2-categories or ∞-categories. Then we treat them as ordinary categories by working in the *homotopy categories* $\mathrm{Ho}(\mathbf{Art}_{\mathbb{K}}), \ldots, \mathrm{Ho}(\mathbf{dSt}_{\mathbb{K}})$ (see Definition 2.18).

2.3.2 Basics of 2-categories, the 2-category of Artin stacks

We review 2-categories, as in Borceux [17, §7], Kelly and Street [87], and Behrend et al. [14, App. B]. A (*strict*) 2-*category* $\boldsymbol{\mathcal{C}}$ has objects $X, Y, \ldots$, and two kinds of morphisms, 1-morphisms $f : X \to Y$ between objects, and 2-morphisms $\eta : f \Rightarrow g$ between 1-morphisms $f, g : X \to Y$. One can also consider *weak* 2-*categories*, or *bicategories*, in which composition of 1-morphisms is associative only up to 2-isomorphisms, but we will not discuss these.

There are three kinds of composition in a 2-category, satisfying various associativity relations. If $f : X \to Y$ and $g : Y \to Z$ are 1-morphisms in $\boldsymbol{\mathcal{C}}$ then $g \circ f : X \to Z$ is the *composition of* 1-*morphisms*. If $f, g, h : X \to Y$ are 1-morphisms and $\eta : f \Rightarrow g$, $\zeta : g \Rightarrow h$ are 2-morphisms in $\boldsymbol{\mathcal{C}}$ then $\zeta \odot \eta : f \Rightarrow h$ is the *vertical composition of* 2-*morphisms*, as a diagram

$$\begin{tikzcd}
X \arrow[r, bend left=50, "f"] \arrow[r, "g" description] \arrow[r, bend right=50, "h"'] \arrow[r, phantom, shift left=3, "\Downarrow\eta"] \arrow[r, phantom, shift right=3, "\Downarrow\zeta"] & Y & \rightsquigarrow & X \arrow[r, bend left, "f"] \arrow[r, bend right, "h"'] \arrow[r, phantom, "\Downarrow\zeta\odot\eta"] & Y.
\end{tikzcd}$$

If $f, \dot{f} : X \to Y$ and $g, \dot{g} : Y \to Z$ are 1-morphisms and $\eta : f \Rightarrow \dot{f}$, $\zeta : g \Rightarrow \dot{g}$ are 2-morphisms in $\boldsymbol{\mathcal{C}}$ then $\zeta * \eta : g \circ f \Rightarrow \dot{g} \circ \dot{f}$ is the *horizontal composition of* 2-*morphisms*, as a diagram

$$\begin{tikzcd}
X \arrow[r, bend left, "f"] \arrow[r, bend right, "\dot{f}"'] \arrow[r, phantom, "\Downarrow\eta"] & Y \arrow[r, bend left, "g"] \arrow[r, bend right, "\dot{g}"'] \arrow[r, phantom, "\Downarrow\zeta"] & Z & \rightsquigarrow & X \arrow[r, bend left, "g\circ f"] \arrow[r, bend right, "\dot{g}\circ\dot{f}"'] \arrow[r, phantom, "\Downarrow\zeta*\eta"] & Z.
\end{tikzcd}$$

There are also two kinds of identity: *identity* 1-*morphisms* $\mathrm{id}_X : X \to X$ of objects and *identity* 2-*morphisms* $\mathrm{id}_f : f \Rightarrow f$ of 1-morphisms. A 2-morphism $\eta : f \Rightarrow g$ is a 2-*isomorphism* if it is invertible under vertical composition.

Commutative diagrams in 2-categories should in general only commute *up to (specified)* 2-*isomorphisms*, rather than strictly. A simple example of a commutative diagram in a 2-category $\boldsymbol{\mathcal{C}}$ is

$$\begin{tikzcd} & Y \arrow[dr, "g"] \arrow[d, phantom, "\Downarrow\eta"] & \\ X \arrow[ur, "f"] \arrow[rr, "h"'] & & Z, \end{tikzcd}$$

which means that X, Y, Z are objects of $\boldsymbol{\mathcal{C}}$, $f : X \to Y$, $g : Y \to Z$ and $h : X \to Z$ are 1-morphisms in $\boldsymbol{\mathcal{C}}$, and $\eta : g \circ f \Rightarrow h$ is a 2-isomorphism.

In a 2-category $\boldsymbol{\mathcal{C}}$, there are three notions of when objects X, Y in $\boldsymbol{\mathcal{C}}$ are 'the same': *equality* $X = Y$, and 1-*isomorphism*, that is we have 1-morphisms $f : X \to Y$, $g : Y \to X$ with $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$, and *equivalence*, that is, we have 1-morphisms $f : X \to Y$, $g : Y \to X$ and 2-isomorphisms $\eta : g \circ f \Rightarrow \mathrm{id}_X$ and $\zeta : f \circ g \Rightarrow \mathrm{id}_Y$. Usually equivalence is the correct notion.

Definition 2.18. Let $\boldsymbol{\mathcal{C}}$ be a 2-category. The *homotopy category* $\mathrm{Ho}(\boldsymbol{\mathcal{C}})$ of $\boldsymbol{\mathcal{C}}$ is the category whose objects are objects of $\boldsymbol{\mathcal{C}}$, and whose morphisms $[f] : X \to Y$ are 2-isomorphism classes $[f]$ of 1-morphisms $f : X \to Y$ in $\boldsymbol{\mathcal{C}}$. Then equivalences in $\boldsymbol{\mathcal{C}}$ become isomorphisms in $\mathrm{Ho}(\boldsymbol{\mathcal{C}})$, 2-commutative diagrams in $\boldsymbol{\mathcal{C}}$ become commutative diagrams in $\mathrm{Ho}(\boldsymbol{\mathcal{C}})$, and so on.

Artin $\mathbb{K}$-stacks $\mathbf{Art}_{\mathbb{K}}$ and $\mathbf{Art}_{\mathbb{K}}^{\mathrm{lft}}$ form strict 2-categories. As in Gómez [55, §2.2] and Olsson [127, §8], for us an *Artin* $\mathbb{K}$-*stack* is defined to be a pair (X, p_X) of a category X and a functor $p_X : X \to \mathbf{Sch}_{\mathbb{K}}$, where $\mathbf{Sch}_{\mathbb{K}}$ is the category of $\mathbb{K}$-schemes, such that X, p_X satisfy many complicated conditions which we will not go into. These are the objects in $\mathbf{Art}_{\mathbb{K}}$. If $(X, p_X), (Y, p_Y)$ are Artin $\mathbb{K}$-stacks, a 1-*morphism* $f : (X, p_X) \to (Y, p_Y)$ in $\mathbf{Art}_{\mathbb{K}}$ is a functor $f : X \to Y$ with $p_Y \circ f = p_X$. If $f, g : (X, p_X) \to (Y, p_Y)$ are 1-morphisms, a 2-*morphism* $\eta : f \Rightarrow g$ in $\mathbf{Art}_{\mathbb{K}}$ is a natural transformation of functors $\eta : f \Rightarrow g$ such that $\mathrm{id}_{p_Y} * \eta = \mathrm{id}_{p_X}$. All 2-morphisms in $\mathbf{Art}_{\mathbb{K}}$ are 2-isomorphisms.

Definition 2.19. Let (X, p_X) be an Artin $\mathbb{K}$-stack. A *substack* (Y, p_Y) of X is a subcategory $Y \subseteq X$ which is closed under isomorphisms in X, such that the restriction $p_Y := p_X|_Y : Y \to \mathbf{Sch}_{\mathbb{K}}$ makes (Y, p_Y) into an Artin $\mathbb{K}$-stack. Then the inclusion functor $\iota : Y \hookrightarrow X$ is a representable 1-morphism in $\mathbf{Art}_{\mathbb{K}}$.

2.3.3 Fibre products of stacks

We define fibre products in 2-categories, following [14, Def. B.13], [127, §3.4.9].

Definition 2.20. Let $\boldsymbol{\mathcal{C}}$ be a strict 2-category and $g : X \to Z$, $h : Y \to Z$ be 1-morphisms in $\boldsymbol{\mathcal{C}}$. A *fibre product* in $\boldsymbol{\mathcal{C}}$ consists of an object W, 1-morphisms $e : W \to X$ and $f : W \to Y$ and a 2-isomorphism $\eta : g \circ e \Rightarrow h \circ f$ in $\boldsymbol{\mathcal{C}}$, so that we have a 2-commutative diagram

$$\begin{tikzcd} W \arrow[r, "f"'] \arrow[d, "e"] \arrow[dr, phantom, "\eta\Uparrow"] & Y \arrow[d, "h"'] \\ X \arrow[r, "g"] & Z, \end{tikzcd} \tag{2.18}$$

with the following universal property: suppose $e' : W' \to X$ and $f' : W' \to Y$ are 1-morphisms and $\eta' : g \circ e' \Rightarrow h \circ f'$ is a 2-isomorphism in $\boldsymbol{\mathcal{C}}$. Then there should exist a 1-morphism $b : W' \to W$ and 2-isomorphisms $\zeta : e \circ b \Rightarrow e'$, $\theta : f \circ b \Rightarrow f'$ such that the following diagram of 2-isomorphisms commutes:

$$\begin{tikzcd} g \circ e \circ b \arrow[r, Rightarrow, "\eta * \mathrm{id}_b"'] \arrow[d, Rightarrow, "\mathrm{id}_g * \zeta"] & h \circ f \circ b \arrow[d, Rightarrow, "\mathrm{id}_h * \theta"'] \\ g \circ e' \arrow[r, Rightarrow, "\eta'"] & h \circ f'. \end{tikzcd}$$

Furthermore, if $\tilde{b}, \tilde{\zeta}, \tilde{\theta}$ are alternative choices of b, ζ, θ then there should exist a unique 2-isomorphism $\epsilon : b \Rightarrow \tilde{b}$ with

$$\zeta = \tilde{\zeta} \odot (\mathrm{id}_e * \epsilon) \quad \text{and} \quad \theta = \tilde{\theta} \odot (\mathrm{id}_f * \epsilon).$$

We call such a fibre product diagram (2.18) a 2-*Cartesian square*. We often write $W = X \times_Z Y$ or $W = X \times_{g,Z,h} Y$, and call W the fibre product. If a fibre product $X \times_Z Y$ in $\boldsymbol{\mathcal{C}}$ exists then it is unique up to canonical equivalence in $\boldsymbol{\mathcal{C}}$.

We can also define the dual notion of *pushouts* in 2-category, and 2-*co-Cartesian squares*, by reversing the directions of all 1-morphisms in the above.

All fibre products exist in $\mathbf{Art}_{\mathbb{K}}, \mathbf{Art}_{\mathbb{K}}^{\mathrm{lft}}$, [98, Prop. 4.5(i)], [127, Prop. 8.1.16].

For higher stacks, we must use ∞-category fibre products (homotopy fibre products). All such fibre products exist in $\mathbf{HSt}_{\mathbb{K}}, \mathbf{HSt}_{\mathbb{K}}^{\mathrm{lft}}$.

2.3.4 $\mathbb{K}$-points and isotropy groups

If S is an Artin $\mathbb{K}$-stack, as in [98, §5] a $\mathbb{K}$-*point* of S is a morphism $s : * \to S$ in $\mathrm{Ho}(\mathbf{Art}_{\mathbb{K}})$, where $* = \operatorname{Spec} \mathbb{K}$ is the point, as a $\mathbb{K}$-scheme. We write $S(\mathbb{K})$ for the set of $\mathbb{K}$-points of S. If $s \in S(\mathbb{K})$, then lifting $s : * \to S$ to a 1-morphism in $\mathbf{Art}_{\mathbb{K}}$, the *isotropy group* (or *stabilizer group*, or *automorphism group* [127, Rem. 8.3.4]) $\mathrm{Iso}_S(s)$ is the group of 2-isomorphisms $\lambda : s \Rightarrow s$ in $\mathbf{Art}_{\mathbb{K}}$ under vertical composition. It is the set of $\mathbb{K}$-points of the fibre product $* \times_{s,S,s} *$, which is a $\mathbb{K}$-scheme, and in fact an algebraic $\mathbb{K}$-group.

If $\mathcal{M}$ is a moduli stack of objects in a $\mathbb{K}$-linear abelian category $\mathcal{A}$, and $[E]$ in $\mathcal{M}(\mathbb{K})$ corresponds to E in $\mathcal{A}$, then $\mathrm{Iso}_{\mathcal{M}}([E])$ is isomorphic to the automorphism group $\mathrm{Aut}(E)$, the group of invertible elements in $\mathrm{Hom}_{\mathcal{A}}(E, E)$.

If $f : S \to T$ is a morphism in $\mathrm{Ho}(\mathbf{Art}_{\mathbb{K}})$, we define $f(\mathbb{K}) : S(\mathbb{K}) \to T(\mathbb{K})$ to map $f(\mathbb{K}) : s \mapsto f \circ s$. If $s \in S(\mathbb{K})$ with $f(\mathbb{K})s = t \in T(\mathbb{K})$ there is a morphism $f_* : \mathrm{Iso}_S(s) \to \mathrm{Iso}_T(t)$ of algebraic $\mathbb{K}$-groups given by $f_* : \lambda \mapsto \mathrm{id}_f * \lambda$. As this depends on lifting f, s, t from morphisms in $\mathrm{Ho}(\mathbf{Art}_{\mathbb{K}})$ to 1-morphisms in $\mathbf{Art}_{\mathbb{K}}$, f_* is only canonical up to conjugation in $\mathrm{Iso}_T(t)$. These $f(\mathbb{K})$ and f_* are covariantly functorial in f.

Similar definitions work for $\mathbb{K}$-points and isotropy groups of higher stacks. There are also higher isotropy groups $\pi_n(S, s)$ for $n \geqslant 0$ with $\pi_0(S, s) = \mathrm{Iso}_S(s)$, as in Toën [149, §3.1.6].

If U is a $\mathbb{K}$-scheme and G an algebraic $\mathbb{K}$-group acting on U, we can form the *quotient stack* $[U/G]$. The $\mathbb{K}$-points of $[U/G]$ correspond to G-orbits $G \cdot u$

in $U(\mathbb{K})$, with $\mathrm{Iso}_{[U/G]}(G\cdot u)\cong \mathrm{Stab}_G(u)$ the stabilizer group of u in G. An important example is $[*/\mathbb{G}_m]$, where $*=\operatorname{Spec}\mathbb{K}$ is the point and $\mathbb{G}_m=\mathbb{A}^1\setminus\{0\}$ is the multiplicative group.

2.3.5 Topologies on stacks, and the smooth topology

The appropriate notion of topology on Artin $\mathbb{K}$-stacks is *Grothendieck topologies*, as in [127, §2], including the *Zariski topology*, the *étale topology*, and the *smooth topology*. When we use 'locally', 'locally equivalent', 'locally trivial fibration', and so on, of stacks and their morphisms, we mean *locally in the smooth topology*, unless we say otherwise.

For example, a *vector bundle* $\pi: E\to S$ on an Artin $\mathbb{K}$-stack S is a locally trivial fibre bundle in the smooth topology with fibre $\mathbb{K}^r$. Requiring $\pi: E\to S$ to be locally trivial only in the smooth topology (rather than in the stronger Zariski or étale topologies) means that the isotropy groups $\mathrm{Iso}_S(s)$ of $\mathbb{K}$-points $s\in S(\mathbb{K})$ can have nontrivial linear actions on the fibres $E|_s$ of E.

2.3.6 Vector bundles, coherent sheaves, and complexes on stacks

If S is one of any of the kinds of spaces in §2.3.1, we can consider *vector bundles* $E\to S$ on S, and (*quasi-*)*coherent sheaves* $\mathcal{E}$ on S, and *complexes of* (*quasi-*)*coherent sheaves* $\mathcal{E}^\bullet$ on S, including *perfect complexes.* We write $h^i(\mathcal{E}^\bullet)$ for the i^{th} cohomology sheaf of $\mathcal{E}^\bullet$, as an object in $\mathrm{coh}(S)$ or $\mathrm{qcoh}(S)$. We usually consider vector bundles to be examples of coherent sheaves. (Quasi-) coherent sheaves on schemes are discussed by Hartshorne [62, §II.5] and Huybrechts and Lehn [68]. Some references on derived categories of complexes of sheaves $D^b\mathrm{coh}(S)$ on schemes are Gelfand and Manin [53] and Huybrechts [67]. Sheaves and complexes on (derived) (Artin) stacks are covered in Laumon and Moret-Bailly [98, §15], Olsson [125, §9], [127, §9], and Toën and Vezzosi [151,152].

We will need the following notation:

Definition 2.21. Let S be a $\mathbb{K}$-scheme, or Artin $\mathbb{K}$-stack, or higher Artin $\mathbb{K}$-stack. An object $\mathcal{E}^\bullet$ in the unbounded derived category of quasicoherent sheaves $D(\mathrm{qcoh}(S))$ is called *perfect* if it is equivalent locally on S to a complex $\cdots\to E_i\to E_{i+1}\to\cdots$ with E_i a vector bundle in degree i, with $E_i=0$ for $|i|\gg 0$. It is called *perfect in the interval* $[a,b]$ for $a\leqslant b$ in $\mathbb{Z}$ if it is locally equivalent to such a complex with $E_i=0$ for $i\notin[a,b]$. Write $\mathrm{Perf}(S)\subset D(\mathrm{qcoh}(S))$ for the $\mathbb{K}$-linear triangulated subcategory of perfect complexes.

Write $\mathrm{Vect}(S)$ for the exact category of vector bundles on S. Then $\mathrm{Vect}(S)$ embeds as a full subcategory of $\mathrm{Perf}(S)$, by regarding a vector bundle $\mathcal{E}$ as a complex $\cdots\to 0\to\mathcal{E}\to 0\to\cdots$ with $\mathcal{E}$ in degree 0.

Write $K_0(\mathrm{Perf}(S))$ for the Grothendieck group of $\mathrm{Perf}(S)$, that is, the abelian group generated by quasi-isomorphism classes $[\mathcal{E}^\bullet]$ of objects $\mathcal{E}^\bullet$ in $\mathrm{Perf}(S)$, subject to the relations that $[\mathcal{F}^\bullet]=[\mathcal{E}^\bullet]+[\mathcal{G}^\bullet]$ whenever $\mathcal{E}^\bullet\to\mathcal{F}^\bullet\to\mathcal{G}^\bullet\to\mathcal{E}^\bullet[+1]$ is a distinguished triangle in $\mathrm{Perf}(S)$.

Note that for direct sums $\mathcal{E}^\bullet\oplus\mathcal{F}^\bullet$ in $\mathrm{Perf}(S)$ we have $[\mathcal{E}^\bullet\oplus\mathcal{F}^\bullet]=[\mathcal{E}^\bullet]+[\mathcal{F}^\bullet]$.

Write $\mathrm{LCon}(S,\mathbb{Z})$ for the abelian group of locally constant functions $f : S(\mathbb{K}) \to \mathbb{Z}$. Then there is a group morphism $\mathrm{rank} : K_0(\mathrm{Perf}(S)) \to \mathrm{LCon}(S,\mathbb{Z})$ called the *rank*, such that if $\mathcal{E}^\bullet \in \mathrm{Perf}(S)$, and $s \in S(\mathbb{K})$, and $\mathcal{E}^\bullet$ is quasi-isomorphic near s to a finite complex of vector bundles $\cdots \mathcal{F}_i \to \mathcal{F}_{i+1} \to \cdots$ with $\mathcal{F}_i$ in degree i, then $\mathrm{rank}([\mathcal{E}^\bullet]) : s \mapsto \sum_{i\in\mathbb{Z}}(-1)^i \,\mathrm{rank}\,\mathcal{F}_i$. If $\mathcal{E}$ is a vector bundle of rank r then $\mathrm{rank}([\mathcal{E}]) \equiv r$.

There is a natural biadditive operation $\otimes : K_0(\mathrm{Perf}(S)) \times K_0(\mathrm{Perf}(S)) \to K_0(\mathrm{Perf}(S))$, which acts by $[\mathcal{E}]\otimes[\mathcal{F}] = [\mathcal{E}\otimes\mathcal{F}]$ on vector bundles $\mathcal{E},\mathcal{F} \in \mathrm{Vect}(S)$, and $[\mathcal{E}^\bullet]\otimes[\mathcal{F}^\bullet] = [\mathcal{E}^\bullet \otimes^L \mathcal{F}^\bullet]$ on perfect complexes $\mathcal{E}^\bullet, \mathcal{F}^\bullet \in \mathrm{Perf}(S)$, where $\otimes^L$ is the left derived tensor product. Then $\otimes$ is commutative and associative, and makes $K_0(\mathrm{Perf}(S))$ into a commutative ring with identity $[\mathcal{O}_S]$, where $\mathcal{O}_S$ is the structure sheaf of S, a trivial line bundle of rank 1. We also have $\mathrm{rank}(\alpha\otimes\beta) = \mathrm{rank}\,\alpha\cdot\mathrm{rank}\,\beta$ for all $\alpha,\beta \in K_0(\mathrm{Perf}(S))$.

There is an additive operation of *duality* $^\vee : K_0(\mathrm{Perf}(S)) \to K_0(\mathrm{Perf}(S))$, written $\alpha \mapsto \alpha^\vee$ for $\alpha \in K_0(\mathrm{Perf}(S))$, which acts by $[\mathcal{E}]^\vee = [\mathcal{E}^*]$ on vector bundles $\mathcal{E} \to S$, where $\mathcal{E}^*$ is the dual vector bundle, and by $[\mathcal{E}^\bullet]^\vee = [(\mathcal{E}^\bullet)^\vee]$ on perfect complexes $\mathcal{E}^\bullet$, where $(\mathcal{E}^\bullet)^\vee$ is the dual complex. We have $(\alpha^\vee)^\vee = \alpha$, so duality is an isomorphism. Also $\mathrm{rank}\,\alpha^\vee = \mathrm{rank}\,\alpha$, and $(\alpha\otimes\beta)^\vee = (\alpha^\vee)\otimes(\beta^\vee)$.

If $f : S \to T$ is a morphism of $\mathbb{K}$-schemes, etc., we have *pullback morphisms* $K_0(f) : K_0(\mathrm{Perf}(T)) \to K_0(\mathrm{Perf}(S))$ acting by $K_0(f) : [\mathcal{E}^\bullet] \mapsto [f^*(\mathcal{E}^\bullet)]$ for $\mathcal{E}^\bullet \in \mathrm{Perf}(T)$. Pullbacks are contravariantly functorial and commute with all the structures above.

Example 2.22. Let G be an algebraic $\mathbb{K}$-group, and consider the quotient Artin $\mathbb{K}$-stack $[*/G]$, where $* = \mathrm{Spec}\,\mathbb{K}$. Then vector bundles on $[*/G]$ are equivalent to finite-dimensional G-representations, that is, we have an equivalence of categories $\mathrm{Vect}([*/G]) \simeq \mathrm{Rep}^{\mathrm{fd}}(G)$, and $\mathrm{Perf}([*/G])$ is equivalent to the bounded derived category $D^b\,\mathrm{Rep}^{\mathrm{fd}}(G)$.

When $G = \mathbb{G}_m$, the irreducible $\mathbb{G}_m$-representations are E_k for $k \in \mathbb{Z}$, where $E_k = \mathbb{K}$ with $\mathbb{G}_m$-action $\lambda : e \mapsto \lambda^k e$ for $\lambda \in \mathbb{G}_m$ and $e \in \mathbb{K}$, and finite-dimensional $\mathbb{G}_m$-representations are isomorphic to finite direct sums of the E_k. Hence $K_0(\mathrm{Perf}([*/\mathbb{G}_m]))$ is the free abelian group generated by classes $[E_k]$ for $k \in \mathbb{Z}$. We have $\mathrm{rank}[E_k] = 1$ and $[E_k]\otimes[E_l] = [E_{k+l}]$ for $k,l \in \mathbb{Z}$. Thus as a ring we have $K_0(\mathrm{Perf}([*/\mathbb{G}_m])) \cong \mathbb{Z}[\tau,\tau^{-1}]$ with $\tau = [E_1]$.

If S is a $\mathbb{K}$-scheme, or an Artin $\mathbb{K}$-stack, or a higher or derived $\mathbb{K}$-stack, then under some assumptions we can define the *cotangent complex* $\mathbb{L}_S$, an object in $D(\mathrm{qcoh}(S))$. See Illusie [69, 70] for $\mathbb{K}$-schemes, Laumon and Moret-Bailly [98, §17] and Olsson [125, §8] for Artin $\mathbb{K}$-stacks, and Toën and Vezzosi [151, §1.4], [149, §4.2.4–§4.2.5] for derived $\mathbb{K}$-stacks. We have $h^i(\mathbb{L}_S) = 0$ for $i > 0$ if S is a $\mathbb{K}$-scheme or Deligne–Mumford $\mathbb{K}$-stack, and $h^i(\mathbb{L}_S) = 0$ for $i > 1$ if S is an Artin $\mathbb{K}$-stack. If S is a smooth $\mathbb{K}$-scheme then $\mathbb{L}_S = T^*S$ is the usual cotangent bundle. If $\boldsymbol{S}$ is a derived stack locally of finite presentation then $\mathbb{L}_{\boldsymbol{S}}$ is perfect.

2.3.7 $[*/\mathbb{G}_m]$-actions, and principal $[*/\mathbb{G}_m]$-bundles

Suppose S is a $\mathbb{K}$-scheme, and $\mu : \mathbb{G}_m \times S \to S$ an action of the algebraic $\mathbb{K}$-group $\mathbb{G}_m = \mathbb{K} \setminus \{0\}$ on S. Then μ is a free action if it acts freely on $\mathbb{K}$-points $S(\mathbb{K})$. If μ is free we can form a quotient $\mathbb{K}$-scheme $T = S/\mathbb{G}_m$ with projection $\pi : S \to T$, which is a principal $\mathbb{G}_m$-bundle. We will generalize all this to Artin $\mathbb{K}$-stacks, replacing $\mathbb{G}_m$ by the group stack $[*/\mathbb{G}_m]$.

Definition 2.23. Let $\Omega : [*/\mathbb{G}_m] \times [*/\mathbb{G}_m] \to [*/\mathbb{G}_m]$ be the stack morphism induced by the group morphism $\mathbb{G}_m \times \mathbb{G}_m \to \mathbb{G}_m$ mapping $(\lambda, \mu) \mapsto \lambda\mu$, and $\iota : [*/\mathbb{G}_m] \to [*/\mathbb{G}_m]$ be induced by the group morphism $\mathbb{G}_m \to \mathbb{G}_m$ mapping $\lambda \mapsto \lambda^{-1}$, and $1 : * \to [*/\mathbb{G}_m]$ be the unique $\mathbb{K}$-point. Then $[*/\mathbb{G}_m]$ is an *abelian group stack*, an abelian group object in $\mathrm{Ho}(\mathbf{Art}_{\mathbb{K}}^{\mathrm{lft}})$, with multiplication Ω, inverse map ι, and identity 1. That is, writing $\pi : [*/\mathbb{G}_m] \to *$ for the projection and $\sigma : [*/\mathbb{G}_m]^2 \to [*/\mathbb{G}_m]^2$ for exchange of factors, we have

$$\begin{aligned} \Omega = \Omega \circ \sigma : [*/\mathbb{G}_m]^2 &\longrightarrow [*/\mathbb{G}_m], && \text{(commutative)},\\ \Omega \circ (\Omega \times \mathrm{id}) = \Omega \circ (\mathrm{id} \times \Omega) : [*/\mathbb{G}_m]^3 &\longrightarrow [*/\mathbb{G}_m], && \text{(associative)},\\ \Omega \circ (1 \circ \pi, \mathrm{id}) = \Omega \circ (\mathrm{id}, 1 \circ \pi) = \mathrm{id} : [*/\mathbb{G}_m] &\longrightarrow [*/\mathbb{G}_m], && \text{(identity)},\\ \Omega \circ (\iota, \mathrm{id}) = \Omega \circ (\mathrm{id}, \iota) = 1 \circ \pi : [*/\mathbb{G}_m] &\longrightarrow [*/\mathbb{G}_m], && \text{(inverses)}. \end{aligned} \tag{2.19}$$

We can then define an *action of the group stack* $[*/\mathbb{G}_m]$ *on an object* S *in* $\mathrm{Ho}(\mathbf{Art}_{\mathbb{K}}^{\mathrm{lft}})$ to be a morphism $\Psi : [*/\mathbb{G}_m] \times S \to S$ in $\mathrm{Ho}(\mathbf{Art}_{\mathbb{K}}^{\mathrm{lft}})$ such that the following commute in $\mathrm{Ho}(\mathbf{Art}_{\mathbb{K}}^{\mathrm{lft}})$:

$$\begin{tikzcd} {[*/\mathbb{G}_m] \times [*/\mathbb{G}_m] \times S} \arrow[r, "\Omega \times \mathrm{id}_S"'] \arrow[d, "\mathrm{id}_{[*/\mathbb{G}_m]} \times \Psi"] & {[*/\mathbb{G}_m] \times S} \arrow[d, "\Psi"'] \\ {[*/\mathbb{G}_m] \times S} \arrow[r, "\Psi"] & S, \end{tikzcd} \tag{2.20}$$

$$\begin{tikzcd} & {[*/\mathbb{G}_m] \times S} \arrow[dr, "\Psi"] & \\ * \times S \arrow[ur, "1 \times \mathrm{id}_S"] \arrow[rr, "\pi_S"] & & S. \end{tikzcd} \tag{2.21}$$

We call a $[*/\mathbb{G}_m]$-action $\Psi : [*/\mathbb{G}_m] \times S \to S$ *free* if for all $\mathbb{K}$-points $s \in S(\mathbb{K})$, the induced morphism on isotropy groups as in §2.3.4

$$\begin{aligned} &\Psi_* : \mathrm{Iso}_{[*/\mathbb{G}_m] \times S}((*, s)) \cong \mathbb{G}_m \times \mathrm{Iso}_S(s) \longrightarrow \mathrm{Iso}_S(s) \\ \text{has injective restriction} \quad &\Psi_*|_{\mathbb{G}_m \times \{1\}} : \mathbb{G}_m \times \{1\} \longrightarrow \mathrm{Iso}_S(s). \end{aligned} \tag{2.22}$$

A $[*/\mathbb{G}_m]$-*principal bundle* is a morphism $\rho : S \to T$ in $\mathrm{Ho}(\mathbf{Art}_{\mathbb{K}}^{\mathrm{lft}})$ and a $[*/\mathbb{G}_m]$-action $\Psi : [*/\mathbb{G}_m] \times S \to S$, such that ρ is a locally trivial fibre bundle in the smooth topology with fibre $[*/\mathbb{G}_m]$, and the following commutes

$$\begin{tikzcd} {[*/\mathbb{G}_m] \times S} \arrow[r, "\Psi"'] \arrow[d, "\pi_S"] & S \arrow[d, "\rho"'] \\ S \arrow[r, "\rho"] & T \end{tikzcd} \tag{2.23}$$

in $\mathrm{Ho}(\mathbf{Art}^{\mathrm{lft}}_{\mathbb{K}})$, and locally over T equation (2.23) is equivalent to the diagram

$$\begin{tikzcd}
{[*/\mathbb{G}_m] \times [*/\mathbb{G}_m] \times T} \arrow[rr, "\Omega\times\mathrm{id}_T"'] \arrow[d, "\Pi_1\times\Pi_3"] && {[*/\mathbb{G}_m] \times T} \arrow[d, "\pi_T"'] \\
{[*/\mathbb{G}_m] \times T} \arrow[rr, "\pi_T"] && T.
\end{tikzcd} \tag{2.24}$$

This implies that Ψ is a free $[*/\mathbb{G}_m]$-action on S.

We can also make the same definition for higher stacks.

We can also generalize Definition 2.23 to actions $\Psi : [*/\mathbb{G}_m]^k \times S \to S$ of $[*/\mathbb{G}_m]^k$ on S for $k = 2, 3, \ldots$, equivalent to k commuting $[*/\mathbb{G}_m]$-actions $\Psi_1, \ldots, \Psi_k$ on S, and principal $[*/\mathbb{G}_m]^k$-bundles, in the obvious way.

Remark 2.24. We have written Definition 2.23 solely in the ordinary category $\mathrm{Ho}(\mathbf{Art}^{\mathrm{lft}}_{\mathbb{K}})$. However, sometimes (e.g. in the proof of Proposition 2.29 below) we need to lift the ideas to the 2-category $\mathbf{Art}^{\mathrm{lft}}_{\mathbb{K}}$. To do this, first note that the morphisms $\Omega, \iota, 1, \pi, \sigma, \Psi, \rho, \ldots$ in $\mathrm{Ho}(\mathbf{Art}^{\mathrm{lft}}_{\mathbb{K}})$ are 2-isomorphism classes of 1-morphisms in $\mathbf{Art}^{\mathrm{lft}}_{\mathbb{K}}$, and choose a 1-morphism in each 2-isomorphism class, which by abuse of notation we also denote by $\Omega, \iota, \ldots$.

Then an equation such as $\Omega = \Omega \circ \sigma$ in $\mathrm{Ho}(\mathbf{Art}^{\mathrm{lft}}_{\mathbb{K}})$ in (2.19) means that *there exists* a 2-isomorphism $\eta : \Omega \Rightarrow \Omega \circ \sigma$ in $\mathbf{Art}^{\mathrm{lft}}_{\mathbb{K}}$. We must make a *particular choice* of such a 2-isomorphism, for each equation in morphisms in $\mathrm{Ho}(\mathbf{Art}^{\mathrm{lft}}_{\mathbb{K}})$ in Definition 2.23, which become part of the data we work with. These particular choices of 2-morphisms *are then required to satisfy a large number of identities.* We will not write these identities down explicitly, but the general rule is that if by using vertical and horizontal compositions of our 2-morphism data, we can make two 2-morphisms $\eta, \eta' : f \Rightarrow g$ mapping between the same 1-morphisms f, g, then we require that $\eta = \eta'$.

So, for instance, $[*/\mathbb{G}_m]$ is a 2-*group object* in the strict 2-category $\mathbf{Art}^{\mathrm{lft}}_{\mathbb{K}}$. The 2-morphisms lifting the equalities (2.19) are part of the data of a 2-group, and the identities they satisfy follow from the axioms of a 2-group.

Now it turns out that every $[*/\mathbb{G}_m]$-action $\Psi : [*/\mathbb{G}_m] \times S \to S$ in $\mathrm{Ho}(\mathbf{Art}^{\mathrm{lft}}_{\mathbb{K}})$ can be lifted to a 2-category $[*/\mathbb{G}_m]$-action in $\mathbf{Art}^{\mathrm{lft}}_{\mathbb{K}}$, which is unique up to the appropriate notion of equivalence. So the distinction will not be important to us, and we will mostly work with the simpler $\mathrm{Ho}(\mathbf{Art}^{\mathrm{lft}}_{\mathbb{K}})$ notion.

To prove this, observe that to lift a $[*/\mathbb{G}_m]$-action $\Psi : [*/\mathbb{G}_m] \times S \to S$ from $\mathrm{Ho}(\mathbf{Art}^{\mathrm{lft}}_{\mathbb{K}})$ to $\mathbf{Art}^{\mathrm{lft}}_{\mathbb{K}}$, in (2.20)–(2.21) we must choose 2-isomorphisms

$$\zeta : \Psi \circ (\mathrm{id}_{[*/\mathbb{G}_m]} \times \Psi) \Longrightarrow \Psi \circ (\Omega \times \mathrm{id}_S), \quad \eta : \Psi \circ (1 \times \mathrm{id}_S) \Longrightarrow \pi_S, \tag{2.25}$$

which must satisfy some identities. Even without imposing identities, ζ, η in (2.25) are close to being unique: from the definition of 2-morphisms in $\mathbf{Art}^{\mathrm{lft}}_{\mathbb{K}}$ in §2.3.2 we can show that if ζ', η' are alternative choices then $\zeta' = \alpha * \zeta$, $\eta' = \beta * \eta$ for unique 2-morphisms $\alpha, \beta : \mathrm{id}_S \Rightarrow \mathrm{id}_S$, where $*$ is horizontal composition in the strict 2-category $\mathbf{Art}^{\mathrm{lft}}_{\mathbb{K}}$, and the 2-morphisms $\alpha : \mathrm{id}_S \Rightarrow \mathrm{id}_S$ form an abelian group under vertical composition.

One of the identities that ζ,η must satisfy for the 2-category $[*/\mathbb{G}_m]$-action is that the following should commute, in 1- and 2-morphisms $*\times S\rightarrow S$:

$$\begin{tikzcd}
\Psi\times(\mathrm{id}_{[*/\mathbb{G}_m]}\times\Psi)\circ((1,1)\times\mathrm{id}_S) \arrow[d, Rightarrow, "\zeta*\mathrm{id}_{(1,1)\times\mathrm{id}_S}"'] \arrow[dr, Rightarrow, "\mathrm{id}_\Psi*(\mathrm{id}_{\mathrm{id}_{[*/\mathbb{G}_m]}}\times\eta)"] & \\
\Psi\circ(\Omega\times\mathrm{id}_S)\circ((1,1)\times\mathrm{id}_S) \arrow[r, Rightarrow, "\mathrm{id}_\Psi*(\theta\times\mathrm{id}_{\mathrm{id}_S})"] & \Psi\circ(1\times\mathrm{id}_S),
\end{tikzcd} \tag{2.26}$$

where $\theta:\Omega\circ(1,1)\Rightarrow 1$ is the 2-morphism of 1-morphisms $*\rightarrow[*/\mathbb{G}_m]$ in the 2-group structure on $[*/\mathbb{G}_m]$. For an arbitrary choice of ζ,η in (2.25), equation (2.26) need not hold, but as for the almost uniqueness of ζ,η above, there exists a unique 2-morphism $\alpha:\mathrm{id}_S\Rightarrow\mathrm{id}_S$ such that

$$\bigl(\alpha*\mathrm{id}_{\Psi\circ(1\times\mathrm{id}_S)}\bigr)\odot\bigl(\mathrm{id}_\Psi*(\theta\times\mathrm{id}_{\mathrm{id}_S})\bigr)\odot\bigl(\zeta*\mathrm{id}_{(1,1)\times\mathrm{id}_S}\bigr)=\mathrm{id}_\Psi*(\mathrm{id}_{\mathrm{id}_{[*/\mathbb{G}_m]}}\times\eta).$$

Then replacing ζ,η by $\zeta'=\alpha*\zeta$ and $\eta'=\eta$, we find that (2.26) holds.

We can now show that with these ζ',η', all the other required identities automatically hold. This can be reduced to considering the following situation: as in §2.3.2, $p_S:S\rightarrow\mathbf{Sch}_{\mathbb{K}}$ is a category fibred in groupoids. Let s be an object in S, and write Σ_s for the group of isomorphisms $\sigma:s\rightarrow s$ in S with $p_S(\sigma)=\mathrm{id}_{p_S(s)}$, and G_s for the abelian group $\mathbb{G}_m(p_S(s))$. Then after making some simplifying choices, Ψ induces a group morphism $\Psi_s:G_s\times\Sigma_s\rightarrow\Sigma_s$, and ζ,η induce elements $\zeta_s,\eta_s\in\Sigma_s$ satisfying $\zeta_s\Psi_s(\gamma,\Psi_s(\delta,\sigma))\zeta_s^{-1}=\Psi_s(\gamma\delta,\sigma)$ and $\eta_s\Psi_s(1,\sigma)\eta_s^{-1}=\sigma$ for all $\gamma,\delta\in G_s$ and $\sigma\in\Sigma_s$, and (2.26) forces $\zeta_s=\eta_s$. By elementary group theory we see that $\Psi_s(\alpha,\sigma)=\rho_s(\alpha)\cdot\zeta_s^{-1}\sigma\zeta_s$ for $\rho_s:G_s\rightarrow Z(\Sigma_s)\subseteq\Sigma_s$ a group morphism to the centre $Z(\Sigma_s)$ of Σ_s. We then use this formula to check the remaining identities hold.

Proposition 2.25. **(a)** *Suppose $\Psi:[*/\mathbb{G}_m]\times S\rightarrow S$ is a free $[*/\mathbb{G}_m]$-action in $\mathrm{Ho}(\mathbf{Art}_{\mathbb{K}}^{\mathrm{lft}})$. Then there exists a morphism $\rho:S\rightarrow T$ in $\mathrm{Ho}(\mathbf{Art}_{\mathbb{K}}^{\mathrm{lft}})$ with ρ,Ψ a principal $[*/\mathbb{G}_m]$-bundle, and T,ρ are unique up to isomorphism, and equation* (2.23) *is both homotopy Cartesian and homotopy co-Cartesian in $\mathbf{Art}_{\mathbb{K}}^{\mathrm{lft}}$. We regard T as the quotient $S/[*/\mathbb{G}_m]$ of S by the free $[*/\mathbb{G}_m]$-action Ψ. Here T is known as a* ***rigidification*** *of S, and written $T=S\,/\!\!/\,\mathbb{G}_m$ in* [3, §A], [139, §5].

(b) *The analogue of* **(a)** *also holds in $\mathrm{Ho}(\mathbf{HSt}_{\mathbb{K}}^{\mathrm{lft}})$. More generally, if $\Psi:[*/\mathbb{G}_m]\times S\rightarrow S$ is a $[*/\mathbb{G}_m]$-action in $\mathbf{HSt}_{\mathbb{K}}^{\mathrm{lft}}$ which need not be free, there still exists a morphism $\rho:S\rightarrow T$ in $\mathrm{Ho}(\mathbf{HSt}_{\mathbb{K}}^{\mathrm{lft}})$ such that* (2.23) *is homotopy co-Cartesian in $\mathbf{HSt}_{\mathbb{K}}^{\mathrm{lft}}$. We regard T as the quotient $S/[*/\mathbb{G}_m]$.*

Proof. Part (a) follows from Abramovich, Corti and Vistoli [2, Th. 5.1.5] (see also Abramovich, Olsson and Vistoli [3, §A] and Romagny [139, §5]). These concern 'rigidification', in which one modifies an Artin stack S by quotienting out a subgroup G from all its isotropy groups to get a new Artin stack $T=S\,/\!\!/\,G$. It is used, for example, to rigidify the Picard stack $\mathrm{Pic}(X)$ of line bundles on a projective scheme X to get the Picard scheme $\mathrm{Pic}(X)\,/\!\!/\,\mathbb{G}_m$.

Part (b) is straightforward because quotienting by a $[*/\mathbb{G}_m]$-action is an allowed operation for higher stacks. As in Toën [150, Def. 3.2] (see also Toën

[149, §3.1.2] and Toën and Vezzosi [151, §1.3.4]), we can define Artin 1-stacks to be Artin stacks, and then define Artin n-stacks inductively on n by saying an ∞-stack is an Artin $(n+1)$-stack if it is associated to a smooth groupoid $(X_1 \rightrightarrows X_0)$ in Artin n-stacks. A $[*/\mathbb{G}_m]$-action on an Artin n-stack X induces a smooth groupoid $([*/\mathbb{G}_m] \times X \rightrightarrows X)$, which is associated to the quotient Artin $(n+1)$-stack $X/[*/\mathbb{G}_m]$. For us, a higher stack is an Artin n-stack for any n. $\square$

We can *pull back* principal $[*/\mathbb{G}_m]$-bundles: if $\rho : S \to T$ is a principal $[*/\mathbb{G}_m]$-bundle with $[*/\mathbb{G}_m]$-action Ψ, and $f : T' \to T$ is a morphism in $\mathrm{Ho}(\mathbf{Art}_{\mathbb{K}}^{\mathrm{lft}})$, then we can form the 2-Cartesian square in $\mathbf{Art}_{\mathbb{K}}^{\mathrm{lft}}$:

$$\begin{tikzcd} S' = S \times_{\rho,T,f} T' \arrow[r, "\rho'"'] \arrow[d, "f'"] & T' \arrow[d, "f"'] \\ S \arrow[r, "\rho"] & T, \end{tikzcd} \tag{2.27}$$

and there is a natural $[*/\mathbb{G}_m]$-action $\Psi' : [*/\mathbb{G}_m] \times S' \to S'$ making $\rho' : S' \to T'$ into a principal $[*/\mathbb{G}_m]$-bundle. The next definition will be important in §3.4.

Definition 2.26. Let $\rho : S \to T$ be a principal $[*/\mathbb{G}_m]$-bundle with $[*/\mathbb{G}_m]$-action Ψ in $\mathrm{Ho}(\mathbf{Art}_{\mathbb{K}}^{\mathrm{lft}})$ or $\mathrm{Ho}(\mathbf{HSt}_{\mathbb{K}}^{\mathrm{lft}})$.

(i) We call ρ *trivial* if there exists an isomorphism $S \cong [*/\mathbb{G}_m] \times T$ which identifies $\rho : S \to T$ with $\pi_T : [*/\mathbb{G}_m] \times T \to T$ and $\Psi : [*/\mathbb{G}_m] \times S \to S$ with $\Omega \times \mathrm{id}_T : [*/\mathbb{G}_m] \times [*/\mathbb{G}_m] \times T \to [*/\mathbb{G}_m] \times T$, for $\Omega : [*/\mathbb{G}_m] \times [*/\mathbb{G}_m] \to [*/\mathbb{G}_m]$ as in Definition 2.23.

(ii) We call ρ *rationally trivial* if there exists a surjective morphism $f : T' \to T$ which over each connected component of T is a locally trivial fibration with fibre $[*/\mathbb{Z}_n]$, where $n = 1, 2, \ldots$ may depend on the connected component, such that the pullback principal $[*/\mathbb{G}_m]$-bundle $\rho' : S' = S \times_{\rho,T,f} T' \to T'$ is trivial, as in (i).

Rationally trivial $[*/\mathbb{G}_m]$-bundles behave like trivial $[*/\mathbb{G}_m]$-bundles from the point of view of homology $H_*(-)$ over a $\mathbb{Q}$-algebra R, which will be useful in §3.4. Proposition 2.29 below gives a criterion for when a principal $[*/\mathbb{G}_m]$-bundle is rationally trivial.

2.3.8 $[*/\mathbb{G}_m]$-actions on coherent sheaves and complexes

Definition 2.27. Let S be an Artin $\mathbb{K}$-stack locally of finite type, and $\Psi : [*/\mathbb{G}_m] \times S \to S$ be a $[*/\mathbb{G}_m]$-action, as in Definition 2.23, and $F \to S$ be a vector bundle, or coherent sheaf, or a complex in $\mathrm{Perf}(S)$, $D^b\mathrm{coh}(X)$ or $D(\mathrm{qcoh}(X))$.

An *action of* $[*/\mathbb{G}_m]$ *on* F *compatible with* Ψ, of *weight* $n \in \mathbb{Z}$, is an isomorphism of vector bundles, sheaves or complexes on $[*/\mathbb{G}_m] \times S$:

$$\Psi_F : \Psi^*(F) \longrightarrow \pi^*_{[*/\mathbb{G}_m]}(E_n) \otimes \pi^*_S(F), \tag{2.28}$$

where $\pi_{[*/\mathbb{G}_m]} : [*/\mathbb{G}_m] \times S \to [*/\mathbb{G}_m]$, $\pi_S : [*/\mathbb{G}_m] \times S \to S$ are the projections, and E_n is as in Example 2.22, regarded as a line bundle on $[*/\mathbb{G}_m]$, such that the following diagram of isomorphisms over $[*/\mathbb{G}_m] \times [*/\mathbb{G}_m] \times S$ commutes:

$$
\begin{tikzcd}
(\Omega \times \mathrm{id}_S)^* \circ \Psi^*(F) \arrow[r, "\text{from (2.20)}"'] \arrow[d, "(\Omega\times\mathrm{id}_S)^*(\Psi_F)"] & (\mathrm{id}_{[*/\mathbb{G}_m]} \times \Psi)^* \circ \Psi^*(F) \arrow[d, "(\mathrm{id}_{[*/\mathbb{G}_m]}\times\Psi)^*(\Psi_F)"'] \\
(\Omega\times\mathrm{id}_S)^*(\pi^*_{[*/\mathbb{G}_m]}(E_n)\otimes\pi^*_S(F)) \arrow[d, "\text{natural isomorphism}"] & (\mathrm{id}_{[*/\mathbb{G}_m]}\times\Psi)^*(\pi^*_{[*/\mathbb{G}_m]}(E_n)\otimes\pi^*_S(F)) \arrow[d, "\text{natural isomorphism}"'] \\
\pi^*_{[*/\mathbb{G}_m]_1\times[*/\mathbb{G}_m]_2}(\Omega^*(E_n))\otimes\pi^*_S(F) \arrow[d, "\pi^*_{[*/\mathbb{G}_m]_1\times[*/\mathbb{G}_m]_2}(\theta_n)\otimes\mathrm{id}_{\pi^*_S(F)}"] & \pi^*_{[*/\mathbb{G}_m]_1}(E_n) \otimes \pi^*_{[*/\mathbb{G}_m]_2\times S}(\Psi^*(F)) \arrow[d, "\mathrm{id}_{\pi^*_{[*/\mathbb{G}_m]_1}(E_n)}\otimes\pi^*_{[*/\mathbb{G}_m]_2\times S}(\Psi_F)"'] \\
\pi^*_{[*/\mathbb{G}_m]_1}(E_n)\otimes\pi^*_{[*/\mathbb{G}_m]_2}(E_n)\otimes\pi^*_S(F) \arrow[r, equal] & \pi^*_{[*/\mathbb{G}_m]_1}(E_n)\otimes\pi^*_{[*/\mathbb{G}_m]_2}(E_n)\otimes\pi^*_S(F).
\end{tikzcd}
\tag{2.29}
$$

Here we write $[*/\mathbb{G}_m] \times [*/\mathbb{G}_m] \times S$ as $[*/\mathbb{G}_m]_1 \times [*/\mathbb{G}_m]_2 \times S$ to distinguish the factors, and $\Omega : [*/\mathbb{G}_m]_1 \times [*/\mathbb{G}_m]_2 \to [*/\mathbb{G}_m]$ is as in Definition 2.23, and $\theta_n : \Omega^*(E_n) \to \pi^*_{[*/\mathbb{G}_m]_1}(E_n)\otimes\pi^*_{[*/\mathbb{G}_m]_2}(E_n)$ is the natural isomorphism.

When $n = 0$, E_0 and $\pi^*_{[*/\mathbb{G}_m]}(E_0)$ are trivial line bundles, so we can omit them from the tensor products, and take Ψ_F to map $\Psi_F : \Psi^*(F) \to \pi^*_S(F)$.

Example 2.28. Let $\rho = \pi_T : S = [*/\mathbb{G}_m] \times T \to T$ be the trivial principal $[*/\mathbb{G}_m]$-bundle over an Artin $\mathbb{K}$-stack T with $[*/\mathbb{G}_m]$-action $\Psi = \Omega \times \mathrm{id}_T : [*/\mathbb{G}_m]^2 \times T \to [*/\mathbb{G}_m] \times T$. Suppose $G \to T$ is a vector bundle, or coherent sheaf, or complex on T, and $n \in \mathbb{Z}$. Then $F = \pi^*_{[*/\mathbb{G}_m]}(E_n) \otimes \pi^*_T(G)$ is a vector bundle, or coherent sheaf, or complex on $S = [*/\mathbb{G}_m] \times T$ with a natural $[*/\mathbb{G}_m]$-action Ψ_F compatible with Ψ, of weight n.

Conversely, if $F \to S$ has a $[*/\mathbb{G}_m]$-action Ψ_F compatible with Ψ of weight n, set $G = (1 \circ \pi, \mathrm{id}_T)^*(F)$, where $\pi : T \to *$ and $1 : * \to [*/\mathbb{G}_m]$ are the unique morphisms. Then we can use (2.29) to show that $F \cong \pi^*_{[*/\mathbb{G}_m]}(E_n) \otimes \pi^*_T(G)$.

If $\mathcal{F}^\bullet$ is a perfect complex on S of rank r, equipped with a $[*/\mathbb{G}_m]$-action $\Psi_{\mathcal{F}^\bullet}$ of weight n, it is easy to see that the determinant line bundle $\det(\mathcal{F}^\bullet)$ has a natural $[*/\mathbb{G}_m]$-action $\Psi_{\det(\mathcal{F}^\bullet)}$ of weight $n \cdot r$.

If Ψ is the $[*/\mathbb{G}_m]$-action of a principal $[*/\mathbb{G}_m]$-bundle $\rho : S \to T$, one can show that if G is a vector bundle, sheaf or complex on T then $F = \rho^*(G)$ has a natural weight zero $[*/\mathbb{G}_m]$-action Ψ_F, and conversely, if $F \to S$ has a weight zero $[*/\mathbb{G}_m]$-action then $F \cong \rho^*(G)$ for some $G \to T$.

Proposition 2.29. *Suppose the field $\mathbb{K}$ is algebraically closed. Let $\rho : S \to T$ be a principal $[*/\mathbb{G}_m]$-bundle with $[*/\mathbb{G}_m]$-action $\Psi : [*/\mathbb{G}_m] \times S \to S$. Suppose $L \to S$ is a line bundle, with a $[*/\mathbb{G}_m]$-action of weight $n \neq 0$ compatible with Ψ. Then $\rho : S \to T$ is rationally trivial, in the sense of Definition 2.26. For example, if $\mathcal{F}^\bullet$ is a perfect complex on S of rank $r \neq 0$, equipped with a $[*/\mathbb{G}_m]$-action of weight $d \neq 0$ compatible with Ψ, we can take $L = \det(\mathcal{F}^\bullet)$ and $n = d \cdot r$.*

Proof. Considering L as a coherent sheaf over S, write $\pi : \bar{L} \to S$ for the corresponding $\mathbb{A}^1$-bundle in $\mathrm{Ho}(\mathbf{Art}_{\mathbb{K}}^{\mathrm{lft}})$, and write $\bar{L}'$ for the complement of the zero section in $\bar{L}$, so that $\pi' = \pi|_{\bar{L}'} : \bar{L}' \to S$ is a bundle with fibre $\mathbb{A}^1 \setminus \{0\} = \mathbb{G}_m$.

Consider the diagram, in both $\mathrm{Ho}(\mathbf{Art}_{\mathbb{K}}^{\mathrm{lft}})$ and $\mathbf{Art}_{\mathbb{K}}^{\mathrm{lft}}$:

$$\begin{tikzcd}
{[*/\mathbb{G}_m]^2 \times \bar{L}'} \arrow[r, "\Omega\times\mathrm{id}_{\bar{L}'}"'] \arrow[d, "\mathrm{id}_{[*/\mathbb{G}_m]^2}\times\pi'"] & {[*/\mathbb{G}_m] \times \bar{L}'} \arrow[r, "\pi_{\bar{L}'}"'] \arrow[d, "\mathrm{id}_{[*/\mathbb{G}_m]}\times\pi'"] & \bar{L}' \arrow[d, "\pi'"'] \\
{[*/\mathbb{G}_m]^2 \times S} \arrow[r, "\Omega\times\mathrm{id}_S"] \arrow[d, "\mathrm{id}_{[*/\mathbb{G}_m]}\times\Psi"] & {[*/\mathbb{G}_m] \times S} \arrow[r, "\pi_S"] \arrow[d, "\Psi"] & S \arrow[d, "\rho"'] \\
{[*/\mathbb{G}_m] \times S} \arrow[r, "\Psi"] & S \arrow[r, "\rho"] & T,
\end{tikzcd} \tag{2.30}$$

where $\Omega : [*/\mathbb{G}_m]^2 \to [*/\mathbb{G}_m]$ is as in Definition 2.23. The top two squares obviously commute in $\mathrm{Ho}(\mathbf{Art}_{\mathbb{K}}^{\mathrm{lft}})$, and are 2-Cartesian in $\mathbf{Art}_{\mathbb{K}}^{\mathrm{lft}}$. The bottom two squares are (2.20) and (2.23), so they commute in $\mathrm{Ho}(\mathbf{Art}_{\mathbb{K}}^{\mathrm{lft}})$. When we lift to 2-categories as in Remark 2.24, they also become 2-Cartesian.

As all the squares in (2.30) are 2-Cartesian, we see that $\pi_{\bar{L}'} : [*/\mathbb{G}_m] \times \bar{L}' \to \bar{L}'$ is the pullback principal $[*/\mathbb{G}_m]$-bundle of $\rho : S \to T$ by $\rho \circ \pi' : \bar{L}' \to T$. The left hand side of (2.30) gives the $[*/\mathbb{G}_m]$-action on this pullback principal $[*/\mathbb{G}_m]$-bundle, and shows it is trivial, in the sense of Definition 2.26(i).

Now $\rho \circ \pi' : L' \to T$ is the composition of a $[*/\mathbb{G}_m]$-fibration and a $\mathbb{G}_m$-fibration. As L has a $[*/\mathbb{G}_m]$-action of weight $n \neq 0$, considering local models shows that $\rho \circ \pi'$ is a fibration with fibre $[\mathbb{G}_m/\mathbb{G}_m]$, where $\mathbb{G}_m$ acts on $\mathbb{G}_m$ by $\lambda : \mu \mapsto \lambda^n \mu$. Since $\mathbb{K}$ is algebraically closed we see that $[\mathbb{G}_m/\mathbb{G}_m] \cong [*/\mathbb{Z}_{|n|}]$, where $\mathbb{Z}_{|n|}$ is the group of $|n|^{\mathrm{th}}$ roots of 1 in $\mathbb{K}$, and $\rho \circ \pi'$ is a locally trivial $[*/\mathbb{Z}_{|n|}]$-fibration. (See the proof of Theorem 3.47(b) in §4.11 for more details on this argument.) Hence ρ is rationally trivial by Definition 2.26(ii). □

The analogue of Proposition 2.29 also works for higher stacks.

2.4 (Co)homology of (higher) Artin $\mathbb{K}$-stacks

2.4.1 Assumptions on homology and cohomology theories

For a (higher) Artin $\mathbb{K}$-stack S, we will need good notions of *homology* $H_*(S)$, *cohomology* $H^*(S)$, and *Chern class maps* $c_i : K_0(\mathrm{Perf}(S)) \to H^{2i}(S)$, with the usual structures and properties. The next assumption gives the properties we will need for all our vertex algebra and Lie algebra constructions in §3.

Assumption 2.30. (a) We fix a commutative ring R, such as $R = \mathbb{Z}, \mathbb{Q}, \mathbb{C}$ or $\mathbb{K}$, which will be the coefficients for our (co)homology theories. Write R-mod for the category of R-modules.

(i) We should be given covariant functors $H_i : \mathrm{Ho}(\mathbf{Art}_{\mathbb{K}}^{\mathrm{lft}}) \to R$-mod for $i = 0, 1, \ldots$ called *homology*, and contravariant functors $H^i : \mathrm{Ho}(\mathbf{Art}_{\mathbb{K}}^{\mathrm{lft}}) \to R$-mod for $i = 0, 1, \ldots$ called *cohomology*. We set $H_i(S) = H^i(S) = 0$ if $i < 0$.

That is, whenever S is an Artin $\mathbb{K}$-stack, we are given R-modules $H_i(S)$ for $i = 0, 1, \ldots$ called the *homology groups* of S, and R-modules $H^i(S)$ for $i = 0, 1, \ldots$ called the *cohomology groups* of S.

Whenever $f : S \to T$ is a morphism of such stacks, we are given functorial R-module morphisms $H_i(f) : H_i(S) \to H_i(T)$ and $H^i(f) : H^i(T) \to H^i(S)$ for $i = 0, 1, \ldots$.

For the triangulated category versions of our theory, involving moduli of complexes, we require these (co)homology theories also to be defined for *higher* Artin $\mathbb{K}$-stacks, that is, we should be given covariant functors $H_i : \mathrm{Ho}(\mathbf{HSt}_{\mathbb{K}}^{\mathrm{lft}}) \to R\text{-mod}$ and contravariant functors $H^i : \mathrm{Ho}(\mathbf{HSt}_{\mathbb{K}}^{\mathrm{lft}}) \to R\text{-mod}$ for $i = 0, 1, \ldots$.

When we say 'for all S' below, we mean either for all Artin $\mathbb{K}$-stacks S, or for all higher Artin $\mathbb{K}$-stacks S, depending on the domain $\mathrm{Ho}(\mathbf{Art}_{\mathbb{K}}^{\mathrm{lft}})$ or $\mathrm{Ho}(\mathbf{HSt}_{\mathbb{K}}^{\mathrm{lft}})$ of the H_i, H^i.

If $\boldsymbol{S}$ is a derived Artin stack, or a derived stack, we define the (co)homology groups $H_i(\boldsymbol{S}), H^i(\boldsymbol{S})$ to be the (co)homology of the classical truncation $H_i(t_0(\boldsymbol{S})), H^i(t_0(\boldsymbol{S}))$.

(ii) There are R-bilinear, functorial *cup products* $\cup : H^i(S) \times H^j(S) \to H^{i+j}(S)$ and *cap products* $\cap : H_i(S) \times H^j(S) \to H_{i-j}(S)$ and an *identity element* $1_S \in H^0(S)$ for all S, with the usual properties of cup and cap products in classical (co)homology [63,108], so that $\cup, 1_S$ make $H^*(S)$ into a supercommutative, associative, graded, unital R-algebra, and $\cap$ makes $H_*(S)$ into a graded module over $H^*(S)$. If $f : S \to T$ is a morphism and $\alpha \in H_i(S)$, $\beta \in H^j(T)$ then $H_{i-j}(f)(\alpha \cap H^j(f)(\beta)) = (H_i(f)(\alpha)) \cap \beta$ in $H_{i-j}(T)$.

(iii) For all S, T and $i, j \geqslant 0$ there are R-bilinear, functorial *external products* $\boxtimes : H_i(S) \times H_j(T) \to H_{i+j}(S \times T)$, $\boxtimes : H^i(S) \times H^j(T) \to H^{i+j}(S \times T)$, with the usual properties of external products in classical (co)homology, including being associative and supercommutative.

For cohomology we have $\alpha \boxtimes \beta = \pi_S^*(\alpha) \cup \pi_T^*(\beta)$ when $\alpha \in H^i(S)$ and $\beta \in H^j(T)$, so $\boxtimes$ is not extra data in this case.

(iv) When S is the point $* = \operatorname{Spec} \mathbb{K}$, there are canonical isomorphisms:

$$H_i(*) \cong \begin{cases} R, & i = 0, \\ 0, & i > 0, \end{cases} \qquad H^i(*) \cong \begin{cases} R, & i = 0, \\ 0, & i > 0, \end{cases}$$

identifying $1_* \in H^0(*)$ with $1 \in R$ and $\cup, \cap$ with multiplication in R.

These conditions do not determine the isomorphism $H_0(*) \cong R$ uniquely. We do this by requiring that for all S and $\alpha \in H_k(S)$ we have $\alpha \boxtimes 1 \cong \alpha$ under the isomorphism $H_k(S \times *) \cong H_k(S)$ coming from $S \times * \cong S$.

(v) Let $\{S_a : a \in A\}$ be a family of (higher) Artin $\mathbb{K}$-stacks. Then we can form the disjoint union $\coprod_{a \in A} S_a$, with inclusion morphisms $\iota_b : S_b \hookrightarrow$

$\coprod_{a\in A} S_a$ for $b \in A$, inducing maps $H_i(\iota_b) : H_i(S_b) \to H_i(\coprod_{a\in A} S_a)$, $H^i(\iota_b) : H^i(\coprod_{a\in A} S_a) \to H^i(S_b)$ for all $i = 0, 1, \ldots$. These $H_i(\iota_b), H^i(\iota_b)$ for all $b \in A$ should induce isomorphisms of R-modules:

$$H_i(\textstyle\coprod_{a\in A} S_a) \cong \bigoplus_{a\in A} H_i(S_a), \tag{2.31}$$

$$H^i(\textstyle\coprod_{a\in A} S_a) \cong \prod_{a\in A} H^i(S_a). \tag{2.32}$$

The difference between $\bigoplus_{a\in A}$ and $\prod_{a\in A}$ is that $(\lambda_a)_{a\in A} \in \bigoplus_{a\in A} H_i(S_a)$ has $\lambda_a \neq 0$ in $H_i(S_a)$ for only finitely many $a \in A$, but $(\mu_a)_{a\in A} \in \prod_{a\in A} H^i(S_a)$ can have $\mu_a \neq 0$ in $H^i(S_a)$ for arbitrarily many $a \in A$.

(vi) $H_i(\pi_S) : H_i(S \times \mathbb{A}^1) \to H_i(S)$ and $H^i(\pi_S) : H^i(S) \to H^i(S \times \mathbb{A}^1)$ are isomorphisms for all S, i, where $\pi_S : S \times \mathbb{A}^1 \to S$ is the projection.

(b) Use the notation of Definition 2.21. We should be given *Chern class maps* $c_i : K_0(\mathrm{Perf}(S)) \to H^{2i}(S)$ for all $i = 1, 2, \ldots$ and for all S. We also write $c_0(\alpha) = 1_S$ in $H^0(S)$ for all $\alpha \in K_0(\mathrm{Perf}(S))$.

These maps c_i have the usual properties of Chern classes in classical algebraic topology and intersection theory, as in Hirzebruch [64], Milnor and Stasheff [114] and Hartshorne [62, App. A], for instance. In particular:

(i) If $\mathcal{E}$ is a vector bundle of rank r on S then $c_i([\mathcal{E}]) = 0$ for $i > r$.

(ii) If $\alpha, \beta \in K_0(\mathrm{Perf}(S))$ and $j \geqslant 0$ then

$$c_j(\alpha + \beta) = \sum_{i=0}^{j} c_i(\alpha) \cup c_{j-i}(\beta). \tag{2.33}$$

(iii) If $\alpha, \beta \in K_0(\mathrm{Perf}(S))$, for all $k = 1, 2, \ldots$ we have a formula

$$c_k(\alpha\otimes\beta) = P_k^{\otimes}\bigl(\operatorname{rank}\alpha, c_1(\alpha), \ldots, c_k(\alpha), \operatorname{rank}\beta, c_1(\beta), \ldots, c_k(\beta)\bigr), \tag{2.34}$$

where $P_k^{\otimes}(a_0, \ldots, a_k, b_0, \ldots, b_k)$ are universal polynomials with rational coefficients (though for $a_0, b_0 \in \mathbb{Z}$ they have integral coeffcients as polynomials in $a_1, \ldots, a_k, b_1, \ldots, b_k$), with $P_k^{\otimes}(a_0, \ldots, a_k, b_0, \ldots, b_k) = P_k^{\otimes}(b_0, \ldots, b_k, a_0, \ldots, a_k)$, such that if a_i, b_i are graded of degree $2i$ then $P_k^{\otimes}$ is graded of degree $2k$. The $P_k^{\otimes}$ may be computed as in Hartshorne [62, p. 430] and Hirzebruch [64, §4.4], and the first few are

$$\begin{aligned}
P_1^{\otimes}(a_0, a_1, b_0, b_1) &= a_0 b_1 + a_1 b_0, \\
P_2^{\otimes}(a_0, a_1, a_2, b_0, b_1, b_2) &= \tbinom{a_0}{2} b_1^2 + a_0 b_2 + \tbinom{b_0}{2} a_1^2 + b_0 a_2 \\
&\quad + (a_0 b_0 - 1) a_1 b_1, \\
P_3^{\otimes}(a_0, a_1, a_2, a_3, b_0, b_1, b_2, b_3) &= \tbinom{a_0}{3} b_1^3 + 2\tbinom{a_0}{2} b_1 b_2 + a_0 b_3 \\
+\tbinom{b_0}{3} a_1^3 + 2\tbinom{b_0}{2} a_1 a_2 &+ b_0 a_3 + (a_0 - 1)(\tfrac{1}{2} a_0 b_0 - 1) a_1 b_1^2 \\
+(b_0 - 1)(\tfrac{1}{2} a_0 b_0 - 1) a_1^2 b_1 &+ (a_0 b_0 - 2)(a_1 b_2 + a_2 b_1).
\end{aligned} \tag{2.35}$$

In the special case in which $\beta = [L]$ for $L \to S$ a line bundle, so that $\operatorname{rank}\beta = 1$ and $c_i(\beta) = 0$ for $i > 1$, equation (2.34) may be written

$$\begin{aligned} c_j(\alpha\otimes\beta) &= \sum_{i=0}^{j} \binom{\operatorname{rank}\alpha - i}{j-i} c_i(\alpha)\cup c_1(\beta)^{j-i} \\ &= \sum_{i=0}^{j} (-1)^{j-i}\binom{j-\operatorname{rank}\alpha - 1}{j-i} c_i(\alpha)\cup c_1(\beta)^{j-i}, \end{aligned} \tag{2.36}$$

where binomial coefficients $\binom{m}{n}$ for $m,n\in\mathbb{Z}$ are discussed in Appendix A, and the two expressions are equal by (A.4). See §2.4.2 for an alternative approach to Chern classes of tensor products using *Chern characters*.

(iv) Under duality we have $c_i(\alpha^\vee) = (-1)^i c_i(\alpha)$ in $H^{2i}(S)$ for all $i = 1,2,\ldots$ and $\alpha\in K_0(\operatorname{Perf}(S))$.

(v) If $f : S\to T$ is a morphism and $\alpha\in K_0(\operatorname{Perf}(T))$ then $c_i(K_0(f)(\alpha)) = H^{2i}(f)(c_i(\alpha))$ for all $i = 1,2,\ldots$.

(c) When S is the quotient Artin $\mathbb{K}$-stack $[*/\mathbb{G}_m]$, there are canonical isomorphisms of graded R-modules $H_*([*/\mathbb{G}_m])\cong R[t]$ and $H^*([*/\mathbb{G}_m])\cong R[\tau]$, where t and τ are formal variables of degree 2. That is, we have $H_{2i}([*/\mathbb{G}_m]) = R\cdot t^i$ and $H^{2i}([*/\mathbb{G}_m]) = R\cdot\tau^i$ for $i = 0,1,2,\ldots$, where $t^i\in H_{2i}([*/\mathbb{G}_m])$ and $\tau^i\in H^{2i}([*/\mathbb{G}_m])$ are elements which freely generate $H_{2i}([*/\mathbb{G}_m]), H^{2i}([*/\mathbb{G}_m])$ as R-modules, and $H_k([*/\mathbb{G}_m]) = H^k([*/\mathbb{G}_m]) = 0$ for $k\in\mathbb{Z}\setminus 2\mathbb{N}$.

These canonical isomorphisms are characterized uniquely by the following:

(i) $\tau^0 = 1_{[*/\mathbb{G}_m]}$ in $H^0([*/\mathbb{G}_m])$.

(ii) $\tau^i\cup\tau^j = \tau^{i+j}$ in $H^{2i+2j}([*/\mathbb{G}_m])$ for all $i,j\geqslant 0$.

(iii) Example 2.22 describes $\operatorname{Vect}([*/\mathbb{G}_m])$, $\operatorname{Perf}([*/\mathbb{G}_m])$, $K_0(\operatorname{Perf}([*/\mathbb{G}_m]))$ explicitly. Using the notation of Example 2.22 we have $\tau = c_1([E_1])$, where E_1 is the standard representation of $\mathbb{G}_m$ on $\mathbb{K}$, regarded as a line bundle on $[*/\mathbb{G}_m]$.

(iv) $t^i\cap\tau^j = t^{i-j}$ in $H_{2i-2j}([*/\mathbb{G}_m])$ whenever $0\leqslant j\leqslant i$.

(v) The projection $\pi : [*/\mathbb{G}_m]\to *$ induces $H_0(\pi) : H_0([*/\mathbb{G}_m])\to H_0(*)$. Under the isomorphisms $H_0([*/\mathbb{G}_m]) = R\cdot t^0$ above and $H_0(*)\cong R$ in (a)(iv), we have $H_0(\pi) : t^0\mapsto 1$.

Here is an extra assumption for when R is a $\mathbb{Q}$-algebra:

Assumption 2.31. Suppose Assumption 2.30 holds for (co)homology theories $H_*(-), H^*(-)$ over R of Artin $\mathbb{K}$-stacks (or higher Artin $\mathbb{K}$-stacks), where R must be a $\mathbb{Q}$-*algebra.* Then

(a) **(Homology of $[*/\mathbb{Z}_n]$-fibrations.)** Let $f : S\to T$ be a locally trivial fibre bundle in $\operatorname{Ho}(\mathbf{Art}^{\mathrm{lft}}_{\mathbb{K}})$ (or $\operatorname{Ho}(\mathbf{HSt}^{\mathrm{lft}}_{\mathbb{K}})$) with fibres $[*/\mathbb{Z}_n]$ for some $n\geqslant 1$. Then $H_k(f) : H_k(S)\to H_k(T)$ is an isomorphism for all k.

(b) (Künneth Theorem.) The exterior product maps $\boxtimes$ of Assumption 2.30(a)(iii) induce an isomorphism

$$\bigoplus_{i,j\geqslant 0:i+j=k}\boxtimes:\bigoplus_{i,j\geqslant 0:i+j=k}H_i(S)\otimes_R H_j(T)\longrightarrow H_k(S\times T).$$

Remark 2.32. (Coarse moduli spaces.) As in Olsson [127, §11], under good conditions a Deligne–Mumford or Artin $\mathbb{K}$-stack S has a *coarse moduli space* $S_{\rm coa}$, which is a $\mathbb{K}$-scheme or algebraic $\mathbb{K}$-space with a morphism $\pi:S\to S_{\rm coa}$ which is universal for morphisms $S\to T$ for T an algebraic $\mathbb{K}$-space. The coarse moduli space $S_{\rm coa}$ forgets the isotropy groups $\mathrm{Iso}_S(x)$ of points x in S.

It is important that by the (co)homology groups $H_i(S),H^i(S)$ we do *not* mean the (co)homology groups $H_i(S_{\rm coa}),H^i(S_{\rm coa})$ of the coarse moduli space. Rather, $H_i(S),H^i(S)$ *really do depend on the stack structure and the isotropy groups* $\mathrm{Iso}_S(x)$ *in a nontrivial way.* For example, $[*/\mathbb{G}_m]$ has coarse moduli space $*$, but Assumption 2.30(a)(iv) and (c) show that the (co)homologies of $[*/\mathbb{G}_m]$ and $*$ are different.

When S is a quotient stack $[X/G]$ we should think of $H_*(S),H^*(S)$ as the equivariant (co)homology $H_*^G(X),H^*_G(X)$. These need not agree with the (co)homology $H_*(X/G),H^*(X/G)$ of the topological quotient X/G, if G does not act freely on X.

Remark 2.33. (Different types of (co)homology.) For homology and cohomology of ordinary topological spaces X, there are four main types of theory:

(i) *Homology* $H_*(X,R)$, as in [22, 41, 63, 108, 117, 145]. This is covariantly functorial under all continuous maps $f:X\to Y$. It is homotopy invariant.

(ii) *Cohomology* $H^*(X,R)$, as in [22, 41, 63, 108, 117, 145]. This is contravariantly functorial under all continuous maps $f:X\to Y$. It is homotopy invariant.

(iii) *Compactly-supported cohomology* $H^*_{\rm cs}(X,R)$, as in [22, 63, 108, 145]. This is contravariantly functorial under all proper maps $f:X\to Y$. It is not homotopy invariant.

(iv) *Locally finite homology* $H^{\rm lf}_*(X,R)$ as in [65], also known as *homology with closed supports* [36], or *Borel–Moore homology* [19]. We recommend Hughes and Ranicki [65, §3] for an introduction. This is covariantly functorial under all proper maps $f:X\to Y$. It is not homotopy invariant.

If R is a field then $H^k(X,R)\cong H_k(X,R)^*$ and $H^{\rm lf}_k(X,R)\cong H^k_{\rm cs}(X,R)^*$.

In Assumption 2.30 we need theories with the properties of homology and cohomology. Some homology theories of stacks in the literature are analogues of locally finite homology $H^{\rm lf}_*(-)$ rather than homology $H_*(-)$, so that pushforward maps $H_i(f)$ are defined only for proper (or proper and representable, or projective) morphisms. We will call such theories 'of type $H^{\rm lf}_*(-)$'. *Homology theories of this type are unsuitable for our purposes*, as we need pushforward maps $H_i(f)$ along *non-proper* morphisms $f:X\to Y$. However, as in Remark 2.34(a), given a theory of type $H^{\rm lf}_*(-)$, we can define another theory of the type we want by a direct limit.

Remark 2.34. (Constructing (co)homology theories from other (co)homology theories.) Suppose that we have found some (co)homology theory of $\mathbb{K}$-schemes or $\mathbb{K}$-stacks in the literature, and we would like to use it in our vertex algebra and Lie algebra constructions, but it does not have all the properties required in Assumption 2.30. For example, we might have:

(i) We are given a homology theory $H_*(-)$ and a compatible cohomology theory $H^*(-)$, but the homology theory is of type $H_*^{\mathrm{lf}}(-)$, with pushforwards defined only for proper morphisms (or proper and representable, etc.).

(ii) We are given (co)homology theories $H_*(S), H^*(S)$ which are not defined for all (higher) Artin $\mathbb{K}$-stacks S, but only for a subclass, for instance, only for $\mathbb{K}$-schemes, or Deligne–Mumford $\mathbb{K}$-stacks, or finite type Artin stacks.

(iii) We are given a homology theory $H_*(-)$ (possibly of type $H_*^{\mathrm{lf}}(-)$), but no matching cohomology theory.

In this case Chern classes $c_i(\mathcal{E})$ for vector bundles $\mathcal{E} \to X$ may still be defined as *operations on homology* $\cap\, c_i(\mathcal{E}) : H_k(X) \to H_{k-2i}(X)$.

It is often possible to use the given theories $H_*(-), H^*(-)$ to construct (co)homology theories $\hat{H}_*(-), \hat{H}^*(-)$ with all the properties we want in a purely formal way, by a limiting procedure. This is discussed by Fulton and MacPherson [51, §3.3, §8] in the context of bivariant theories. It has the disadvantage that the new $\hat{H}_*(-), \hat{H}^*(-)$ may be more difficult to compute in examples.

(a) (Converting locally finite homology to homology.) Suppose we are given a homology theory $H_*(-)$ of Artin $\mathbb{K}$-stacks which is of type $H_*^{\mathrm{lf}}(-)$, with pushforwards defined only for proper morphisms; we may also be given a compatible cohomology theory $H^*(-)$.

As in [51, §3.3], for each Artin $\mathbb{K}$-stack S define

$$\hat{H}_k(S) = \varinjlim_{\substack{\varphi:P\to S,\\ P \text{ proper}}} H_k(P), \tag{2.37}$$

where the objects in the direct limit in R-mod are $H_k(P)$ for all morphisms $\varphi : P \to S$ in $\mathrm{Ho}(\mathbf{Art}_{\mathbb{K}}^{\mathrm{lft}})$ with P proper, and the morphisms in the direct limit are $H_k(\psi) : H_k(P_1) \to H_k(P_2)$ for all commutative triangles in $\mathrm{Ho}(\mathbf{Art}_{\mathbb{K}}^{\mathrm{lft}})$ with P_1, P_2 proper (which implies ψ is proper, so $H_k(\psi)$ is well-defined):

$$\begin{tikzcd} P_1 \arrow[r, "\psi"] \arrow[dr, "\varphi_1"'] & P_2 \arrow[d, "\varphi_2"] \\ & S. \end{tikzcd} \tag{2.38}$$

Then $\hat{H}_k(S)$ is well defined, and has a morphism $\Pi_{P,\varphi} : H_k(P) \to \hat{H}_k(S)$ for all $\varphi : P \to S$ with P proper, such that $\Pi_{P_1,\varphi_1} = \Pi_{P_2,\varphi_2} \circ H_k(\psi)$ for all commutative diagrams (2.38), and is universal with this property. If $f : S \to T$ is any morphism in $\mathrm{Ho}(\mathbf{Art}_{\mathbb{K}}^{\mathrm{lft}})$ (not necessarily proper), there is a unique morphism $\hat{H}_k(f) : \hat{H}_k(S) \to \hat{H}_k(T)$ such that $\hat{H}_k(f) \circ \Pi_{P,\varphi} = \Pi_{P,f\circ\varphi}$ for all $\varphi : P \to S$. If S is proper then $\Pi_{S,\mathrm{id}_S} : H_k(S) \to \hat{H}_k(S)$ is an isomorphism.

Then $\hat{H}_*(-)$ should be a homology theory of the type we need (one should verify the appropriate parts of Assumption 2.30), and will be compatible with the cohomology theory $H^*(-)$, if this is given.

If instead $H_*(-)$ has pushforwards $H_*(f)$ defined only for proper and representable morphisms, we can define $\hat{H}_*(-)$ as in (2.37)–(2.38), but over $\varphi : P \to S$ with P a proper $\mathbb{K}$-scheme or algebraic $\mathbb{K}$-space, so that the morphisms $\psi : P \to Q$ in (2.38) are automatically proper and representable.

(b) (Extending the domains of (co)homology theories.) We can also use the direct limit trick in **(a)** to extend the domain of a (co)homology theory. Suppose, for example, we are given a homology theory $H_*(P)$ defined for $\mathbb{K}$-schemes P, with pushforwards $H_*(f)$ for arbitrary $\mathbb{K}$-scheme morphisms $f : P \to Q$. Then we can define a homology theory $\hat{H}_*(S)$ on Artin $\mathbb{K}$-stacks S by $\hat{H}_k(S) = \varinjlim_{\varphi:P\to S} H_k(P)$, where the direct limit is over morphisms $\varphi : P \to S$ in $\mathrm{Ho}(\mathbf{Art}_{\mathbb{K}}^{\mathrm{lft}})$ with P a $\mathbb{K}$-scheme. Then $\hat{H}_k(P) \cong H_k(P)$ if P is a $\mathbb{K}$-scheme.

Similarly, we may extend a cohomology theory $H^*(-)$ on $\mathbb{K}$-schemes to a cohomology theory $\hat{H}^*(S)$ on Artin $\mathbb{K}$-stacks S by $\hat{H}^k(S) = \varprojlim_{\varphi:P\to S} H^k(P)$, where the inverse limit is over morphisms $\varphi : P \to S$ in $\mathrm{Ho}(\mathbf{Art}_{\mathbb{K}}^{\mathrm{lft}})$ with P a $\mathbb{K}$-scheme. In the same way, we can extend a (co)homology theory from Artin $\mathbb{K}$-stacks to higher Artin $\mathbb{K}$-stacks, or from finite type Artin $\mathbb{K}$-stacks to locally of finite type Artin $\mathbb{K}$-stacks, and so on.

(c) (Defining a cohomology theory from a homology theory.) Suppose we have a homology theory $H_*(-)$ of Artin $\mathbb{K}$-stacks (possibly of type $H_*^{\mathrm{lf}}(-)$), but no matching cohomology theory. Following Fulton and MacPherson [51, §8], we can construct a compatible cohomology theory $\hat{H}^*(-)$. We define elements ϵ of $\hat{H}^k(S)$ to be families $(\epsilon_{\varphi,i})_{\varphi,i}$ of R-module morphisms $\epsilon_{\varphi,i} : H_{i+k}(P) \to H_i(P)$ for all morphisms $\varphi : P \to S$ in $\mathrm{Ho}(\mathbf{Art}_{\mathbb{K}}^{\mathrm{lft}})$ and $i = 0, 1, \ldots$, such that if $\varphi = \xi \circ \psi$ for $\psi : P \to Q$, $\xi : Q \to S$ then $H_i(\psi) \circ \epsilon_{\varphi,i} = \epsilon_{\xi,i} \circ H_{i+k}(\psi) : H_{i+k}(P) \to H_i(Q)$.

If there was a compatible cohomology theory $H^*(-)$, then an element $\epsilon \in H^k(S)$ would define such a family $(\epsilon_{\varphi,i})_{\varphi,i}$ by defining $\epsilon_{\varphi,i}(\alpha) = \alpha \cap H^k(\varphi)(\epsilon)$ for all $\varphi : P \to S$, $i \geqslant 0$ and $\alpha \in H_{i+k}(P)$. It is now straightforward to define pullback morphisms $\hat{H}^k(f) : \hat{H}^k(T) \to \hat{H}^k(S)$ for morphisms $f : S \to T$ in $\mathrm{Ho}(\mathbf{Art}_{\mathbb{K}}^{\mathrm{lft}})$, cup, cap and exterior products, and so on.

(d) (Defining a homology theory from a cohomology theory.) Suppose we are given a cohomology theory $H^*(S)$ over a *field* R (such as $\mathbb{Q}_l$ or $\mathbb{K}$), defined for *finite type* Artin $\mathbb{K}$-stacks S, with the property that $H^k(S)$ is finite-dimensional over R for each $k = 0, 1, \ldots$ for finite type S. Then we can just define homology $H_k(S) = H^k(S)^*$ to be the dual R-vector spaces. This extends easily to a homology theory $H_*(-)$ of finite type Artin $\mathbb{K}$-stacks, compatible with $H^*(-)$. Since $H^k(S)$ is finite-dimensional we have $H^k(S) \cong H_k(S)^*$, the usual duality relation for (co)homology over a field. We can then extend the domains of $H^*(-), H_*(-)$ to $\mathrm{Ho}(\mathbf{Art}_{\mathbb{K}}^{\mathrm{lft}})$ or $\mathrm{Ho}(\mathbf{HSt}_{\mathbb{K}}^{\mathrm{lft}})$ as in **(b)**.

2.4.2 Chern classes and Chern characters

Suppose $H_*(-), H^*(-)$ are (co)homology theories of (higher) Artin $\mathbb{K}$-stacks S satisfying Assumption 2.30 over a commutative ring R. Then as in Assumption 2.30(b) we have Chern classes $c_i(\alpha) \in H^{2i}(S)$ for $i = 1, 2, \ldots$ and $\alpha \in K_0(\mathrm{Perf}(S))$, such as $\alpha = [E]$ for a vector bundle $E \to S$. There is a formula (2.34) for Chern classes $c_k(\alpha \otimes \beta)$ of a tensor product $\alpha \otimes \beta$, but it involves universal polynomials $P_k^{\otimes}$ that are inconvenient to work with.

Let R be a $\mathbb{Q}$-algebra. Then as in Milnor and Stasheff [114], Hirzebruch [64, §4 & §10.1], and Hartshorne [62, App. A], we can rewrite Chern classes $c_i(\alpha)$ in terms of Chern characters $\mathrm{ch}_j(\alpha)$, which have much simpler behaviour under tensor products. If S is a (higher) Artin $\mathbb{K}$-stack and $\alpha \in K_0(\mathrm{Perf}(S))$, the *Chern characters* $\mathrm{ch}_i(\alpha) \in H^{2i}(S)$ for $i = 0, 1, 2, \ldots$, are defined by

$$\mathrm{ch}_0(\alpha) = \mathrm{rank}\,\alpha, \quad \mathrm{ch}_i(\alpha) = \mathrm{Ch}_i\bigl(c_1(\alpha), c_2(\alpha), \ldots, c_i(\alpha)\bigr), \quad i \geqslant 1, \tag{2.39}$$

where $\mathrm{Ch}_1, \mathrm{Ch}_2, \ldots$ are a family of universal polynomials over $\mathbb{Q}$ given by

$$\mathrm{Ch}_r(c_1, \ldots, c_r) = \sum_{\substack{a_1,\ldots,a_r \geqslant 0:\\ a_1+2a_2+\cdots+ra_r = r}} (-1)^{r+a_1+\cdots+a_r} \frac{(a_1+\cdots+a_r-1)!}{(r-1)!a_1!\cdots a_r!}\, c_1^{a_1}\cdots c_r^{a_r}. \tag{2.40}$$

The first few polynomials Ch_i are

$$\begin{aligned}&\mathrm{Ch}_1(c_1) = c_1,\ \mathrm{Ch}_2(c_1, c_2) = \tfrac{1}{2}(c_1^2 - 2c_2),\ \mathrm{Ch}_3(c_1, c_2, c_3) = \tfrac{1}{6}(c_1^3 - 3c_1c_2 + 3c_3),\\ &\mathrm{Ch}_4(c_1, c_2, c_3, c_4) = \tfrac{1}{24}(c_1^4 - 4c_1^2c_2 + 4c_1c_3 + 2c_2^2 - 4c_4),\ \ldots .\end{aligned} \tag{2.41}$$

If each c_j is graded of degree $2j$, then $\mathrm{Ch}_i(c_1, c_2, \ldots, c_i)$ is graded of degree $2i$. Note that $\mathrm{ch}_i(E)$ only makes sense in $H^{2i}(S)$ over a $\mathbb{Q}$-*algebra* R such as $R = \mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$, because of the rational factors in (2.40)–(2.41).

We can invert (2.39) and write $c_i(E)$ in $H^{2i}(S)$ in terms of the $\mathrm{ch}_j(E)$ by

$$c_i(E) = \mathrm{C}_i\bigl(\mathrm{ch}_1(E), \mathrm{ch}_2(E), \ldots, \mathrm{ch}_i(E)\bigr), \quad i \geqslant 1, \tag{2.42}$$

where $\mathrm{C}_1, \mathrm{C}_2, \ldots$ are another family of universal polynomials over $\mathbb{Q}$, given by

$$\mathrm{C}_r\,(b_1, \ldots, b_r) = \sum_{\substack{a_1,\ldots,a_r \geqslant 0:\\ a_1+2a_2+\cdots+ra_r = r}} (-1)^{r+a_1+\cdots+a_r} \prod_{i=1}^{r} \frac{((i-1)!)^{a_i}}{a_i!}\, b_1^{a_1}\cdots b_r^{a_r}. \tag{2.43}$$

The first few polynomials C_i are

$$\begin{aligned}&\mathrm{C}_1\,(b_1) = b_1,\ \ \mathrm{C}_2(b_1, b_2) = \tfrac{1}{2}(b_1^2 - 2b_2),\ \ \mathrm{C}_3(b_1, b_2, b_3) = \tfrac{1}{6}(b_1^3 - 6b_1b_2 + 12b_3),\\ &\mathrm{C}_4\,(b_1, b_2, b_3, b_4) = \tfrac{1}{24}(b_1^4 - 12b_1^2b_2 + 48b_1b_3 + 12b_2^2 - 144b_4),\ \ldots .\end{aligned} \tag{2.44}$$

We can also relate $c_i(\alpha)$ and $\mathrm{ch}_j(\alpha)$ by the generating function formulae in $H^*(S)[[z]]$, where z is a formal variable, noting that $c_0(\alpha) = 1$:

$$\textstyle\sum_{i\geqslant 0} z^i c_i(\alpha) = \exp\bigl[\sum_{j\geqslant 1}(-1)^{j-1}(j-1)!z^j\,\mathrm{ch}_j(\alpha)\bigr], \tag{2.45}$$

$$\textstyle\sum_{j\geqslant 1}(-1)^{j-1}(j-1)!z^j\,\mathrm{ch}_j(\alpha) = \log\bigl[\sum_{i\geqslant 0} z^i c_i(\alpha)\bigr]. \tag{2.46}$$

Chern characters have the very useful property that for $\alpha, \beta \in K_0(\mathrm{Perf}(S))$, Chern characters of sums $\alpha+\beta$ and tensor products $\alpha\otimes\beta$ are given by

$$\mathrm{ch}_i(\alpha+\beta)=\mathrm{ch}_i(\alpha)+\mathrm{ch}_i(\beta),\;\; \mathrm{ch}_i(\alpha\otimes\beta)=\sum_{j,k\geqslant 0: i=j+k}\mathrm{ch}_j(\alpha)\cup\mathrm{ch}_k(\beta). \tag{2.47}$$

2.4.3 Examples of (co)homology theories satisfying the Assumptions

The next example explains how to define data $H_*(-), H^*(-), c_i$ satisfying Assumptions 2.30 and 2.31 over the field $\mathbb{K}=\mathbb{C}$.

Example 2.35. (Homology, cohomology of (higher) Artin $\mathbb{C}$-stacks.)
(a) (Topological stacks and classifying spaces.) There is a notion of *topological stack*, a kind of stack in topological spaces, developed by Metzler [113] and Noohi [123]. They form a 2-category **TopSta**, which includes topological spaces as a full discrete (2-)subcategory $\mathbf{Top}\subset\mathbf{TopSta}$.

Noohi [123, §20] defines a 2-functor $F_{\mathbf{Art}_{\mathbb{C}}}^{\mathbf{TopSta}}:\mathbf{Art}_{\mathbb{C}}^{\mathrm{lft}}\to\mathbf{TopSta}$ from the 2-category of Artin $\mathbb{C}$-stacks locally of finite type to topological stacks, which preserves fibre products. If X is a $\mathbb{C}$-scheme locally of finite type, $F_{\mathbf{Art}_{\mathbb{C}}}^{\mathbf{TopSta}}$ maps X to the set $X(\mathbb{C})$ of $\mathbb{C}$-points of X with the complex analytic topology.

Noohi [123, 124] shows that topological stacks have a good notion of homotopy theory. He proves [124] that each topological stack $\mathfrak{X}$ has an atlas $\varphi:\mathfrak{X}^{\mathrm{cla}}\to\mathfrak{X}$ in **TopSta** such that $\mathfrak{X}^{\mathrm{cla}}\in\mathbf{Top}\subset\mathbf{TopSta}$, and if $\psi:T\to\mathfrak{X}$ is a morphism in **TopSta** with $T\in\mathbf{Top}\subset\mathbf{TopSta}$ then the projection $\mathfrak{X}^{\mathrm{cla}}\times_{\mathfrak{X}}T\to T$ is a weak homotopy equivalence in **Top**. This topological space $\mathfrak{X}^{\mathrm{cla}}$ is unique up to weak homotopy equivalence (and unique up to homotopy equivalence if $\mathfrak{X}$ has a paracompact atlas). We call $\mathfrak{X}^{\mathrm{cla}}$ the *classifying space* of $\mathfrak{X}$.

If $\mathfrak{X}$ is a topological space we may choose $\mathfrak{X}^{\mathrm{cla}}=\mathfrak{X}$, and if $\mathfrak{X}$ is a quotient stack $[T/G]$ we may choose $\mathfrak{X}^{\mathrm{cla}}=(T\times EG)/G$. If $f:\mathfrak{X}\to\mathfrak{Y}$ is a morphism in **TopSta**, we may lift f to a morphism $f^{\mathrm{cla}}:\mathfrak{X}^{\mathrm{cla}}\to\mathfrak{Y}^{\mathrm{cla}}$, unique up to (weak) homotopy.

Combining these results allows us to define the homology and cohomology of an Artin $\mathbb{C}$-stack X, over a commutative ring R: we set $H_i(X)=H_i(F_{\mathbf{Art}_{\mathbb{C}}}^{\mathbf{TopSta}}(X)^{\mathrm{cla}},R)$ and $H^i(X)=H^i(F_{\mathbf{Art}_{\mathbb{C}}}^{\mathbf{TopSta}}(X)^{\mathrm{cla}},R)$. These are well defined as $F_{\mathbf{Art}_{\mathbb{C}}}^{\mathbf{TopSta}}(X)^{\mathrm{cla}}$ is unique up to (weak) homotopy equivalence, and (co)homology is homotopy invariant. For a morphism $f:X\to Y$ in $\mathbf{Art}_{\mathbb{C}}^{\mathrm{lft}}$ we set $H_i(f)=(F_{\mathbf{Art}_{\mathbb{C}}}^{\mathbf{TopSta}}(f)^{\mathrm{cla}})_*:H_i(X)\to H_i(Y)$ and $H^i(f)=(F_{\mathbf{Art}_{\mathbb{C}}}^{\mathbf{TopSta}}(f)^{\mathrm{cla}})^*:H^i(Y)\to H^i(X)$, which are well defined as $F_{\mathbf{Art}_{\mathbb{C}}}^{\mathbf{TopSta}}(f)^{\mathrm{cla}}$ is unique up to (weak) homotopy. This gives (co)homology theories satisfying Assumption 2.30(a),(c).

Note that this approach does not work for $H^*_{\mathrm{cs}}(-)$ or $H^{\mathrm{lf}}_*(-)$, as these are not homotopy invariant, but $\mathfrak{X}^{\mathrm{cla}}$ is only unique up to (weak) homotopy.

We can also discuss Chern classes of vector bundles on Artin $\mathbb{C}$-stacks in this language. Suppose X is an Artin $\mathbb{C}$-stack and $E\to X$ is a vector bundle of rank r. Then E determines a morphism $\phi_E:X\to[*/\mathrm{GL}(r,\mathbb{C})]$ in $\mathbf{Art}_{\mathbb{C}}^{\mathrm{lft}}$, so in the above notation we have morphisms $H^k(\phi_E):H^k([*/\mathrm{GL}(r,\mathbb{C})])\to H^k(X)$

on cohomology. As in Milnor and Stasheff [114, §14] and §5.2 below, we have an isomorphism $H^k([*/\mathrm{GL}(r,\mathbb{C})])\cong R[\gamma_1,\ldots,\gamma_r]$, with γ_i a generator in degree $2i$, and we set $c_i(E)=H^{2i}(\phi_E)(\gamma_i)\in H^{2i}(X)$. These Chern classes factor through maps $c_i:K_0(\mathrm{Vect}(X))\to H^{2i}(X)$ with $c_i([E])=c_i(E)$, which have the properties required in Assumption 2.30(b),(c).

If R is a $\mathbb{Q}$-algebra then Assumption 2.31 also holds in this case.

(b) (Singular (co)homology of a topological stack.) Behrend [13, §2] defines the singular homology and cohomology of a topological stack $\mathfrak{X}$ directly, without first constructing a classifying space $\mathfrak{X}^{\mathrm{cla}}$, using a presentation of $\mathfrak{X}$ as a topological groupoid. He also discusses Chern classes of complex vector bundles. Combining this with $F_{\mathbf{Art}_{\mathbb{C}}}^{\mathbf{TopSta}}$ in **(a)** yields an alternative definition of (co)homology of Artin $\mathbb{C}$-stacks. Noohi's homotopy theory [124] implies that this is equivalent to the definition in **(a)**.

(c) (Topological realization functors.) As for the classifying space approach in **(a)**, there is an another way to associate a topological space to a (higher) Artin $\mathbb{C}$-stack, due to Simpson [142], Morel and Voevodsky [116, §3.3], Dugger and Isaksen [40], and Blanc [16, §3]. Consider the functor $F_{\mathbf{Aff}_{\mathbb{C}}}^{\mathbf{Top}}:\mathbf{Aff}_{\mathbb{C}}^{\mathrm{ft}}\to\mathbf{Top}$ taking a finite type affine $\mathbb{C}$-scheme X to its $\mathbb{C}$-points $X(\mathbb{C})$ with the complex analytic topology. By simplicially-enriched left Kan extension, one defines an ∞-functor $\mathbf{SPr}(\mathbf{Aff}_{\mathbb{C}}^{\mathrm{ft}})\to\mathbf{Top}_\infty$, the *topological realization functor*, where $\mathbf{SPr}(\mathbf{Aff}_{\mathbb{C}}^{\mathrm{ft}})$ is the ∞-category of simplicial presheaves on $\mathbf{Aff}_{\mathbb{C}}^{\mathrm{ft}}$, and $\mathbf{Top}_\infty$ the ∞-category of topological spaces up to homotopy.

Now $\mathbf{SPr}(\mathbf{Aff}_{\mathbb{C}}^{\mathrm{ft}})$ includes the 2-category of Artin $\mathbb{C}$-stacks $\mathbf{Art}_{\mathbb{C}}^{\mathrm{lft}}$ locally of finite type, and the ∞-subcategory of higher Artin $\mathbb{C}$-stacks $\mathbf{HSt}_{\mathbb{C}}^{\mathrm{lft}}$ locally of finite type, as full ∞-subcategories. So restriction gives ∞-functors $\mathbf{Art}_{\mathbb{C}}^{\mathrm{lft}}\to\mathbf{Top}_\infty$ and $F_{\mathbf{HSt}_{\mathbb{C}}}^{\mathbf{Top}_\infty}:\mathbf{HSt}_{\mathbb{C}}^{\mathrm{lft}}\to\mathbf{Top}_\infty$, where the former is equivalent to the map $X\mapsto F_{\mathbf{Art}_{\mathbb{C}}}^{\mathbf{TopSta}}(X)^{\mathrm{cla}}$ in **(a)**.

We then compose with the homology and cohomology functors over a commutative ring R. The compositions factor through the homotopy categories, yielding covariant functors $H_i:\mathrm{Ho}(\mathbf{Art}_{\mathbb{C}}^{\mathrm{lft}}),\mathrm{Ho}(\mathbf{HSt}_{\mathbb{C}}^{\mathrm{lft}})\to R\text{-mod}$ and contravariant functors $H^i:\mathrm{Ho}(\mathbf{Art}_{\mathbb{C}}^{\mathrm{lft}}),\mathrm{Ho}(\mathbf{HSt}_{\mathbb{C}}^{\mathrm{lft}})\to R\text{-mod}$ for $i=0,1,\ldots.$ The Artin stack versions are equivalent to those in **(a)**,**(b)**.

We can also use these ideas to define Chern classes of perfect complexes, as in §5.2 below. As in Toën and Vezzosi [151, Def. 1.3.7.5], there is a higher Artin $\mathbb{C}$-stack $\mathbf{Perf}_{\mathbb{C}}$ which classifies perfect complexes, in the same way that $[*/\mathrm{GL}(r,\mathbb{C})]$ classifies rank r vector bundles in **(a)** above. Suppose X is a (higher) Artin $\mathbb{C}$-stack, and $\mathcal{E}^\bullet$ is a perfect complex on X. Then $\mathcal{E}^\bullet$ determines a morphism $\phi_{\mathcal{E}^\bullet}:X\to\mathbf{Perf}_{\mathbb{C}}$ in $\mathbf{HSt}_{\mathbb{C}}^{\mathrm{lft}}$, so we have morphisms $H^k(\phi_{\mathcal{E}^\bullet}):H^k(\mathbf{Perf}_{\mathbb{C}})\to H^k(X)$ on cohomology.

Now $\mathbf{Perf}_{\mathbb{C}}=\coprod_{r\in\mathbb{Z}}\mathbf{Perf}_{\mathbb{C}}^r$, where $\mathbf{Perf}_{\mathbb{C}}^r$ classifies complexes of rank r, with $[*/\mathrm{GL}(r,\mathbb{C})]\subset\mathbf{Perf}_{\mathbb{C}}^r$ an open substack for $r\geqslant 0$, and $H^*(\mathbf{Perf}_{\mathbb{C}}^r)\cong R[\gamma_1,\gamma_2,\ldots]$, with γ_i in degree $2i$. The Chern class $c_i(\mathcal{E}^\bullet)$ is $H^{2i}(\phi_{\mathcal{E}^\bullet})(\gamma_i)$ in $H^{2i}(X)$. It factors through a map $c_i:K_0(\mathrm{Perf}(X))\to H^{2i}(X)$ with $c_i([\mathcal{E}^\bullet])=c_i(\mathcal{E}^\bullet)$, which has the properties required in Assumption 2.30(b),(c).

Next we consider (co)homology for stacks over other fields $\mathbb{K}$.

Example 2.36. (a) (Étale and ℓ-adic cohomology of stacks.) *Étale cohomology* is a cohomology theory of $\mathbb{K}$-schemes introduced by Grothendieck for general fields $\mathbb{K}$. It is used to construct *ℓ-adic cohomology* $H^i(S,\mathbb{Z}_\ell), H^i(S,\mathbb{Q}_\ell)$ of $\mathbb{K}$-schemes S, for ℓ a prime number different to $\operatorname{char}\mathbb{K}$. This has many important applications, including Deligne's proof of the Weil Conjecture. A good reference is Freitag and Kiehl [45].

The theory has been extended to Artin stacks, see for example Behrend [12], Gaitsgory and Lurie [52, §3.2], Laszlo and Olsson [96], Laumon and Moret-Bailly [98], Olsson [125], and Liu and Zheng [102, 103], who also cover higher Artin stacks. Chern classes can be constructed in étale and ℓ-adic cohomology.

There are also matching theories of *étale homology* and *ℓ-adic homology*, described by Laumon [97] and Li [101] for $\mathbb{K}$-schemes, by Olsson [126] for Deligne–Mumford stacks, and separately by Lurie [104] for Artin $\mathbb{K}$-stacks. The versions of [97, 101, 126] are of type $H_*^{\mathrm{lf}}(-)$, with pushforwards $H_i(f)$ defined only for proper morphisms f, but we can use them to construct homology theories of (higher) Artin $\mathbb{K}$-stacks of the kind we need as in Remark 2.34(a),(b). Lurie's version may work for our theory in its current form.

(b) (Algebraic de Rham cohomology.) Let $\mathbb{K}$ be a field of characteristic zero. Hartshorne [60, 61] develops theories of *algebraic de Rham homology and cohomology* $H_*(S), H^*(S)$ for finite type $\mathbb{K}$-schemes S, over $R=\mathbb{K}$. Toën [147, §3.1.1] extends them to finite type Artin $\mathbb{K}$-stacks. The homology theories are of type $H_*^{\mathrm{lf}}(-)$, with pushforwards $H_i(f)$ defined only for proper representable morphisms f, but we can use them to construct homology theories of the kind we need as in Remark 2.34(a),(b). Chern classes are defined, [147, §3.1.1].

(c) (Chow groups of schemes and stacks.) *Chow homology groups* $A_*(S)$ of $\mathbb{K}$-schemes S are studied in intersection theory, as in Fulton [50]. They have been extended to Artin stacks by Kresch [94] and Joshua [72, 73].

As in [50, p. 370], Chow groups $A_k(S)$ should be understood as a type of locally finite homology group $H_{2k}^{\mathrm{lf}}(S)$, with pushforwards only defined along projective morphisms (projective implies proper). Without constructing a cohomology theory, Kresch [94, §3.6] defines Chern classes of vector bundles $E\rightarrow S$ as maps $\cap\, c_i(E): A_{k+i}(S)\rightarrow A_i(S)$.

We can use Remark 2.34(a) to construct a homology theory $\hat H_{2*}(-)$ of Artin $\mathbb{K}$-stacks from $A_*(-)$, with $\hat H_{2k}(S)=\varinjlim_{\varphi:P\rightarrow S}A_k(P)$ a direct limit over $\varphi: P\rightarrow S$ in $\mathrm{Ho}(\mathbf{Art}_{\mathbb{K}}^{\mathrm{lft}})$ with P projective. We can also use Remark 2.34(c) to construct a compatible cohomology theory $\hat H^{2*}(-)$ from $A_*(-)$. Then Kresch's Chern class maps $\cap\, c_i(E): A_{k+i}(S)\rightarrow A_i(S)$ induce elements $c_i(E)$ in $\hat H^{2i}(S)$.

We define homotopy equivalences of stacks:

Definition 2.37. A morphism $f:S\rightarrow T$ in $\mathrm{Ho}(\mathbf{Art}_{\mathbb{K}}^{\mathrm{lft}})$ or $\mathrm{Ho}(\mathbf{HSt}_{\mathbb{K}}^{\mathrm{lft}})$ is called a *homotopy equivalence* if there exist morphisms $g:T\rightarrow S$, $F:S\times\mathbb{A}^1\rightarrow S$, $G:T\times\mathbb{A}^1\rightarrow T$, with $F|_{\{0\}\times S}=g\circ f$, $F|_{\{1\}\times S}=\mathrm{id}_S$, $G|_{\{0\}\times T}=f\circ g$, and $G|_{\{1\}\times T}=\mathrm{id}_T$. We say that S,T are *homotopic*, written $S\simeq T$, if there exists a homotopy equivalence $f:S\rightarrow T$.

Here we use the $\mathbb{A}^1$ in $S \times \mathbb{A}^1$ (rather than a connected algebraic curve, for example), as we want to use Assumption 2.30(a)(vi) in the next proof.

Lemma 2.38. *Let Assumption* 2.30 *hold for (co)homology theories* $H_*(-)$, $H^*(-)$ *of (higher) Artin* $\mathbb{K}$*-stacks over a commutative ring* R*, and suppose* $f : S \to T$ *is a homotopy equivalence of (higher) Artin* $\mathbb{K}$*-stacks. Then* $H_*(f) : H_*(S) \to H_*(T)$ *and* $H^*(f) : H^*(T) \to H^*(S)$ *are isomorphisms.*

Proof. For g, F, G as in Definition 2.37, we have a commutative diagram

$$\begin{tikzcd} S \arrow[r, "{(\mathrm{id}_S,0)}"'] \arrow[dr, bend right=15, "g\circ f"'] & S \times \mathbb{A}^1 \arrow[d, "F"'] & S, \arrow[l, "{(\mathrm{id}_S,1)}"] \arrow[dl, bend left=15, "\mathrm{id}_S"] \\ & S & \end{tikzcd}$$

so applying functorial homology $H_*(-)$ gives a commutative diagram

$$\begin{tikzcd} H_*(S) \arrow[r, "{H_*((\mathrm{id}_S,0))}"'] \arrow[dr, bend right=15, "H_*(g)\circ H_*(f)"'] & H_*(S \times \mathbb{A}^1) \arrow[l, dotted, bend right=0, "H_*(\pi_S)"'] \arrow[r, dotted, "H_*(\pi_S)"] \arrow[d, "H_*(F)"'] & H_*(S). \arrow[l, "{H_*((\mathrm{id}_S,1))}"] \arrow[dl, bend left=15, "H_*(\mathrm{id}_S)=\mathrm{id}"] \\ & H_*(S) & \end{tikzcd}$$

Now Assumption 2.30(a)(vi) implies that the morphisms $H_*(\pi_S)$ indicated '$\dashrightarrow$' are isomorphisms. Since they are left inverse to $H_*((\mathrm{id}_S, 0)), H_*((\mathrm{id}_S, 1))$, it follows that $H_*((\mathrm{id}_S, 0)) = H_*(\pi_S)^{-1} = H_*((\mathrm{id}_S, 1))$, so the diagram implies that $H_*(g) \circ H_*(f) = \mathrm{id}$. Similarly $H_*(f) \circ H_*(g) = \mathrm{id}$, so $H_*(f) : H_*(S) \to H_*(T)$ is an isomorphism. The argument for $H^*(f)$ is the same. $\square$

2.5 Assumptions on the 'projective Euler class'

For our 'projective linear' Lie algebras in §3.4, given a principal $[*/\mathbb{G}_m]$-bundle $\rho : S \to T$ and a weight one $[*/\mathbb{G}_m]$-equivariant perfect complex $\mathcal{E}^\bullet$ on E, we will need a (new, partially conjectural) kind of characteristic class $\mathrm{PE}([\mathcal{E}^\bullet])$, as a map $H_*(T) \to H_*(S)$, which we call the 'projective Euler class'.

Assumption 2.39. Let $\rho : S \to T$ be a principal $[*/\mathbb{G}_m]$-bundle in $\mathrm{Ho}(\mathbf{Art}_{\mathbb{K}}^{\mathrm{lft}})$ with $[*/\mathbb{G}_m]$-action $\Psi : [*/\mathbb{G}_m] \times S \to S$, as in §2.3.7. Then as in §2.3.8 we can consider perfect complexes $\mathcal{E}^\bullet$ on S with $[*/\mathbb{G}_m]$-actions $\Psi_{\mathcal{E}^\bullet}$ compatible with Ψ of weight 1. These form a triangulated category $\mathrm{Perf}(S)^{\mathrm{wt}1}$, with $[*/\mathbb{G}_m]$-equivariant morphisms. Thus we can form the Grothendieck group $K_0(\mathrm{Perf}(S)^{\mathrm{wt}1})$. There is a morphism $\mathrm{rank} : K_0(\mathrm{Perf}(S)^{\mathrm{wt}1}) \to \mathrm{LCon}(S, \mathbb{Z})$ mapping $[\mathcal{E}^\bullet] \mapsto \mathrm{rank}\, \mathcal{E}^\bullet$.

Suppose Assumption 2.30 holds for (co)homology theories $H_*(-), H^*(-)$ of Artin $\mathbb{K}$-stacks. Then for all principal $[*/\mathbb{G}_m]$-bundles $\rho : S \to T$ as above and all $\theta \in K_0(\mathrm{Perf}(S)^{\mathrm{wt}1})$ with constant rank on S and all $k \geqslant 0$ we should be given R-module morphisms

$$\mathrm{PE}(\theta) : H_k(T) \longrightarrow H_{k-2\,\mathrm{rank}\,\theta-2}(S), \tag{2.48}$$

which we call the *projective Euler class*. These should satisfy:

(a) As in Example 2.28, let $\rho = \pi_T : S = [*/\mathbb{G}_m] \times T \to T$ be the trivial principal $[*/\mathbb{G}_m]$-bundle over an Artin $\mathbb{K}$-stack T with $[*/\mathbb{G}_m]$-action $\Psi = \Omega \times \mathrm{id}_T : [*/\mathbb{G}_m]^2 \times T \to [*/\mathbb{G}_m] \times T$, let $\mathcal{F}^\bullet \in \mathrm{Perf}(T)$ have constant rank, and set $\mathcal{E}^\bullet = \pi^*_{[*/\mathbb{G}_m]}(E_1) \otimes \pi_T^*(\mathcal{F}^\bullet)$ in $\mathrm{Perf}(S)^{\mathrm{wt}1}$. Then we have

$$\mathrm{PE}([\mathcal{E}^\bullet]) : \zeta \longmapsto \sum_{\substack{i \geqslant 0:\, 2i \leqslant k, \\ i \geqslant \mathrm{rank}\,\theta + 1}} t^{i - \mathrm{rank}\,\mathcal{F}^\bullet - 1} \boxtimes (\zeta \cap c_i([\mathcal{F}^\bullet])), \tag{2.49}$$

where $\zeta \in H_k(T)$, and we use the notation of Assumption 2.30.

(b) Let $\rho : S \to T$ be a principal $[*/\mathbb{G}_m]$-bundle in $\mathrm{Ho}(\mathbf{Art}^{\mathrm{lft}}_{\mathbb{K}})$ with $[*/\mathbb{G}_m]$-action $\Psi : [*/\mathbb{G}_m] \times S \to S$, and $f : T' \to T$ be a morphism in $\mathrm{Ho}(\mathbf{Art}^{\mathrm{lft}}_{\mathbb{K}})$. Then as in §2.3.7 we may form the pullback principal $[*/\mathbb{G}_m]$-bundle $\rho' : S' = S \times_{\rho,T,f} T' \to T'$, with $[*/\mathbb{G}_m]$-action Ψ', in a commutative square (2.27) in $\mathrm{Ho}(\mathbf{Art}^{\mathrm{lft}}_{\mathbb{K}})$ which is 2-Cartesian in $\mathbf{Art}^{\mathrm{lft}}_{\mathbb{K}}$. The projection $f' = \pi_S : S' \to S$ induces a pullback functor $f'^* : \mathrm{Perf}(S)^{\mathrm{wt}1} \to \mathrm{Perf}(S')^{\mathrm{wt}1}$, and a morphism $K_0(f') : K_0(\mathrm{Perf}(S)^{\mathrm{wt}1}) \to K_0(\mathrm{Perf}(S')^{\mathrm{wt}1})$.

Then for all constant rank $\theta \in K_0(\mathrm{Perf}(S)^{\mathrm{wt}1})$ and $k \geqslant 0$, the following commutes:

$$\begin{tikzcd} H_k(T') \arrow[rr, "\mathrm{PE}(K_0(f')\theta)"'] \arrow[d, "H_k(f)"] && H_{k-2\,\mathrm{rank}\,\theta-2}(S') \arrow[d, "H_{k-2\,\mathrm{rank}\,\theta-2}(f')"'] \\ H_k(T) \arrow[rr, "\mathrm{PE}(\theta)"] && H_{k-2\,\mathrm{rank}\,\theta-2}(S). \end{tikzcd} \tag{2.50}$$

(c) Let $\rho : S \to T$ be a principal $[*/\mathbb{G}_m]$-bundle in $\mathrm{Ho}(\mathbf{Art}^{\mathrm{lft}}_{\mathbb{K}})$ with $[*/\mathbb{G}_m]$-action $\Psi : [*/\mathbb{G}_m] \times S \to S$. Define $\Psi^\vee : [*/\mathbb{G}_m] \times S \to S$ by $\Psi^\vee = \Psi \circ (\iota \times \mathrm{id}_S)$, where the inverse map $\iota : [*/\mathbb{G}_m] \to [*/\mathbb{G}_m]$ is induced by the group morphism $\mathbb{G}_m \to \mathbb{G}_m$ mapping $\lambda \mapsto \lambda^{-1}$, as in Definition 2.23. Then $\Psi^\vee$ is also a $[*/\mathbb{G}_m]$-action on S, and $\rho : S \to T$ is also a principal $[*/\mathbb{G}_m]$-bundle for the $[*/\mathbb{G}_m]$-action $\Psi^\vee$.

Write $\mathrm{Perf}(S)^{\mathrm{wt}1}_\Psi$ and $\mathrm{Perf}(S)^{\mathrm{wt}1}_{\Psi^\vee}$ for the categories of perfect complexes on S with weight one $[*/\mathbb{G}_m]$-actions compatible with Ψ and $\Psi^\vee$, respectively. Then the duality $\mathcal{E}^\bullet \mapsto (\mathcal{E}^\bullet)^\vee$ induces a contravariant equivalence of categories $(-)^\vee : \mathrm{Perf}(S)^{\mathrm{wt}1}_\Psi \to \mathrm{Perf}(S)^{\mathrm{wt}1}_{\Psi^\vee}$, since if $\mathcal{E}^\bullet$ has weight 1 for Ψ then $(\mathcal{E}^\bullet)^\vee$ has weight -1 for Ψ, and thus weight 1 for $\Psi^\vee$. This equivalence descends to an isomorphism $(-)^\vee : K_0(\mathrm{Perf}(S)^{\mathrm{wt}1}_\Psi) \to K_0(\mathrm{Perf}(S)^{\mathrm{wt}1}_{\Psi^\vee})$ mapping $[\mathcal{E}^\bullet] \mapsto [\mathcal{E}^\bullet]^\vee = [(\mathcal{E}^\bullet)^\vee]$.

Then for all constant rank $\theta \in K_0(\mathrm{Perf}(S)^{\mathrm{wt}1}_\Psi)$ and $k \geqslant 0$, we require that

$$\mathrm{PE}(\theta)_\Psi = (-1)^{\mathrm{rank}\,\theta+1}\,\mathrm{PE}(\theta^\vee)_{\Psi^\vee} : H_k(T) \longrightarrow H_{k-2\,\mathrm{rank}\,\theta-2}(S). \tag{2.51}$$

(d) Let $\rho : S \to T$ be a principal $[*/\mathbb{G}_m]$-bundle with $[*/\mathbb{G}_m]$-action $\Psi : [*/\mathbb{G}_m] \times S \to S$, and U be an Artin $\mathbb{K}$-stack. Then $\rho \times \mathrm{id}_U : S \times U \to T \times U$ is a principal $[*/\mathbb{G}_m]$-bundle with $[*/\mathbb{G}_m]$-action $\Psi \times \mathrm{id}_U$. Pullback by

$\pi_S : S \times U \to S$ induces a functor $\pi_S^* : \mathrm{Perf}(S)^{\mathrm{wt1}} \to \mathrm{Perf}(S \times U)^{\mathrm{wt1}}$, and a morphism $K_0(\pi_S) : K_0(\mathrm{Perf}(S)^{\mathrm{wt1}}) \to K_0(\mathrm{Perf}(S \times U)^{\mathrm{wt1}})$. Then for all constant rank $\theta \in K_0(\mathrm{Perf}(S)^{\mathrm{wt1}})$ and $\zeta \in H_k(T)$, $\eta \in H_l(U)$, we require that in $H_*(S \times U)$ we have

$$(\mathrm{PE}(\theta)\zeta) \boxtimes \eta = \mathrm{PE}(K_0(\pi_S)(\theta)(\zeta \boxtimes \eta).$$

(e) As in §2.3.7, the material of §2.3.7–§2.3.8 on principal principal $[*/\mathbb{G}_m]$-bundles generalizes to $[*/\mathbb{G}_m]^k$-bundles on an obvious way.

Suppose $\rho : S \to V$ is a principal $[*/\mathbb{G}_m]^3$-bundle with $[*/\mathbb{G}_m]^3$-action $\Psi : [*/\mathbb{G}_m]^3 \times S \to S$. Form the commutative diagram in $\mathrm{Ho}(\mathbf{Art}_{\mathbb{K}}^{\mathrm{lft}})$:

$$\begin{tikzcd}
& & U_{12} \arrow[dr, "\upsilon_{12}"] & \\
S \arrow[r, "\sigma"] & T \arrow[ur, "\tau_{12}"] \arrow[r, "\tau_{23}"] \arrow[dr, "\tau_{31}"'] & U_{23} \arrow[r, "\upsilon_{23}"] & V, \\
& & U_{31} \arrow[ur, "\upsilon_{31}"'] &
\end{tikzcd} \tag{2.52}$$

where T is the quotient of S by the free $[*/\mathbb{G}_m]$-action $\Psi \circ (\Delta_{123} \times \mathrm{id}_S) : [*/\mathbb{G}_m] \times S \to S$ for $\Delta_{123} : [*/\mathbb{G}_m] \to [*/\mathbb{G}_m]^3$ the diagonal morphism, and U_{ij} is the the quotient of S by the free $[*/\mathbb{G}_m]^2$-action $\Psi \circ (\Delta_{ij} \times \mathrm{id}_S) : [*/\mathbb{G}_m]^2 \times S \to S$, where $\Delta_{ij} : [*/\mathbb{G}_m]^2 \to [*/\mathbb{G}_m]^2$ is induced by the morphism $\delta_{ij} : \mathbb{G}_m^2 \to \mathbb{G}_m^3$ given by

$$\delta_{12} : (\lambda, \mu) \mapsto (\lambda, \lambda, \mu), \quad \delta_{23} : (\lambda, \mu) \mapsto (\mu, \lambda, \lambda), \quad \delta_{31} : (\lambda, \mu) \mapsto (\lambda, \mu, \lambda).$$

The morphisms $\sigma, \tau_{ij}, \upsilon_{ij}$ in (2.52) are the natural projections coming from compatibility of the quotient $[*/\mathbb{G}_m]^k$-actions. The composition of morphisms $S \to V$ in (2.52) is ρ.

Each morphism in (2.52) is a principal $[*/\mathbb{G}_m]$-bundle for an appropriate $[*/\mathbb{G}_m]$-action on $S, T, U_{12}, U_{23}, U_{31}$ descending from Ψ. We write the $[*/\mathbb{G}_m]$-actions on T for $\tau_{12}, \tau_{23}, \tau_{31}$ as $\Phi_{12}, \Phi_{23}, \Phi_{31}$, and the $[*/\mathbb{G}_m]$-actions on U_{12}, U_{23}, U_{31} for $\upsilon_{12}, \upsilon_{23}, \upsilon_{31}$ as $\Xi_{12}, \Xi_{23}, \Xi_{31}$, respectively. We determine the signs of these $[*/\mathbb{G}_m]$-actions by taking Φ_{ij} to be induced from Ψ by the k^{th} $[*/\mathbb{G}_m]$ factor in $[*/\mathbb{G}_m]^3$ and Ξ_{ij} to be induced from Ψ by the j^{th} $[*/\mathbb{G}_m]$ factor, where $\{i, j, k\} = \{1, 2, 3\}$.

Suppose now that $\mathcal{E}_{12}^\bullet$, $\mathcal{E}_{23}^\bullet$, $\mathcal{E}_{31}^\bullet$ are constant rank perfect complexes on U_{12}, U_{23}, U_{31} respectively, which have weight one $[*/\mathbb{G}_m]$-actions compatible with $\Xi_{12}, \Xi_{23}, \Xi_{31}$. Then we see that:

- $\tau_{12}^*(\mathcal{E}_{12}^\bullet)$ has $[*/\mathbb{G}_m]$-actions compatible with the $[*/\mathbb{G}_m]$-actions Φ_{12}, Φ_{23}, Φ_{31} on T of weights $0, -1, 1$, respectively.
- $\tau_{23}^*(\mathcal{E}_{23}^\bullet)$ has $[*/\mathbb{G}_m]$-actions compatible with the $[*/\mathbb{G}_m]$-actions Φ_{12}, Φ_{23}, Φ_{31} on T of weights $1, 0, -1$, respectively.
- $\tau_{31}^*(\mathcal{E}_{31}^\bullet)$ has $[*/\mathbb{G}_m]$-actions compatible with the $[*/\mathbb{G}_m]$-actions Φ_{12}, Φ_{23}, Φ_{31} on T of weights $-1, 0, 1$, respectively.

Hence $\tau_{23}^*(\mathcal{E}_{23}^\bullet)\oplus\tau_{31}^*(\mathcal{E}_{31}^\bullet)^\vee$ has weight 1 for the $[*/\mathbb{G}_m]$-action Φ_{12}, and $\tau_{31}^*(\mathcal{E}_{31}^\bullet)\oplus\tau_{12}^*(\mathcal{E}_{12}^\bullet)^\vee$ has weight 1 for the $[*/\mathbb{G}_m]$-action Φ_{23}, and $\tau_{12}^*(\mathcal{E}_{12}^\bullet)\oplus\tau_{23}^*(\mathcal{E}_{23}^\bullet)^\vee$ has weight 1 for the $[*/\mathbb{G}_m]$-action Φ_{31}. We require that

$$\begin{aligned}0=&(-1)^{\operatorname{rank}\mathcal{E}_{31}^\bullet}\cdot\mathrm{PE}\bigl([\tau_{23}^*(\mathcal{E}_{23}^\bullet)\oplus\tau_{31}^*(\mathcal{E}_{31}^\bullet)^\vee]\bigr)\circ\mathrm{PE}\bigl([\mathcal{E}_{12}^\bullet]\bigr)+\\ &(-1)^{\operatorname{rank}\mathcal{E}_{12}^\bullet}\cdot\mathrm{PE}\bigl([\tau_{31}^*(\mathcal{E}_{31}^\bullet)\oplus\tau_{12}^*(\mathcal{E}_{12}^\bullet)^\vee]\bigr)\circ\mathrm{PE}\bigl([\mathcal{E}_{23}^\bullet]\bigr)+\\ &(-1)^{\operatorname{rank}\mathcal{E}_{23}^\bullet}\cdot\mathrm{PE}\bigl([\tau_{12}^*(\mathcal{E}_{12}^\bullet)\oplus\tau_{23}^*(\mathcal{E}_{23}^\bullet)^\vee]\bigr)\circ\mathrm{PE}\bigl([\mathcal{E}_{31}^\bullet]\bigr):\\ &\quad H_k(V)\longrightarrow H_{k-2(\operatorname{rank}\mathcal{E}_{12}^\bullet+\operatorname{rank}\mathcal{E}_{23}^\bullet+\operatorname{rank}\mathcal{E}_{31}^\bullet)-4}(T).\end{aligned}\tag{2.53}$$

So far as the author can tell these ideas are new, and we will not actually prove Assumption 2.39 in the general case, though we do for rationally trivial $[*/\mathbb{G}_m]$-bundles when R is a $\mathbb{Q}$-algebra in the next proposition.

Proposition 2.40. *Suppose Assumptions* 2.30 *and* 2.31 *hold for (co)homology theories $H_*(-),H^*(-)$ of Artin $\mathbb{K}$-stacks over a $\mathbb{Q}$-algebra R. Then the restriction of Assumption* 2.39 *to principal $[*/\mathbb{G}_m]$-bundles $\rho:S\to T$ which are **rationally trivial** in the sense of Definition* 2.26 *holds.*

Note that Proposition 2.29 *implies that if $\mathbb{K}$ is algebraically closed and $\mathcal{E}^\bullet$ is a perfect complex on S with a weight one $[*/\mathbb{G}_m]$-action and* $\operatorname{rank}\mathcal{E}^\bullet\neq 0$, *then ρ is rationally trivial.*

Proof. Suppose $\rho:S\to T$ is a rationally trivial $[*/\mathbb{G}_m]$-bundle with $[*/\mathbb{G}_m]$-action $\Psi:[*/\mathbb{G}_m]\times S\to S$. Then by Definition 2.26 there exists a surjective morphism $f:T'\to T$ which is a $[*/\mathbb{Z}_n]$-fibration over each connected component of T, such that the pullback principal $[*/\mathbb{G}_m]$-bundle $\rho':S'=S\times_{\rho,T,f}T'\to T'$ is trivial. Assumption 2.39(b) says that (2.50) should commute.

The columns of (2.50) are isomorphisms by Assumption 2.31(a) as f,f' are $[*/\mathbb{Z}_n]$-fibrations over each connected component. Thus in (2.50), the morphism $\mathrm{PE}(\theta)$ is determined uniquely by $\mathrm{PE}(K_0(f')\theta)$. Also $\rho':S'\to T'$ is a trivial $[*/\mathbb{G}_m]$-fibration, so $S'\cong[*/\mathbb{G}_m]\times T'$, and writing $K_0(f')\theta=[\mathcal{E}^\bullet]$ for $\mathcal{E}^\bullet$ in $\mathrm{Perf}(S')^{\mathrm{wt}1}$, as in Example 2.28 we can show that $\mathcal{E}^\bullet\cong\pi_{[*/\mathbb{G}_m]}^*(E_1)\otimes\pi_{T'}^*(\mathcal{F}^\bullet)$ for $\mathcal{F}^\bullet\in\mathrm{Perf}(T')$, and then $\mathrm{PE}(K_0(f')\theta)$ in (2.50) is given explicitly by (2.49).

We claim that these morphisms $\mathrm{PE}(\theta)$ are independent of the choices of local $[*/\mathbb{Z}_n]$-fibration $f:T'\to T$ and $[*/\mathbb{G}_m]$-equivariant trivialization $S'\cong[*/\mathbb{G}_m]\times T'$. Two such trivializations differ by an isomorphism $\bigl(\Omega\circ(\mathrm{id}_{[*/\mathbb{G}_m]},\alpha),\pi_{T'}\bigr):[*/\mathbb{G}_m]\times T'\to[*/\mathbb{G}_m]\times T'$ for any morphism $\alpha:T'\to[*/\mathbb{G}_m]$, where $\Omega:[*/\mathbb{G}_m]^2\to[*/\mathbb{G}_m]$ is as in Definition 2.23, and one can show by calculation that (2.49) is invariant under such isomorphisms. Then independence of $f:T'\to T$ is easy to show by considering $T_1'\times_{f_1,T,f_2}T_2'$ for alternative choices $f_1:T_1'\to T$ and $f_2:T_2'\to T$.

Thus, Assumption 2.39(a),(b) determine unique morphisms $\mathrm{PE}(\theta)$ for rationally trivial $[*/\mathbb{G}_m]$-bundles $\rho:S\to T$. We can then check that these $\mathrm{PE}(\theta)$ satisfy Assumption 2.39(a)–(e). For (b)–(e), we use rational triviality as above to reduce to the cases when $\rho:S\to T$ in (b)–(d) is a trivial $[*/\mathbb{G}_m]$-bundle and $\rho:S\to V$ in (e) is a trivial $[*/\mathbb{G}_m]^3$-bundle, and then we prove (2.50), (2.51) and (2.53) in these cases using the formula (2.49). $\square$

Question 2.41. *Can we construct morphisms* $\mathrm{PE}(\theta)$ *satisfying Assumption* 2.39 *for general* R, *and for* $\rho : S \to T$ *which are not rationally trivial?*

The author conjectures the answer is yes for $\mathbb{K} = \mathbb{C}$ and (co)homology $H_*(-)$, $H^*(-)$ over any commutative ring R defined as in Example 2.35.

Remark 2.42. (Relation to Fulton–MacPherson's bivariant theories.) Fulton and MacPherson [51] introduce *bivariant theories*, which generalize both homology and cohomology. Given a category such as $\mathrm{Ho}(\mathbf{Art}_{\mathbb{K}}^{\mathrm{lft}})$ and a commutative ring R, a bivariant theory gives an R-module $B^k(f : S \to T)$ for each morphism $f : S \to T$ in $\mathrm{Ho}(\mathbf{Art}_{\mathbb{K}}^{\mathrm{lft}})$ and $k \in \mathbb{Z}$, which have R-bilinear products

$$\cdot : B^k(f : S \to T) \times B^l(g : T \to U) \longrightarrow B^{k+l}(g \circ f : S \to U)$$

and 'pushforward' and 'pullback' operations, satisfying some axioms. A bivariant theory defines homology and cohomology theories $H_*(-), H^*(-)$ by

$$H_k(S) = B^{-k}(\pi : S \to *) \qquad \text{and} \qquad H^k(S) = B^k(\mathrm{id} : S \to S).$$

Bivariant theories in algebraic geometry are discussed by Fulton and MacPherson [51, §9], Olsson [126, §1.3], and Li [101, 3.1].

If the (co)homology theories $H_*(-), H^*(-)$ in Assumption 2.39 come from a bivariant theory $B^*(-)$, it would be natural to define $\mathrm{PE}(\theta)$ as a bivariant class

$$\mathrm{PE}(\theta) \in B^{2\,\mathrm{rank}\,\theta+2}(\rho : S \to T), \tag{2.54}$$

and then take (2.48) to map $\zeta \mapsto \mathrm{PE}(\theta) \cdot \zeta$ using the bivariant product

$$\cdot : B^{2\,\mathrm{rank}\,\theta+2}(\rho : S \to T) \times B^{-k}(\pi : T \to *) \longrightarrow B^{2\,\mathrm{rank}\,\theta+2-k}(\pi : S \to *).$$

The author imagines answering Question 2.41 as follows. We should construct a homotopy Cartesian square in $\mathbf{HSt}_{\mathbb{K}}$:

$$\begin{tikzcd} E\,\mathrm{Perf}^{\mathrm{wt1}} \arrow[r, "\beta"] \arrow[d, "\alpha"] & E[*/\mathbb{G}_m] \arrow[d, "\delta"] \\ B\,\mathrm{Perf}^{\mathrm{wt1}} \arrow[r, "\gamma"] & B[*/\mathbb{G}_m], \end{tikzcd}$$

where α, δ are principal $[*/\mathbb{G}_m]$-bundles. Here $B[*/\mathbb{G}_m]$ should be the *classifying stack for principal* $[*/\mathbb{G}_m]$*-bundles*, so that if $\rho : S \to T$ is a principal $[*/\mathbb{G}_m]$-bundle in $\mathbf{Art}_{\mathbb{K}}^{\mathrm{lft}}$ or $\mathbf{HSt}_{\mathbb{K}}$ then there is a natural morphism $\epsilon : T \to B[*/\mathbb{G}_m]$ with $S \cong E[*/\mathbb{G}_m] \times_{\delta, B[*/\mathbb{G}_m], \epsilon} T$ the pullback principal $[*/\mathbb{G}_m]$-bundle.

Also $B\,\mathrm{Perf}^{\mathrm{wt1}}$ should be the *relative classifying stack for weight one perfect complexes on principal* $[*/\mathbb{G}_m]$*-bundles*, so that if $\rho : S \to T$, ϵ are as above and $\mathcal{E}^\bullet$ is a weight one $[*/\mathbb{G}_m]$-equivariant perfect complex on S then there is a natural morphism $\zeta : T \to B\,\mathrm{Perf}^{\mathrm{wt1}}$ with $\epsilon = \gamma \circ \zeta$ such that if $\eta : S \to E\,\mathrm{Perf}^{\mathrm{wt1}}$ is induced by ζ, ϵ and the isomorphism $S \cong E[*/\mathbb{G}_m] \times_{\delta, B[*/\mathbb{G}_m], \epsilon} T$ then $\mathcal{E}^\bullet \cong \eta^*(\mathcal{U}^\bullet)$ for a 'universal' weight one complex $\mathcal{U}^\bullet$ on $E\,\mathrm{Perf}^{\mathrm{wt1}}$.

Then there should exist a class PE in $B^*(\alpha : E\,\mathrm{Perf}^{\mathrm{wt1}} \to B\,\mathrm{Perf}^{\mathrm{wt1}})$, such that $\mathrm{PE}([\mathcal{E}^\bullet])$ in (2.54) is obtained by bivariant pullback of PE by (ζ, η).

3 Vertex algebras and Lie algebras in Algebraic Geometry

Throughout this section we fix a field $\mathbb{K}$ and work with (higher) Artin $\mathbb{K}$-stacks as discussed in §2.3.

3.1 Assumptions on the abelian or triangulated category

The next assumption sets out the notation, extra data, and properties we need for the $\mathbb{K}$-linear abelian category $\mathcal{A}$ in all the versions of our construction. The symmetric product χ in (c) and perfect complex $\Theta^\bullet$ in (i)–(l) will be explained in Remark 3.3, and the signs $\epsilon_{\alpha,\beta}$ in (d) in Remark 3.4.

Assumption 3.1. **(a)** We are given a $\mathbb{K}$-linear abelian category $\mathcal{A}$.

(b) The *Grothendieck group* of $\mathcal{A}$ is the abelian group generated by isomorphism classes $[E]$ of objects E of $\mathcal{A}$, with a relation $[F]=[E]+[G]$ for each exact sequence $0\rightarrow E\rightarrow F\rightarrow G\rightarrow 0$ in $\mathcal{A}$. We are given a quotient group $K(\mathcal{A})$ of $K_0(\mathcal{A})$, with surjective projection $K_0(\mathcal{A})\twoheadrightarrow K(\mathcal{A})$. Thus, each object $E\in\mathcal{A}$ has a class in $K(\mathcal{A})$, which we will write as $[\![E]\!]$.

(c) We are given a biadditive map $\chi:K(\mathcal{A})\times K(\mathcal{A})\rightarrow\mathbb{Z}$, which is symmetric, that is, $\chi(\alpha,\beta)=\chi(\beta,\alpha)$ for all $\alpha,\beta\in K(\mathcal{A})$.

(d) We are given signs $\epsilon_{\alpha,\beta}\in\{1,-1\}$ for all $\alpha,\beta\in K(\mathcal{A})$, which satisfy

$$\epsilon_{\alpha,\beta}\cdot\epsilon_{\beta,\alpha}=(-1)^{\chi(\alpha,\beta)+\chi(\alpha,\alpha)\chi(\beta,\beta)}, \tag{3.1}$$

$$\epsilon_{\alpha,\beta}\cdot\epsilon_{\alpha+\beta,\gamma}=\epsilon_{\alpha,\beta+\gamma}\cdot\epsilon_{\beta,\gamma}, \tag{3.2}$$

$$\epsilon_{\alpha,0}=\epsilon_{0,\alpha}=1, \tag{3.3}$$

for all $\alpha,\beta,\gamma\in K(\mathcal{A})$. Note that if the map $K(\mathcal{A})\times K(\mathcal{A})\rightarrow\{\pm1\}$ taking $(\alpha,\beta)\mapsto\epsilon_{\alpha,\beta}$ is *biadditive* (i.e. $\epsilon_{\alpha+\beta,\gamma}=\epsilon_{\alpha,\gamma}\cdot\epsilon_{\beta,\gamma}$ and $\epsilon_{\alpha,\beta+\gamma}=\epsilon_{\alpha,\beta}\cdot\epsilon_{\alpha,\gamma}$) then (3.2) is automatic, as both sides are $\epsilon_{\alpha,\beta}\cdot\epsilon_{\alpha,\gamma}\cdot\epsilon_{\beta,\gamma}$, and so is (3.3).

(e) We can form a moduli stack $\mathcal{M}$ of all objects in $\mathcal{A}$, which is an Artin $\mathbb{K}$-stack, locally of finite type. $\mathbb{K}$-points $x\in\mathcal{M}(\mathbb{K})$ correspond naturally to isomorphism classes $[E]$ of objects E in $\mathcal{A}$, and we will write points of $\mathcal{M}(\mathbb{K})$ as $[E]$. There are isomorphisms of algebraic $\mathbb{K}$-groups $\mathrm{Aut}(E)\cong\mathrm{Iso}_{\mathcal{M}}([E])$, natural up to conjugation in $\mathrm{Aut}(E)$, between the automorphism group $\mathrm{Aut}(E)$ of E in $\mathcal{A}$ and the isotropy group $\mathrm{Iso}_{\mathcal{M}}([E])$ of $[E]$ in $\mathcal{M}$.

We also write $\mathcal{M}'=\mathcal{M}\setminus\{[0]\}$ for the open substack $\mathcal{M}'\subset\mathcal{M}$ which is the moduli stack of nonzero objects in $\mathcal{A}$.

If S is a $\mathbb{K}$-scheme, a stack morphism $\mathcal{E}:S\rightarrow\mathcal{M}$ should be interpreted as a 'family of objects $\mathcal{E}$ in $\mathcal{A}$ over the base $\mathbb{K}$-scheme S'. In our examples, morphisms $S\rightarrow\mathcal{M}$ will be equivalent to objects of an $\mathcal{O}_S$-linear exact category $\mathcal{A}(S)$. That is, in these examples we can enhance the Artin $\mathbb{K}$-stack $\mathcal{M}$ to a *stack in exact categories on* $\mathbf{Sch}_{\mathbb{K}}$.

We do not assume this, but we will use these ideas in **(g)**,**(h)** below to better explain the stack morphisms Φ,Ψ, using the operations of direct sum $\oplus$ and tensor product $L\otimes-$ by an S-line bundle L in the exact category $\mathcal{A}(S)$.

(f) The map $\mathcal{M}(\mathbb{K}) \to K(\mathcal{A})$ mapping $[E] \mapsto [\![E]\!]$ is locally constant in the Zariski topology. Thus there is a decomposition $\mathcal{M} = \coprod_{\alpha \in K(\mathcal{A})} \mathcal{M}_\alpha$, where $\mathcal{M}_\alpha \subset \mathcal{M}$ is the open and closed $\mathbb{K}$-substack of points $[E] \in \mathcal{M}(\mathbb{K})$ with $[\![E]\!] = \alpha$ in $K(\mathcal{A})$. We have $\mathcal{M}_\alpha = \emptyset$ if there are no objects $E \in \mathcal{A}$ with $[\![E]\!] = \alpha$. We write $\mathcal{M}'_\alpha = \mathcal{M}_\alpha$ if $\alpha \neq 0$ and $\mathcal{M}'_0 = \mathcal{M}_0 \setminus \{[0]\}$, so that $\mathcal{M}' = \coprod_{\alpha \in K(\mathcal{A})} \mathcal{M}'_\alpha$.

(g) There is a natural morphism of Artin stacks $\Phi : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ which on $\mathbb{K}$-points acts by $\Phi(\mathbb{K}) : ([E],[F]) \mapsto [E \oplus F]$, for all objects $E, F \in \mathcal{A}$, and on isotropy groups acts by $\Phi_* : \mathrm{Iso}_{\mathcal{M}\times\mathcal{M}}([E],[F]) \cong \mathrm{Aut}(E) \times \mathrm{Aut}(F) \to \mathrm{Iso}_{\mathcal{M}}([E \oplus F]) \cong \mathrm{Aut}(E \oplus F)$ by $(\lambda, \mu) \mapsto \left(\begin{smallmatrix} \lambda & 0 \\ 0 & \mu \end{smallmatrix}\right)$ for $\lambda \in \mathrm{Aut}(E)$ and $\mu \in \mathrm{Aut}(F)$, using the obvious matrix notation for $\mathrm{Aut}(E \oplus F)$. That is, Φ is the morphism of moduli stacks induced by direct sum in the abelian category $\mathcal{A}$.

As $[\![E \oplus F]\!] = [\![E]\!] + [\![F]\!]$, we see that Φ maps $\mathcal{M}_\alpha \times \mathcal{M}_\beta \to \mathcal{M}_{\alpha+\beta}$ for $\alpha, \beta \in K(\mathcal{A})$, and we write $\Phi_{\alpha,\beta} := \Phi|_{\mathcal{M}_\alpha \times \mathcal{M}_\beta} : \mathcal{M}_\alpha \times \mathcal{M}_\beta \to \mathcal{M}_{\alpha+\beta}$.

These morphisms $\Phi_{\alpha,\beta}$ satisfy the following identities in $\mathrm{Ho}(\mathbf{Art}^{\mathrm{lft}}_{\mathbb{K}})$:

$$\Phi_{\beta,\alpha} \circ \sigma_{\alpha,\beta} = \Phi_{\alpha,\beta} : \mathcal{M}_\alpha \times \mathcal{M}_\beta \longrightarrow \mathcal{M}_{\alpha+\beta}, \tag{3.4}$$

$$\begin{aligned} &\Phi_{\alpha+\beta,\gamma} \circ \bigl(\Phi_{\alpha,\beta} \times \mathrm{id}_{\mathcal{M}_\gamma}\bigr) = \Phi_{\alpha,\beta+\gamma} \circ \bigl(\mathrm{id}_{\mathcal{M}_\alpha} \times \Phi_{\beta,\gamma}\bigr) : \\ &\qquad \mathcal{M}_\alpha \times \mathcal{M}_\beta \times \mathcal{M}_\gamma \longrightarrow \mathcal{M}_{\alpha+\beta+\gamma}, \end{aligned} \tag{3.5}$$

where $\sigma_{\alpha,\beta} : \mathcal{M}_\alpha \times \mathcal{M}_\beta \to \mathcal{M}_\beta \times \mathcal{M}_\alpha$ exchanges the factors. That is, Φ is commutative and associative.

If $\mathcal{M}$ comes from a stack in exact categories on $\mathbf{Sch}_{\mathbb{K}}$, as in **(e)**, we can provide an explicit description of Φ. Then $\mathcal{M}(S) := \mathbf{Hom}(S, \mathcal{M})$ is the groupoid of objects $\mathcal{E}$ in the exact category $\mathcal{A}(S)$ and their isomorphisms for S in $\mathbf{Sch}_{\mathbb{K}}$, and $\Phi(S) : (\mathcal{M} \times \mathcal{M})(S) = \mathcal{M}(S) \times \mathcal{M}(S) \to \mathcal{M}(S)$ is the functor of groupoids mapping $\Phi(S) : (\mathcal{E}, \mathcal{F}) \mapsto \mathcal{E} \oplus \mathcal{F}$, using direct sum in $\mathcal{A}(S)$.

(h) There is a natural morphism of Artin stacks $\Psi : [*/\mathbb{G}_m] \times \mathcal{M} \to \mathcal{M}$ which on $\mathbb{K}$-points acts by $\Psi(\mathbb{K}) : (*, [E]) \mapsto [E]$, for all objects E in $\mathcal{A}$, and on isotropy groups acts by $\Psi_* : \mathrm{Iso}_{[*/\mathbb{G}_m]\times\mathcal{M}}(*, [E]) \cong \mathbb{G}_m \times \mathrm{Aut}(E) \to \mathrm{Iso}_{\mathcal{M}}([E]) \cong \mathrm{Aut}(E)$ by $(\lambda, \mu) \mapsto \lambda\mu = (\lambda \cdot \mathrm{id}_E) \circ \mu$ for $\lambda \in \mathbb{G}_m$ and $\mu \in \mathrm{Aut}(E)$. Note that Ψ is *not* the same as the projection $\pi_{\mathcal{M}} : [*/\mathbb{G}_m] \times \mathcal{M} \to \mathcal{M}$ from the product $[*/\mathbb{G}_m] \times \mathcal{M}$, which acts on isotropy groups as $(\pi_{\mathcal{M}})_* : (\lambda, \mu) \mapsto \mu$.

Clearly Ψ maps $[*/\mathbb{G}_m] \times \mathcal{M}_\alpha \to \mathcal{M}_\alpha$ for $\alpha \in K(\mathcal{A})$, and we write $\Psi_\alpha := \Psi|_{[*/\mathbb{G}_m]\times\mathcal{M}_\alpha} : [*/\mathbb{G}_m] \times \mathcal{M}_\alpha \to \mathcal{M}_\alpha$.

These morphisms $\Phi_{\alpha,\beta}, \Psi_\alpha$ satisfy the following identities in $\mathrm{Ho}(\mathbf{Art}^{\mathrm{lft}}_{\mathbb{K}})$:

$$\begin{aligned} &\Psi_{\alpha+\beta} \circ (\mathrm{id}_{[*/\mathbb{G}_m]} \times \Phi_{\alpha,\beta}) = \Phi_{\alpha,\beta} \circ \bigl((\Psi_\alpha \circ (\Pi_{[*/\mathbb{G}_m]\times\mathcal{M}_\alpha})), (\Psi_\beta \circ (\Pi_{[*/\mathbb{G}_m]\times\mathcal{M}_\beta}))\bigr) : \\ &\qquad [*/\mathbb{G}_m] \times \mathcal{M}_\alpha \times \mathcal{M}_\beta \longrightarrow \mathcal{M}_{\alpha+\beta}, \end{aligned} \tag{3.6}$$

$$\begin{aligned} &\Psi_\alpha \circ (\mathrm{id}_{[*/\mathbb{G}_m]} \times \Psi_\alpha) = \Psi_\alpha \circ (\Omega \times \mathrm{id}_{\mathcal{M}_\alpha}) : \\ &\qquad [*/\mathbb{G}_m] \times [*/\mathbb{G}_m] \times \mathcal{M}_\alpha \longrightarrow \mathcal{M}_\alpha, \end{aligned} \tag{3.7}$$

where $\Omega : [*/\mathbb{G}_m] \times [*/\mathbb{G}_m] \to [*/\mathbb{G}_m]$ is induced by the group morphism $\mathbb{G}_m \times \mathbb{G}_m \to \mathbb{G}_m$ mapping $(\lambda, \mu) \mapsto \lambda\mu$.

Note that (3.7) says Ψ_α is a $[*/\mathbb{G}_m]$-*action* on $\mathcal{M}_\alpha$ in the sense of Definition 2.23, so Ψ is a $[*/\mathbb{G}_m]$-action on $\mathcal{M}$. The action of Ψ on isotropy groups above implies that Ψ is a *free* $[*/\mathbb{G}_m]$-action, as in (2.22), except over $[0] \in \mathcal{M}(\mathbb{K})$, so $\Psi' := \Psi|_{[*/\mathbb{G}_m]\times\mathcal{M}'} : [*/\mathbb{G}_m] \times \mathcal{M}' \to \mathcal{M}'$ is a free $[*/\mathbb{G}_m]$-action on $\mathcal{M}'$.

If $\mathcal{M}$ comes from a stack in exact categories on $\mathbf{Sch}_{\mathbb{K}}$, as in **(e)**, we can provide an explicit description of Ψ. Then $[*/\mathbb{G}_m](S) = \mathrm{Pic}(S)$ is the groupoid of line bundles $L \to S$ and their isomorphisms for $S \in \mathbf{Sch}_{\mathbb{K}}$, and $\mathcal{M}(S)$ is the groupoid of objects $\mathcal{E}$ in the exact category $\mathcal{A}(S)$ and their isomorphisms, and $\Psi(S) : \mathrm{Pic}(S) \times \mathcal{M}(S) \to \mathcal{M}(S)$ is the functor of groupoids mapping $\Psi(S) : (L, \mathcal{E}) \mapsto L \otimes \mathcal{E}$, noting that we may tensor by line bundles on S in the $\mathcal{O}_S$-linear exact category $\mathcal{A}(S)$.

(i) We are given a perfect complex $\Theta^\bullet$ on $\mathcal{M} \times \mathcal{M}$, as in §2.3.6. We write $\Theta^\bullet_{\alpha,\beta}$ for the restriction of $\Theta^\bullet$ to $\mathcal{M}_\alpha \times \mathcal{M}_\beta \subset \mathcal{M} \times \mathcal{M}$ for $\alpha, \beta \in K(\mathcal{A})$. Then $\operatorname{rank} \Theta^\bullet_{\alpha,\beta} = \chi(\alpha, \beta)$ for all $\alpha, \beta \in K(\mathcal{A})$.

(j) Write $\sigma_{\alpha,\beta} : \mathcal{M}_\alpha \times \mathcal{M}_\beta \to \mathcal{M}_\beta \times \mathcal{M}_\alpha$ for the isomorphism exchanging the factors. Then for some $n \in \mathbb{Z}$, and for all $\alpha, \beta \in K(\mathcal{A})$, we are given isomorphisms in $\mathrm{Perf}(\mathcal{M}_\alpha \times \mathcal{M}_\beta)$:

$$\sigma^*_{\alpha,\beta}(\Theta^\bullet_{\beta,\alpha}) \cong (\Theta^\bullet_{\alpha,\beta})^\vee[2n]. \tag{3.8}$$

Here $(\cdots)^\vee$ is duality and $[2n]$ is shift by $2n$ in $\mathrm{Perf}(\mathcal{M}_\alpha \times \mathcal{M}_\beta)$. Equation (3.8) is consistent with $\chi(\alpha, \beta) = \chi(\beta, \alpha)$ in **(c)** and $\operatorname{rank} \Theta^\bullet_{\alpha,\beta} \equiv \chi(\alpha, \beta)$ in **(i)**.

(k) For all $\alpha, \beta, \gamma \in K(\mathcal{A})$, we are given isomorphisms in $\mathrm{Perf}(\mathcal{M}_\alpha \times \mathcal{M}_\beta \times \mathcal{M}_\gamma)$:

$$(\Phi_{\alpha,\beta} \times \mathrm{id}_{\mathcal{M}_\gamma})^*(\Theta^\bullet_{\alpha+\beta,\gamma}) \cong \Pi^*_{\mathcal{M}_\alpha\times\mathcal{M}_\gamma}(\Theta^\bullet_{\alpha,\gamma}) \oplus \Pi^*_{\mathcal{M}_\beta\times\mathcal{M}_\gamma}(\Theta^\bullet_{\beta,\gamma}), \tag{3.9}$$

$$(\mathrm{id}_{\mathcal{M}_\alpha} \times \Phi_{\beta,\gamma})^*(\Theta^\bullet_{\alpha,\beta+\gamma}) \cong \Pi^*_{\mathcal{M}_\alpha\times\mathcal{M}_\beta}(\Theta^\bullet_{\alpha,\beta}) \oplus \Pi^*_{\mathcal{M}_\alpha\times\mathcal{M}_\gamma}(\Theta^\bullet_{\alpha,\gamma}). \tag{3.10}$$

The isomorphisms (3.8) identify (3.9) with the dual of (3.10), and vice versa.

(l) For all $\alpha, \beta \in K(\mathcal{A})$, we are given isomorphisms in $\mathrm{Perf}([*/\mathbb{G}_m] \times \mathcal{M}_\alpha \times \mathcal{M}_\beta)$:

$$(\Psi_\alpha \times \mathrm{id}_{\mathcal{M}_\beta})^*(\Theta^\bullet_{\alpha,\beta}) \cong \Pi^*_{[*/\mathbb{G}_m]}(E_1) \otimes \Pi^*_{\mathcal{M}_\alpha\times\mathcal{M}_\beta}(\Theta^\bullet_{\alpha,\beta}), \tag{3.11}$$

$$(\Pi_{\mathcal{M}_\alpha}, (\Psi_\beta \circ \Pi_{[*/\mathbb{G}_m]\times\mathcal{M}_\beta}))^*(\Theta^\bullet_{\alpha,\beta}) \cong \Pi^*_{[*/\mathbb{G}_m]}(E_{-1}) \otimes \Pi^*_{\mathcal{M}_\alpha\times\mathcal{M}_\beta}(\Theta^\bullet_{\alpha,\beta}). \tag{3.12}$$

Here E_1, E_{-1} are as in Example 2.22, regarded as line bundles on $[*/\mathbb{G}_m]$. By **(h)**, Ψ_α, Ψ_β are $[*/\mathbb{G}_m]$-actions on $\mathcal{M}_\alpha, \mathcal{M}_\beta$ as in §2.3.7, which lift to two commuting $[*/\mathbb{G}_m]$-actions on $\mathcal{M}_\alpha \times \mathcal{M}_\beta$. We require that (3.11)–(3.12) are $[*/\mathbb{G}_m]$-actions on $\Theta^\bullet_{\alpha,\beta}$ of weights $1, -1$ compatible with these $[*/\mathbb{G}_m]$-actions on $\mathcal{M}_\alpha \times \mathcal{M}_\beta$ in the sense of Definition 2.27, that is, the analogues of (2.29) commute.

The isomorphisms (3.11)–(3.12) should be compatible with each other, that is, they define a $[*/\mathbb{G}_m]^2$-action on $\Theta^\bullet_{\alpha,\beta}$ with multi-weight $(1, -1)$. Also the isomorphisms (3.8)–(3.10) should be equivariant under these $[*/\mathbb{G}_m]^2$-actions, in the appropriate sense.

Here is the analogue for triangulated categories $\mathcal{T}$:

Assumption 3.2. Suppose are given a $\mathbb{K}$-linear triangulated category $\mathcal{T}$. Assume the analogue of Assumption 3.1(a)–(l), replacing $\mathcal{A}$ by $\mathcal{T}$ throughout, and with the following changes:

(i) To define the Grothendieck group $K_0(\mathcal{T})$ in (a), we impose a relation $[F]=[E]+[G]$ for each distinguished triangle $E\to F\to G\to E[+1]$ in $\mathcal{T}$.

(ii) In (d), the moduli stack $\mathcal{M}$ of all objects in $\mathcal{T}$ is a *higher* Artin $\mathbb{K}$-stack locally of finite type.

(iii) In (e), in our examples morphisms $S\to\mathcal{M}$ are equivalent to objects of an $\mathcal{O}_S$-linear *triangulated* category $\mathcal{T}(S)$. That is, we can enhance the higher Artin $\mathbb{K}$-stack $\mathcal{M}$ to a *stack in triangulated categories on* $\mathbf{Sch}_{\mathbb{K}}$.

(iv) For each object E in $\mathcal{T}$, there is a morphism $\xi_E:\mathbb{A}^1\to\mathcal{M}_0$ in $\mathrm{Ho}(\mathbf{HSt}_{\mathbb{K}}^{\mathrm{lft}})$, which on $\mathbb{K}$-points maps $x\in\mathbb{K}$ to $\bigl[\mathrm{Cone}(x\cdot\mathrm{id}_E:E\to E)\bigr]$, taking cones of morphisms in the triangulated category $\mathcal{T}$. Note that

$$\bigl[\mathrm{Cone}(x\cdot\mathrm{id}_E:E\to E)\bigr]=\begin{cases}\bigl[E\oplus E[1]\bigr], & x=0,\\ [0], & x\neq 0.\end{cases}$$

Remark 3.3. Assumptions 3.1 and 3.2 include a symmetric product $\chi:K(\mathcal{A})\times K(\mathcal{A})\to\mathbb{Z}$ in (c) and a perfect complex $\Theta^\bullet\in\mathrm{Perf}(\mathcal{M}\times\mathcal{M})$ in (i). We now explain where these come from in the examples of §5–§6.

In the examples of abelian categories $\mathcal{A}$ we are interested in, for objects E,F in $\mathcal{A}$ we can define Ext groups $\mathrm{Ext}^i(E,F)$ for $i=0,1,\ldots,$ which are finite-dimensional $\mathbb{K}$-vector spaces with $\mathrm{Ext}^0(E,F)=\mathrm{Hom}_{\mathcal{A}}(E,F)$ and $\mathrm{Ext}^i(E,F)=0$ for $i\gg 0$. For example, if $\mathcal{A}=\mathrm{coh}(X)$ for X a smooth projective $\mathbb{K}$-scheme then $\mathrm{Ext}^i(E,F)$ is as in Hartshorne [62, §III.6]. In the triangulated category case, for $E,F\in\mathcal{T}$ we define $\mathrm{Ext}^i(E,F)=\mathrm{Hom}_{\mathcal{T}}(E,F[i])$ for $i\in\mathbb{Z}$.

The *Euler form* of $\mathcal{A}$ is the bilinear map $\chi_{\mathcal{A}}:K_0(\mathcal{A})\times K_0(\mathcal{A})\to\mathbb{Z}$ acting by $\chi_{\mathcal{A}}([E],[F])=\sum_i(-1)^i\dim_{\mathbb{K}}\mathrm{Ext}^i(E,F)$. The *numerical Grothendieck group* is $K^{\mathrm{num}}(\mathcal{A})=K_0(\mathcal{A})/\{\alpha\in K_0(\mathcal{A}):\chi_{\mathcal{A}}(\alpha,\beta)=0$ for all $\beta\in K_0(\mathcal{A})\}$. Then $\chi_{\mathcal{A}}$ descends to $\chi_{\mathcal{A}}:K^{\mathrm{num}}(\mathcal{A})\times K^{\mathrm{num}}(\mathcal{A})\to\mathbb{Z}$. In our examples we will often choose $K(\mathcal{A})=K^{\mathrm{num}}(\mathcal{A})$ in Assumption 3.1(b).

Under good conditions, there should exist a natural perfect complex $\mathcal{E}xt^\bullet$ in $\mathrm{Perf}(\mathcal{M}\times\mathcal{M})$, which at each $\mathbb{K}$-point $([E],[F])$ of $\mathcal{M}\times\mathcal{M}$ has cohomology $H^i(\mathcal{E}xt^\bullet\vert_{([E],[F])})\cong\mathrm{Ext}^i(E,F)$ for all i. Thus $\mathrm{rank}\,\mathcal{E}xt^\bullet\vert_{([E],[F])}=\chi_{\mathcal{A}}([E],[F])$.

We can write $\mathcal{E}xt^\bullet$ explicitly in terms of cotangent complexes of derived moduli spaces as follows. Suppose that $\mathcal{M}=t_0(\boldsymbol{\mathcal{M}})$ for a derived moduli stack $\boldsymbol{\mathcal{M}}$, and also that we have a derived moduli stack $\boldsymbol{\mathfrak{Exact}}$ of exact sequences $0\to E\to F\to G\to 0$ in $\mathcal{A}$ (or of distinguished triangles in $\mathcal{T}$), with $\boldsymbol{\mathcal{M}},\boldsymbol{\mathfrak{Exact}}$ locally of finite presentation. Define a morphism $\Upsilon:\boldsymbol{\mathcal{M}}\times\boldsymbol{\mathcal{M}}\to\boldsymbol{\mathfrak{Exact}}$ to map (E,F) to the exact sequence $0\to F\to E\oplus F\to E\to 0$. Then $(\mathcal{E}xt^\bullet)^\vee$ should be the restriction of $\mathbb{L}_{\boldsymbol{\mathcal{M}}\times\boldsymbol{\mathcal{M}}/\boldsymbol{\mathfrak{Exact}}}[1]\in\mathrm{Perf}(\boldsymbol{\mathcal{M}}\times\boldsymbol{\mathcal{M}})$ to the classical truncation $\mathcal{M}\times\mathcal{M}$, where $\mathbb{L}_{\boldsymbol{\mathcal{M}}\times\boldsymbol{\mathcal{M}}/\boldsymbol{\mathfrak{Exact}}}$ is the relative cotangent complex of Υ.

Without using derived algebraic geometry, one can define $\mathcal{E}xt^\bullet$ by the techniques used to construct obstruction theories, as in Behrend and Fantechi [15].

There will be two main versions of our construction, both with interesting applications, with different definitions of $\chi,\Theta^\bullet$ in each case:

(A) **(The even Calabi–Yau case.)** Suppose that $\mathcal{A}$ or $\mathcal{T}$ is a $2n$-Calabi–Yau category, for instance, $\mathcal{A} = \mathrm{coh}(X)$ or $\mathcal{T} = D^b\mathrm{coh}(X)$ for X a projective Calabi–Yau $2n$-fold over $\mathbb{K}$. Then we have isomorphisms $\mathrm{Ext}^i(F,E)^* \cong \mathrm{Ext}^{2n-i}(E,F)$ for all E,F,i, which should be induced by an isomorphism $\sigma^*((\mathcal{E}xt^\bullet)^\vee) \cong \mathcal{E}xt^\bullet[2n]$ in $\mathrm{Perf}(\mathcal{M}\times\mathcal{M})$, where $\sigma : \mathcal{M}\times\mathcal{M} \to \mathcal{M}\times\mathcal{M}$ exchanges the two factors of $\mathcal{M}$ as in Assumption 3.1(j).

In this case we define $\chi = \chi_{\mathcal{A}}$ and $\Theta^\bullet = (\mathcal{E}xt^\bullet)^\vee$ in $\mathrm{Perf}(\mathcal{M}\times\mathcal{M})$. Then $\mathrm{Ext}^i(F,E)^* \cong \mathrm{Ext}^{2n-i}(E,F)$ implies that $\chi(\alpha,\beta) = \chi(\beta,\alpha)$ in Assumption 3.1(c), and $\sigma^*((\mathcal{E}xt^\bullet)^\vee) \cong \mathcal{E}xt^\bullet[2n]$ implies that $\sigma^*_{\alpha,\beta}(\Theta^\bullet_{\beta,\alpha}) \cong (\Theta^\bullet_{\alpha,\beta})^\vee[2n]$ in Assumption 3.1(j).

(B) **(The general case.)** If $\mathcal{A},\mathcal{T}$ are not $2n$-Calabi–Yau, then in general we cannot take $\chi = \chi_{\mathcal{A}}$ and $\Theta^\bullet = (\mathcal{E}xt^\bullet)^\vee$ as in (A), since we might have $\chi(\alpha,\beta) \neq \chi(\beta,\alpha)$ and $\sigma^*_{\alpha,\beta}(\Theta^\bullet_{\beta,\alpha}) \ncong (\Theta^\bullet_{\alpha,\beta})^\vee[2n]$. Then we define $\chi(\alpha,\beta) = \chi_{\mathcal{A}}(\alpha,\beta) + \chi_{\mathcal{A}}(\beta,\alpha)$ and $\Theta^\bullet = (\mathcal{E}xt^\bullet)^\vee \oplus \sigma^*(\mathcal{E}xt^\bullet)[2n]$, for some $n \in \mathbb{Z}$, so $\chi(\alpha,\beta) = \chi(\beta,\alpha)$ and $\sigma^*_{\alpha,\beta}(\Theta^\bullet_{\beta,\alpha}) \cong (\Theta^\bullet_{\alpha,\beta})^\vee[2n]$ hold trivially.

If $\mathcal{A}$ or $\mathcal{T}$ is $2n$-Calabi–Yau then both (A) and (B) work, but usually yield rather different Lie algebra structures on $H_*(\mathcal{M})$ under our constructions.

We have justified Assumption 3.1(c),(i),(j) in both cases (A),(B). Part (k) holds because of the analogue for the complex $\mathcal{E}xt^\bullet$ of

$$\mathrm{Ext}^i(E,F\oplus G) \cong \mathrm{Ext}^i(E,G)\oplus\mathrm{Ext}^i(E,F) \quad \text{and}$$
$$\mathrm{Ext}^i(E\oplus F,G) \cong \mathrm{Ext}^i(E,G)\oplus\mathrm{Ext}^i(F,G).$$

Part (l) basically holds because in the action of $\mathrm{Aut}(E)\times\mathrm{Aut}(F)$ on $\mathrm{Ext}^i(E,F)^*$, $(\lambda\,\mathrm{id}_E,\mu\,\mathrm{id}_F)$ acts by multiplication by $\lambda\mu^{-1}$, for all $\lambda,\mu \in \mathbb{G}_m$.

Remark 3.4. **(a)** The signs $\epsilon_{\alpha,\beta}$ in Assumption 3.1(d) will be needed in the definitions of our Lie brackets $[\,,\,]$ on $H_*(\mathcal{M})$ to make $[\,,\,]$ graded antisymmetric for a certain (nonstandard) grading on $H_*(\mathcal{M})$. Lemma 3.5 shows we can always choose $\epsilon_{\alpha,\beta}$ satisfying (3.1)–(3.3). In Remark 3.3(B), when $\chi(\alpha,\beta) = \chi_{\mathcal{A}}(\alpha,\beta) + \chi_{\mathcal{A}}(\beta,\alpha)$, there is a natural choice $\epsilon_{\alpha,\beta} = (-1)^{\chi_{\mathcal{A}}(\alpha,\beta)}$.

We will explain in §7 that the $\epsilon_{\alpha,\beta}$ are related to the problem of choosing 'orientation data' on the category $\mathcal{A}$, as in Cao and Leung [31] for instance.

(b) Rather than taking $\epsilon_{\alpha,\beta} = \pm 1$, it is sometimes natural to define $\epsilon_{\alpha,\beta}$ to be a *locally constant function* $\epsilon_{\alpha,\beta} : \mathcal{M}_\alpha \times \mathcal{M}_\beta \to \{\pm 1\}$, and then to require (3.1)–(3.3) to hold in functions $\mathcal{M}_\alpha\times\mathcal{M}_\beta \to \{\pm1\}$ and $\mathcal{M}_\alpha\times\mathcal{M}_\beta\times\mathcal{M}_\gamma \to \{\pm1\}$.

(c) Very similar signs $\epsilon_{\alpha,\beta}$ occur in the theory of affine Lie algebras, as in Kac [84, §7.8], where they are called *asymmetry functions*, and in the theory of vertex algebras, as in [85, (5.14)], [46, (5.4)] and (2.10)–(2.12) below.

Lemma 3.5. *Let $K(\mathcal{A})$ be a finitely generated abelian group, and $\chi : K(\mathcal{A}) \times K(\mathcal{A}) \to \mathbb{Z}$ be a symmetric biadditive map. Then we can choose $\epsilon_{\alpha,\beta} \in \{1,-1\}$ for all $\alpha,\beta \in K(\mathcal{A})$ satisfying* (3.1)–(3.3)*. Furthermore, we can do this so that $(\alpha,\beta) \mapsto \epsilon_{\alpha,\beta}$ is a biadditive map of abelian groups $K(\mathcal{A})\times K(\mathcal{A}) \to \{\pm1\}$.*

Proof. Write $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z} = \{0,1\}$ for the field with two elements, and set $V = K(\mathcal{A})\otimes_{\mathbb{Z}}\mathbb{Z}_2$ as a $\mathbb{Z}_2$-vector space, with surjective projection $\pi : K(\mathcal{A}) \to V$. Then V is finite-dimensional, as $K(\mathcal{A})$ is finitely generated, so we can choose a basis $v_1,\ldots,v_n$ for V. Define $Q : V \times V \to \mathbb{Z}_2$ by $Q(v,w) = \chi(v',w') + \chi(v',v')\chi(w',w') \mod 2$ for $v',w' \in K(\mathcal{A})$ with $\pi(v') = v$, $\pi(w') = w$. This is independent of the choices of v',w'.

Now the map $v \mapsto \chi(v',v') \mod 2$ is actually an additive, and hence linear, map $V \to \mathbb{Z}_2$, since $\chi(v_1'+v_2', v_1'+v_2') \equiv \chi(v_1',v_1') + \chi(v_2',v_2') + 2\chi(v_1',v_2') \equiv \chi(v_1',v_1')+\chi(v_2',v_2') \mod 2$. Hence $Q : V\times V \to \mathbb{Z}_2$ is bilinear (although it looks quartic). Also $Q(v,v) \equiv \chi(v',v') + \chi(v',v')\chi(v',v') \equiv 0 \mod 2$ for any $v \in V$.

Write $q_{ij} = Q(v_i,v_j)$ for $i,j = 1,\ldots,n$. Then $q_{ii} = 0$ for $i = 1,\ldots,n$. For $\alpha,\beta \in K(\mathcal{A})$, write $\pi(\alpha) = a_1v_1 + \cdots + a_nv_n$ and $\pi(\beta) = b_1v_1 + \cdots + b_nv_n$ using the basis $v_1,\ldots,v_n$, for $a_i,b_i \in \mathbb{Z}_2$. Then define $\epsilon_{\alpha,\beta} = (-1)^{\sum_{1\leqslant i<j\leqslant n} a_ib_jq_{ij}}$. It is easy to check that these $\epsilon_{\alpha,\beta}$ satisfy (3.1), and $(\alpha,\beta) \mapsto \epsilon_{\alpha,\beta}$ is biadditive, so (3.2)–(3.3) also hold as in Assumption 3.1(d). $\square$

The next proposition will be proved in §4.1.

Proposition 3.6. *Let Assumption* 3.2 *hold for $\mathcal{T}$. Then:*

- **(a)** *If $\mathcal{T} \ncong 0$ (i.e. $\mathcal{T}$ contains nonzero objects) the inclusions $\mathcal{M}' \hookrightarrow \mathcal{M}$ and $\mathcal{M}_0' \hookrightarrow \mathcal{M}_0$ are homotopy equivalences, as in Definition* 2.37*.*
- **(b)** *For each $\alpha \in K(\mathcal{T})$, choose a $\mathbb{K}$-point $[E] \in \mathcal{M}_\alpha(\mathbb{K})$ (this is possible as $K_0(\mathcal{T}) \twoheadrightarrow K(\mathcal{T})$ is surjective by Assumption* 3.1(b)*). Then the morphism $\Phi_{0,\alpha}|_{\mathcal{M}_0\times\{[E]\}} : \mathcal{M}_0 \cong \mathcal{M}_0 \times \{[E]\} \to \mathcal{M}_\alpha$, which maps $[F] \mapsto [F \oplus E]$ on $\mathbb{K}$-points, is a homotopy equivalence.*

Thus, if Assumption 2.30 *holds for (co)homology theories $H_*(-), H^*(-)$ of higher Artin $\mathbb{K}$-stacks, Lemma* 2.38 *gives isomorphisms for all $\alpha \in K(\mathcal{T})$:*

$$\begin{aligned} &H_*(\mathcal{M}') \cong H_*(\mathcal{M}), \quad H_*(\mathcal{M}_0') \cong H_*(\mathcal{M}_0), \quad H_*(\mathcal{M}_0) \cong H_*(\mathcal{M}_\alpha), \\ &H^*(\mathcal{M}') \cong H^*(\mathcal{M}), \quad H^*(\mathcal{M}_0') \cong H^*(\mathcal{M}_0), \quad H^*(\mathcal{M}_0) \cong H^*(\mathcal{M}_\alpha). \end{aligned} \tag{3.13}$$

3.2 Operations $[\,,\,]_n$ on $H_*(\mathcal{M})$, and vertex algebras

We will work in the situation of Assumption 3.1, and define some algebraic structures on the homology groups $H_*(\mathcal{M})$ and $H_*(\mathcal{M}_\alpha)$ for $\alpha \in K(\mathcal{A})$. We first make $H_*([*/\mathbb{G}_m])$ into a commutative graded R-algebra:

Definition 3.7. Let Assumption 2.30 hold for (co)homology $H_*(-), H^*(-)$ of Artin $\mathbb{K}$-stacks over an arbitrary commutative ring R. Then Assumption 2.30(c) writes $H_*([*/\mathbb{G}_m]) = R[t]$ and $H^*([*/\mathbb{G}_m]) = R[\tau]$.

Define a stack morphism $\Omega : [*/\mathbb{G}_m] \times [*/\mathbb{G}_m] \to [*/\mathbb{G}_m]$ to be induced by the group morphism $\mathbb{G}_m \times \mathbb{G}_m \to \mathbb{G}_m$ mapping $(\mu,\nu) \mapsto \mu\nu$, as in Assumption 3.1(h). Define an R-bilinear operation

$$\begin{aligned} &\star : H_a([*/\mathbb{G}_m]) \times H_b([*/\mathbb{G}_m]) \longrightarrow H_{a+b}([*/\mathbb{G}_m]) \quad \text{by} \\ &\kappa \star \lambda = H_{a+b}(\Omega)(\kappa \boxtimes \lambda) \quad \text{for } \kappa \in H_a([*/\mathbb{G}_m]) \text{ and } \lambda \in H_b([*/\mathbb{G}_m]). \end{aligned} \tag{3.14}$$

From the definition of Ω we see that $H^2(\Omega)(\tau) = \tau_1 \boxtimes 1 + 1 \boxtimes \tau_2$. By considering $(t_1^p \boxtimes t_2^q) \cap (H^2(\Omega)(\tau))^{p+q}$ in $H_*([*/\mathbb{G}_m]\times[*/\mathbb{G}_m])$ we find that for all $p,q \geqslant 0$ we have

$$t^p \star t^q = \binom{p+q}{p} t^{p+q}. \tag{3.15}$$

Hence induction on $n = 0,1,\ldots$ gives

$$t^{\star^n} = \overbrace{t \star t \star \cdots \star t}^{n \text{ copies}} = n! \cdot t^n. \tag{3.16}$$

Equation (3.15) shows $\star$ is a commutative, associative product on $H^*([*/\mathbb{G}_m])$, making $H^*([*/\mathbb{G}_m]) = R[t]$ into a commutative graded R-algebra, with identity $t^0 = 1$. We can also see this geometrically, e.g. associativity of $\star$ follows from

$$\Omega \circ (\Omega \times \mathrm{id}_{[*/\mathbb{G}_m]}) = \Omega \circ (\mathrm{id}_{[*/\mathbb{G}_m]} \times \Omega) : [*/\mathbb{G}_m]\times[*/\mathbb{G}_m]\times[*/\mathbb{G}_m] \longrightarrow [*/\mathbb{G}_m].$$

Remark 3.8. Readers are warned that Definition 3.7 is potentially confusing for two reasons. Firstly, usually cohomology $H^*(S)$ is an R-algebra, but here we make the *homology* $H_*([*/\mathbb{G}_m])$ into an R-algebra. This works because $[*/\mathbb{G}_m]$ is a group object in $\mathrm{Ho}(\mathbf{Art}_{\mathbb{K}})$, with multiplication $\Omega : [*/\mathbb{G}_m]\times[*/\mathbb{G}_m] \to [*/\mathbb{G}_m]$.

Secondly, as Assumption 2.30(c) writes $H_*([*/\mathbb{G}_m]) = R[t]$, the obvious product would be $t^p \star t^q = t^{p+q}$ and $t^{\star^n} = t^n$, but this is *not* what (3.15)–(3.16) give us. Here t^n is just notation, a choice of generator of $H_{2n}([*/\mathbb{G}_m])$, and should not be thought of as the n^{th} power of t under a product on $H_*([*/\mathbb{G}_m])$.

Now t^n is a free basis element for $H_{2n}([*/\mathbb{G}_m]) = R \cdot t^n$. If R is not a $\mathbb{Q}$-algebra (e.g. if $R = \mathbb{Z}$) then in general $n!$ is not invertible in R, so $t^{\star^n}$ is not a basis element for $H_{2n}([*/\mathbb{G}_m])$. This is why we chose the elements t^n, rather than $t^{\star^n}$, to describe $H_{2n}([*/\mathbb{G}_m])$. If R is a $\mathbb{Q}$-algebra then $t^{\star^n}$ is an alternative basis element for $H_{2n}([*/\mathbb{G}_m])$, which is more convenient for some formulae.

Then $H_*(\mathcal{M}_\alpha)$ and $H_*(\mathcal{M})$ are graded modules over $H_*([*/\mathbb{G}_m])$:

Definition 3.9. Let Assumption 2.30 hold for (co)homology theories $H_i, H^i : \mathrm{Ho}(\mathbf{Art}_{\mathbb{K}}^{\mathrm{lft}}) \to R$-mod of Artin $\mathbb{K}$-stacks, let Assumption 3.1 hold for the abelian category $\mathcal{A}$, and use the notation of §2 and §3.1.

In a similar way to (3.14), for each $\alpha \in K(\mathcal{A})$ and $a,b \geqslant 0$, define an R-bilinear operation

$$\begin{aligned} &\diamond : H_a([*/\mathbb{G}_m]) \times H_b(\mathcal{M}_\alpha) \longrightarrow H_{a+b}(\mathcal{M}_\alpha) \quad \text{by} \\ &\kappa \diamond \zeta = H_{a+b}(\Psi_\alpha)(\kappa \boxtimes \zeta) \quad \text{for } \kappa \in H_a([*/\mathbb{G}_m]) \text{ and } \zeta \in H_b(\mathcal{M}_\alpha), \end{aligned} \tag{3.17}$$

where $\Psi_\alpha : [*/\mathbb{G}_m]\times\mathcal{M}_\alpha \to \mathcal{M}_\alpha$ is as in Assumption 3.1(h). Then (3.7) implies that for all $\kappa \in H_a([*/\mathbb{G}_m])$, $\lambda \in H_b([*/\mathbb{G}_m])$, and $\zeta \in H_c(\mathcal{M}_\alpha)$ we have

$$\kappa \diamond (\lambda \diamond \zeta) = (\kappa \star \lambda) \diamond \zeta \qquad \text{in } H_{a+b+c}(\mathcal{M}_\alpha). \tag{3.18}$$

That is, $\diamond$ makes $H_*(\mathcal{M}_\alpha)$ into a *graded module* over the graded R-algebra $\bigl(H_*([*/\mathbb{G}_m]), \star\bigr)$ from Definition 3.9.

We make $H_*(\mathcal{M})$ into a graded $H_*([*/\mathbb{G}_m])$-module in the same way, by $\kappa \diamond \zeta = H_{a+b}(\Psi)(\kappa \boxtimes \zeta)$ for $\kappa \in H_a([*/\mathbb{G}_m])$ and $\zeta \in H_b(\mathcal{M})$. Since $\mathcal{M} = \coprod_{\alpha \in K(\mathcal{A})} \mathcal{M}_\alpha$ by Assumption 3.1(f), equation (2.31) gives an isomorphism

$$H_*(\mathcal{M}) \cong \bigoplus\nolimits_{\alpha \in K(\mathcal{A})} H_*(\mathcal{M}_\alpha), \tag{3.19}$$

which is an isomorphism of graded $H_*([*/\mathbb{G}_m])$-modules.

We define a family of R-bilinear operations $[\,,\,]_n : H_*(\mathcal{M}_\alpha) \times H_*(\mathcal{M}_\beta) \to H_*(\mathcal{M}_{\alpha+\beta})$ for $\alpha, \beta \in K(\mathcal{A})$ and $n = 0, 1, \ldots$. Theorem 3.14 below shows they are equivalent to the structure of a *graded vertex algebra* on $H_*(\mathcal{M})$, as in §2.2.1. Later we will use the $[\,,\,]_n$ to define our Lie brackets on $H_*(\mathcal{M})$.

Definition 3.10. Let Assumption 2.30 hold for (co)homology theories $H_i, H^i : \mathrm{Ho}(\mathbf{Art}_{\mathbb{K}}^{\mathrm{lft}}) \to R$-mod of Artin $\mathbb{K}$-stacks, let Assumption 3.1 hold for the abelian category $\mathcal{A}$, and use the notation of §2 and §3.1.

For $\alpha, \beta \in K(\mathcal{A})$, define $\Xi_{\alpha,\beta} : [*/\mathbb{G}_m] \times \mathcal{M}_\alpha \times \mathcal{M}_\beta \to \mathcal{M}_{\alpha+\beta}$ to be the composition of morphisms of Artin $\mathbb{K}$-stacks

$$[*/\mathbb{G}_m] \times \mathcal{M}_\alpha \times \mathcal{M}_\beta \xrightarrow{\Psi_\alpha \times \mathrm{id}_{\mathcal{M}_\beta}} \mathcal{M}_\alpha \times \mathcal{M}_\beta \xrightarrow{\Phi_{\alpha,\beta}} \mathcal{M}_{\alpha+\beta}. \tag{3.20}$$

Here $\Phi_{\alpha,\beta}, \Psi_\alpha, \Psi_\beta$ are as in Assumption 3.1(g),(h).

With the convention that $H_j(\cdots) = 0$ if $j < 0$, for all $\alpha, \beta \in K(\mathcal{A})$, $a, b = 0, 1, \ldots$ and $n \in \mathbb{Z}$ define an R-bilinear map

$$[\,,\,]_n : H_a(\mathcal{M}_\alpha) \times H_b(\mathcal{M}_\beta) \longrightarrow H_{a+b-2n-2\chi(\alpha,\beta)-2}(\mathcal{M}_{\alpha+\beta}) \tag{3.21}$$

by, for all $\zeta \in H_a(\mathcal{M}_\alpha)$ and $\eta \in H_b(\mathcal{M}_\beta)$,

$$[\zeta, \eta]_n = \sum_{\substack{i \geqslant 0:\ 2i \leqslant a+b, \\ i \geqslant n+\chi(\alpha,\beta)+1}} \begin{aligned}[t] &\epsilon_{\alpha,\beta}(-1)^{a\chi(\beta,\beta)} \cdot H_{a+b-2n-2\chi(\alpha,\beta)-2}(\Xi_{\alpha,\beta}) \\ &\bigl(t^{i-n-\chi(\alpha,\beta)-1} \boxtimes \bigl[(\zeta \boxtimes \eta) \cap c_i([\Theta^\bullet_{\alpha,\beta}])\bigr]\bigr). \end{aligned} \tag{3.22}$$

Here $H_*(\cdots), \boxtimes, \cap$ are the pushforward maps, external product, and cap product on (co)homology from Assumption 2.30(a), and $c_i(\cdots)$ is the Chern class map from Assumption 2.30(b), and $t^k \in H_{2k}([*/\mathbb{G}_m])$ is as in Assumption 2.30(c), the sign $\epsilon_{\alpha,\beta} = \pm 1$ is as in Assumption 3.1(d), and the perfect complex $\Theta^\bullet_{\alpha,\beta}$ is as in Assumption 3.1(i).

With respect to the obvious gradings on $H_*(\mathcal{M}_\alpha), H_*(\mathcal{M}_\beta), H_*(\mathcal{M}_{\alpha+\beta})$, the bracket $[\,,\,]_n$ is graded of degree $-2n - 2\chi(\alpha, \beta) - 2$. However, we often prefer to work with an alternative grading. For each $i \in \mathbb{Z}$ and $\alpha \in K(\mathcal{A})$, define

$$\tilde{H}_i(\mathcal{M}_\alpha) = H_{i+2-\chi(\alpha,\alpha)}(\mathcal{M}_\alpha). \tag{3.23}$$

That is, $\tilde{H}_*(\mathcal{M}_\alpha)$ is just $H_*(\mathcal{M}_\alpha)$ with grading shifted by $2 - \chi(\alpha, \alpha)$. Then as χ is biadditive with $\chi(\alpha, \beta) = \chi(\beta, \alpha)$ by Assumption 3.1(c), we see that (3.21) is equivalent to

$$[\,,\,]_n : \tilde{H}_{\tilde{a}}(\mathcal{M}_\alpha) \times \tilde{H}_{\tilde{b}}(\mathcal{M}_\beta) \longrightarrow \tilde{H}_{\tilde{a}+\tilde{b}-2n}(\mathcal{M}_{\alpha+\beta}), \tag{3.24}$$

where $\tilde{a} = a - 2 + \chi(\alpha,\alpha)$ and $\tilde{b} = b - 2 + \chi(\beta,\beta)$. Thus, $[\,,\,]_n$ is graded of degree $-2n$ on $\tilde{H}_*(\mathcal{M}_\alpha), \tilde{H}_*(\mathcal{M}_\beta), \tilde{H}_*(\mathcal{M}_{\alpha+\beta})$.

The zero object $0 \in \mathcal{A}$ gives a $\mathbb{K}$-point $[0] \in \mathcal{M}_0(\mathbb{K}) \subset \mathcal{M}(\mathbb{K})$, which we can regard as a morphism $[0] : * \to \mathcal{M}_0$ in $\mathrm{Ho}(\mathbf{Art}_{\mathbb{K}}^{\mathrm{lft}})$, where $* = \operatorname{Spec}\mathbb{K}$ is the point. Thus we have the pushforward $H_0([0]) : H_0(*) \to H_0(\mathcal{M}_0) = \tilde{H}_{-2}(\mathcal{M}_0)$, where Assumption 2.30(a)(iv) identifies $H_0(*) \cong R$. Write $\mathbb{1} = H_0([0])(1)$ in $H_0(\mathcal{M}_0) = \tilde{H}_{-2}(\mathcal{M}_0) \subseteq \tilde{H}_{-2}(\mathcal{M})$. We call $\mathbb{1}$ the *identity element*, or *vacuum element*, of $\tilde{H}_*(\mathcal{M})$.

We can generalize all the above to the triangulated category case. Suppose instead that Assumption 2.30 holds for (co)homology theories $H_i, H^i : \mathrm{Ho}(\mathbf{HSt}_{\mathbb{K}}^{\mathrm{lft}}) \to R\text{-mod}$ of *higher* Artin $\mathbb{K}$-stacks, and Assumption 3.2 holds for the $\mathbb{K}$-linear triangulated category $\mathcal{T}$. We replace $K(\mathcal{A})$ by $K(\mathcal{T})$ throughout, but otherwise the definition of $[\,,\,]_n$ works without change.

Remark 3.11. The operations $[\,,\,]_n$ are not Lie brackets, though we will use them to define Lie brackets later. Note that the definition of $[\,,\,]_n$ is not (anti)symmetric between $\mathcal{M}_\alpha$ and $\mathcal{M}_\beta$, as it involves Ψ_α but not Ψ_β.

The next theorem, proved in §4.2, gives some identities satisfied by the $[\,,\,]_n$.

Theorem 3.12. *In the situation of Definition* 3.10, *let* $\alpha, \beta, \gamma \in K(\mathcal{A})$, *and* $\zeta \in \tilde{H}_{\tilde{a}}(\mathcal{M}_\alpha)$, $\eta \in \tilde{H}_{\tilde{b}}(\mathcal{M}_\beta)$, $\theta \in \tilde{H}_{\tilde{c}}(\mathcal{M}_\gamma)$ *for* $\tilde{a}, \tilde{b}, \tilde{c} \in \mathbb{Z}$. *Then using the notation of Definition* 3.9, *for all* $n \in \mathbb{Z}$ *we have*

$$[\zeta, \mathbb{1}]_n = \begin{cases} t^{-n-1} \diamond \zeta, & n < 0, \\ 0, & n \geqslant 0, \end{cases} \quad [\mathbb{1}, \zeta]_n = \begin{cases} \zeta, & n = -1, \\ 0 & \textit{otherwise}. \end{cases} \tag{3.25}$$

For all $n \in \mathbb{Z}$ *and* $p \geqslant 0$, *we have*

$$[t^p \diamond \zeta, \eta]_n = (-1)^p \binom{n}{p} \cdot [\zeta, \eta]_{n-p}, \tag{3.26}$$

$$[\zeta, t^p \diamond \eta]_n = \sum_{k=0}^{p} \binom{n}{p-k} \cdot t^k \diamond [\zeta, \eta]_{n+k-p}. \tag{3.27}$$

For all $n \in \mathbb{Z}$ *we have*

$$[\eta, \zeta]_n = \sum_{k \geqslant 0:\ 2k \leqslant \tilde{a}+\tilde{b}-2n+2-\chi(\alpha+\beta,\alpha+\beta)} (-1)^{1+\tilde{a}\tilde{b}+k+n} \cdot t^k \diamond [\zeta, \eta]_{k+n}. \tag{3.28}$$

And for all $l, m \in \mathbb{Z}$ *we have*

$$\begin{aligned} &[[\zeta, \eta]_l, \theta]_m - \sum_{\substack{n \geqslant 0:\ 2m+2n \leqslant \tilde{b}+\tilde{c} \\ +2-\chi(\beta+\gamma,\beta+\gamma)}} (-1)^n \binom{l}{n} \cdot [\zeta, [\eta, \theta]_{m+n}]_{l-n} \\ &+ \sum_{\substack{n \geqslant 0:\ 2n \leqslant \tilde{a}+\tilde{c} \\ +2-\chi(\alpha+\gamma,\alpha+\gamma)}} (-1)^{n+l+\tilde{a}\tilde{b}} \binom{l}{n} \cdot [\eta, [\zeta, \theta]_n]_{l+m-n} = 0. \end{aligned} \tag{3.29}$$

The analogue of all the above holds in the triangulated category case, replacing $K(\mathcal{A})$ by $K(\mathcal{T})$.

Remark 3.13. **(a)** We can replace $\mathcal{M}$ by $\mathcal{M}' = \mathcal{M} \setminus \{[0]\}$ throughout §3.2, which will be useful in §3.4. This works as Φ, Ψ in Assumption 3.1(g),(h) restrict to $\Phi' := \Phi|_{\mathcal{M}'\times\mathcal{M}'} : \mathcal{M}' \times \mathcal{M}' \to \mathcal{M}'$ and $\Psi' := \Psi|_{[*/\mathbb{G}_m]\times\mathcal{M}'} : [*/\mathbb{G}_m] \times \mathcal{M}' \to \mathcal{M}'$ with the same properties.

(b) Equation (3.28) will be used to prove graded antisymmetry of our Lie brackets later. The equation looks rather asymmetric, but we have

$$\begin{aligned}
[\eta,\zeta]_n &= \sum_{k\geqslant 0}(-1)^{1+\tilde a\tilde b+k+n}\cdot t^k\diamond[\zeta,\eta]_{k+n}\\
&= \sum_{k\geqslant 0}(-1)^{1+\tilde a\tilde b+k+n}\cdot t^k\diamond\Bigl[\sum_{l\geqslant 0}(-1)^{1+\tilde a\tilde b+k+l+n}t^l\diamond[\eta,\zeta]_{k+l+n}\Bigr]\\
&= \sum_{k,l\geqslant 0}(-1)^l\cdot t^k\diamond\bigl(t^l\diamond[\eta,\zeta]_{k+l+n}\bigr) = \sum_{j\geqslant 0}\sum_{l=0}^{j}(-1)^l\cdot(t^{j-l}\star t^l)\diamond[\eta,\zeta]_{j+n}\\
&= \sum_{j\geqslant 0}\Biggl[\sum_{l=0}^{j}(-1)^l\binom{l}{j}\Biggr]\cdot t^j\diamond[\eta,\zeta]_{j+n} = \sum_{j\geqslant 0}(1-1)^j\cdot t^j\diamond[\eta,\zeta]_{j+n} = [\eta,\zeta]_n,
\end{aligned}$$

using (3.28) in the first and second steps, equation (3.18) and changing variables from k to $j = k + l$ in the fourth, equation (3.15) in the fifth, and the binomial theorem in the sixth. So in fact (3.28) has a hidden symmetry, and is self-inverse.

(c) When $k = l = 0$, equation (3.29) becomes

$$[[\zeta,\eta]_0,\theta]_0 - [\zeta,[\eta,\theta]_0]_0 + (-1)^{\tilde a\tilde b}[\eta,[\zeta,\theta]_0]_0 = 0. \tag{3.30}$$

This will give the graded Jacobi identity for the '$t = 0$' Lie algebra in §3.3.

With a change in notation, Theorem 3.12 says that $H_*(\mathcal{M})$ is a *graded vertex algebra*, in the sense of §2.2.1.

Theorem 3.14. *In the situation of Definition* 3.10, *for all $a \in \frac{1}{2}\mathbb{Z}$ define*

$$\check H_a(\mathcal{M}) = \tilde H_{2a-2}(\mathcal{M}) = \textstyle\bigoplus_{\alpha\in K(\mathcal{A})} H_{2a-\chi(\alpha,\alpha)}(\mathcal{M}_\alpha), \tag{3.31}$$

so that $\check H_*(\mathcal{M}) = \bigoplus_{a\in\frac{1}{2}\mathbb{Z}}\check H_a(\mathcal{M})$ is just $\tilde H_*(\mathcal{M})$ or $H_*(\mathcal{M})$ with an alternative grading over $\frac{1}{2}\mathbb{Z}$. Define $D^{(n)} : \check H_*(\mathcal{M}) \to \check H_*(\mathcal{M})$ for $n = 0, 1, 2, \ldots$ by $D^{(n)}(v) = t^n \diamond v$. Define $u_n : \check H_*(\mathcal{M}) \to \check H_*(\mathcal{M})$ for all $u \in \check H_*(\mathcal{M})$ and $n \in \mathbb{Z}$ by $u_n(v) = [u, v]_n$. Let $\mathbb{1} \in \check H_0(\mathcal{M})$ be as in Definition 3.10. *These make $\check H_*(\mathcal{M})$ into a* ***graded vertex algebra*** *over R, in the sense of Definition* 2.8.

Proof. Clearly $D^{(n)}, u_n, \mathbb{1}$ have the R-linearity and grading properties required. For Definition 2.8(i), if $u \in H_a(\mathcal{M}_\alpha)$ and $v \in H_b(\mathcal{M}_\beta)$ then $u_n(v) = 0$ if $n > \frac{1}{2}(a+b) - \chi(\alpha,\beta) - 1$ by (3.21). Definition 2.8(ii)–(v) follow from equations (3.25), (3.28) and (3.29). □

Remark 3.15. (a) It seems very likely that Theorem 3.14 has an explanation in Mathematical Physics and String Theory, at least when $\mathcal{A}$ or $\mathcal{T}$ is of physical interest, e.g. if $\mathcal{T} = D^b\mathrm{coh}(X)$ for X a Calabi–Yau $2n$-fold.

(b) As in Definition 2.9, *vertex operator (super)algebras* are a class of graded vertex algebras of particular importance in Conformal Field Theory, so it seems natural to ask whether our examples in Theorem 3.14 have this structure.

For all our examples from abelian categories $\mathcal{A}$ with $\mathcal{A} \neq 0$, the answer to this is no. The conformal vector ω must live in $H_4(\mathcal{M}_0)$, but in the abelian case $\mathcal{M}_0 = *$ and $H_4(\mathcal{M}_0) = 0$, so $\omega = 0$, contradicting Definition 2.9(ii),(iii).

In the triangulated category case, conformal vectors $\omega \in H_4(\mathcal{M}_0)$ exist in many of our examples, but the author does not know a good geometric explanation for where they come from in our context.

(c) Because of Theorem 3.14, any standard construction for graded vertex algebras, such as Definition 2.11, can be applied in our situation.

Remark 3.16. In the situation of Definition 3.10, motivated by Theorem 3.14 and the definition of fields $Y(u,z)$ for a vertex algebra in (2.9), for each ζ in $\tilde H_*(\mathcal{M})$ we define an R-linear map

$$Y(\zeta,z): \tilde H_*(\mathcal{M}) \longrightarrow \tilde H_*(\mathcal{M})[[z,z^{-1}]],\;\; Y(\zeta,z):\eta \longmapsto \sum_{n\in\mathbb{Z}}[\zeta,\eta]_n z^{-n-1}, \tag{3.32}$$

where z is a formal variable of degree -2. If $\zeta \in \tilde H_{\tilde a}(\mathcal{M})$ then $Y(\zeta,z)$ is graded of degree $\tilde a + 2$. From (3.22) we see that if $\zeta \in H_a(\mathcal{M}_\alpha)$ and $\eta \in H_b(\mathcal{M}_\beta)$ then

$$\begin{aligned} Y(\zeta,z)\eta = \textstyle\sum_{n\in\mathbb{Z}}[\zeta,\eta]_n z^{-n-1} = \epsilon_{\alpha,\beta}(-1)^{a\chi(\beta,\beta)} z^{\chi(\alpha,\beta)}\cdot\\ H_*(\Xi_{\alpha,\beta})\bigl\{\bigl(\textstyle\sum_{i\geqslant 0} z^i t^i\bigr)\boxtimes\bigl[(\zeta\boxtimes\eta)\cap\bigl(\textstyle\sum_{j\geqslant 0} z^{-j}c_j([\Theta^\bullet_{\alpha,\beta}])\bigr)\bigr]\bigr\}. \end{aligned}\tag{3.33}$$

Now suppose R is a $\mathbb{Q}$-algebra. Then as in §2.4.2 we may transform from Chern classes $c_i(-)$ to Chern characters $\mathrm{ch}_i(-)$. Equations (2.45) and (3.33) give

$$\begin{aligned} &Y(\zeta,z)\eta = \textstyle\sum_{n\in\mathbb{Z}}[\zeta,\eta]_n z^{-n-1} = \epsilon_{\alpha,\beta}(-1)^{a\chi(\beta,\beta)} z^{\chi(\alpha,\beta)}\cdot\\ &H_*(\Xi_{\alpha,\beta})\bigl\{\bigl(\sum_{i\geqslant 0} z^i t^i\bigr)\boxtimes\bigl[(\zeta\boxtimes\eta)\cap\exp\bigl(\sum_{j\geqslant 1}(-1)^{j-1}(j-1)!z^{-j}\,\mathrm{ch}_j([\Theta^\bullet_{\alpha,\beta}])\bigr)\bigr]\bigr\}. \end{aligned}\tag{3.34}$$

Remark 3.17 (Commutative vertex algebras and the odd Calabi–Yau case)**.** In Definition 3.10 suppose the complex $\Theta^\bullet$ in Assumption 3.1(i) has $[\Theta^\bullet] = 0$ in $K_0(\mathrm{Perf}(\mathcal{M}\times\mathcal{M}))$. For instance we could take $\Theta^\bullet = 0$, which trivially satisfies Assumption 3.1(i)–(l). Then $\chi(\alpha,\beta) = 0$ and $c_i([\Theta^\bullet_{\alpha,\beta}]) = 0$ for all $\alpha,\beta \in K(\mathcal{A})$ and $i \geqslant 1$, so from (3.22) we see that $[\zeta,\eta]_n = 0$ for all $\zeta,\eta \in \tilde H_*(\mathcal{M})$ and $n \geqslant 0$. Hence in Theorem 3.14 we have $u_n(v) = 0$ for all $u,v \in \check H_*(\mathcal{M})$ and $n \geqslant 0$. That is, $\check H_*(\mathcal{M})$ is a *commutative* graded vertex algebra, as in Definition 2.12.

As in Remark 3.3(B), our standard definition of $\Theta^\bullet$ for categories $\mathcal{A},\mathcal{T}$ which are not $2n$-Calabi–Yau is to set $\Theta^\bullet = (\mathcal{E}xt^\bullet)^\vee \oplus \sigma^*(\mathcal{E}xt^\bullet)[2n]$. Now if $\mathcal{A},\mathcal{T}$ are $(2n+1)$-Calabi–Yau then $(\mathcal{E}xt^\bullet)^\vee \cong \sigma^*(\mathcal{E}xt^\bullet)[2n+1]$, so $[\Theta^\bullet] = 0$ in $K_0(\mathrm{Perf}(\mathcal{M}\times\mathcal{M}))$. Thus, under our standard construction, odd Calabi–Yau categories $\mathcal{A},\mathcal{T}$ yield commutative graded vertex algebras, and abelian graded Lie algebras in §3.3–§3.8, which are basically trivial and boring.

The next proposition will be proved in §4.3, and will be used to prove the graded Jacobi identity in the 'positive rank' version of §3.5. The proof uses only equations (3.15), (3.18), (3.26), (3.28) and (3.29), and so is valid in any graded vertex (Lie) algebra, with the appropriate notation changes as in Theorem 3.14.

Proposition 3.18. *In the situation of Definition* 3.10, *let* $\alpha,\beta,\gamma\in K(\mathcal{A})$, *and* $\zeta\in\tilde H_{\tilde a}(\mathcal{M}_\alpha)$, $\eta\in\tilde H_{\tilde b}(\mathcal{M}_\beta)$, $\theta\in\tilde H_{\tilde c}(\mathcal{M}_\gamma)$ *for* $\tilde a,\tilde b,\tilde c\in\mathbb{Z}$. *Then for all* $l\in\mathbb{Z}$ *and all* x,y,z *with* $x+y+z=1$ *we have*

$$
\begin{aligned}
&\sum_{\substack{m,n\geqslant 0:\ m+n\geqslant l,\ 2(m+n)\leqslant\\ \tilde a+\tilde b+\tilde c+2-\chi(\alpha+\beta+\gamma,\alpha+\beta+\gamma)}}(-1)^{\tilde c\tilde a+m+n}\frac{(m+n-l)!}{m!n!}x^m(x+y)^n\cdot t^{m+n-l}\diamond[[\zeta,\eta]_m,\theta]_n+\\
&\sum_{\substack{m,n\geqslant 0:\ m+n\geqslant l,\ 2(m+n)\leqslant\\ \tilde a+\tilde b+\tilde c+2-\chi(\alpha+\beta+\gamma,\alpha+\beta+\gamma)}}(-1)^{\tilde a\tilde b+m+n}\frac{(m+n-l)!}{m!n!}y^m(y+z)^n\cdot t^{m+n-l}\diamond[[\eta,\theta]_m,\zeta]_n+\\
&\sum_{\substack{m,n\geqslant 0:\ m+n\geqslant l,\ 2(m+n)\leqslant\\ \tilde a+\tilde b+\tilde c+2-\chi(\alpha+\beta+\gamma,\alpha+\beta+\gamma)}}(-1)^{\tilde b\tilde c+m+n}\frac{(m+n-l)!}{m!n!}z^m(z+x)^n\cdot t^{m+n-l}\diamond[[\theta,\zeta]_m,\eta]_n=0.
\end{aligned}
\tag{3.35}
$$

Here we can interpret (3.35) *as an equation in* $\tilde H_{\tilde a+\tilde b+\tilde c}(\mathcal{M}_{\alpha+\beta+\gamma})$ *which holds for all* $x,y,z\in R$ *with* $x+y+z=1$. *Alternatively we can take* x,y,z *to be formal variables, and interpret* (3.35) *as an equation in*

$$\tilde H_{\tilde a+\tilde b+\tilde c}(\mathcal{M}_{\alpha+\beta+\gamma})\otimes_R\big(R(x,y,z)/(x+y+z-1)\big),$$

where $R(x,y,z)$ *is the ring of polynomials in* x,y,z *over* R, *and* $(x+y+z-1)\subset R(x,y,z)$ *is the ideal generated by* $x+y+z-1$. *The analogue holds in the triangulated category case, replacing* $K(\mathcal{A})$ *by* $K(\mathcal{T})$.

3.3 The '$t=0$' version

Here is the first version of our Lie algebra construction, which we call the '$t=0$' version, as it involves quotienting $H_*(\mathcal{M})$ by the action of t in $H_*([*/\mathbb{G}_m])=R[t]$, in effect setting $t=0$. This is an example of a well known construction for vertex algebras, given in Definition 2.11, applied to the vertex algebra of Theorem 3.14. We give a geometric interpretation of $H_*(\mathcal{M})^{t=0}$ as the homology $H_*(\mathcal{M}^{\rm pl})$ of a stack $\mathcal{M}^{\rm pl}$ in the 'projective linear' version of §3.4.

Definition 3.19. Let Assumption 2.30 hold for (co)homology theories $H_i,H^i:\mathrm{Ho}(\mathbf{Art}_{\mathbb{K}}^{\rm lft})\to R$-mod of Artin $\mathbb{K}$-stacks over an arbitrary commutative ring R. Let Assumption 3.1 hold for the abelian category $\mathcal{A}$, and use the notation of §2 and §3.1–§3.2.

Definition 3.7 makes $H_*([*/\mathbb{G}_m])=R[t]$ into a commutative R-algebra, with multiplication $\star$. Write

$$I_t=\langle t,t^2,t^3,\ldots\rangle_R$$

for the ideal in $H_*([*/\mathbb{G}_m])$ spanned over R by all positive powers of t. If R is a $\mathbb{Q}$-algebra then I_t is the ideal generated by t. For each $\alpha\in K(\mathcal{A})$, define

$$H_*(\mathcal{M}_\alpha)^{t=0}=H_*(\mathcal{M}_\alpha)/(I_t\diamond H_*(\mathcal{M}_\alpha)), \tag{3.36}$$

as a quotient graded R-module, where $\diamond$ is the $H_*([*/\mathbb{G}_m])$-action from Definition 3.9. Equivalently, we have

$$H_a(\mathcal{M}_\alpha)^{t=0}=H_a(\mathcal{M}_\alpha)/\bigl(\textstyle\sum_{1\leqslant i\leqslant a/2}t^i\diamond H_{a-2i}(\mathcal{M}_\alpha)\bigr).$$

Similarly set $H_*(\mathcal{M})^{t=0}=H_*(\mathcal{M})/(I_t\diamond H_*(\mathcal{M}))$, so that as in (3.19) we have

$$H_*(\mathcal{M})^{t=0}\cong\textstyle\bigoplus_{\alpha\in K(\mathcal{A})}H_*(\mathcal{M}_\alpha)^{t=0}. \tag{3.37}$$

We will write $\Pi:H_*(\mathcal{M}_\alpha)\to H_*(\mathcal{M}_\alpha)^{t=0}$ and $\Pi:H_*(\mathcal{M})\to H_*(\mathcal{M})^{t=0}$ for the projections. We will say that an equation in $H_*(\mathcal{M}_\alpha)$ 'holds modulo I_t' if it holds up to addition of an element of $I_t\diamond H_*(\mathcal{M}_\alpha)$, that is, if the image under Π of the equation holds in $H_*(\mathcal{M}_\alpha)^{t=0}$.

Reducing equations (3.26)–(3.28) of Theorem 3.12 with $n=0$ and $p>0$ and (3.30) modulo I_t shows that for all $\alpha,\beta,\gamma\in K(\mathcal{A})$ and $\zeta\in\tilde H_{\tilde a}(\mathcal{M}_\alpha)$, $\eta\in\tilde H_{\tilde b}(\mathcal{M}_\beta)$, $\theta\in\tilde H_{\tilde c}(\mathcal{M}_\gamma)$ with $\tilde a,\tilde b,\tilde c\in\mathbb{Z}$, we have

$$[t^p\diamond\zeta,\eta]_0=0\qquad\mod I_t\quad\text{if }p>0, \tag{3.38}$$

$$[\zeta,t^p\diamond\eta]_0=0\qquad\mod I_t\quad\text{if }p>0, \tag{3.39}$$

$$[\eta,\zeta]_0=(-1)^{1+\tilde a\tilde b}[\zeta,\eta]_0\qquad\mod I_t, \tag{3.40}$$

$$[[\zeta,\eta]_0,\theta]_0-[\zeta,[\eta,\theta]_0]_0+(-1)^{\tilde a\tilde b}[\eta,[\zeta,\theta]_0]_0=0\quad\mod I_t. \tag{3.41}$$

For $\alpha,\beta\in K(\mathcal{A})$, define an R-bilinear map

$$\begin{aligned}&[\,,\,]^{t=0}:H_a(\mathcal{M}_\alpha)^{t=0}\times H_b(\mathcal{M}_\beta)^{t=0}\longrightarrow H_{a+b-2-2\chi(\alpha,\beta)}(\mathcal{M}_{\alpha+\beta})^{t=0}\\ &\text{by}\qquad\bigl[\zeta+(I_t\diamond H_*(\mathcal{M}_\alpha))_a,\eta+(I_t\diamond H_*(\mathcal{M}_\beta))_b\bigr]^{t=0}\\ &\qquad\qquad=\Pi([\zeta,\eta]_0)=[\zeta,\eta]_0+(I_t\diamond H_*(\mathcal{M}_{\alpha+\beta}))_{a+b-2-2\chi(\alpha,\beta)},\end{aligned} \tag{3.42}$$

where $(I_t\diamond H_*(\mathcal{M}_\alpha))_a$ means the a^{th} graded piece of $I_t\diamond H_*(\mathcal{M}_\alpha)\subseteq H_*(\mathcal{M}_\alpha)$. Equations (3.38)–(3.39) imply that the last line of (3.42) is independent of the choices of representatives ζ,η for the equivalence classes $\zeta+(I_t\diamond H_*(\mathcal{M}_\alpha))_a$ and $\eta+(I_t\diamond H_*(\mathcal{M}_\beta))_b$, and so $[\,,\,]^{t=0}$ is well defined.

As in (3.23), we define an alternative grading on $H_*(\mathcal{M}_\alpha)^{t=0}$ by

$$\tilde H_i(\mathcal{M}_\alpha)^{t=0}=H_{i+2-\chi(\alpha,\alpha)}(\mathcal{M}_\alpha)^{t=0}. \tag{3.43}$$

Then as in (3.24), we see that $[\,,\,]^{t=0}$ in (3.42) maps

$$[\,,\,]^{t=0}:\tilde H_{\tilde a}(\mathcal{M}_\alpha)^{t=0}\times\tilde H_{\tilde b}(\mathcal{M}_\beta)^{t=0}\longrightarrow\tilde H_{\tilde a+\tilde b}(\mathcal{M}_{\alpha+\beta})^{t=0}, \tag{3.44}$$

where $\tilde a=a-2+\chi(\alpha,\alpha)$ and $\tilde b=b-2+\chi(\beta,\beta)$. Thus, $[\,,\,]^{t=0}$ preserves gradings on $\tilde H_*(\mathcal{M}_\alpha),\tilde H_*(\mathcal{M}_\beta),\tilde H_*(\mathcal{M}_{\alpha+\beta})$.

Define $[\,,\,]^{t=0} : H_*(\mathcal{M})^{t=0} \times H_*(\mathcal{M})^{t=0} \to H_*(\mathcal{M})^{t=0}$ to be the R-bilinear map which is identified with $[\,,\,]^{t=0}$ in (3.42) on each component $H_a(\mathcal{M}_\alpha)^{t=0} \times H_b(\mathcal{M}_\beta)^{t=0}$ under the canonical isomorphism (3.37).

Write $\tilde H_i(\mathcal{M})^{t=0}$ for $i \in \mathbb{Z}$ for the subspace of $\bigoplus_{j\geqslant 0} H_j(\mathcal{M})^{t=0}$ corresponding to $\bigoplus_{\alpha\in K(\mathcal{A})} \tilde H_i(\mathcal{M}_\alpha)^{t=0} = \bigoplus_{\alpha\in K(\mathcal{A})} H_{i+2-\chi(\alpha,\alpha)}(\mathcal{M}_\alpha)^{t=0}$ under the isomorphism (3.37), using the alternative gradings (3.43). Then $\tilde H_*(\mathcal{M})^{t=0}$ is just $H_*(\mathcal{M})^{t=0}$ with a different grading, and (3.44) implies that $[\,,\,]^{t=0}$ preserves the grading on $\tilde H_*(\mathcal{M})^{t=0}$.

We can also generalize the above to triangulated categories. Suppose instead that Assumption 2.30 holds for (co)homology theories $H_i, H^i : \mathrm{Ho}(\mathbf{HSt}_{\mathbb{K}}^{\mathrm{lft}}) \to R$-mod of *higher* Artin $\mathbb{K}$-stacks over R, and Assumption 3.2 holds for the triangulated category $\mathcal{T}$. We replace $K(\mathcal{A})$ by $K(\mathcal{T})$ throughout, but otherwise the definition of $[\,,\,]^{t=0}$ on $\tilde H_*(\mathcal{M})^{t=0}$ works without change.

The next important theorem follows immediately from (3.40)–(3.41), which came from (3.28)–(3.29) in Theorem 3.12. Equation (3.45) is *graded antisymmetry* for the graded Lie bracket $[\,,\,]^{t=0}$, and (3.46) is the *graded Jacobi identity.*

Theorem 3.20. *In Definition* 3.19, *if* $\zeta \in \tilde H_{\tilde a}(\mathcal{M})^{t=0}$, $\eta \in \tilde H_{\tilde b}(\mathcal{M})^{t=0}$ *and* $\theta \in \tilde H_{\tilde c}(\mathcal{M})^{t=0}$ *then*

$$[\eta,\zeta]^{t=0} = (-1)^{\tilde a\tilde b+1}[\zeta,\eta]^{t=0}, \tag{3.45}$$

$$[[\zeta,\eta]^{t=0},\theta]^{t=0} - [\zeta,[\eta,\theta]^{t=0}]^{t=0} + (-1)^{\tilde a\tilde b}[\eta,[\zeta,\theta]^{t=0}]^{t=0} = 0, \tag{3.46}$$

in both the abelian category and triangulated category cases. That is, $[\,,\,]^{t=0}$ *is a* ***graded (super) Lie bracket*** *on* $\tilde H_*(\mathcal{M})^{t=0}$, *making* $\tilde H_*(\mathcal{M})^{t=0}$ *into a* ***graded Lie algebra*** *(sometimes called a* ***graded Lie superalgebra****).*

Hence $\big(\tilde H_0(\mathcal{M})^{t=0}, [\,,\,]^{t=0}\big)$ *is an ordinary Lie algebra over R.*

Remark 3.21. (a) As in Remark 3.13(a), we can replace $\mathcal{M}$ by $\mathcal{M}' = \mathcal{M}\setminus\{[0]\}$ throughout §3.3, and so define a graded Lie bracket $[\,,\,]^{t=0}$ on $\tilde H_*(\mathcal{M}')^{t=0}$.

(b) Observe that by (3.43) we have

$$\tilde H_0(\mathcal{M})^{t=0} = \bigoplus\nolimits_{\alpha\in K(\mathcal{A}):\chi(\alpha,\alpha)\leqslant 2} H_{2-\chi(\alpha,\alpha)}(\mathcal{M}_\alpha)^{t=0}.$$

If $K(\mathcal{A})$ is of finite rank and $\chi(\,,\,)$ is positive definite, there will be only finitely many classes $\alpha \in K(\mathcal{A})$ with $\chi(\alpha,\alpha) \leqslant 2$, so the Lie algebra $\tilde H_0(\mathcal{M})^{t=0}$ may be finite-dimensional. This happens for representations of ADE quivers.

3.4 The 'projective linear' version

The '$t = 0$' version in §3.3 has the disadvantage that $H_*(\mathcal{M})^{t=0}$ is not presented as the homology of an interesting space, but as a quotient of the homology $H_*(\mathcal{M})$. The 'projective linear' version of our construction remedies this, by interpreting $H_*(\mathcal{M}')^{t=0}$ as the homology $H_*(\mathcal{M}^{\mathrm{pl}})$ of a modified version $\mathcal{M}^{\mathrm{pl}}$ of the moduli stack $\mathcal{M}'$ of nonzero objects in $\mathcal{A}$ or $\mathcal{T}$. We must assume the coefficient ring R is a $\mathbb{Q}$-algebra to prove the isomorphism $H_*(\mathcal{M}')^{t=0} \cong H_*(\mathcal{M}^{\mathrm{pl}})$.

Definition 3.22. Let Assumption 3.1 hold for the abelian category $\mathcal{A}$. Then Assumption 3.1(h) says that $\Psi' = \Psi|_{[*/\mathbb{G}_m]\times\mathcal{M}'} : [*/\mathbb{G}_m] \times \mathcal{M}' \to \mathcal{M}'$ is a free $[*/\mathbb{G}_m]$-action on $\mathcal{M}' = \mathcal{M} \setminus \{[0]\}$, as in Definition 2.23. Write $\Pi^{\mathrm{pl}} : \mathcal{M}' \to \mathcal{M}^{\mathrm{pl}}$ for the principal $[*/\mathbb{G}_m]$-bundle with $[*/\mathbb{G}_m]$-action Ψ' given by Proposition 2.25(a). Then $\mathcal{M}^{\mathrm{pl}}$ is an Artin $\mathbb{K}$-stack, locally of finite type.

We regard $\mathcal{M}^{\mathrm{pl}}$ as the moduli stack of all nonzero objects in $\mathcal{A}$ 'up to projective linear isomorphisms'. Since $\Pi^{\mathrm{pl}} : \mathcal{M}' \to \mathcal{M}^{\mathrm{pl}}$ is a $[*/\mathbb{G}_m]$-bundle it is an isomorphism on $\mathbb{K}$-points. Thus, $\mathbb{K}$-points $x \in \mathcal{M}^{\mathrm{pl}}(\mathbb{K})$ correspond naturally to isomorphism classes $[E]$ of nonzero objects $E \in \mathcal{A}$, as for $\mathcal{M}'(\mathbb{K})$, and we will write points of $\mathcal{M}^{\mathrm{pl}}(\mathbb{K})$ as $[E]$, and then $\Pi^{\mathrm{pl}}(\mathbb{K})$ maps $[E] \mapsto [E]$.

The isotropy groups of $\mathcal{M}^{\mathrm{pl}}$ satisfy $\mathrm{Iso}_{\mathcal{M}^{\mathrm{pl}}}([E]) \cong \mathrm{Iso}_{\mathcal{M}'}([E])/\mathbb{G}_m$, where the $\mathbb{G}_m$-subgroup of $\mathrm{Iso}_{\mathcal{M}'}([E])$ is determined by the action of Ψ' on isotropy groups. Thus by Assumption 3.1(e),(h) we see that

$$\mathrm{Iso}_{\mathcal{M}^{\mathrm{pl}}}([E]) \cong \mathrm{Aut}(E)/(\mathbb{G}_m \cdot \mathrm{id}_E). \tag{3.47}$$

The action of Π^{pl} on isotropy groups is given by the commutative diagram

$$\begin{tikzcd}
\mathrm{Iso}_{\mathcal{M}'}([E]) \arrow[r, "\Pi^{\mathrm{pl}}_*"'] \arrow[d, "\cong"] & \mathrm{Iso}_{\mathcal{M}^{\mathrm{pl}}}([E]) \arrow[d, "\cong", "(3.47)"'] \\
\mathrm{Aut}(E) \arrow[r, "\epsilon \longmapsto \epsilon\mathbb{G}_m"] & \mathrm{Aut}(E)/(\mathbb{G}_m \cdot \mathrm{id}_E).
\end{tikzcd}$$

In Assumption 3.1(e) we explained that if S is a $\mathbb{K}$-scheme, a stack morphism $e : S \to \mathcal{M}$ should be heuristically interpreted as a 'family of objects $\mathcal{E}$ in $\mathcal{A}$ over the base $\mathbb{K}$-scheme S'. But for an Artin $\mathbb{K}$-stack X, $\mathbf{Hom}(S, X)$ is a *groupoid*, with objects 1-morphisms $e, f : S \to X$ and (iso)morphisms 2-morphisms $\lambda : f \Rightarrow g$. Thus, to fully describe X we should specify both objects and morphisms in $\mathbf{Hom}(S, X)$. For $\mathcal{M}', \mathcal{M}^{\mathrm{pl}}$ we have:

(i) Objects of $\mathbf{Hom}(S, \mathcal{M}')$ (that is, 1-morphisms $e : S \to \mathcal{M}'$) correspond to 'families of nonzero objects $\mathcal{E}$ in $\mathcal{A}$ over the base $\mathbb{K}$-scheme S'. Morphisms of $\mathbf{Hom}(S, \mathcal{M}')$ (that is, 2-morphisms $\lambda : e \Rightarrow f$ of 1-morphisms $e, f : S \to \mathcal{M}'$) correspond to isomorphisms $\Lambda : \mathcal{E} \to \mathcal{F}$ of such families.

(ii) Objects of $\mathbf{Hom}(S, \mathcal{M}^{\mathrm{pl}})$ correspond to 'families of nonzero objects $\mathcal{E}$ in $\mathcal{A}$ over the base $\mathbb{K}$-scheme S', as for $\mathbf{Hom}(S, \mathcal{M}')$. But morphisms of $\mathbf{Hom}(S, \mathcal{M}^{\mathrm{pl}})$ correspond to equivalence classes $[L, \Lambda]$ of pairs (L, Λ), where $L \to S$ is a line bundle and $\Lambda : \mathcal{E} \to \pi_S^*(L) \otimes \mathcal{F}$ is an isomorphism of families. Two such pairs $(L, \Lambda), (L', \Lambda')$ are equivalent if there exists an isomorphism of line bundles $\iota : L \to L'$ with $\Lambda' = (\pi_S^*(\iota) \otimes \mathrm{id}_{\mathcal{F}}) \circ \Lambda$.

We use *'projective linear'*, and *'up to projective linear isomorphisms'*, to mean the use of isomorphisms up to $\mathbb{G}_m$ rescalings as in (3.47), or up to tensor product with a line bundle as in (ii). For example, if $\mathrm{Iso}_{\mathcal{M}'}(E) \cong \mathrm{GL}(n, \mathbb{K})$ then $\mathrm{Iso}_{\mathcal{M}^{\mathrm{pl}}}(E) \cong \mathrm{PGL}(n, \mathbb{K})$ is the corresponding projective linear group by (3.47).

As in Assumption 3.1(f), for $\alpha \in K(\mathcal{A})$ write $\mathcal{M}^{\mathrm{pl}}_\alpha \subset \mathcal{M}^{\mathrm{pl}}$ for the open and closed $\mathbb{K}$-substack of points $[E] \in \mathcal{M}^{\mathrm{pl}}(\mathbb{K})$ with $[\![E]\!] = \alpha$ in $K(\mathcal{A})$. Then

$\mathcal{M}^{\rm pl}=\coprod_{\alpha\in K(\mathcal{A})}\mathcal{M}_\alpha^{\rm pl}$, and we write $\Pi_\alpha^{\rm pl}=\Pi^{\rm pl}|_{\mathcal{M}'_\alpha}:\mathcal{M}'_\alpha\to\mathcal{M}_\alpha^{\rm pl}$, for $\mathcal{M}'_\alpha$ as in Assumption 3.1(f).

Since $\Pi^{\rm pl}$ and $\Pi_\alpha^{\rm pl}$ are principal $[*/\mathbb{G}_m]$-bundles, as in (2.23) and Proposition 2.25(a), we have a diagram which is commutative $\mathrm{Ho}(\mathbf{Art}_{\mathbb{K}}^{\rm lft})$, and 2-Cartesian and 2-co-Cartesian in $\mathbf{Art}_{\mathbb{K}}^{\rm lft}$:

$$\begin{tikzcd}[*/\mathbb{G}_m]\times\mathcal{M}'_\alpha \arrow[r, "\Psi'_\alpha"'] \arrow[d, "\pi_{\mathcal{M}'_\alpha}"] & \mathcal{M}'_\alpha \arrow[d, "\Pi_\alpha^{\rm pl}"'] \\ \mathcal{M}'_\alpha \arrow[r, "\Pi_\alpha^{\rm pl}"] & \mathcal{M}_\alpha^{\rm pl}.\end{tikzcd} \tag{3.48}$$

Now suppose also that Assumption 2.30 holds for a homology theory $H_*(-)$ of Artin $\mathbb{K}$-stacks over a commutative ring R, and use the notation of §3.3. Then for $\alpha\in K(\mathcal{A})$ and $a=0,1,\ldots$ we can consider the sequence

$$0\longrightarrow(I_t\diamond H_*(\mathcal{M}'_\alpha))_a\xhookrightarrow{\rm inc}H_a(\mathcal{M}'_\alpha)\xrightarrow{H_a(\Pi_\alpha^{\rm pl})}H_a(\mathcal{M}_\alpha^{\rm pl})\longrightarrow 0. \tag{3.49}$$

If $p>0$ and $\zeta\in H_{a-2p}(\mathcal{M}'_\alpha)$ then

$$H_a(\Pi_\alpha^{\rm pl})(t^p\diamond\zeta)=H_a(\Pi_\alpha^{\rm pl})\circ H_a(\Psi'_\alpha)(t^p\boxtimes\zeta)=H_a(\Pi_\alpha^{\rm pl})\circ H_a(\pi_{\mathcal{M}'_\alpha})(t^p\boxtimes\zeta)=0,$$

using (3.17) in the first step, commutativity of (3.48) in $\mathrm{Ho}(\mathbf{Art}_{\mathbb{K}}^{\rm lft})$ and functoriality of $H_a(-)$ in the second, and $H_a(\pi_{\mathcal{M}'_\alpha})(t^p\boxtimes\zeta)=0$ for $p>0$ in the third. Hence $H_a(\Pi_\alpha^{\rm pl})\circ\mathrm{inc}=0$ in (3.49), that is, (3.49) is a complex of R-modules. The analogue of (3.49) with $\mathcal{M}',\mathcal{M}^{\rm pl}$ in place of $\mathcal{M}'_\alpha,\mathcal{M}_\alpha^{\rm pl}$ is also a complex. Thus by (3.36) there are unique R-module morphisms

$$\Pi_{t=0}^{\rm pl}:H_*(\mathcal{M}'_\alpha)^{t=0}\longrightarrow H_*(\mathcal{M}_\alpha^{\rm pl}),\quad \Pi_{t=0}^{\rm pl}:H_*(\mathcal{M}')^{t=0}\longrightarrow H_*(\mathcal{M}^{\rm pl}), \tag{3.50}$$

such that $\Pi_{t=0}^{\rm pl}\circ\Pi=H_*(\Pi_\alpha^{\rm pl})$ or $\Pi_{t=0}^{\rm pl}\circ\Pi=H_*(\Pi^{\rm pl})$. These are isomorphisms if and only if (3.49) and its analogue for $\mathcal{M}'$ are exact.

We can also extend all the above to triangulated categories, supposing Assumption 3.2 instead of Assumption 3.1 and replacing $\mathcal{A},K(\mathcal{A})$ by $\mathcal{T},K(\mathcal{T})$ throughout, and taking $\mathcal{M},\mathcal{M}^{\rm pl}$ to be higher $\mathbb{K}$-stacks.

Remark 3.23. We chose to define $\mathcal{M}^{\rm pl}$ as a $[*/\mathbb{G}_m]$-quotient of $\mathcal{M}'=\mathcal{M}\setminus\{[0]\}$, deleting the point $[0]$ in $\mathcal{M}$, as the $[*/\mathbb{G}_m]$-action Ψ on $\mathcal{M}$ is not free over $[0]$, and we needed a free $[*/\mathbb{G}_m]$-action to apply Proposition 2.25(a). This seems to be the most natural thing to do in the abelian category version.

Now for higher stacks, Proposition 2.25(b) gives $[*/\mathbb{G}_m]$-quotients for non-free $[*/\mathbb{G}_m]$-actions. Thus in the triangulated category version, we could instead define $\mathcal{M}^{\rm pl}=\mathcal{M}/[*/\mathbb{G}_m]$ as a higher stack. But as in Proposition 3.6(b), this would not change the homology $H_*(\mathcal{M}^{\rm pl})$, so it makes little difference to us.

The next proposition is proved in §4.4. Part (a) gives a sufficient condition for $H_*(\mathcal{M}'_\alpha)^{t=0}$ to be isomorphic to $H_*(\mathcal{M}_\alpha^{\rm pl})$, at least when R is a $\mathbb{Q}$-algebra. Parts (b),(c) show that this condition often holds automatically.

Proposition 3.24. **(a)** *Let Assumptions* 2.30 *and* 2.31 *hold for (co)homology theories* $H_*(-), H^*(-)$ *over a* $\mathbb{Q}$*-algebra* R*, and Assumption* 3.1 *hold for the abelian category* $\mathcal{A}$*. Suppose that for some* $\alpha \in K(\mathcal{A})$*, the principal* $[*/\mathbb{G}_m]$*-bundle* $\Pi_\alpha^{\mathrm{pl}} : \mathcal{M}'_\alpha \to \mathcal{M}_\alpha^{\mathrm{pl}}$ *in Definition* 3.22 *is rationally trivial, as in Definition* 2.26*. Then* $\Pi_{t=0}^{\mathrm{pl}} : H_*(\mathcal{M}'_\alpha)^{t=0} \to H_*(\mathcal{M}_\alpha^{\mathrm{pl}})$ *in* (3.50) *is an isomorphism.*

(b) *Suppose the field* $\mathbb{K}$ *is algebraically closed,* $0 \neq \alpha \in K(\mathcal{A})$*, and there exists* $0 \neq \beta \in K(\mathcal{A})$ *such that* $\chi(\alpha,\beta) \neq 0$ *and* $\mathcal{M}_\beta \neq \emptyset$*. Then* $\Pi_\alpha^{\mathrm{pl}} : \mathcal{M}'_\alpha \to \mathcal{M}_\alpha^{\mathrm{pl}}$ *is rationally trivial.*

(c) *Suppose* $\mathbb{K}$ *is algebraically closed,* $K(\mathcal{A})$ *is a free abelian group,* $\chi : K(\mathcal{A}) \times K(\mathcal{A}) \to \mathbb{Z}$ *is nondegenerate, and* $\mathcal{M}_0(\mathbb{K}) = \{[0]\}$*. Then* $\Pi^{\mathrm{pl}} : \mathcal{M}' \to \mathcal{M}^{\mathrm{pl}}$ *is rationally trivial. Hence, if Assumptions* 2.30 *and* 2.31 *hold over a* $\mathbb{Q}$*-algebra* R*, then* $\Pi_{t=0}^{\mathrm{pl}} : H_*(\mathcal{M}')^{t=0} \longrightarrow H_*(\mathcal{M}^{\mathrm{pl}})$ *in* (3.50) *is an isomorphism by* **(a)**.

Parts **(a)**,**(b)** *extend to the triangulated category case in the obvious way.*

Remark 3.25. Proposition 3.24(c) does not work in the triangulated category case. If Assumption 3.2 holds for the triangulated category $\mathcal{T}$, and $\mathcal{E}^\bullet$ is any object in $\mathcal{T}$, then $[\mathcal{E}^\bullet \oplus \mathcal{E}^\bullet[1]]$ lies in $\mathcal{M}_0(\mathbb{K})$, so we never have $\mathcal{M}_0(\mathbb{K}) = \{[0]\}$ unless $\mathcal{T} = 0$. The author expects the principal $[*/\mathbb{G}_m]$-bundle $\Pi_0^{\mathrm{pl}} : \mathcal{M}'_0 \to \mathcal{M}_0^{\mathrm{pl}}$ *not* to be rationally trivial, and $\Pi_{t=0}^{\mathrm{pl}} : H_k(\mathcal{M}'_0)^{t=0} \to H_k(\mathcal{M}_0^{\mathrm{pl}})$ *not* to be an isomorphism for $k \geqslant 3$, in almost all interesting triangulated category examples, and we illustrate this when $k = 3$ in Example 5.23 with $\mathcal{T} = D^b \operatorname{Vect}_{\mathbb{C}}$. But $\Pi_{t=0}^{\mathrm{pl}}$ is an isomorphism for $\mathbb{K} = \mathbb{C}$ when $k = 0, 1, 2$ by the next result.

The next proposition will be proved in §4.5, using the Leray–Serre spectral sequence for the homology of fibrations.

Proposition 3.26. *Work over the field* $\mathbb{K} = \mathbb{C}$*, with the (co)homology theories of (higher) Artin* $\mathbb{C}$*-stacks described in Example* 2.35*, over any commutative ring* R*. Then the morphism* $\Pi_{t=0}^{\mathrm{pl}} : H_k(\mathcal{M}')^{t=0} \to H_k(\mathcal{M}^{\mathrm{pl}})$ *in* (3.50) *is an isomorphism when* $k = 0, 1$ *or* 2*, in both abelian and triangulated category cases.*

Combining Propositions 3.24 and 3.26 yields:

Corollary 3.27. *Work over the field* $\mathbb{K} = \mathbb{C}$*, with the (co)homology theories of (higher) Artin* $\mathbb{C}$*-stacks described in Example* 2.35*, over a* $\mathbb{Q}$*-algebra* R*. Let Assumption* 3.1 *hold for* $\mathcal{A}$*, or Assumption* 3.2 *hold for* $\mathcal{T}$*. Then* $\Pi_{t=0}^{\mathrm{pl}} : \tilde{H}_0(\mathcal{M}')^{t=0} \longrightarrow \tilde{H}_0(\mathcal{M}^{\mathrm{pl}})$ *in* (3.50) *is an isomorphism.*

Proof. Let $\alpha \in K(\mathcal{A})$ or $K(\mathcal{T})$. Divide into cases (a) $\mathcal{M}_\alpha = \emptyset$, (b) $\mathcal{M}_\alpha \neq \emptyset$ and $\chi(\alpha,\alpha) \neq 0$, and (c) $\mathcal{M}_\alpha \neq \emptyset$ and $\chi(\alpha,\alpha) = 0$. Then $\Pi_{t=0}^{\mathrm{pl}} : \tilde{H}_0(\mathcal{M}'_\alpha)^{t=0} \longrightarrow \tilde{H}_0(\mathcal{M}_\alpha^{\mathrm{pl}})$ is an isomorphism in case (a) trivially, in case (b) by Proposition 3.24(b) with $\beta = \alpha$, and in case (c) by Proposition 3.26 as $\tilde{H}_0(\mathcal{M}'_\alpha)^{t=0} = H_2(\mathcal{M}'_\alpha)^{t=0}$ and $\tilde{H}_0(\mathcal{M}^{\mathrm{pl}}) = H_2(\mathcal{M}^{\mathrm{pl}})$. $\square$

If we have an isomorphism $\Pi_{t=0}^{\mathrm{pl}} : H_*(\mathcal{M}')^{t=0} \to H_*(\mathcal{M}^{\mathrm{pl}})$ as in Proposition 3.24(c) (or just on $\tilde{H}_0(\mathcal{M}^{\mathrm{pl}})$, as in Corollary 3.27), then there is a natural graded Lie bracket $[\,,\,]^{\mathrm{pl}}$ on $\tilde{H}_*(\mathcal{M}^{\mathrm{pl}})$ (or just on $\tilde{H}_0(\mathcal{M}^{\mathrm{pl}})$) identified with $[\,,\,]^{t=0}$ on

$\tilde{H}_*(\mathcal{M}')^{t=0}$ in §3.3 by $\Pi^{\rm pl}_{t=0}$. Using the 'projective Euler class' in §2.5, we will define a graded Lie bracket $[\,,\,]^{\rm pl}$ on $\tilde{H}_*(\mathcal{M}^{\rm pl})$ in the general case.

Definition 3.28. Let Assumptions 2.30, 2.39 and 3.1 hold. For $\alpha,\beta\in K(\mathcal{A})$ we have a free $[*/\mathbb{G}_m]$-action on $\mathcal{M}'_\alpha\times\mathcal{M}'_\beta$, as in §2.3.7:

$$\begin{aligned}&\bigl((\Psi'_\alpha\circ\Pi_{[*/\mathbb{G}_m]\times\mathcal{M}'_\alpha}),(\Psi'_\beta\circ\Pi_{[*/\mathbb{G}_m]\times\mathcal{M}'_\beta})\bigr):\\ &\qquad [*/\mathbb{G}_m]\times\mathcal{M}'_\alpha\times\mathcal{M}'_\beta\longrightarrow\mathcal{M}'_\alpha\times\mathcal{M}'_\beta,\end{aligned}\tag{3.51}$$

the diagonal action of the $[*/\mathbb{G}_m]$-actions Ψ'_α on $\mathcal{M}'_\alpha$ and Ψ'_β on $\mathcal{M}'_\beta$. As for $\mathcal{M}^{\rm pl}_\alpha$ in Definition 3.22, write $\Pi^{\rm pl}_{\alpha,\beta}:\mathcal{M}'_\alpha\times\mathcal{M}'_\beta\to(\mathcal{M}_\alpha\times\mathcal{M}_\beta)^{\rm pl}$ for the principal $[*/\mathbb{G}_m]$-bundle with $[*/\mathbb{G}_m]$-action (3.51) given by Proposition 2.25(a). Then $(\mathcal{M}_\alpha\times\mathcal{M}_\beta)^{\rm pl}$ is an Artin $\mathbb{K}$-stack, locally of finite type. The following commutes in $\mathrm{Ho}(\mathbf{Art}^{\rm lft}_{\mathbb{K}})$, and is 2-Cartesian and 2-co-Cartesian in $\mathbf{Art}^{\rm lft}_{\mathbb{K}}$:

$$\begin{tikzcd}[*/\mathbb{G}_m]\times\mathcal{M}'_\alpha\times\mathcal{M}'_\beta \arrow[r, "((\Psi'_\alpha\circ\Pi_{[*/\mathbb{G}_m]\times\mathcal{M}'_\alpha}),(\Psi'_\beta\circ\Pi_{[*/\mathbb{G}_m]\times\mathcal{M}'_\beta}))"'] \arrow[d, "\Pi_{\mathcal{M}'_\alpha\times\mathcal{M}'_\beta}"] & \mathcal{M}'_\alpha\times\mathcal{M}'_\beta \arrow[d, "\Pi^{\rm pl}_{\alpha,\beta}"'] \\ \mathcal{M}'_\alpha\times\mathcal{M}'_\beta \arrow[r, "\Pi^{\rm pl}_{\alpha,\beta}"] & (\mathcal{M}_\alpha\times\mathcal{M}_\beta)^{\rm pl}.\end{tikzcd}\tag{3.52}$$

The morphism $\Pi^{\rm pl}_{\alpha,\beta}$ induces a bijection on $\mathbb{K}$-points. We write elements of $(\mathcal{M}_\alpha\times\mathcal{M}_\beta)^{\rm pl}(\mathbb{K})$ as $([E],[F])$ for $[E]\in\mathcal{M}'_\alpha(\mathbb{K})$ and $[F]\in\mathcal{M}'_\beta(\mathbb{K})$, so that $\Pi^{\rm pl}_{\alpha,\beta}(\mathbb{K}):([E],[F])\mapsto([E],[F])$. As for (3.47), the isotropy groups are given by

$$\mathrm{Iso}_{(\mathcal{M}_\alpha\times\mathcal{M}_\beta)^{\rm pl}}([E],[F])\cong\bigl(\mathrm{Aut}(E)\times\mathrm{Aut}(F)\bigr)/\bigl(\mathbb{G}_m\cdot(\mathrm{id}_E,\mathrm{id}_F)\bigr).$$

By lifting the $[*/\mathbb{G}_m]$-actions to the 2-category $\mathbf{Art}^{\rm lft}_{\mathbb{K}}$ as in Remark 2.24, and using the 2-co-Cartesian property of (3.52), we can construct natural morphisms:

$$\begin{aligned}&\check{\Pi}^{\rm pl}_{\alpha,\beta}:(\mathcal{M}_\alpha\times\mathcal{M}_\beta)^{\rm pl}\longrightarrow\mathcal{M}^{\rm pl}_\alpha\times\mathcal{M}^{\rm pl}_\beta,\quad \Phi^{\rm pl}_{\alpha,\beta}:(\mathcal{M}_\alpha\times\mathcal{M}_\beta)^{\rm pl}\longrightarrow\mathcal{M}^{\rm pl}_{\alpha+\beta},\\ &\Psi^{\rm pl}_{\alpha,\beta}:[*/\mathbb{G}_m]\times(\mathcal{M}_\alpha\times\mathcal{M}_\beta)^{\rm pl}\longrightarrow(\mathcal{M}_\alpha\times\mathcal{M}_\beta)^{\rm pl},\end{aligned}\tag{3.53}$$

which are analogues of $\Pi^{\rm pl}_\alpha$ in Definition 3.22 and $\Phi_{\alpha,\beta},\Psi_\alpha$ in Assumption 3.1, such that the following diagrams 2-commute in $\mathbf{Art}^{\rm lft}_{\mathbb{K}}$:

$$\begin{tikzcd} & \mathcal{M}'_\alpha\times\mathcal{M}'_\beta \arrow[dl, "\Pi^{\rm pl}_\alpha\times\Pi^{\rm pl}_\beta"'] \arrow[r, "\Phi'_{\alpha,\beta}"'] \arrow[d, "\Pi^{\rm pl}_{\alpha,\beta}"] & \mathcal{M}'_{\alpha+\beta} \arrow[d, "\Pi^{\rm pl}_{\alpha+\beta}"'] \\ \mathcal{M}^{\rm pl}_\alpha\times\mathcal{M}^{\rm pl}_\beta & (\mathcal{M}_\alpha\times\mathcal{M}_\beta)^{\rm pl} \arrow[l, "\check{\Pi}^{\rm pl}_{\alpha,\beta}"'] \arrow[r, "\Phi^{\rm pl}_{\alpha,\beta}"] & \mathcal{M}^{\rm pl}_{\alpha+\beta},\end{tikzcd}$$

$$\begin{tikzcd}[*/\mathbb{G}_m]\times\mathcal{M}'_\alpha\times\mathcal{M}'_\beta \arrow[r, "\Psi'_\alpha\times\mathrm{id}_{\mathcal{M}'_\beta}"'] \arrow[d, "\mathrm{id}_{[*/\mathbb{G}_m]}\times\Pi^{\rm pl}_{\alpha,\beta}"] & \mathcal{M}'_\alpha\times\mathcal{M}'_\beta \arrow[d, "\Pi^{\rm pl}_{\alpha,\beta}"'] \\ [*/\mathbb{G}_m]\times(\mathcal{M}_\alpha\times\mathcal{M}_\beta)^{\rm pl} \arrow[r, "\Psi^{\rm pl}_{\alpha,\beta}"] & (\mathcal{M}_\alpha\times\mathcal{M}_\beta)^{\rm pl},\end{tikzcd}$$

$$\begin{tikzcd}
{[*/\mathbb{G}_m]\times\mathcal{M}'_\alpha\times\mathcal{M}'_\beta} \arrow[rr, "{(\Pi_{\mathcal{M}'_\alpha},\Psi'_\beta\circ(\Upsilon_{-1}\circ\Pi_{[*/\mathbb{G}_m]\times\mathcal{M}'_\beta}))}"'] \arrow[d, "\mathrm{id}_{[*/\mathbb{G}_m]}\times\Pi^{\rm pl}_{\alpha,\beta}"] && \mathcal{M}'_\alpha\times\mathcal{M}'_\beta \arrow[d, "\Pi^{\rm pl}_{\alpha,\beta}"'] \\
{[*/\mathbb{G}_m]\times(\mathcal{M}_\alpha\times\mathcal{M}_\beta)^{\rm pl}} \arrow[rr, "\Psi^{\rm pl}_{\alpha,\beta}"] && (\mathcal{M}_\alpha\times\mathcal{M}_\beta)^{\rm pl}.
\end{tikzcd}$$

Here $\Upsilon_{-1}:[*/\mathbb{G}_m]\to[*/\mathbb{G}_m]$ is induced by $\upsilon_{-1}:\mathbb{G}_m\to\mathbb{G}_m$ mapping $\upsilon_{-1}:\lambda\mapsto\lambda^{-1}$.

The morphism $\Psi^{\rm pl}_{\alpha,\beta}$ is a free $[*/\mathbb{G}_m]$-action on $(\mathcal{M}_\alpha\times\mathcal{M}_\beta)^{\rm pl}$, and makes $\check\Pi^{\rm pl}_{\alpha,\beta}:(\mathcal{M}_\alpha\times\mathcal{M}_\beta)^{\rm pl}\to\mathcal{M}^{\rm pl}_\alpha\times\mathcal{M}^{\rm pl}_\beta$ into a principal $[*/\mathbb{G}_m]$-bundle. Here $\Pi^{\rm pl}_\alpha\times\Pi^{\rm pl}_\beta:\mathcal{M}'_\alpha\times\mathcal{M}'_\beta\to\mathcal{M}^{\rm pl}_\alpha\times\mathcal{M}^{\rm pl}_\beta$ is a principal $[*/\mathbb{G}_m]^2$-bundle, and we have factorized it into two principal $[*/\mathbb{G}_m]^2$-bundles $\Pi^{\rm pl}_{\alpha,\beta}:\mathcal{M}'_\alpha\times\mathcal{M}'_\beta\to(\mathcal{M}_\alpha\times\mathcal{M}_\beta)^{\rm pl}$ and $\check\Pi^{\rm pl}_{\alpha,\beta}:(\mathcal{M}_\alpha\times\mathcal{M}_\beta)^{\rm pl}\to\mathcal{M}^{\rm pl}_\alpha\times\mathcal{M}^{\rm pl}_\beta$.

The perfect complex $\Theta^\bullet_{\alpha,\beta}|_{\mathcal{M}'_\alpha\times\mathcal{M}'_\beta}$ on $\mathcal{M}'_\alpha\times\mathcal{M}'_\beta$ has a $[*/\mathbb{G}_m]^2$-action of multi-weight $(1,-1)$ compatible with the $[*/\mathbb{G}_m]^2$-action on $\mathcal{M}'_\alpha\times\mathcal{M}'_\beta$, by Assumption 3.1(l). Hence $\Theta^\bullet_{\alpha,\beta}|_{\mathcal{M}'_\alpha\times\mathcal{M}'_\beta}$ has weight 0 for the diagonal $[*/\mathbb{G}_m]$-action (3.51), so as in §2.3.8 we have $\Theta^\bullet_{\alpha,\beta}|_{\mathcal{M}'_\alpha\times\mathcal{M}'_\beta}\cong(\Pi^{\rm pl}_{\alpha,\beta})^*(\check\Theta^\bullet_{\alpha,\beta})$ for a perfect complex $\check\Theta^\bullet_{\alpha,\beta}$ on $(\mathcal{M}_\alpha\times\mathcal{M}_\beta)^{\rm pl}$, with $\operatorname{rank}\check\Theta^\bullet_{\alpha,\beta}=\operatorname{rank}\Theta^\bullet_{\alpha,\beta}=\chi(\alpha,\beta)$. The second $[*/\mathbb{G}_m]$-action on $\Theta^\bullet_{\alpha,\beta}|_{\mathcal{M}'_\alpha\times\mathcal{M}'_\beta}$ descends to $\check\Theta^\bullet_{\alpha,\beta}$, so $\check\Theta^\bullet_{\alpha,\beta}$ has a weight one $[*/\mathbb{G}_m]$-action compatible with the $[*/\mathbb{G}_m]$-action $\Psi^{\rm pl}_{\alpha,\beta}$ on $(\mathcal{M}_\alpha\times\mathcal{M}_\beta)^{\rm pl}$.

Since $\check\Pi^{\rm pl}_{\alpha,\beta}:(\mathcal{M}_\alpha\times\mathcal{M}_\beta)^{\rm pl}\to\mathcal{M}^{\rm pl}_\alpha\times\mathcal{M}^{\rm pl}_\beta$ is a principal $[*/\mathbb{G}_m]$-bundle with $[*/\mathbb{G}_m]$-action $\Psi^{\rm pl}_{\alpha,\beta}$, Assumption 2.39 now gives projective Euler class maps

$$\mathrm{PE}([\check\Theta^\bullet_{\alpha,\beta}]):H_k\bigl(\mathcal{M}^{\rm pl}_\alpha\times\mathcal{M}^{\rm pl}_\beta\bigr)\longrightarrow H_{k-2\chi(\alpha,\beta)-2}\bigl((\mathcal{M}_\alpha\times\mathcal{M}_\beta)^{\rm pl}\bigr).$$

Define an R-bilinear map

$$[\,,\,]^{\rm pl}:H_a(\mathcal{M}^{\rm pl}_\alpha)\times H_b(\mathcal{M}^{\rm pl}_\beta)\longrightarrow H_{a+b-2\chi(\alpha,\beta)-2}(\mathcal{M}^{\rm pl}_{\alpha+\beta}) \tag{3.54}$$

by, for all $\zeta\in H_a(\mathcal{M}^{\rm pl}_\alpha)$ and $\eta\in H_b(\mathcal{M}^{\rm pl}_\beta)$,

$$[\zeta,\eta]^{\rm pl}=\epsilon_{\alpha,\beta}(-1)^{a\chi(\beta,\beta)}\cdot H_{a+b-2\chi(\alpha,\beta)-2}(\Phi^{\rm pl}_{\alpha,\beta})\circ\mathrm{PE}([\check\Theta^\bullet_{\alpha,\beta}])(\zeta\boxtimes\eta). \tag{3.55}$$

As in (3.23) and (3.43), we define an alternative grading on $H_*(\mathcal{M}^{\rm pl}_\alpha)$ by

$$\tilde H_i(\mathcal{M}^{\rm pl}_\alpha)=H_{i+2-\chi(\alpha,\alpha)}(\mathcal{M}^{\rm pl}_\alpha). \tag{3.56}$$

Then as in (3.44), $[\,,\,]^{\rm pl}$ in (3.54)–(3.55) maps

$$[\,,\,]^{\rm pl}:\tilde H_{\tilde a}(\mathcal{M}^{\rm pl}_\alpha)\times\tilde H_{\tilde b}(\mathcal{M}^{\rm pl}_\beta)\longrightarrow\tilde H_{\tilde a+\tilde b}(\mathcal{M}^{\rm pl}_{\alpha+\beta}). \tag{3.57}$$

Define $[\,,\,]^{\rm pl}:H_*(\mathcal{M}^{\rm pl})\times H_*(\mathcal{M}^{\rm pl})\to H_*(\mathcal{M})^{\rm pl}$ to be the R-bilinear map which is identified with $[\,,\,]^{\rm pl}$ in (3.54)–(3.55) on each component $H_a(\mathcal{M}^{\rm pl}_\alpha)\times H_b(\mathcal{M}^{\rm pl}_\beta)$ under the canonical isomorphism

$$H_*(\mathcal{M}^{\rm pl})\cong\textstyle\bigoplus_{\alpha\in K(\mathcal{A})}H_*(\mathcal{M}^{\rm pl}_\alpha), \tag{3.58}$$

induced by $\mathcal{M}^{\rm pl}=\coprod_{\alpha\in K(\mathcal{A})}\mathcal{M}_\alpha^{\rm pl}$ and (2.31), as in (3.19) and (3.37).

Write $\tilde H_i(\mathcal{M}^{\rm pl})$ for $i\in\mathbb{Z}$ for the subspace of $\bigoplus_{j\geqslant 0}H_j(\mathcal{M}^{\rm pl})$ corresponding to $\bigoplus_{\alpha\in K(\mathcal{A})}\tilde H_i(\mathcal{M}_\alpha^{\rm pl})=\bigoplus_{\alpha\in K(\mathcal{A})}H_{i+2-\chi(\alpha,\alpha)}(\mathcal{M}_\alpha^{\rm pl})$ under the isomorphism (3.58). Then $\tilde H_*(\mathcal{M}^{\rm pl})$ is just $H_*(\mathcal{M}^{\rm pl})$ with a different grading, and (3.57) implies that $[\,,]^{\rm pl}$ preserves the grading on $\tilde H_*(\mathcal{M}^{\rm pl})$.

We can also generalize all the above to the triangulated category case, in a straightforward way.

The next theorem will be proved in §4.6.

Theorem 3.29. *Work in the situation of Definition* 3.28*.*

(a) *The R-bilinear bracket $[\,,]^{\rm pl}$ in* (3.54)–(3.55) *is a **graded Lie bracket** on $\tilde H_*(\mathcal{M}^{\rm pl})$, making $\tilde H_*(\mathcal{M}^{\rm pl})$ into a **graded Lie algebra**.*

(b) *The morphism $\Pi_{t=0}^{\rm pl}:\tilde H_*(\mathcal{M}')^{t=0}\rightarrow\tilde H_*(\mathcal{M}^{\rm pl})$ in* (3.50) *is a morphism of graded Lie algebras over R, for $\bigl(\tilde H_*(\mathcal{M}')^{t=0},[\,,]^{t=0}\bigr)$ as in* §3.3*.*

Remark 3.30. (a) As we will explain in §7, the 'projective linear' version has important applications in areas involving *virtual classes* of moduli spaces.

In areas such as Mochizuki's invariants counting coherent sheaves on surfaces [115], or Donaldson–Thomas invariants counting coherent sheaves on Calabi–Yau 3-folds or Fano 3-folds [81,146], given a suitable abelian category $\mathcal{A}$, a stability condition τ on $\mathcal{A}$, and $\alpha\in K(\mathcal{A})$, one forms moduli schemes $\underline{\mathcal{M}}_\alpha^{\rm st}(\tau),\underline{\mathcal{M}}_\alpha^{\rm ss}(\tau)$ of τ-stable and τ-semistable objects $E\in\mathcal{A}$ with $[\![E]\!]=\alpha$.

Under good conditions, $\underline{\mathcal{M}}_\alpha^{\rm ss}(\tau)$ is a proper $\mathbb{K}$-scheme, and $\underline{\mathcal{M}}_\alpha^{\rm st}(\tau)\subseteq\underline{\mathcal{M}}_\alpha^{\rm ss}(\tau)$ an open $\mathbb{K}$-subscheme, and $\underline{\mathcal{M}}_\alpha^{\rm st}(\tau)$ has a natural perfect obstruction theory. Thus, if $\underline{\mathcal{M}}_\alpha^{\rm st}(\tau)=\underline{\mathcal{M}}_\alpha^{\rm ss}(\tau)$ then $\underline{\mathcal{M}}_\alpha^{\rm st}(\tau)$ is proper with a perfect obstruction theory, so by Behrend and Fantechi [15] it has a virtual class $[\underline{\mathcal{M}}_\alpha^{\rm st}(\tau)]_{\rm virt}$ in a suitable homology theory $H_*(\underline{\mathcal{M}}_\alpha^{\rm st}(\tau))$ (e.g. Chow homology $A_*(\underline{\mathcal{M}}_\alpha^{\rm st}(\tau))$).

Any τ-stable object $E\in\mathcal{A}$ has $\mathrm{Aut}(E)=\mathbb{G}_m$. Thus in the 'projective linear' moduli stack $\mathcal{M}_\alpha^{\rm pl}$, with isotropy groups $\mathrm{Aut}(E)/\mathbb{G}_m$, we have $\mathrm{Iso}_{\mathcal{M}_\alpha^{\rm pl}}([E])=\{1\}$. Because of this, the $\mathbb{K}$-scheme $\underline{\mathcal{M}}_\alpha^{\rm st}(\tau)$ should be an *open* $\mathbb{K}$*-substack* of $\mathcal{M}_\alpha^{\rm pl}$. Hence $[\underline{\mathcal{M}}_\alpha^{\rm st}(\tau)]_{\rm virt}$ pushes forward to $[\underline{\mathcal{M}}_\alpha^{\rm st}(\tau)]_{\rm virt}$ in $H_*(\mathcal{M}_\alpha^{\rm pl})$. This does not work for $H_*(\mathcal{M}_\alpha)$. As in §7, we propose to use our Lie bracket $[\,,]^{\rm pl}$ to express relationships between virtual classes $[\underline{\mathcal{M}}_\alpha^{\rm st}(\tau)]_{\rm virt}$ in $H_*(\mathcal{M}^{\rm pl})$, including a wall-crossing formula for change of stability condition τ.

(b) The shift $2-\chi(\alpha,\alpha)$ in the grading $\tilde H_i(\mathcal{M}_\alpha^{\rm pl})=H_{i+2-\chi(\alpha,\alpha)}(\mathcal{M}_\alpha^{\rm pl})$ in (3.23), (3.43) and (3.56) may be understood as follows. We define $\chi(-,-)$ such that the moduli stack $\mathcal{M}_\alpha^{\rm pl}$ has (real/homological) virtual dimension $\mathrm{vdim}\,\mathcal{M}_\alpha^{\rm pl}=2-\chi(\alpha,\alpha)$, and therefore virtual classes $[\underline{\mathcal{M}}_\alpha^{\rm st}(\tau)]_{\rm virt}$ lie in $H_{2-\chi(\alpha,\alpha)}(\mathcal{M}_\alpha^{\rm pl})=\tilde H_0(\mathcal{M}_\alpha^{\rm pl})$. Thus, as $[\,,]^{\rm pl}$ maps $\tilde H_0(\mathcal{M}_\alpha^{\rm pl})\times\tilde H_0(\mathcal{M}_\beta^{\rm pl})\rightarrow\tilde H_0(\mathcal{M}_{\alpha+\beta}^{\rm pl})$ we can express relations between virtual classes on $\mathcal{M}_\alpha^{\rm pl},\mathcal{M}_\beta^{\rm pl},\mathcal{M}_{\alpha+\beta}^{\rm pl}$ in terms of $[\,,]^{\rm pl}$.

3.5 The 'positive rank' version

The third version of our Lie algebra construction is called the 'positive rank' version. We will need an extra piece of data:

Assumption 3.31. Let Assumption 3.1 hold. We should be given a group morphism $\mathrm{rk}: K(\mathcal{A}) \to \mathbb{Z}$ that we will call the *rank*. Write $\mathcal{M}^{\mathrm{rk}>0}$ for the open substack of $\mathcal{M}$ representing objects $E \in \mathcal{A}$ with $\mathrm{rk}(\llbracket E\rrbracket) > 0$, so that under the decomposition $\mathcal{M} = \coprod_{\alpha\in K(\mathcal{A})} \mathcal{M}_\alpha$ we have $\mathcal{M}^{\mathrm{rk}>0} = \coprod_{\alpha\in K(\mathcal{A}):\mathrm{rk}\,\alpha>0} \mathcal{M}_\alpha$.

If instead Assumption 3.2 holds, we replace $K(\mathcal{A})$ by $K(\mathcal{T})$, so we have a group morphism $\mathrm{rk}: K(\mathcal{T}) \to \mathbb{Z}$.

Definition 3.32. Suppose Assumption 2.30 holds for (co)homology theories $H_i, H^i : \mathrm{Ho}(\mathbf{Art}_{\mathbb{K}}^{\mathrm{lft}}) \to R\text{-mod}$ of Artin $\mathbb{K}$-stacks over a $\mathbb{Q}$-algebra R, suppose Assumptions 3.1 and 3.31 hold for the $\mathbb{K}$-linear abelian category $\mathcal{A}$, and use the notation of §2 and §3.1–§3.2.

For $\alpha, \beta \in K(\mathcal{A})$ with $\mathrm{rk}\,\alpha > 0$ and $\mathrm{rk}\,\beta > 0$, define an R-bilinear map

$$[\,,\,]^{\mathrm{rk}>0} : H_a(\mathcal{M}_\alpha) \times H_b(\mathcal{M}_\beta) \longrightarrow H_{a+b-2\chi(\alpha,\beta)-2}(\mathcal{M}_{\alpha+\beta})$$

by $$[\zeta,\eta]^{\mathrm{rk}>0} = \sum_{n\geqslant 0:\, 2n\leqslant a+b-2\chi(\alpha,\beta)-2} \Bigl(\frac{-\mathrm{rk}\,\alpha}{\mathrm{rk}(\alpha+\beta)}\Bigr)^n \cdot t^n \diamond [\zeta,\eta]_n. \tag{3.59}$$

We need R to be a $\mathbb{Q}$-algebra because of the rational factors in (3.59). Using the alternative gradings $\tilde H_*(\mathcal{M}_\alpha)$ of (3.23), this maps

$$[\,,\,]^{\mathrm{rk}>0} : \tilde H_{\tilde a}(\mathcal{M}_\alpha) \times \tilde H_{\tilde b}(\mathcal{M}_\beta) \longrightarrow \tilde H_{\tilde a+\tilde b}(\mathcal{M}_{\alpha+\beta}). \tag{3.60}$$

Define $[\,,\,]^{\mathrm{rk}>0} : H_*(\mathcal{M}^{\mathrm{rk}>0}) \times H_*(\mathcal{M}^{\mathrm{rk}>0}) \to H_*(\mathcal{M}^{\mathrm{rk}>0})$ to be the R-bilinear map which is identified with $[\,,\,]^{\mathrm{rk}>0}$ in (3.59) on each component $H_a(\mathcal{M}_\alpha) \times H_b(\mathcal{M}_\beta)$ under the canonical isomorphism from (2.31)

$$H_*(\mathcal{M}^{\mathrm{rk}>0}) \cong \textstyle\bigoplus_{a=0}^\infty \bigoplus_{\alpha\in K(\mathcal{A}):\mathrm{rk}\,\alpha>0} H_a(\mathcal{M}_\alpha). \tag{3.61}$$

Write $\tilde H_i(\mathcal{M}^{\mathrm{rk}>0})$ for $i \in \mathbb{Z}$ for the subspace of $\bigoplus_{j\geqslant 0} H_j(\mathcal{M}^{\mathrm{rk}>0})$ corresponding to $\bigoplus_{\alpha:\mathrm{rk}\,\alpha>0} \tilde H_i(\mathcal{M}_\alpha) = \bigoplus_{\alpha:\mathrm{rk}\,\alpha>0} H_{i+2-\chi(\alpha,\alpha)}(\mathcal{M}_\alpha)$ under the isomorphism (3.61), using the alternative gradings (3.23). Then $\tilde H_i(\mathcal{M}^{\mathrm{rk}>0}) \cong \bigoplus_{\alpha:\mathrm{rk}\,\alpha>0} \tilde H_i(\mathcal{M}_\alpha)$, so $\tilde H_*(\mathcal{M}^{\mathrm{rk}>0})$ is just $H_*(\mathcal{M}^{\mathrm{rk}>0})$ with a different grading, and (3.60) implies that $[\,,\,]^{\mathrm{rk}>0}$ preserves the grading on $\tilde H_*(\mathcal{M}^{\mathrm{rk}>0})$.

We can also generalize all the above to the triangulated category case. Suppose instead that Assumption 2.30 holds for (co)homology theories $H_i, H^i : \mathrm{Ho}(\mathbf{HSt}_{\mathbb{K}}^{\mathrm{lft}}) \to R\text{-mod}$ of *higher* Artin $\mathbb{K}$-stacks over the $\mathbb{Q}$-algebra R, and Assumptions 3.2 and 3.31 hold for the $\mathbb{K}$-linear triangulated category $\mathcal{T}$. We replace $K(\mathcal{A})$ by $K(\mathcal{T})$ throughout, but otherwise the definition of $[\,,\,]^{\mathrm{rk}>0}$ on $\tilde H_*(\mathcal{M}^{\mathrm{rk}>0})$ works without change.

Here is the analogue of Theorems 3.20 and 3.29(a), proved in §4.7:

Theorem 3.33. *In Definition* 3.32, *the R-bilinear bracket* $[\,,\,]^{\mathrm{rk}>0}$ *is a* ***graded Lie bracket*** *on* $\tilde H_*(\mathcal{M}^{\mathrm{rk}>0})$, *making* $\tilde H_*(\mathcal{M}^{\mathrm{rk}>0})$ *into a* ***graded Lie algebra***.

Remark 3.34. (a) The Lie bracket $[\,,\,]^{\mathrm{rk}>0}$ in Definition 3.32 is defined only on the homology $H_*(\mathcal{M}^{\mathrm{rk}>0})$ of the open substack $\mathcal{M}^{\mathrm{rk}>0} \subset \mathcal{M}$ of objects with positive rank, and it does depend on the function rk. We admit this seems unnatural. Here are the two main examples of rank functions we have in mind:

(i) Let X be a connected smooth projective $\mathbb{K}$-scheme and $\mathcal{A} = \mathrm{coh}(X)$ be the abelian category of coherent sheaves on X, and $K(\mathcal{A}) = K^{\mathrm{num}}(\mathcal{A})$ be the numerical Grothendieck group of $\mathcal{A}$, and $\mathrm{rk} : K(\mathcal{A}) \to \mathbb{Z}$ map $\mathrm{rk} : [\![E]\!] \mapsto \mathrm{rank}\, E$, so that if E is a vector bundle of rank n then $\mathrm{rk}([\![E]\!]) = n$.

(ii) Let $Q = (Q_0, Q_1, h, t)$ be a finite quiver, and $\mathcal{A} = \text{mod-}\mathbb{K}Q$, and $K(\mathcal{A}) = \mathbb{Z}^{Q_0}$ be the lattice of dimension vectors $\boldsymbol{d} : Q_0 \to \mathbb{Z}$ for Q, and $\mathrm{rk} : K(\mathcal{A}) \to \mathbb{Z}$ be the total dimension, so that $\mathrm{rk}(\boldsymbol{d}) = \sum_{v \in Q_0} \boldsymbol{d}(v)$. Note that if $E \in \mathcal{A}$ is nonzero then $\mathrm{rk}([\![E]\!]) > 0$, so $\mathcal{M}^{\mathrm{rk}>0} = \mathcal{M} \setminus \{[0]\}$.

In abelian category problems, restricting to positive rank may not matter that much, e.g. in (ii) it only excludes the zero object. However, for triangulated categories, as the shift functor $[1] : \mathcal{T} \to \mathcal{T}$ changes the sign of rk, the substack $\mathcal{M}^{\mathrm{rk}>0}$ is less than half of $\mathcal{M}$, so we lose a lot.

(b) The definition of 'positive rank' Lie algebra also makes sense for any graded vertex algebra V_* as in §2.2.1 with an additional grading $V_* = \bigoplus_{\alpha \in L} V_*^\alpha$ over an abelian group L compatible with the vertex algebra structure, and a morphism $\mathrm{rk} : L \to \mathbb{Z}$. But the author has not found this construction in the vertex algebra literature. The same applies to the 'mixed' Lie algebra in §3.6.

The next proposition, proved in §4.8, gives an alternative expression for $[\,,\,]^{\mathrm{rk}>0}$ in (3.59) which is more symmetric in $\mathcal{M}_\alpha, \mathcal{M}_\beta$.

Proposition 3.35. *In the situation of Definition* 3.32, *for* $\alpha, \beta \in K(\mathcal{A})$ *with* $\mathrm{rk}\,\alpha, \mathrm{rk}\,\beta > 0$, *define* $\mathrm{X}_{\alpha,\beta} : [*/\mathbb{G}_m] \times [*/\mathbb{G}_m] \times \mathcal{M}_\alpha \times \mathcal{M}_\beta \to \mathcal{M}_{\alpha+\beta}$ *to be the composition of morphisms of Artin* $\mathbb{K}$*-stacks*

$$[*/\mathbb{G}_m]\times[*/\mathbb{G}_m]\times\mathcal{M}_\alpha\times\mathcal{M}_\beta \xrightarrow{\substack{(\Psi_\alpha\circ(\Pi_1\times\Pi_3))\times\\ (\Psi_\beta\circ(\Pi_2\times\Pi_4))}} \mathcal{M}_\alpha\times\mathcal{M}_\beta \xrightarrow{\Phi_{\alpha,\beta}} \mathcal{M}_{\alpha+\beta}. \qquad (3.62)$$

Here Π_i *is the projection to the* i^{th} *factor of* $[*/\mathbb{G}_m]\times[*/\mathbb{G}_m]\times\mathcal{M}_\alpha\times\mathcal{M}_\beta$, *and* $\Phi_{\alpha,\beta}, \Psi_\alpha, \Psi_\beta$ *are as in Assumption* 3.1(g),(h). *Then* $[\,,\,]^{\mathrm{rk}>0}$ *in* (3.59) *satisfies*

$$\begin{aligned}[\zeta,\eta]^{\mathrm{rk}>0} = \sum_{\substack{i,p,q\geqslant 0:\ 2i\leqslant a+b,\\ i=p+q+\chi(\alpha,\beta)+1}} & \epsilon_{\alpha,\beta}(-1)^{q+a\chi(\beta,\beta)}\frac{(\mathrm{rk}\,\beta)^p(\mathrm{rk}\,\alpha)^q}{(\mathrm{rk}(\alpha+\beta))^{p+q}}\cdot \\ & H_{a+b-2\chi(\alpha,\beta)-2}(\mathrm{X}_{\alpha,\beta}) \\ & \bigl(t_1^p\boxtimes t_2^q\boxtimes\bigl[(\zeta\boxtimes\eta)\cap c_i([\Theta^\bullet_{\alpha,\beta}])\bigr]\bigr).\end{aligned} \qquad (3.63)$$

Definition 3.36. Work in the situation of Definition 3.32. Consider the polynomial algebra $R[s]$ for a formal variable s of degree 2. For each $\alpha \in K(\mathcal{A})$ with $\mathrm{rk}\,\alpha > 0$, define an R-bilinear map $\heartsuit : R[s] \times H_*(\mathcal{M}_\alpha) \to H_*(\mathcal{M}_\alpha)$ by

$$s^n \heartsuit \zeta = n!(\mathrm{rk}\,\alpha)^{-n}\cdot t^n \diamond \zeta \qquad (3.64)$$

in $H_{a+2n}(\mathcal{M}_\alpha)$ for all $n \geqslant 0$ and $\zeta \in H_a(\mathcal{M}_\alpha)$, where $t^n \in H_{2n}([*/\mathbb{G}_m])$ and $t^n \diamond \zeta$ is defined in (3.17). Then $\heartsuit$ is graded, and also graded as a R-bilinear map $\heartsuit : R[s] \times \tilde{H}_*(\mathcal{M}_\alpha) \to \tilde{H}_*(\mathcal{M}_\alpha)$. We have

$$
\begin{aligned}
s^m \heartsuit (s^n \heartsuit \zeta) &= s^n \heartsuit \big(n!(\operatorname{rk}\alpha)^{-n} \cdot t^n \diamond \zeta\big) = m!n!(\operatorname{rk}\alpha)^{-m-n} \cdot t^m \diamond (t^n \diamond \zeta) \\
&= m!n!(\operatorname{rk}\alpha)^{-m-n} \cdot [t^m \star t^n] \diamond \zeta = m!n!(\operatorname{rk}\alpha)^{-m-n} \cdot \big[\tbinom{m+n}{n} t^{m+n}\big] \diamond \zeta \\
&= (m+n)!(\operatorname{rk}\alpha)^{-m-n} \cdot t^{m+n} \diamond \zeta = s^{m+n} \heartsuit \zeta,
\end{aligned}
$$

using (3.64) in the first, second and sixth steps, (3.18) in the third, and (3.15) in the fourth. This implies that $\heartsuit$ makes $H_*(\mathcal{M}_\alpha)$, and also $\tilde{H}_*(\mathcal{M}_\alpha)$, into a *graded $R[s]$-module*. We also write $\heartsuit$ for the graded $R[s]$-action on $H_*(\mathcal{M}^{\operatorname{rk}>0}) = \tilde{H}_*(\mathcal{M}^{\operatorname{rk}>0})$ which restricts to $\heartsuit$ in (3.64) on each subspace $H_*(\mathcal{M}_\alpha)$ in (3.61).

The next result, proved in §4.9, shows $[\,,\,]^{\operatorname{rk}>0}$ is *$R[s]$-bilinear* for the $R[s]$-module action $\heartsuit$ on $\tilde{H}_*(\mathcal{M}^{\operatorname{rk}>0})$, so $\tilde{H}_*(\mathcal{M}^{\operatorname{rk}>0})$ is a *graded Lie algebra over $R[s]$*.

Proposition 3.37. *In Definition* 3.36, *if* $\alpha, \beta \in K(\mathcal{A})$ *with* $\operatorname{rk}\alpha, \operatorname{rk}\beta > 0$, *and* $\zeta \in H_a(\mathcal{M}_\alpha)$, $\eta \in H_b(\mathcal{M}_\beta)$, *and* $m, n \geqslant 0$, *we have*

$$
[s^m \heartsuit \zeta, s^n \heartsuit \eta]^{\operatorname{rk}>0} = s^{m+n} \heartsuit [\zeta, \eta]^{\operatorname{rk}>0}. \tag{3.65}
$$

We can relate the 'positive rank' and '$t = 0$' Lie algebras by a morphism:

Definition 3.38. Suppose Assumption 2.30 holds for (co)homology theories $H_i, H^i : \operatorname{Ho}(\mathbf{Art}_{\mathbb{K}}^{\mathrm{lft}}) \to R$-mod of Artin $\mathbb{K}$-stacks over a $\mathbb{Q}$-algebra R, suppose Assumptions 3.1 and 3.31 hold for the $\mathbb{K}$-linear abelian category $\mathcal{A}$, and use the notation of §2 and §3.1–§3.3. Then Definition 3.19 and Theorem 3.20 give a graded Lie bracket $[\,,\,]^{t=0}$ on $\tilde{H}_*(\mathcal{M})^{t=0}$, and Definition 3.32 and Theorem 3.33 give a graded Lie bracket $[\,,\,]^{\operatorname{rk}>0}$ on $\tilde{H}_*(\mathcal{M}^{\operatorname{rk}>0})$, depending on the rank function $\operatorname{rk} : K(\mathcal{A}) \to \mathbb{Z}$ (note that $[\,,\,]^{t=0}$ is independent of rk).

Define $\Pi^{t=0}_{\operatorname{rk}>0} : \tilde{H}_*(\mathcal{M}^{\operatorname{rk}>0}) \to \tilde{H}_*(\mathcal{M})^{t=0}$ by $\Pi^{t=0}_{\operatorname{rk}>0}(\zeta) = \zeta + (t \diamond H_*(\mathcal{M}))$. By comparing (3.42) with (3.59), and noting that the $n > 0$ terms in (3.59) map to zero in $\tilde{H}_*(\mathcal{M})^{t=0}$, we see that $\Pi^{t=0}_{\operatorname{rk}>0} : \tilde{H}_*(\mathcal{M}^{\operatorname{rk}>0}) \to \tilde{H}_*(\mathcal{M})^{t=0}$ is an *R-linear morphism of graded Lie algebras*.

From (3.64) we see that the kernel of $\Pi^{t=0}_{\operatorname{rk}>0}$ is $s \heartsuit \tilde{H}_*(\mathcal{M}^{\operatorname{rk}>0}) \subset \tilde{H}_*(\mathcal{M}^{\operatorname{rk}>0})$, and the image of $\Pi^{t=0}_{\operatorname{rk}>0}$ is $\tilde{H}_*(\mathcal{M}^{\operatorname{rk}>0})^{t=0} \subset \tilde{H}_*(\mathcal{M})^{t=0}$.

If also Assumption 2.39 holds then §3.4 gives a graded Lie algebra $(\tilde{H}_*(\mathcal{M}^{\mathrm{pl}}), [\,,\,]^{\mathrm{pl}})$ with a morphism $\Pi^{\mathrm{pl}}_{t=0} : \tilde{H}_*(\mathcal{M}') \to \tilde{H}_*(\mathcal{M}^{\mathrm{pl}})$, so we obtain a morphism of graded Lie algebras $\Pi^{\mathrm{pl}}_{\operatorname{rk}>0} = \Pi^{\mathrm{pl}}_{t=0} \circ \Pi^{t=0}_{\operatorname{rk}>0} : \tilde{H}_*(\mathcal{M}^{\operatorname{rk}>0}) \to \tilde{H}_*(\mathcal{M}^{\mathrm{pl}})$.

The analogue also works in the triangulated category case.

3.6 The 'mixed' version

In the 'positive rank' version in §3.5 we restricted to the substack $\mathcal{M}^{\operatorname{rk}>0} \subset \mathcal{M}$ of $\mathcal{M}_\alpha$ with $\operatorname{rk}\alpha > 0$ because equation (3.59) is undefined if $\operatorname{rk}(\alpha + \beta) = 0$, and (3.64) is undefined if $\operatorname{rk}\alpha = 0$. The 'mixed' version combines the '$t = 0$' and 'positive rank' versions, by defining $H_*(\mathcal{M}_\alpha)^{\mathrm{mix}}$ to be $H_*(\mathcal{M}_\alpha)$ if $\operatorname{rk}\alpha \neq 0$, and $R[s] \otimes_R H_*(\mathcal{M}_\alpha)^{t=0}$ if $\operatorname{rk}\alpha = 0$, so the construction works over the whole of $\mathcal{M}$.

Definition 3.39. Suppose Assumption 2.30 holds for (co)homology theories $H_i, H^i : \mathrm{Ho}(\mathbf{Art}_{\mathbb{K}}^{\mathrm{lft}}) \to R\text{-mod}$ of Artin $\mathbb{K}$-stacks over a $\mathbb{Q}$-algebra R, suppose Assumptions 3.1 and 3.31 hold for the $\mathbb{K}$-linear abelian category $\mathcal{A}$, and use the notation of §2 and §3.1–§3.5.

For each $\alpha \in K(\mathcal{A})$, define a graded R-module $H_*(\mathcal{M}_\alpha)^{\mathrm{mix}}$ by

$$H_*(\mathcal{M}_\alpha)^{\mathrm{mix}} = \begin{cases} H_*(\mathcal{M}_\alpha), & \mathrm{rk}\,\alpha \neq 0, \\ R[s] \otimes_R H_*(\mathcal{M}_\alpha)^{t=0}, & \mathrm{rk}\,\alpha = 0, \end{cases}$$

where s is a formal variable of degree 2, and $H_*(\mathcal{M}_\alpha)^{t=0}$ is as in (3.36). That is, $H_a(\mathcal{M}_\alpha)^{\mathrm{mix}} = \bigoplus_{0 \leqslant n \leqslant a/2} \langle s^n \rangle_R \otimes_R H_{a-2n}(\mathcal{M}_\alpha)^{t=0}$ when $\mathrm{rk}\,\alpha = 0$.

Define an R-bilinear map $\heartsuit : R[s] \times H_*(\mathcal{M}_\alpha)^{\mathrm{mix}} \to H_*(\mathcal{M}_\alpha)^{\mathrm{mix}}$ by (3.64) if $\mathrm{rk}\,\alpha \neq 0$ and $s^m \heartsuit (s^n \otimes \bar\zeta) = s^{m+n} \otimes \bar\zeta$ for all $m, n \geqslant 0$ and $\bar\zeta \in H_*(\mathcal{M}_\alpha)^{t=0}$ if $\mathrm{rk}\,\alpha = 0$. By Definition 3.36, $\heartsuit$ makes $H_*(\mathcal{M}_\alpha)^{\mathrm{mix}}$ into a *graded* $R[s]$*-module.*

For all $\alpha, \beta \in K(\mathcal{A})$, define an R-bilinear map

$$[\,,\,]^{\mathrm{mix}} : H_a(\mathcal{M}_\alpha)^{\mathrm{mix}} \times H_b(\mathcal{M}_\beta)^{\mathrm{mix}} \longrightarrow H_{a+b-2\chi(\alpha,\beta)-2}(\mathcal{M}_{\alpha+\beta})^{\mathrm{mix}} \tag{3.66}$$

in cases, according to whether $\mathrm{rk}\,\alpha, \mathrm{rk}\,\beta, \mathrm{rk}(\alpha+\beta)$ are zero or nonzero, by:

(a) If $\mathrm{rk}\,\alpha, \mathrm{rk}\,\beta, \mathrm{rk}(\alpha+\beta)$ are all nonzero then $[\,,\,]^{\mathrm{mix}}$ equals $[\,,\,]^{\mathrm{rk}>0}$ in (3.59).

(b) If $\mathrm{rk}\,\alpha = \mathrm{rk}\,\beta = \mathrm{rk}(\alpha+\beta) = 0$ then for all $m, n \geqslant 0$ and $\bar\zeta \in H_*(\mathcal{M}_\alpha)^{t=0}$, $\bar\eta \in H_*(\mathcal{M}_\beta)^{t=0}$ we have

$$[s^m \otimes \bar\zeta, s^n \otimes \bar\eta]^{\mathrm{mix}} = s^{m+n} \otimes [\bar\zeta, \bar\eta]^{t=0}, \tag{3.67}$$

where $[\,,\,]^{t=0}$ is as in (3.42).

(c) If $\mathrm{rk}\,\alpha = 0$ and $\mathrm{rk}\,\beta = \mathrm{rk}(\alpha+\beta) \neq 0$ then

$$\bigl[s^m \otimes (\zeta + I_t \diamond H_*(\mathcal{M}_\alpha)), \eta\bigr]^{\mathrm{mix}} = s^m \heartsuit [\zeta, \eta]_0. \tag{3.68}$$

Equation (3.26) implies that this is independent of the choice of representative ζ for $\bar\zeta = \zeta + I_t \diamond H_*(\mathcal{M}_\alpha)$ in $H_*(\mathcal{M}_\alpha)^{t=0}$, so this is well defined.

(d) If $\mathrm{rk}\,\beta = 0$ and $\mathrm{rk}\,\alpha = \mathrm{rk}(\alpha+\beta) \neq 0$ then

$$\bigl[\zeta, s^m \otimes (\eta + I_t \diamond H_*(\mathcal{M}_\beta))\bigr]^{\mathrm{mix}} = s^m \heartsuit \Bigl(\sum_{n \geqslant 0:\ 2n \leqslant a+b-2\chi(\alpha,\beta)-2} (-1)^n t^n \diamond [\zeta, \eta]_n \Bigr). \tag{3.69}$$

To see that this is independent of the choice of representative η for $\bar\eta = \eta + I_t \diamond H_*(\mathcal{M}_\beta)$ in $H_*(\mathcal{M}_\beta)^{t=0}$, observe that for $p > 0$ we have

$$\begin{aligned} &\sum_{n \geqslant 0:\ 2n \leqslant a+b-2\chi(\alpha,\beta)-2} (-1)^n t^n \diamond [\zeta, t^p \diamond \eta]_n = \sum_{k,n \geqslant 0:\ k \leqslant p,\ 2(k+n) \leqslant a+b+2p-2\chi(\alpha,\beta)-2} (-1)^n \tbinom{n}{p-k} \cdot t^n \diamond \bigl(t^k \diamond [\zeta, \eta]_{n+k-p}\bigr) \\ &= \sum_{k,n \geqslant 0:\ k \leqslant p,\ 2(k+n) \leqslant a+b+2p-2\chi(\alpha,\beta)-2} (-1)^n \tbinom{n}{p-k} \tbinom{k+n}{k} \cdot t^{k+n} \diamond [\zeta, \eta]_{n+k-p} \\ &= \sum_{k,m \geqslant 0:\ k \leqslant p,\ 2m \leqslant a+b+2p-2\chi(\alpha,\beta)-2} (-1)^{m+k} \tbinom{m}{p} \tbinom{p}{k} \cdot t^m \diamond [\zeta, \eta]_{m-p} = 0, \end{aligned}$$

using (3.27) in the first step, (3.15) and (3.18) in the second, changing variables from n to $m=k+n$ in the third, and $\sum_{k=0}^{p}(-1)^k\binom{p}{k}=(1-1)^p=0$ as $p>0$ in the fourth. Thus (3.69) is well defined.

(e) If $\operatorname{rk}\alpha=-\operatorname{rk}\beta\neq 0$ and $\operatorname{rk}(\alpha+\beta)=0$ then for all $\zeta\in H_*(\mathcal{M}_\alpha)$, $\eta\in H_*(\mathcal{M}_\beta)$ we have

$$[\zeta,\eta]^{\mathrm{mix}}=\sum_{\substack{n\geqslant 0:\\ 2n\leqslant a+b-2\chi(\alpha,\beta)-2}}\frac{(\operatorname{rk}\beta)^n}{n!}\cdot s^n\otimes\bigl([\zeta,\eta]_n+I_t\diamond H_*(\mathcal{M}_0)\bigr),\tag{3.70}$$

where $[\,,\,]_n$ is as in (3.22).

Define a graded R-module

$$H_*(\mathcal{M})^{\mathrm{mix}}=\bigoplus\nolimits_{\alpha\in K(\mathcal{A})}H_*(\mathcal{M}_\alpha)^{\mathrm{mix}}.\tag{3.71}$$

Then $\heartsuit$ above on each $H_*(\mathcal{M}_\alpha)^{\mathrm{mix}}$ makes $H_*(\mathcal{M})^{\mathrm{mix}}$ into a graded $R[s]$-module. Define $[\,,\,]^{\mathrm{mix}}:H_*(\mathcal{M})^{\mathrm{mix}}\times H_*(\mathcal{M})^{\mathrm{mix}}\rightarrow H_*(\mathcal{M})^{\mathrm{mix}}$ to be the R-bilinear map identified with $[\,,\,]^{\mathrm{mix}}$ in (3.66) on each component $H_*(\mathcal{M}_\alpha)^{\mathrm{mix}}\times H_*(\mathcal{M}_\beta)^{\mathrm{mix}}$ in (3.71). As in (3.23) and (3.43), define an alternative grading on $H_*(\mathcal{M}_\alpha)^{\mathrm{mix}}$ by

$$\tilde H_i(\mathcal{M}_\alpha)^{\mathrm{mix}}=H_{i+2-\chi(\alpha,\alpha)}(\mathcal{M}_\alpha)^{\mathrm{mix}}.$$

Write $\tilde H_i(\mathcal{M})^{\mathrm{mix}}$ for $i\in\mathbb{Z}$ for the subspace of $\bigoplus_{j\geqslant 0}H_j(\mathcal{M})^{\mathrm{mix}}$ corresponding to $\bigoplus_{\alpha\in K(\mathcal{A})}\tilde H_i(\mathcal{M}_\alpha)^{\mathrm{mix}}=\bigoplus_{\alpha\in K(\mathcal{A})}H_{i+2-\chi(\alpha,\alpha)}(\mathcal{M}_\alpha)^{\mathrm{mix}}$ under (3.71). Then $\tilde H_*(\mathcal{M})^{\mathrm{mix}}$ is just $H_*(\mathcal{M})^{\mathrm{mix}}$ with a different grading, and (3.66) implies that $[\,,\,]^{\mathrm{mix}}$ preserves the grading on $\tilde H_*(\mathcal{M})^{\mathrm{mix}}$.

We can also generalize all the above to the triangulated category case. Suppose instead that Assumption 2.30 holds for (co)homology theories $H_i,H^i:\mathrm{Ho}(\mathbf{HSt}_{\mathbb{K}}^{\mathrm{lft}})\rightarrow R$-mod of *higher* Artin $\mathbb{K}$-stacks over the $\mathbb{Q}$-algebra R, and Assumptions 3.2 and 3.31 hold for the $\mathbb{K}$-linear triangulated category $\mathcal{T}$. We replace $K(\mathcal{A})$ by $K(\mathcal{T})$ throughout, but otherwise the definition of $[\,,\,]^{\mathrm{mix}}$ on $\tilde H_*(\mathcal{M})^{\mathrm{mix}}$ works without change.

The next theorem will be proved in §4.10:

Theorem 3.40. *In Definition* 3.39, *the R-bilinear bracket $[\,,\,]^{\mathrm{mix}}$ is a **graded Lie bracket** on $\tilde H_*(\mathcal{M})^{\mathrm{mix}}$, making $\tilde H_*(\mathcal{M})^{\mathrm{mix}}$ into a **graded Lie algebra**. Also, for $\alpha,\beta\in K(\mathcal{A})$, $\zeta\in\tilde H_*(\mathcal{M}_\alpha)^{\mathrm{mix}}$, $\eta\in\tilde H_*(\mathcal{M}_\beta)^{\mathrm{mix}}$ and $m,n\geqslant 0$ we have*

$$[s^m\heartsuit\zeta,s^n\heartsuit\eta]^{\mathrm{mix}}=s^{m+n}\heartsuit[\zeta,\eta]^{\mathrm{mix}},\tag{3.72}$$

*so $[\,,\,]^{\mathrm{mix}}$ is $R[s]$-**bilinear**, and $\tilde H_*(\mathcal{M})^{\mathrm{mix}}$ is a **graded Lie algebra over** $R[s]$.*

Remark 3.41. (a) As in Remarks 3.13(a) and 3.21(a), we can replace $\mathcal{M}$ by $\mathcal{M}'=\mathcal{M}\setminus\{[0]\}$ throughout §3.6.

(b) If we also suppose Assumption 2.31 for $H_*(-)$ over a $\mathbb{Q}$-algebra R, and that if $\operatorname{rk}\alpha=0$ then the principal $[*/\mathbb{G}_m]$-bundle $\Pi_\alpha^{\mathrm{pl}}:\mathcal{M}'_\alpha\rightarrow\mathcal{M}_\alpha^{\mathrm{pl}}$ is rationally trivial, as in §2.3.7 (this often holds automatically, by Proposition 3.24(b),(c)), then $H_*(\mathcal{M}'_\alpha)^{t=0} \cong H_*(\mathcal{M}^{\rm pl}_\alpha)$ by Proposition 3.24(a), so when $\mathrm{rk}\,\alpha = 0$ we could replace $H_*(\mathcal{M}'_\alpha)^{t=0}$ by $H_*(\mathcal{M}^{\rm pl}_\alpha)$ in Definition 3.39, which would be more natural. Also $R[s] \otimes_R H_*(\mathcal{M}^{\rm pl}_\alpha) \cong H_*([*/\mathbb{G}_m] \times \mathcal{M}^{\rm pl}_\alpha)$. Thus we could write

$$\mathcal{M}^{\rm mix} = \bigl(\textstyle\coprod_{\alpha\in K(\mathcal{A}):\mathrm{rk}\,\alpha=0}[*/\mathbb{G}_m] \times \mathcal{M}^{\rm pl}_\alpha\bigr) \amalg \bigl(\textstyle\coprod_{\alpha\in K(\mathcal{A}):\mathrm{rk}\,\alpha\neq 0}\mathcal{M}_\alpha\bigr),$$

and then $H_*(\mathcal{M}')^{\rm mix} \cong H_*(\mathcal{M}^{\rm mix})$, so that $H_*(\mathcal{M}')^{\rm mix}$ is the homology of a geometric space. Actually in the rationally trivial case we have isomorphisms $R[s] \otimes_R H_*(\mathcal{M}'_\alpha)^{t=0} \cong H_*(\mathcal{M}'_\alpha)$, but these are not canonical.

(c) Equations (3.68) and (3.69) are based on (3.59) with $\mathrm{rk}\,\alpha = 0$, $\mathrm{rk}\,\beta \neq 0$, and with $\mathrm{rk}\,\alpha \neq 0$, $\mathrm{rk}\,\beta = 0$, respectively. We can heuristically derive Definition 3.39(a)–(e) from (3.59) by allowing rk to map $K(\mathcal{A}) \to \mathbb{R}$ rather than $K(\mathcal{A}) \to \mathbb{Z}$, and considering what happens as $\mathrm{rk}\,\alpha \to 0$, or $\mathrm{rk}\,\beta \to 0$, or $\mathrm{rk}(\alpha+\beta) \to 0$ in $\mathbb{R}$, as we vary the function rk for fixed α, β, and regard $R[s] \otimes_R H_*(\mathcal{M}_\alpha)^{t=0}$ as the associated graded module of the filtration

$$H_*(\mathcal{M}_\alpha) \supset t \diamond H_*(\mathcal{M}_\alpha) \supset t^2 \diamond H_*(\mathcal{M}_\alpha) \supset t^3 \diamond H_*(\mathcal{M}_\alpha) \supset \cdots.$$

We relate the 'mixed' Lie algebra to those of §3.3–§3.5 by morphisms:

Definition 3.42. Work in the situation of Definition 3.39. Define an R-linear map $\Pi^{\rm mix}_{\mathrm{rk}>0} : \tilde H_*(\mathcal{M}^{\mathrm{rk}>0}) \to \tilde H_*(\mathcal{M})^{\rm mix}$ to map $\Pi^{\rm mix}_{\mathrm{rk}>0} : \zeta \mapsto \zeta$ for all $\alpha \in K(\mathcal{A})$ with $\mathrm{rk}\,\alpha > 0$ and $\zeta \in \tilde H_*(\mathcal{M}_\alpha)$.

Define an R-linear map $\Pi^{t=0}_{\rm mix} : \tilde H_*(\mathcal{M})^{\rm mix} \to \tilde H_*(\mathcal{M})^{t=0}$ by

$$\Pi^{t=0}_{\rm mix}(\zeta) = \zeta + I_t \diamond H_*(\mathcal{M}_\alpha) \quad \forall \alpha \in K(\mathcal{A}) \text{ with } \mathrm{rk}\,\alpha \neq 0,\ \zeta \in H_*(\mathcal{M}_\alpha),$$

$$\Pi^{t=0}_{\rm mix}(s^n \otimes \bar\zeta) = \begin{cases} \bar\zeta, & n=0 \\ 0, & n>0 \end{cases} \quad \forall \alpha \in K(\mathcal{A}) \text{ with } \mathrm{rk}\,\alpha = 0,\ \bar\zeta \in H_*(\mathcal{M}_\alpha)^{t=0}.$$

Then Definition 3.39 implies that $\Pi^{\rm mix}_{\mathrm{rk}>0}$ and $\Pi^{t=0}_{\rm mix}$ are *R-linear morphisms of graded Lie algebras*, and $\Pi^{\rm mix}_{\mathrm{rk}>0}$ is $R[s]$-linear. Also $\Pi^{t=0}_{\rm mix} \circ \Pi^{\rm mix}_{\mathrm{rk}>0} = \Pi^{t=0}_{\mathrm{rk}>0}$, for $\Pi^{t=0}_{\mathrm{rk}>0}$ as in Definition 3.38. Here $\Pi^{\rm mix}_{\mathrm{rk}>0}$ is injective, and $\Pi^{t=0}_{\rm mix}$ is surjective, with kernel $s\heartsuit\tilde H_*(\mathcal{M})^{\rm mix} \subset \tilde H_*(\mathcal{M})^{\rm mix}$.

If also Assumption 2.39 holds then §3.4 gives a graded Lie algebra $(\tilde H_*(\mathcal{M}^{\rm pl}), [\,,\,]^{\rm pl})$ with a morphism $\Pi^{\rm pl}_{t=0} : \tilde H_*(\mathcal{M}') \to \tilde H_*(\mathcal{M}^{\rm pl})$, so we obtain a morphism of graded Lie algebras $\Pi^{\rm pl}_{\rm mix} = \Pi^{\rm pl}_{t=0} \circ \Pi^{t=0}_{\rm mix} : \tilde H_*(\mathcal{M}')^{\rm mix} \to \tilde H_*(\mathcal{M}^{\rm pl})$.

The analogue also works in the triangulated category case.

3.7 The 'fixed determinant' versions

Let X be a smooth projective $\mathbb{K}$-scheme. Then each coherent sheaf E, or complex E in $D^b\mathrm{coh}(X)$, has a *determinant* $\det E$, a line bundle on X. It is common to consider moduli spaces $\mathcal{M}$ of sheaves with *fixed determinant*, that is, moduli spaces of pairs (E, ι) for E a coherent sheaf and $\iota : L \to \det E$ an isomorphism, for $L \to X$ a fixed line bundle. For example, *Hilbert schemes* of subschemes of codimension at least 2 are moduli spaces of rank 1 torsion-free sheaves with fixed determinant $\mathcal{O}_X$.

We will show that all the Lie algebra constructions of §3.3–§3.6 also work for moduli stacks of objects with fixed determinant. The next assumption gives a notion of determinant for our abelian category $\mathcal{A}$. Example 3.45 explains how it relates to determinants of coherent sheaves.

Assumption 3.43. Let Assumptions 3.1 and 3.31 hold for the abelian category $\mathcal{A}$. Then:

(a) We are given an Artin $\mathbb{K}$-stack $\mathcal{P}$ locally of finite type, which we will call the *Picard stack*. There is a canonical isomorphism $\mathrm{Iso}_{\mathcal{P}}(L) \cong \mathbb{G}_m$ for every $\mathbb{K}$-point $L \in \mathcal{P}(\mathbb{K})$.

(b) We are given a morphism $\det : \mathcal{M} \to \mathcal{P}$ in $\mathrm{Ho}(\mathbf{Art}_{\mathbb{K}}^{\mathrm{lft}})$, called the *determinant*. Write $\det_\alpha = \det|_{\mathcal{M}_\alpha} : \mathcal{M}_\alpha \to \mathcal{P}$.

(c) We are given a morphism $\hat{\Phi} : \mathcal{P} \times \mathcal{P} \to \mathcal{P}$ in $\mathrm{Ho}(\mathbf{Art}_{\mathbb{K}}^{\mathrm{lft}})$. The following diagram commutes in $\mathrm{Ho}(\mathbf{Art}_{\mathbb{K}}^{\mathrm{lft}})$ for all $\alpha, \beta \in K(\mathcal{A})$:

$$\begin{tikzcd} \mathcal{M}_\alpha \times \mathcal{M}_\beta \arrow[r, "\Phi_{\alpha,\beta}"'] \arrow[d, "\det_\alpha \times \det_\beta"] & \mathcal{M}_{\alpha+\beta} \arrow[d, "\det_{\alpha+\beta}"'] \\ \mathcal{P} \times \mathcal{P} \arrow[r, "\hat{\Phi}"] & \mathcal{P}. \end{tikzcd} \tag{3.73}$$

(d) We are given a morphism $\hat{\Psi} : [*/\mathbb{G}_m] \times \mathcal{P} \to \mathcal{P}$ in $\mathrm{Ho}(\mathbf{Art}_{\mathbb{K}}^{\mathrm{lft}})$. On $\mathbb{K}$-points $L \in \mathcal{P}(\mathbb{K})$ it acts by $\hat{\Psi}(\mathbb{K}) : (*, L) \mapsto L$, and on isotropy groups

$$\begin{aligned} &\hat{\Psi}_* : \mathrm{Iso}_{[*/\mathbb{G}_m]\times\mathcal{P}}(*, L) \cong \mathbb{G}_m \times \mathbb{G}_m \longrightarrow \mathrm{Iso}_{\mathcal{P}}(L) \cong \mathbb{G}_m \\ &\text{acts by} \qquad \hat{\Psi}_* : (\lambda, \mu) \longmapsto \lambda\mu \qquad \text{for } \lambda, \mu \in \mathbb{G}_m. \end{aligned}$$

The analogues of (3.6)–(3.7) hold for $\hat{\Phi}, \hat{\Psi}$. Thus $\hat{\Psi}$ is a $[*/\mathbb{G}_m]$-*action* on $\mathcal{P}$ in the sense of Definition 2.23, so $\hat{\Psi}$ is a $[*/\mathbb{G}_m]$-action on $\mathcal{P}$. The action of $\hat{\Psi}$ on isotropy groups above implies that $\hat{\Psi}$ is a *free* $[*/\mathbb{G}_m]$-action, as in (2.22). The following commute in $\mathrm{Ho}(\mathbf{Art}_{\mathbb{K}}^{\mathrm{lft}})$ for all $\alpha \in K(\mathcal{A})$ and $k, l \in \mathbb{Z}$:

$$\begin{tikzcd} {[*/\mathbb{G}_m] \times \mathcal{M}_\alpha} \arrow[r, "\Psi_\alpha"'] \arrow[d, "\Upsilon_{\mathrm{rk}\,\alpha} \times \det_\alpha"] & \mathcal{M}_\alpha \arrow[d, "\det_\alpha"'] \\ {[*/\mathbb{G}_m] \times \mathcal{P}} \arrow[r, "\hat{\Psi}"] & \mathcal{P}, \end{tikzcd} \tag{3.74}$$

$$\begin{tikzcd} {[*/\mathbb{G}_m] \times \mathcal{P} \times \mathcal{P}} \arrow[r, "{(\hat{\Psi}\circ(\Upsilon_k\circ\Pi_1,\Pi_2),\hat{\Psi}\circ(\Upsilon_l\circ\Pi_1,\Pi_3))}"'] \arrow[d, "\mathrm{id}_{[*/\mathbb{G}_m]} \times \hat{\Phi}"] & \mathcal{P} \times \mathcal{P} \arrow[d, "\hat{\Phi}"'] \\ {[*/\mathbb{G}_m] \times \mathcal{P}} \arrow[r, "\hat{\Psi}\circ(\Upsilon_{k+l} \times \mathrm{id}_{\mathcal{P}})"] & \mathcal{P}, \end{tikzcd} \tag{3.75}$$

where $\Upsilon_k : [*/\mathbb{G}_m] \to [*/\mathbb{G}_m]$ is the stack morphism induced by the group morphism $\upsilon_k : \mathbb{G}_m \to \mathbb{G}_m$ mapping $\upsilon_{\mathrm{rk}\,\alpha} : \lambda \mapsto \lambda^k$.

(e) We are given a $\mathbb{K}$-point $\mathcal{O} \in \mathcal{P}(\mathbb{K})$, which we think of as the *identity morphism* $\mathcal{O} : * \to \mathcal{P}$, where $* = \operatorname{Spec}\mathbb{K}$, and an *inverse morphism* $\hat{\imath} : \mathcal{P} \to \mathcal{P}$, which make $\mathcal{P}$ into an *abelian group stack*, with multiplication $\hat{\Phi} : \mathcal{P} \times \mathcal{P} \to \mathcal{P}$, as in Definition 2.23.

(f) For each $\alpha \in K(\mathcal{A})$ we are given a $\mathbb{K}$-point $L_\alpha \in \mathcal{P}(\mathbb{K})$, with $L_0 = \mathcal{O}$, such that $\hat{\Phi}(\mathbb{K})(L_\alpha, L_\beta) = L_{\alpha+\beta}$ and $\hat{\imath}(\mathbb{K})(L_\alpha) = L_{-\alpha}$ for all $\alpha, \beta \in K(\mathcal{A})$. We regard L_α as a morphism $L_\alpha : * \to \mathcal{P}$ in $\mathrm{Ho}(\mathbf{Art}_{\mathbb{K}}^{\mathrm{lft}})$, where $* = \operatorname{Spec} \mathbb{K}$.

Here is the analogue for triangulated categories $\mathcal{T}$:

Assumption 3.44. Assume the analogue of Assumption 3.43, but replace Assumption 3.1 by Assumption 3.2, and replace $\mathcal{A}, K(\mathcal{A})$ by $\mathcal{T}, K(\mathcal{T})$ throughout. Then $\mathcal{M}$ is a higher stack, but we still suppose $\mathcal{P}$ is an ordinary Artin stack.

The next example motivates Assumption 3.43:

Example 3.45. Let X be a smooth, connected, projective $\mathbb{K}$-scheme. Take $\mathcal{A} = \mathrm{coh}(X)$ and $K(\mathcal{A}) = K^{\mathrm{num}}(\mathcal{A})$. Then:

- Let $\mathcal{P}$ be the moduli stack of line bundles L on X, which is an open substack of $\mathcal{M}$ (we do not assume this in Assumption 3.43). Let $\det : \mathcal{M} \to \mathcal{P}$ map a coherent sheaf E to its determinant line bundle $\det E$.
- The morphism $\hat{\Phi} : \mathcal{P} \times \mathcal{P} \to \mathcal{P}$ maps $(L_1, L_2) \mapsto L_1 \otimes L_2$, using tensor product of line bundles. The morphism $\mathcal{O} : * \to \mathcal{P}$ maps $* \mapsto \mathcal{O}_X$. The morphism $\hat{\imath} : \mathcal{P} \to \mathcal{P}$ maps $L \mapsto L^*$, using dual line bundles. Then $\mathcal{P}$ is an abelian group stack.
- The morphism $\hat{\Psi} : [*/\mathbb{G}_m] \times \mathcal{P} \to \mathcal{P}$ is defined as for Ψ in Assumption 3.1(h), but restricting to $\mathcal{P} \subset \mathcal{M}$.
- Take $L_\alpha = \mathcal{O}_X$ for all $\alpha \in K(\mathcal{A})$.

Then Assumption 3.43 holds.

Readers are advised to familiarize themselves with §2.3.2–§2.3.3 on Artin stacks as a 2-category, substacks, and fibre products, before proceeding further.

Definition 3.46. Let Assumptions 3.1, 3.31 and 3.43 hold. Assumption 3.43(d) says that $\hat{\Psi}$ is a free $[*/\mathbb{G}_m]$-action on $\mathcal{P}$. As for $\mathcal{M}^{\mathrm{pl}}$ in Definition 3.22, write $\hat{\Pi}^{\mathrm{pl}} : \mathcal{P} \to \mathcal{P}^{\mathrm{pl}}$ for the principal $[*/\mathbb{G}_m]$-bundle with $[*/\mathbb{G}_m]$-action $\hat{\Psi}$ given by Proposition 2.25(a). Then $\mathcal{P}^{\mathrm{pl}}$ is an Artin $\mathbb{K}$-stack, locally of finite type. As for (3.48) the following is 2-Cartesian and 2-co-Cartesian in $\mathbf{Art}_{\mathbb{K}}^{\mathrm{lft}}$:

$$
\begin{tikzcd}
{[*/\mathbb{G}_m] \times \mathcal{P}} \arrow[rr, "\hat{\Psi}"] \arrow[d, "\pi_{\mathcal{P}}"] & & \mathcal{P} \arrow[d, "\hat{\Pi}^{\mathrm{pl}}"] \\
\mathcal{P} \arrow[rr, "\hat{\Pi}^{\mathrm{pl}}"] & & \mathcal{P}^{\mathrm{pl}}.
\end{tikzcd}
\tag{3.76}
$$

Since $\hat{\Pi}^{\mathrm{pl}}$ is a bijection on $\mathbb{K}$-points, we write L for both a point in $\mathcal{P}(\mathbb{K})$ and for its image in $\mathcal{P}^{\mathrm{pl}}(\mathbb{K})$, so Assumption 3.43(f) gives $\mathbb{K}$-points $L_\alpha \in \mathcal{P}^{\mathrm{pl}}(\mathbb{K})$. We have $\mathrm{Iso}_{\mathcal{P}^{\mathrm{pl}}}(L) = \{1\}$ for all $L \in \mathcal{P}^{\mathrm{pl}}(\mathbb{K})$, so $\mathcal{P}^{\mathrm{pl}}$ is an *algebraic* $\mathbb{K}$*-space*. We write $\mathcal{O} = \hat{\Pi}^{\mathrm{pl}}(\mathbb{K})(\mathcal{O}) \in \mathcal{P}^{\mathrm{pl}}(\mathbb{K})$.

As for the morphisms (3.53) in Definition 3.28, using the 2-co-Cartesian property of (3.76), we can construct natural morphisms

$$\det{}^{\rm pl}:\mathcal{M}^{\rm pl}\to\mathcal{P}^{\rm pl},\;\; \det{}^{\rm pl}_\alpha=\det{}^{\rm pl}\vert_{\mathcal{M}^{\rm pl}_\alpha}:\mathcal{M}^{\rm pl}_\alpha\to\mathcal{P}^{\rm pl},\;\; \hat\Phi^{\rm pl}:\mathcal{P}^{\rm pl}\times\mathcal{P}^{\rm pl}\to\mathcal{P}^{\rm pl},$$

for $\alpha\in K(\mathcal{A})$, which are 'projective linear' versions of $\det,\det_\alpha,\hat\Phi$ in Assumption 3.43, such that the following commute:

$$\begin{tikzcd}[column sep=huge] \mathcal{M}'_\alpha \arrow[r, "\det'_\alpha:=\det_\alpha\vert_{\mathcal{M}'_\alpha}"'] \arrow[d, "\Pi^{\rm pl}_\alpha"] & \mathcal{P} \arrow[d, "\hat\Pi^{\rm pl}"'] \\ \mathcal{M}^{\rm pl}_\alpha \arrow[r, "\det^{\rm pl}_\alpha"] & \mathcal{P}^{\rm pl}, \end{tikzcd} \tag{3.77}$$

$$\begin{tikzcd}[column sep=huge] \mathcal{P}\times\mathcal{P} \arrow[r, "\hat\Phi"'] \arrow[d, "\hat\Pi^{\rm pl}\times\hat\Pi^{\rm pl}"] & \mathcal{P} \arrow[d, "\hat\Pi^{\rm pl}"'] \\ \mathcal{P}^{\rm pl}\times\mathcal{P}^{\rm pl} \arrow[r, "\hat\Phi^{\rm pl}"] & \mathcal{P}^{\rm pl}. \end{tikzcd} \tag{3.78}$$

For each $\alpha\in K(\mathcal{A})$, define Artin $\mathbb{K}$-stacks $\mathcal{M}^{\rm fd}_\alpha,\mathcal{M}^{\rm fpd}_\alpha,\mathcal{M}^{\rm pfd}_\alpha$ by the 2-category fibre products in $\mathbf{Art}^{\bf lft}_{\mathbb{K}}$, in the sense of Definition 2.20:

$$\begin{gathered} \mathcal{M}^{\rm fd}_\alpha=\mathcal{M}_\alpha\times_{\det_\alpha,\mathcal{P},L_\alpha}*,\qquad \mathcal{M}^{\rm fpd}_\alpha=\mathcal{M}_\alpha\times_{\hat\Pi^{\rm pl}\circ\det_\alpha,\mathcal{P}^{\rm pl},L_\alpha}*,\\ \text{and}\qquad \mathcal{M}^{\rm pfd}_\alpha=\mathcal{M}^{\rm pl}_\alpha\times_{\det^{\rm pl}_\alpha,\mathcal{P}^{\rm pl},L_\alpha}*. \end{gathered} \tag{3.79}$$

These are different moduli stacks of objects in class α in $\mathcal{A}$ with fixed determinant L_α. Here 'fd', 'fpd', 'pfd' stand for 'fixed determinant', 'fixed projective determinant', and 'projective fixed determinant', respectively. They fit into 2-Cartesian squares in $\mathbf{Art}^{\bf lft}_{\mathbb{K}}$, with 2-morphisms $\eta^{\rm fd}_\alpha,\eta^{\rm fpd}_\alpha,\eta^{\rm pfd}_\alpha$:

$$\begin{tikzcd}[column sep=huge] \mathcal{M}^{\rm fd}_\alpha \arrow[r, "\pi"'] \arrow[d, "\pi_{\mathcal{M}_\alpha}"] \arrow[dr, phantom, "\eta^{\rm fd}_\alpha\Uparrow"] & * \arrow[d, "L_\alpha"'] \\ \mathcal{M}_\alpha \arrow[r, "\det_\alpha"] & \mathcal{P}, \end{tikzcd} \quad \begin{tikzcd}[column sep=huge] \mathcal{M}^{\rm fpd}_\alpha \arrow[r, "\pi"'] \arrow[d, "\pi_{\mathcal{M}_\alpha}"] \arrow[dr, phantom, "\eta^{\rm fpd}_\alpha\Uparrow"] & * \arrow[d, "L_\alpha"'] \\ \mathcal{M}_\alpha \arrow[r, "\hat\Pi^{\rm pl}\circ\det_\alpha"] & \mathcal{P}^{\rm pl}, \end{tikzcd}$$
$$\begin{tikzcd}[column sep=huge] \mathcal{M}^{\rm pfd}_\alpha \arrow[r, "\pi"'] \arrow[d, "\pi_{\mathcal{M}^{\rm pl}_\alpha}"] \arrow[dr, phantom, "\eta^{\rm pfd}_\alpha\Uparrow"] & * \arrow[d, "L_\alpha"'] \\ \mathcal{M}^{\rm pl}_\alpha \arrow[r, "\det^{\rm pl}_\alpha"] & \mathcal{P}^{\rm pl}. \end{tikzcd} \tag{3.80}$$

When $\alpha=0$ we have $\mathbb{K}$-points $[0]=[(0,*)]$ in $\mathcal{M}^{\rm fd}_0(\mathbb{K})$, $\mathcal{M}^{\rm fpd}_0(\mathbb{K})$, and we write $\mathcal{M}'^{\rm fd}_0=\mathcal{M}^{\rm fd}_0\setminus[0]$, $\mathcal{M}'^{\rm fpd}_0=\mathcal{M}^{\rm fpd}_0\setminus[0]$, as in Assumption 3.1(e). For $\alpha\neq 0$ we write $\mathcal{M}'^{\rm fd}_\alpha=\mathcal{M}^{\rm fd}_\alpha$ and $\mathcal{M}'^{\rm fpd}_\alpha=\mathcal{M}^{\rm fpd}_\alpha$.

As $\mathcal{P}^{\rm pl}$ is an algebraic $\mathbb{K}$-space by Assumption 3.43(h), $L_\alpha:*\to\mathcal{P}^{\rm pl}$ is a closed immersion, so in the second and third squares in (3.80), $\pi_{\mathcal{M}_\alpha}:\mathcal{M}^{\rm fpd}_\alpha\to\mathcal{M}_\alpha$ and $\pi_{\mathcal{M}^{\rm pl}_\alpha}:\mathcal{M}^{\rm pfd}_\alpha\to\mathcal{M}^{\rm pl}_\alpha$ are closed immersions. This means that $\mathcal{M}^{\rm fpd}_\alpha,\mathcal{M}^{\rm pfd}_\alpha$ are equivalent to *closed substacks* of $\mathcal{M}_\alpha,\mathcal{M}^{\rm pl}_\alpha$, where substacks are defined in Definition 2.19, as subcategories. Since (3.79) only determines $\mathcal{M}^{\rm fpd}_\alpha,\mathcal{M}^{\rm pfd}_\alpha$ up to equivalence anyway, we choose $\mathcal{M}^{\rm fpd}_\alpha,\mathcal{M}^{\rm pfd}_\alpha$ to be substacks of $\mathcal{M}_\alpha,\mathcal{M}_\alpha^{\rm pl}$, and $\pi_{\mathcal{M}_\alpha},\pi_{\mathcal{M}_\alpha^{\rm pl}}$ to be the inclusion morphisms. This determines $\mathcal{M}_\alpha^{\rm fpd},\mathcal{M}_\alpha^{\rm pfd}$ and $\pi_{\mathcal{M}_\alpha},\pi_{\mathcal{M}_\alpha^{\rm pl}}$ uniquely.

The universal property of 2-category fibre products in Definition 2.20 and (3.79) gives natural morphisms

$$\begin{aligned} &\Pi_{\alpha,\rm fd}^{\rm fpd}:\mathcal{M}_\alpha^{\rm fd}\longrightarrow\mathcal{M}_\alpha^{\rm fpd}, && \Pi'^{\rm fpd}_{\alpha,\rm fd}=\Pi_{\alpha,\rm fd}^{\rm fpd}\vert_{\mathcal{M}'^{\rm fd}_\alpha}:\mathcal{M}'^{\rm fd}_\alpha\longrightarrow\mathcal{M}'^{\rm fpd}_\alpha,\\ &\Pi_{\alpha,\rm fd}^{\rm pfd}:\mathcal{M}'^{\rm fd}_\alpha\longrightarrow\mathcal{M}_\alpha^{\rm pfd}, && \Pi_{\alpha,\rm fpd}^{\rm pfd}:\mathcal{M}'^{\rm fpd}_\alpha\longrightarrow\mathcal{M}_\alpha^{\rm pfd}, \end{aligned} \tag{3.81}$$

with $\Pi_{\alpha,\rm fd}^{\rm pfd}=\Pi_{\alpha,\rm fpd}^{\rm pfd}\circ\Pi'^{\rm fpd}_{\alpha,\rm fd}$ in $\mathrm{Ho}(\mathbf{Art}_{\mathbb{K}}^{\rm lft})$. For example, the 2-morphism $\mathrm{id}_{\hat\Pi^{\rm pl}}*\eta_\alpha^{\rm fd}:(\hat\Pi^{\rm pl}\circ\det_\alpha)\circ\pi_{\mathcal{M}_\alpha}\Rightarrow L_\alpha\circ\pi$ of 1-morphisms $\mathcal{M}_\alpha^{\rm fd}\Rightarrow\mathcal{P}^{\rm pl}$ and the universal property of $\mathcal{M}_\alpha^{\rm fpd}$ give a 1-morphism $\Pi_{\alpha,\rm fd}^{\rm fpd}:\mathcal{M}_\alpha^{\rm fd}\to\mathcal{M}_\alpha^{\rm fpd}$ with a 2-isomorphism $\pi_{\mathcal{M}_\alpha}\circ\Pi_{\alpha,\rm fd}^{\rm fpd}\Rightarrow\pi_{\mathcal{M}_\alpha}$.

Properties of fibre products imply that we have a 2-Cartesian square

$$\begin{tikzcd} \mathcal{M}'^{\rm fpd}_\alpha \arrow[rr, "\pi_{\mathcal{M}'_\alpha}"'] \arrow[d, "\Pi^{\rm pfd}_{\alpha,\rm fpd}"] \arrow[drr, phantom, "\Uparrow"] && \mathcal{M}'_\alpha \arrow[d, "\Pi^{\rm pl}_\alpha"'] \\ \mathcal{M}^{\rm pfd}_\alpha \arrow[rr, "\pi_{\mathcal{M}^{\rm pl}_\alpha}"] && \mathcal{M}^{\rm pl}_\alpha. \end{tikzcd} \tag{3.82}$$

Here $\Pi_\alpha^{\rm pl}:\mathcal{M}'_\alpha\to\mathcal{M}_\alpha^{\rm pl}$ is a principal $[*/\mathbb{G}_m]$-fibration by Definition 3.22, so $\Pi_{\alpha,\rm fpd}^{\rm pfd}:\mathcal{M}'^{\rm fpd}_\alpha\to\mathcal{M}_\alpha^{\rm pfd}$ is also a principal $[*/\mathbb{G}_m]$-fibration.

Define Artin stacks $\mathcal{M}^{\rm fd},\mathcal{M}^{\rm fpd},\mathcal{M}^{\rm pfd},\mathcal{M}^{\rm fd}_{\mathrm{rk}>0},\mathcal{M}^{\rm fpd}_{\mathrm{rk}>0},\mathcal{M}^{\rm pfd}_{\mathrm{rk}>0}$ by

$$\begin{gathered} \mathcal{M}^{\rm fd}=\coprod_{\alpha\in K(\mathcal{A})}\mathcal{M}_\alpha^{\rm fd},\qquad \mathcal{M}^{\rm fpd}=\coprod_{\alpha\in K(\mathcal{A})}\mathcal{M}_\alpha^{\rm fpd},\qquad \mathcal{M}^{\rm pfd}=\coprod_{\alpha\in K(\mathcal{A})}\mathcal{M}_\alpha^{\rm pfd},\\ \mathcal{M}^{\rm fd}_{\mathrm{rk}>0}=\coprod_{\substack{\alpha\in K(\mathcal{A}):\\ \mathrm{rk}\,\alpha>0}}\mathcal{M}_\alpha^{\rm fd},\qquad \mathcal{M}^{\rm fpd}_{\mathrm{rk}>0}=\coprod_{\substack{\alpha\in K(\mathcal{A}):\\ \mathrm{rk}\,\alpha>0}}\mathcal{M}_\alpha^{\rm fpd},\qquad \mathcal{M}^{\rm pfd}_{\mathrm{rk}>0}=\coprod_{\substack{\alpha\in K(\mathcal{A}):\\ \mathrm{rk}\,\alpha>0}}\mathcal{M}_\alpha^{\rm pfd}, \end{gathered}$$

and set $\mathcal{M}'^{\rm fd}=\mathcal{M}^{\rm fd}\setminus\{[0]\}$, $\mathcal{M}'^{\rm fpd}=\mathcal{M}^{\rm fpd}\setminus\{[0]\}$. Then $\mathcal{M}^{\rm fpd},\mathcal{M}'^{\rm fpd},\mathcal{M}^{\rm fpd}_{\mathrm{rk}>0}$ are substacks of $\mathcal{M}$, and $\mathcal{M}^{\rm pfd},\mathcal{M}^{\rm pfd}_{\mathrm{rk}>0}$ are substacks of $\mathcal{M}^{\rm pl}$. We write $\Pi^{\rm fpd}_{\rm fd}$, $\Pi^{\rm pfd}_{\rm fd},\Pi^{\rm pfd}_{\rm fpd},\Pi^{\rm fpd}_{\mathrm{rk}>0,\rm fd},\Pi^{\rm pfd}_{\mathrm{rk}>0,\rm fd},\Pi^{\rm pfd}_{\mathrm{rk}>0,\rm fpd}$ for the morphisms between these induced by the morphisms (3.81).

We can also generalize all the above to the triangulated category case. We replace Assumption 3.43 by Assumption 3.44, and $\mathcal{A},K(\mathcal{A})$ by $\mathcal{T},K(\mathcal{T})$, and the 2-category $\mathbf{Art}_{\mathbb{K}}^{\rm lft}$ by the ∞-category $\mathbf{HSt}_{\mathbb{K}}^{\rm lft}$, so that (3.79)–(3.80) are ∞-category fibre products. In fact we do not need any ∞-category techniques, we can treat $\mathbf{HSt}_{\mathbb{K}}^{\rm lft}$ as a 2-category (i.e. work in the 2-category truncation of $\mathbf{HSt}_{\mathbb{K}}^{\rm lft}$), as the arguments above and in the proof of Theorem 3.47 involve 2-morphisms, but no n-morphisms for $n>2$.

The next theorem will be proved in §4.11:

Theorem 3.47. *Work in the situation of Definition* 3.46*. Then:*

(a) *For all $\alpha,\beta \in K(\mathcal{A})$, the morphism $\Phi_{\alpha,\beta} : \mathcal{M}_\alpha \times \mathcal{M}_\beta \to \mathcal{M}_{\alpha+\beta}$ in Assumption* 3.1(g) *maps the substack $\mathcal{M}_\alpha^{\rm fpd} \times \mathcal{M}_\beta^{\rm fpd} \subset \mathcal{M}_\alpha \times \mathcal{M}_\beta$ to the substack $\mathcal{M}_{\alpha+\beta}^{\rm fpd} \subset \mathcal{M}_{\alpha+\beta}$, and so restricts to a unique morphism in* $\mathrm{Ho}(\mathbf{Art}_{\mathbb{K}}^{\rm lft})$:

$$\Phi_{\alpha,\beta}^{\rm fpd} := \Phi_{\alpha,\beta}|_{\mathcal{M}_\alpha^{\rm fpd}\times\mathcal{M}_\beta^{\rm fpd}} : \mathcal{M}_\alpha^{\rm fpd} \times \mathcal{M}_\beta^{\rm fpd} \longrightarrow \mathcal{M}_{\alpha+\beta}^{\rm fpd}. \tag{3.83}$$

Similarly, the morphism Ψ_α in Assumption 3.1(h) *restricts to*

$$\Psi_\alpha^{\rm fpd} := \Psi_\alpha|_{[*/\mathbb{G}_m]\times\mathcal{M}_\alpha^{\rm fpd}} : [*/\mathbb{G}_m] \times \mathcal{M}_\alpha^{\rm fpd} \longrightarrow \mathcal{M}_\alpha^{\rm fpd}. \tag{3.84}$$

Note that these $\Phi_{\alpha,\beta}^{\rm fpd}, \Psi_\alpha^{\rm fpd}$ satisfy the analogues of (3.4)–(3.7), *by restriction. Thus $\Psi_\alpha^{\rm fpd}$ is a $[*/\mathbb{G}_m]$-action on $\mathcal{M}_\alpha^{\rm fpd}$, which is free on $\mathcal{M}_\alpha^{\prime\rm fpd} \subseteq \mathcal{M}_\alpha^{\rm fpd}$. Also $\Pi_{\alpha,\rm fpd}^{\rm pfd} : \mathcal{M}_\alpha^{\prime\rm fpd} \to \mathcal{M}_\alpha^{\rm pfd}$ is a principal $[*/\mathbb{G}_m]$-bundle, with $[*/\mathbb{G}_m]$-action*

$$\Psi_\alpha^{\prime\rm fpd} = \Psi_\alpha^{\rm fpd}|_{[*/\mathbb{G}_m]\times\mathcal{M}_\alpha^{\prime\rm fpd}} : [*/\mathbb{G}_m] \times \mathcal{M}_\alpha^{\prime\rm fpd} \longrightarrow \mathcal{M}_\alpha^{\prime\rm fpd}.$$

(b) *Suppose the field $\mathbb{K}$ is algebraically closed, and let $\alpha \in K(\mathcal{A})$ with $\mathrm{rk}\,\alpha \neq 0$. Then $\Pi_{\alpha,\rm fd}^{\rm pfd} : \mathcal{M}_\alpha^{\rm fd} = \mathcal{M}_\alpha^{\prime\rm fd} \longrightarrow \mathcal{M}_\alpha^{\rm pfd}$ in* (3.81) *is locally trivial with fibre $[*/\mathbb{Z}_n]$ for $n = |\mathrm{rk}\,\alpha|$. Hence, if $H_*(-)$ is a homology theory over a $\mathbb{Q}$-algebra R satisfying Assumptions* 2.30–2.31 *then Assumption* 2.31(a) *shows $H_*(\Pi_{\alpha,\rm fd}^{\rm pfd}) : H_*(\mathcal{M}_\alpha^{\rm fd}) \to H_*(\mathcal{M}_\alpha^{\rm pfd})$ is an isomorphism. Thus $H_*(\Pi_{\mathrm{rk}>0,\rm fd}^{\rm pfd}) : H_*(\mathcal{M}_{\mathrm{rk}>0}^{\rm fd}) \to H_*(\mathcal{M}_{\mathrm{rk}>0}^{\rm pfd})$ is also an isomorphism.*

(c) *Suppose that for some $\alpha \in K(\mathcal{A})$, in morphisms in* $\mathrm{Ho}(\mathbf{Art}_{\mathbb{K}}^{\rm lft})$ *we have*

$$\det\nolimits_\alpha^{\rm pl} = L_\alpha \circ \pi : \mathcal{M}_\alpha^{\rm pl} \longrightarrow \mathcal{P}^{\rm pl}, \tag{3.85}$$

where $\pi : \mathcal{M}_\alpha^{\rm pl} \to *$ is the projection. Then $\mathcal{M}_\alpha^{\rm fpd} = \mathcal{M}_\alpha$ and $\mathcal{M}_\alpha^{\rm pfd} = \mathcal{M}_\alpha^{\rm pl}$.

The generalizations of **(a)**–**(c)** *to the triangulated category case also hold.*

Definition 3.48. In all of §3.2–§3.6, we can make the following substitutions:

- In Assumption 3.1 we replace $\mathcal{M}, \mathcal{M}', \mathcal{M}_\alpha, \mathcal{M}'_\alpha, \Phi_{\alpha,\beta}, \Psi_\alpha, \Theta^\bullet_{\alpha,\beta}$ by $\mathcal{M}^{\rm fpd}$, $\mathcal{M}^{\prime\rm fpd}, \mathcal{M}_\alpha^{\rm fpd}, \mathcal{M}_\alpha^{\prime\rm fpd}, \Phi_{\alpha,\beta}^{\rm fpd}, \Psi_\alpha^{\rm fpd}, \Theta^\bullet_{\alpha,\beta}|_{\mathcal{M}_\alpha^{\rm fpd}\times\mathcal{M}_\beta^{\rm fpd}}$, respectively.
- In Definition 3.22 we replace $\mathcal{M}^{\rm pl}, \mathcal{M}_\alpha^{\rm pl}, \Pi_\alpha^{\rm pl}$ by $\mathcal{M}^{\rm pfd}, \mathcal{M}_\alpha^{\rm pfd}, \Pi_{\alpha,\rm fpd}^{\rm pfd}$.

Then Definition 3.46 and Theorem 3.47(a) imply that all the properties of $\mathcal{M}, \mathcal{M}', \ldots, \Pi_\alpha^{\rm pl}$ used in §3.2–§3.6 and the proofs in §4.1–§4.10 also hold for $\mathcal{M}^{\rm fpd}, \mathcal{M}^{\prime\rm fpd}, \ldots, \Pi_{\alpha,\rm fpd}^{\rm pfd}$. Thus, the constructions of graded Lie algebras in §3.3–§3.6 work with these substitutions. Therefore:

(i) Let Assumption 2.30 hold over a commutative ring R, and Assumptions 3.1 and 3.43 hold. Then as in §3.3, we can define a '$t=0$' graded Lie algebra $\bigl(\tilde H_*(\mathcal{M}^{\rm fpd})^{t=0}, [\,,\,]^{t=0}\bigr)$ on $\tilde H_*(\mathcal{M}^{\rm fpd})^{t=0} = \tilde H_*(\mathcal{M}^{\rm fpd})/I_t \diamond \tilde H_*(\mathcal{M}^{\rm fpd})$.

(ii) Let Assumptions 2.30, 2.39, 3.1 and 3.43 hold. Then as in §3.4, we can define a 'projective linear' graded Lie algebra $\bigl(\tilde H_*(\mathcal{M}^{\rm pfd}), [\,,\,]^{\rm pfd}\bigr)$.

(iii) Let Assumption 2.30 hold over a $\mathbb{Q}$-algebra R, and Assumptions 3.1, 3.31 and 3.43 hold. Then following §3.5 we can define a 'positive rank' graded Lie algebra $\big(\tilde H_*(\mathcal{M}^{\rm fpd}_{{\rm rk}>0}),[\,,\,]^{\rm fpd}_{{\rm rk}>0}\big)$.

(iv) Let Assumption 2.30 hold for a $\mathbb{Q}$-algebra R, and Assumptions 3.1, 3.31 and 3.43 hold. Then as in §3.6 we can define a 'mixed' graded Lie algebra $\big(\tilde H_*(\mathcal{M}^{\rm fpd})^{\rm mix},[\,,\,]^{\rm mix}\big)$ combining (i)–(iii), where $\tilde H_*(\mathcal{M}^{\rm fpd})^{\rm mix}$ is the direct sum over $\alpha\in K(\mathcal{A})$ of $\tilde H_*(\mathcal{M}^{\rm fpd}_\alpha)$ for ${\rm rk}\,\alpha\neq 0$ and $R[s]\otimes_R \tilde H_*(\mathcal{M}^{\rm fpd}_\alpha)^{t=0}$ (or $R[s]\otimes_R\tilde H_*(\mathcal{M}^{\rm pfd}_\alpha)$ in case (ii)) for ${\rm rk}\,\alpha=0$.

There are natural morphisms between these Lie algebras as in Definitions 3.38 and 3.42. The analogue of all the above holds in the triangulated category case.

Remark 3.49. (a) In examples we may be more interested in the fixed determinant moduli stack $\mathcal{M}^{\rm fd}$ than in $\mathcal{M}^{\rm fpd},\mathcal{M}^{\rm pfd}$. Theorem 3.47(b) allows us to identify the graded Lie algebra $\big(\tilde H_*(\mathcal{M}^{\rm pfd}),[\,,\,]^{\rm pfd}\big)$ in Definition 3.48(ii) with $\tilde H_*(\mathcal{M}^{\rm fd})$, at least on the ${\rm rk}\neq 0$ part.

(b) Here is an interesting class of examples in which (3.85) holds. Work in the situation of Example 3.45 with $\mathbb{K}=\mathbb{C}$. If $E\in{\rm coh}(X)$ with $[\![E]\!]=\alpha$ in $K^{\rm num}({\rm coh}(X))$, then ${\rm rank}\,E\in\mathbb{N}$ and $c_i([E])\in H^{2i}(X,\mathbb{Q})$ depend only on α. Choose $\alpha\in K(\mathcal{A})$ with ${\rm rank}\,\alpha=0$ and $c_1(\alpha)=0$. Then for every $[E]\in\mathcal{M}_\alpha(\mathbb{K})$ we have ${\rm rank}\,E=c_1([E])=0$, so E is supported in codimension $\geqslant 2$ in X, and thus $\det E=\mathcal{O}_X$. Hence $\det^{\rm pl}_\alpha\circ\Pi^{\rm pl}_\alpha:\mathcal{M}_\alpha\to\mathcal{P}^{\rm pl}$ factors through $\mathcal{O}:*\to\mathcal{P}^{\rm pl}$.

Theorem 3.47(c) will identify sectors of the 'fixed determinant' Lie algebras $H_*(\mathcal{M}^{\rm fpd})^{t=0},\ldots,H_*(\mathcal{M}^{\rm fpd})^{\rm mix}$ in Definition 3.48 with the corresponding sectors of $H_*(\mathcal{M})^{t=0},\ldots,H_*(\mathcal{M})^{\rm mix}$ in §3.3–§3.6, and so helps us compute them.

3.8 Variations on the constructions

Here are some variations on the constructions of §3.3–§3.7.

3.8.1 Restricting to a substack

In §3.1–§3.7, we suppose we have a moduli stack $\mathcal{M}$ of objects in an abelian category $\mathcal{A}$ or triangulated category $\mathcal{T}$, and we define vertex algebras and Lie brackets on (some modification of) the homology $H_*(\mathcal{M})$.

Suppose we are given a substack $\mathcal{N}\subset\mathcal{M}$ (e.g. an open substack), also a (higher) Artin $\mathbb{K}$-stack locally of finite type, which could for example be the moduli stack of objects in a subcategory $\mathcal{B}\subset\mathcal{A}$ or $\mathcal{U}\subset\mathcal{T}$. Let $\mathcal{N}$ satisfy:

(i) If $[E],[F]\in\mathcal{N}(\mathbb{K})\subset\mathcal{M}(\mathbb{K})$ for $E,F\in\mathcal{A}$ then $[E\oplus F]\in\mathcal{N}(\mathbb{K})$. The morphism $\Phi:\mathcal{M}\times\mathcal{M}\to\mathcal{M}$ in Assumption 3.1(g), when restricted to $\mathcal{N}\times\mathcal{N}$, factors through the inclusion $\mathcal{N}\hookrightarrow\mathcal{M}$, so we have a stack morphism $\Phi|_{\mathcal{N}\times\mathcal{N}}:\mathcal{N}\times\mathcal{N}\to\mathcal{N}$.

(ii) If $[E]\in\mathcal{N}(\mathbb{K})\subset\mathcal{M}(\mathbb{K})$, so that ${\rm Iso}_{\mathcal{N}}([E])\subset{\rm Iso}_{\mathcal{M}}([E])\cong{\rm Aut}(E)$, then the subgroup $\mathbb{G}_m\cdot{\rm id}_E\subseteq{\rm Iso}_{\mathcal{M}}([E])$ lies in ${\rm Iso}_{\mathcal{N}}([E])\subset{\rm Iso}_{\mathcal{M}}([E])$. The morphism $\Psi:[*/\mathbb{G}_m]\times\mathcal{M}\to\mathcal{M}$ in Assumption 3.1(h), when restricted to $[*/\mathbb{G}_m]\times\mathcal{N}$, factors through the inclusion $\mathcal{N}\hookrightarrow\mathcal{M}$, so we have a stack morphism $\Psi|_{[*/\mathbb{G}_m]\times\mathcal{N}}:[*/\mathbb{G}_m]\times\mathcal{N}\to\mathcal{N}$.

Then in all of §3.1–§3.7 we can replace $\mathcal{M}$ by $\mathcal{N}$, and define a graded vertex algebra structure on $\check{H}_*(\mathcal{N})$, and graded Lie brackets on $\tilde{H}_*(\mathcal{N})^{t=0},\tilde{H}_*(\mathcal{N}^{\rm pl})$, $\tilde{H}_*(\mathcal{N}^{{\rm rk}>0}),\ldots$ under appropriate assumptions. The inclusion $\iota:\mathcal{N}\hookrightarrow\mathcal{M}$ induces vertex algebra and Lie algebra morphisms $H_*(\iota):H_*(\mathcal{N})\to H_*(\mathcal{M})$.

Example 3.50. **(a)** As in Remarks 3.13(a) and 3.21(a) we can take $\mathcal{N}=\mathcal{M}'=\mathcal{M}\setminus\{[0]\}$, and this is natural for the 'projective linear' version of §3.4.

(b) The 'fixed determinant' versions of §3.7 are an example of this construction, with $\mathcal{N}=\mathcal{M}^{\rm fpd}$ and $\mathcal{N}^{\rm pl}=\mathcal{M}^{\rm pfd}$.

(c) If $\mathcal{A}=\mathrm{coh}(X)$ is the category of coherent sheaves on a projective $\mathbb{K}$-scheme X, we could take $\mathcal{N}\subset\mathcal{M}$ to be the open substack of vector bundles on X.

(d) Given a stability condition τ on $\mathcal{A}$, such as slope stability or Gieseker stability, we could take $\mathcal{N}\subset\mathcal{M}$ to be the open substack of τ-semistable objects E in $\mathcal{A}$ with fixed 'slope' $\tau(E)=s$.

3.8.2 Restricting to the fixed points of a group

In the situation of §3.1–§3.7, suppose G is a group which acts on $\mathcal{A}$ or $\mathcal{T}$, and so acts on $K(\mathcal{A})$ or $K(\mathcal{T})$ and $\mathcal{M}$ preserving all the structures. Then we can form the substack $\mathcal{N}=\mathcal{M}^G$ of $\mathcal{M}$ fixed by G, and then work with $\mathcal{N}$ and $H_*(\mathcal{N})$ instead of $\mathcal{M}$ and $H_*(\mathcal{M})$, as in §3.8.1.

Alternatively, we can consider the action of G on the homology $H_*(\mathcal{M})$ (here we are thinking mostly of G finite or discrete, such as $G=\mathbb{Z}$) and take the G-invariant subspace $H_*(\mathcal{M})^G$ in $H_*(\mathcal{M})$. Then $\check{H}_*(\mathcal{M})^G$ is closed under operations $t^n\diamond -$, $[\,,\,]_n$ and contains $\mathbb{1}$, so it is a graded vertex subalgebra of $\check{H}_*(\mathcal{M})$ in §3.2, and similarly we get graded Lie subalgebras $(\tilde{H}_*(\mathcal{M})^G)^{t=0},\ldots$ of the graded Lie algebras $\tilde{H}_*(\mathcal{M})^{t=0},\ldots$ in §3.2–§3.7.

Example 3.51. **(a)** The classification of finite-dimensional simple Lie algebras by Dynkin diagrams [66] is divided into the 'simply-laced' cases A_n,D_n,E_6,E_7, E_8 and the 'non-simply-laced' cases B_n,C_n,F_4,G_2. As in Lusztig [106, §12], the non-simply-laced diagrams may be obtained by quotienting simply-laced diagrams by finite automorphism groups G.

We explain in §5 how to obtain the simply-laced simple Lie algebras $\mathfrak{g}$ as '$t=0$' Lie algebras $\tilde{H}_0(\mathcal{M})^{t=0}$ from $\mathcal{T}=D^b\,\text{mod-}\mathbb{C}Q$ for Q a quiver whose underlying graph is the corresponding Dynkin diagram. The non-simply-laced Lie algebras may be constructed as G-invariant '$t=0$' Lie subalgebras $(\tilde{H}_0(\mathcal{M})^G)^{t=0}$ from $\mathcal{T}=D^b\,\text{mod-}\mathbb{C}Q$ for Q a corresponding simply-laced quiver with automorphism group G. See Savage [140] for a similar result for quiver Ringel–Hall algebras and quantum groups.

(b) Let X be a smooth projective $\mathbb{K}$-scheme and $L\to X$ a line bundle (such as the canonical bundle K_X). Take $\mathcal{A}=\mathrm{coh}(X)$ or $\mathcal{T}=D^b\mathrm{coh}(X)$. Define an action of $G=\mathbb{Z}$ on $\mathcal{A}$ or $\mathcal{T}$ by $n:E\mapsto E\otimes L^n$ for $n\in\mathbb{Z}$ and $E\in\mathcal{A},\mathcal{T}$. This induces an action of $\mathbb{Z}$ on $\mathcal{M}$ and on $H_*(\mathcal{M})$, so we can consider the $\mathbb{Z}$-invariant substack $\mathcal{M}^{\mathbb{Z}} \subset \mathcal{M}$, and the $\mathbb{Z}$-invariant subspace $H_*(\mathcal{M})^{\mathbb{Z}} \subseteq H_*(\mathcal{M})$. In §3.8.3 we will explain a construction that involves $\mathcal{M}^{\mathbb{Z}}$ or $H_*(\mathcal{M})^{\mathbb{Z}}$ for $L = K_X$.

3.8.3 Relaxing $\sigma^*_{\alpha,\beta}(\Theta^\bullet_{\beta,\alpha}) \cong (\Theta^\bullet_{\alpha,\beta})^\vee[2n]$ in Assumption 3.1(j)

In Remark 3.3 we explained two methods (A),(B) for defining the data $\Theta^\bullet_{\alpha,\beta}$ in Assumption 3.1(i). In the 'even Calabi–Yau' method (A), when we take $\Theta^\bullet_{\alpha,\beta} = (\mathcal{E}xt^\bullet_{\alpha,\beta})^\vee$, the category $\mathcal{A}$ or $\mathcal{T}$ must satisfy a $2n$-Calabi–Yau condition for $\sigma^*_{\alpha,\beta}(\Theta^\bullet_{\beta,\alpha}) \cong (\Theta^\bullet_{\alpha,\beta})^\vee[2n]$, equation (3.8) in Assumption 3.1(j), to hold.

As most interesting categories $\mathcal{A},\mathcal{T}$ are not $2n$-Calabi–Yau, this means there is a very large supply of natural examples of data $\mathcal{A}, K(\mathcal{A}), \ldots, \Theta^\bullet_{\alpha,\beta}$ satisfying all of Assumption 3.1 except part (j). So if we can find ways to weaken the condition $\sigma^*_{\alpha,\beta}(\Theta^\bullet_{\beta,\alpha}) \cong (\Theta^\bullet_{\alpha,\beta})^\vee[2n]$, such that our vertex algebra and Lie algebra constructions still work, we may much increase the supply of examples.

In §3.1–§3.7, equation (3.8) is only used to prove (3.28)–(3.29), and for these it is enough to replace (3.8) by its consequence in $H^{2i}(\mathcal{M}_\alpha \times \mathcal{M}_\beta)$:

$$H^{2i}(\sigma_{\beta,\alpha})(c_i([\Theta^\bullet_{\alpha,\beta}])) = (-1)^i c_i([\Theta^\bullet_{\beta,\alpha}]), \qquad i = 1, 2, \ldots. \tag{3.86}$$

In examples this may hold under numerical conditions on the data used to build $\mathcal{A}$, without having to construct an isomorphism $\sigma^*_{\alpha,\beta}(\Theta^\bullet_{\beta,\alpha}) \cong (\Theta^\bullet_{\alpha,\beta})^\vee[2n]$. For instance, for the quiver representations $\mathcal{A} = \text{mod-}\mathbb{C}Q$, $\mathcal{T} = D^b\,\text{mod-}\mathbb{C}Q$ discussed in §5.1–§5.4, it is enough for Q to have the same number of edges $\overset{w}{\bullet} \to \overset{v}{\bullet}$ as edges $\overset{v}{\bullet} \to \overset{w}{\bullet}$, for all vertices $v, w \in Q_0$.

As in §3.8.1, we may restrict to a substack $\mathcal{N} \subset \mathcal{M}$. Then it is enough for the restriction of (3.86) to $H^{2i}(\mathcal{N}_\alpha \times \mathcal{N}_\beta)$ to hold. As in §3.8.2, we may restrict to a G-invariant subspace $H_*(\mathcal{M})^G$ of $H_*(\mathcal{M})$. Then it is enough for the cap product of (3.86) with all elements of $H_*(\mathcal{M})^G$ to hold.

Here is an interesting class of examples:

Example 3.52. Let X be a smooth projective $\mathbb{K}$-scheme of even dimension $\dim X = 2n$, and $\mathcal{A} = \text{coh}(X)$ be the category of coherent sheaves on X, and $K(\mathcal{A})$ be the numerical Grothendieck group of $\mathcal{A}$. As in Remark 3.3(A) we define $\Theta^\bullet \in \text{Perf}(\mathcal{M}\times\mathcal{M})$ by $\Theta^\bullet = (\mathcal{E}xt^\bullet)^\vee$, where $\mathcal{E}xt^\bullet$ is a perfect complex on $\mathcal{M}\times\mathcal{M}$ with $H^i(\mathcal{E}xt^\bullet\vert_{([E],[F])}) \cong \text{Ext}^i(E,F)$ for all $i \in \mathbb{Z}$ and $([E],[F]) \in (\mathcal{M}\times\mathcal{M})(\mathbb{K})$.

Serre duality gives natural isomorphisms

$$\text{Ext}^i(F,E)^* \cong \text{Ext}^{2n-i}(E \otimes K_X^{-1}, F), \tag{3.87}$$

where K_X is the canonical line bundle of X, so that $K_X \cong \mathcal{O}_X$ if X is Calabi–Yau. Define $\Upsilon : \mathcal{M} \to \mathcal{M}$ to be the obvious isomorphism of stacks acting by $[E] \mapsto [E \otimes K_X^{-1}]$ on $\mathbb{K}$-points. Then (3.87) is the cohomology at (E,F) of an isomorphism in $\text{Perf}(\mathcal{M}\times\mathcal{M})$:

$$\sigma^*((\mathcal{E}xt^\bullet)^\vee) \cong (\Upsilon \times \text{id})^*(\mathcal{E}xt^\bullet[2n]). \tag{3.88}$$

Write $\upsilon : K(\mathcal{A}) \to K(\mathcal{A})$ for the group isomorphism with $[\![E \otimes K_X^{-1}]\!] = \upsilon(\alpha)$ if $E \in \mathcal{A}$ with $[\![E]\!] = \alpha$, and $\Upsilon_\alpha = \Upsilon|_{\mathcal{M}_\alpha} : \mathcal{M}_\alpha \to \mathcal{M}_{\upsilon(\alpha)}$. Then using $\Theta^\bullet = (\mathcal{E}xt^\bullet)^\vee$ and restricting (3.88) to $\mathcal{M}_\alpha \times \mathcal{M}_\beta$ gives

$$\sigma_{\alpha,\beta}^*(\Theta_{\beta,\alpha}^\bullet) = (\Upsilon_\alpha \times \mathrm{id}_{\mathcal{M}_\beta})^*(\Theta_{\upsilon(\alpha),\beta}^\bullet)^\vee[2n].$$

Applying Chern classes c_i for $i \geqslant 1$ and using Assumption 2.30(b)(iv),(v) gives

$$H^{2i}(\sigma_{\alpha,\beta})(c_i([\Theta_{\beta,\alpha}^\bullet])) = (-1)^i H^{2i}(\Upsilon_\alpha \times \mathrm{id}_{\mathcal{M}_\beta})(c_i([\Theta_{\upsilon(\alpha),\beta}^\bullet])). \tag{3.89}$$

Suppose now that $\alpha \in K(\mathcal{A})$ satisfies $\upsilon(\alpha) = \alpha$, and

$$H^*(\Upsilon_\alpha) = \mathrm{id}_{H^*(\mathcal{M}_\alpha)} : H^*(\mathcal{M}_\alpha) \to H^*(\mathcal{M}_\alpha). \tag{3.90}$$

Then (3.89) reduces to (3.86).

(a) Suppose we can choose a subset $\Lambda \subset K(\mathcal{A})$ closed under addition, such that if $\alpha \in \Lambda$ then $\upsilon(\alpha) = \alpha$ and (3.90) holds. Define $\mathcal{N} = \coprod_{\alpha \in \Lambda} \mathcal{M}_\alpha$. Then the restriction of (3.86) to $H^{2i}(\mathcal{N}_\alpha \times \mathcal{N}_\beta)$ holds for all $\alpha, \beta \in K(\mathcal{A})$, by (3.89) and $H^*(\Upsilon)|_{\mathcal{N}} = \mathrm{id}$ by (3.90). Thus, we can apply the construction of §3.8.1 to $\mathcal{N}$, and obtain a graded vertex algebra structure on $\check{H}_*(\mathcal{N})$, and graded Lie algebra structures on $\tilde{H}_*(\mathcal{N})^{t=0}, \tilde{H}_*(\mathcal{N}^{\mathrm{pl}}), \ldots$.

(b) Define an action of $G = \mathbb{Z}$ on $\mathcal{M}$ such that $n \in \mathbb{Z}$ acts by $\Upsilon^n : \mathcal{M} \to \mathcal{M}$. Consider the $\mathbb{Z}$-invariant subspace $H_*(\mathcal{M})^{\mathbb{Z}}$ in $H_*(\mathcal{M})$. If $\zeta \in H_*(\mathcal{M})^{\mathbb{Z}}$ and $\lambda \in H^*(\mathcal{M})$ then $\zeta \cap \lambda = \zeta \cap H^*(\Upsilon^n)(\lambda)$ for all $n \in \mathbb{Z}$. Hence (3.89) implies that the cap product of (3.86) with all classes ζ in $H_*(\mathcal{M})^{\mathbb{Z}}$ holds, even though (3.86) itself may not hold. Therefore as in §3.8.2, we can define a graded vertex algebra structure on $\check{H}_*(\mathcal{M})^{\mathbb{Z}}$, and graded Lie algebra structures on $(\tilde{H}_*(\mathcal{M})^{\mathbb{Z}})^{t=0}, \ldots$.

We will see in §6 that (3.90) always holds if $\mathcal{A} = \mathrm{coh}(X)$ and $\alpha \in K(\mathcal{A})$ is a class of *dimension zero sheaves* on X. So we can build vertex algebras and Lie algebras from dimension zero sheaves on any smooth projective variety X.

Example 3.52 will be important in our discussion of representations of Lie algebras and vertex algebras in §3.8.4.

3.8.4 Representations of Lie algebras and vertex algebras

Representations of (graded) Lie algebras are defined in §2.1.1, and of graded vertex algebras in §2.2. There are several ways to use our theory to produce representations of the graded vertex algebras in §3.2, and the graded Lie algebras in §3.3–§3.7. Trivially, any Lie algebra or vertex algebra V is a representation of itself, and of any Lie subalgebra or vertex subalgebra of V. We focus on representations that arise in more nontrivial ways.

Example 3.53. In the situation of §3.3 over $\mathcal{A}$ or $\mathcal{T}$, there is a natural representation of the graded Lie algebra $\bigl(\check{H}_*(\mathcal{M})^{t=0}, [\,,\,]^{t=0}\bigr)$ on the graded R-module

$\tilde H_*(\mathcal{M})$, defined by $[\bar\zeta,\eta]=[\zeta,\eta]_0$ for all $\bar\zeta=\zeta+I_t\diamond H_*(\mathcal{M}_\alpha)$ in $\tilde H_*(\mathcal{M})^{t=0}$ and η in $\tilde H_*(\mathcal{M})$. To see this is well defined, note that $[t^p\diamond\epsilon,\eta]_0=0$ for all $\epsilon,\eta\in H_*(\mathcal{M})$ and $p>0$ by (3.26), so $[\zeta,\eta]_0$ is independent of the choice of representative ζ for $\bar\zeta=\zeta+I_t\diamond H_*(\mathcal{M}_\alpha)$. To see it is a graded Lie algebra representation, compare (2.6) and (3.30).

For the next example we illustrate the ideas using the '$t=0$' version of §3.3 on an abelian category $\mathcal{A}$, but the same methods work for vertex algebra representations, the other versions of §3.4–§3.7, and for triangulated categories $\mathcal{T}$.

Example 3.54. Let $\mathcal{A},K(\mathcal{A})$, $\mathcal{M}=\coprod_{\alpha\in K(\mathcal{A})}\mathcal{M}_\alpha$ and the graded Lie algebra $\tilde H_*(\mathcal{M})^{t=0}$ be as in §3.3. Suppose $\Lambda\subset K(\mathcal{A})$ is a subset closed under addition (e.g. a subgroup). Write $\mathcal{N}=\coprod_{\alpha\in\Lambda}\mathcal{M}_\alpha$, as an open substack $\mathcal{N}\subset\mathcal{M}$. Then as in §3.8.1 we obtain a graded Lie subalgebra $\tilde H_*(\mathcal{N})^{t=0}$ in $\tilde H_*(\mathcal{M})^{t=0}$.

For $\rho\in K(\mathcal{A})$, write $\mathcal{R}_\rho=\coprod_{\lambda\in\Lambda}\mathcal{M}_{\rho+\lambda}$, as an open substack $\mathcal{R}_\rho\subset\mathcal{M}$. By (3.37) we have canonical isomorphisms

$$\tilde H_*(\mathcal{N})^{t=0}\cong\textstyle\bigoplus_{\lambda\in\Lambda}\tilde H_*(\mathcal{M}_\lambda)^{t=0},\qquad \tilde H_*(\mathcal{R}_\rho)^{t=0}\cong\textstyle\bigoplus_{\lambda\in\Lambda}\tilde H_*(\mathcal{M}_{\rho+\lambda})^{t=0}.$$

Define a representation of $\bigl(\tilde H_*(\mathcal{N})^{t=0},[\,,\,]^{t=0}\bigr)$ on $\tilde H_*(\mathcal{R}_\rho)^{t=0}$ by the restriction of the Lie bracket $[\,,\,]^{t=0}:\tilde H_*(\mathcal{M})^{t=0}\times\tilde H_*(\mathcal{M})^{t=0}\to\tilde H_*(\mathcal{M})^{t=0}$ from §3.3 to $[\,,\,]^{t=0}:\tilde H_*(\mathcal{N})^{t=0}\times\tilde H_*(\mathcal{R}_\rho)^{t=0}\to\tilde H_*(\mathcal{R}_\rho)^{t=0}$, regarding $\tilde H_*(\mathcal{N})^{t=0},\tilde H_*(\mathcal{R}_\rho)^{t=0}$ as R-submodules of $\tilde H_*(\mathcal{M})^{t=0}$ by (3.37). Then (2.6) for $\tilde H_*(\mathcal{R}_\rho)^{t=0}$ follows from (3.45)–(3.46) for $\tilde H_*(\mathcal{M})^{t=0}$ in Theorem 3.20.

Now consider relaxing $\sigma_{\alpha,\beta}^*(\Theta_{\beta,\alpha}^\bullet)\cong(\Theta_{\alpha,\beta}^\bullet)^\vee[2n]$ in Assumption 3.1(j), as in §3.8.3. The important points are:

(i) To prove $\bigl(\tilde H_*(\mathcal{N})^{t=0},[\,,\,]^{t=0}\bigr)$ is a Lie algebra, we only need (3.8) to hold for all $\alpha,\beta\in\Lambda$. Also, (3.8) can be replaced by (3.86).

(ii) To prove $\tilde H_*(\mathcal{R}_\rho)^{t=0}$ is a representation of $\bigl(\tilde H_*(\mathcal{N})^{t=0},[\,,\,]^{t=0}\bigr)$, we again only need (3.8) to hold for all $\alpha,\beta\in\Lambda$: we *do not need* (3.8) for α or β in $\rho+\Lambda$. This is because the proof of (2.6) involves (3.29) for $\zeta\in H_a(\mathcal{M}_\alpha)$, $\eta\in H_b(\mathcal{M}_\beta)$, $\theta\in H_c(\mathcal{M}_\gamma)$ with $\alpha,\beta\in\Lambda$ and $\gamma\in\rho+\Lambda$, but the proof of (3.29) only uses (3.8) for (α,β), not for (β,γ) or (α,γ).

Also, (3.8) can again be replaced by (3.86) for $\alpha,\beta\in\Lambda$.

In §3.8.3 we explained that there are large classes of interesting examples in which Assumption 3.1(j) does not hold, and so §3.1–§3.7 do not yield vertex algebra or Lie algebras from $H_*(\mathcal{M})$, but we may still be able to find a substack $\mathcal{N}\subset\mathcal{M}$ with such that $H_*(\mathcal{N})$ gives vertex algebras and Lie algebras, or a G-action on $\mathcal{M}$ such that $H_*(\mathcal{M})^G$ gives vertex algebras and Lie algebras.

The argument above shows that in these cases, the graded vertex algebra $\check H_*(\mathcal{N})$ (and similarly $\check H_*(\mathcal{M})^G$) has representations on $\check H_*(\mathcal{R}_\rho)$ and $\check H_*(\mathcal{M})$, and the graded Lie algebra $\tilde H_*(\mathcal{N})^{t=0}$ (and similarly $(\tilde H_*(\mathcal{M})^G)^{t=0}$) has representations on $\tilde H_*(\mathcal{R}_\rho)^{t=0}$ and $\tilde H_*(\mathcal{M})^{t=0}$.

This gives a *large source of interesting representations of vertex algebras and Lie algebras.* As in Example 3.52, if X is any smooth projective $\mathbb{K}$-scheme of even dimension and $\mathcal{A} = \mathrm{coh}(X)$ or $\mathcal{T} = D^b\mathrm{coh}(X)$, we can build vertex and Lie algebras from dimension zero sheaves on X, and these have representations on the homology $\check{H}_*(\mathcal{M})$, $\tilde{H}_*(\mathcal{M})^{t=0}$ of all coherent sheaves and complexes. We will use this in §6 to explain work of Grojnowski [57] and Nakajima [121, 122].

3.8.5 Morphisms of Lie algebras and vertex algebras

Let $\mathcal{A}_1, \mathcal{A}_2$ be $\mathbb{K}$-linear abelian categories satisfying Assumption 3.1 with moduli stacks $\mathcal{M}_1, \mathcal{M}_2$, or $\mathcal{T}_1, \mathcal{T}_2$ be $\mathbb{K}$-linear triangulated categories satisfying Assumption 3.2 with moduli stacks $\mathcal{M}_1, \mathcal{M}_2$, and let Assumption 2.30 hold for (co)homology theories $H_*(-), H^*(-)$ of (higher) Artin $\mathbb{K}$-stacks over R.

Suppose that either $F : \mathcal{A}_1 \to \mathcal{A}_2$, or $F : \mathcal{T}_1 \to \mathcal{T}_2$, or $F : \mathcal{A}_1 \to \mathcal{T}_2$ is a $\mathbb{K}$-linear exact functor, which induces a morphism $f : \mathcal{M}_1 \to \mathcal{M}_2$ on moduli stacks, and suppose that $(f \times f)^*(\Theta_2^\bullet) \cong \Theta_1^\bullet$ in $\mathrm{Perf}(\mathcal{M}_1 \times \mathcal{M}_1)$. Then $H_*(f) : H_*(\mathcal{M}_1) \to H_*(\mathcal{M}_2)$ respects the important structures discussed in §3.1–§3.4, so it induces a morphism $\check{H}_*(f) : \check{H}_*(\mathcal{M}_1) \to \check{H}_*(\mathcal{M}_2)$ of graded vertex algebras in §3.2, and morphisms $\tilde{H}_*(f)^{t=0} : \tilde{H}_*(\mathcal{M}_1)^{t=0} \to \tilde{H}_*(\mathcal{M}_2)^{t=0}$ and $\tilde{H}_*(f^{\mathrm{pl}}) : \tilde{H}_*(\mathcal{M}_1^{\mathrm{pl}}) \to \tilde{H}_*(\mathcal{M}_2^{\mathrm{pl}})$ of graded Lie algebras in §3.3–§3.4.

If we know $\check{H}_*(\mathcal{M}_1)$ is a *simple* vertex algebra, as in §2.2.2 (this happens in Theorem 5.19 below, for instance), then $\check{H}_*(f)$ is automatically injective.

3.9 Open questions

Finally we give some open questions for future work.

3.9.1 The 'supported on indecomposables' version

An object E in the $\mathbb{K}$-linear abelian category $\mathcal{A}$ is *indecomposable* if $E \not\cong 0$ and we cannot write $E \cong E_1 \oplus E_2$ for $E_1, E_2 \not\cong 0$. Equivalently, the algebraic $\mathbb{K}$-group $\mathrm{Aut}(E) = \mathrm{Iso}_{\mathcal{M}}([E])$ has rank 1. Write $\mathcal{M}^{\mathrm{ind}} \subset \mathcal{M}$ for the open substack whose $\mathbb{K}$-points are indecomposable objects in $\mathcal{A}$.

In the study of Ringel–Hall type (Lie) algebras, sometimes one associates a (Lie) algebra $\mathcal{H}$ to an abelian category $\mathcal{A}$, which contains a much smaller, interesting Lie subalgebra $\mathcal{L} \subset \mathcal{H}$ that is in some sense 'supported on indecomposables', where $\mathcal{H}$ may look quite like the universal enveloping algebra $U(\mathcal{L})$.

For Lie algebras actually supported on indecomposables, see Riedtmann [134] and Ringel [135]. The author [76–79, 81] constructed a Ringel–Hall algebra of 'stack functions' $\mathrm{SF}(\mathcal{M})$, with a Lie subalgebra $\mathrm{SF}^{\mathrm{ind}}(\mathcal{M})$ of stack functions 'supported on virtual indecomposables'. Under extra assumptions on $\mathcal{A}$ there is an 'integration morphism' from $\mathrm{SF}^{\mathrm{ind}}(\mathcal{M})$ to an explicit Lie algebra, which is important in wall-crossing formulae for enumerative invariants in $\mathcal{A}$, including Donaldson–Thomas invariants of Calabi–Yau 3-folds X when $\mathcal{A} = \mathrm{coh}(X)$.

The author would like to extend this to our vertex algebra and Lie algebras picture. We illustrate it for the '$t = 0$' version of §3.3.

Question 3.55. **(a)** *In the situation of* §3.3, *consider the graded Lie subalgebra of* $\bigl(\tilde H_*(\mathcal{M})^{t=0},[\,,\,]^{t=0}\bigr)$ *generated by the image of the homology* $H_*(\mathcal{M}^{\mathrm{ind}})$ *of the moduli stack of indecomposable objects under the inclusion* $\mathcal{M}^{\mathrm{ind}}\hookrightarrow\mathcal{M}$.

In examples, is this Lie subalgebra much smaller than $\tilde H_*(\mathcal{M})^{t=0}$, *and is it an interesting graded Lie algebra?*

(b) *Can you construct a moduli stack* $\mathcal{M}^{\mathrm{vi}}$ *of 'virtual indecomposables', with a morphism* $\pi:\mathcal{M}^{\mathrm{vi}}\to\mathcal{M}$, *which is in some sense a refinement of* $\mathcal{M}^{\mathrm{ind}}\hookrightarrow\mathcal{M}$, *an* $R[t]$*-action* $\diamond$ *on* $\tilde H_*(\mathcal{M}^{\mathrm{vi}})$, *and a graded Lie bracket* $[\,,\,]^{\mathrm{vi}}$ *on* $\tilde H_*(\mathcal{M}^{\mathrm{vi}})^{t=0}$, *such that* $\tilde H_*(\pi)^{t=0}:\tilde H_*(\mathcal{M}^{\mathrm{vi}})^{t=0}\to\tilde H_*(\mathcal{M})^{t=0}$ *is a Lie algebra morphism, whose image is roughly the graded Lie subalgebra discussed in* **(a)***?*

(c) *In the vertex algebras setting of Theorem* 3.14, *can one define an interesting* ***vertex Lie subalgebra*** $\check H_*(\mathcal{M})^{\mathrm{ind}}$ *of the vertex algebra* $\check H_*(\mathcal{M})$, *as in* §2.2.3, *which is 'supported on indecomposables'? For example, in* **(b)** *we might hope* $H_*(\mathcal{M}^{\mathrm{vi}})$ *is a vertex Lie algebra, and set* $\check H_*(\mathcal{M})^{\mathrm{ind}}=H_*(\pi)(H_*(\mathcal{M}^{\mathrm{vi}}))$.

In **(b)**, the idea is that $\mathbb{K}$-points of $\mathcal{M}^{\mathrm{vi}}$ should parametrize objects of $\mathcal{A}$, plus some kind of extra data. We mostly care about the homology $H_*(\mathcal{M}^{\mathrm{vi}})$, not the stack $\mathcal{M}^{\mathrm{vi}}$ itself, and we would be happy with a classifying space type construction which is only natural up to homotopy. So, for example, we want the fibre of $\pi:\mathcal{M}^{\mathrm{vi}}\to\mathcal{M}$ over $[E]$ for any indecomposable $E\in\mathcal{A}$ to be contractible, so that $\pi^{-1}(\mathcal{M}^{\mathrm{ind}})\subset\mathcal{M}^{\mathrm{vi}}$ is isomorphic to $\mathcal{M}^{\mathrm{ind}}$ in homology. The author has some ideas on this, and hopes to write about it in future.

The author is uncertain what 'supported on indecomposables' should mean in a triangulated category $\mathcal{T}$, although this is an interesting question.

3.9.2 Restricting to a 'semistable' open substack $\mathcal{M}^{\mathrm{ss}}\subset\mathcal{M}$

The following is closely connected to the ideas of §3.9.1.

In examples, we may wish to study not the homology $H_*(\mathcal{M})$ of the whole moduli stack $\mathcal{M}$, but the homology $H_*(\mathcal{M}^{\mathrm{ss}})$ of an open substack $\mathcal{M}^{\mathrm{ss}}\subset\mathcal{M}$ of 'semistable' points (or 'stable', or 'simple', or 'torsion-free'), where $\mathcal{M}^{\mathrm{ss}}$ does *not* satisfy the conditions on $\mathcal{N}$ in §3.8.1, in particular, if $[E],[F]\in\mathcal{M}^{\mathrm{ss}}(\mathbb{K})\subset\mathcal{M}(\mathbb{K})$ for $E,F\in\mathcal{A}$ then we need not have $[E\oplus F]\in\mathcal{M}^{\mathrm{ss}}(\mathbb{K})$. In particular, the author has in mind cases when $H_*(\mathcal{M}^{\mathrm{ss}})$ has been given the structure of a representation of an interesting Lie algebra, in connection with the ideas of §3.8.4.

In some interesting examples, writing $\iota:\mathcal{M}^{\mathrm{ss}}\hookrightarrow\mathcal{M}$ for the inclusion of substacks, we may be able to show that:

(i) $H_*(\iota):H_*(\mathcal{M}^{\mathrm{ss}})\to H_*(\mathcal{M})$ is injective.

(ii) The image of $H_*(\mathcal{M}^{\mathrm{ss}})$ in $H_*(\mathcal{M})$ is closed under the Lie brackets from §3.3–§3.7, or the Lie algebra representations from §3.8.4.

Then we can replace $H_*(\mathcal{M})$ by $H_*(\mathcal{M}^{\mathrm{ss}})$ in our Lie algebra or representation.

Of course, this is of no use unless we have effective ways of proving (i),(ii). Part (i) is actually a well known problem. For homology over a field R, using the duality between homology and cohomology, (i) is equivalent to asking that

$H^*(\iota) : H^*(\mathcal{M}) \to H^*(\mathcal{M}^{\rm ss})$ should be surjective. This is known as *Kirwan surjectivity*, following work of Atiyah and Bott [5, §9] and Kirwan [88, §14] in Morse theory, as discussed by McGerty and Nevins [112], Harada and Wilkin [58], and Fisher [43], for example.

Suppose $\mathcal{M}_\alpha = [V_\alpha/G_\alpha]$ is a global quotient stack, and $\mathcal{M}_\alpha^{\rm ss} = [V_\alpha^{\rm ss}/G_\alpha]$ the the quotient of a G-invariant open subscheme $V_\alpha^{\rm ss} \subset V_\alpha$. If $\mathcal{M}_\alpha, V_\alpha$ are smooth, and V_α admits a G-equivariantly perfect stratification with $V_\alpha^{\rm ss}$ a union of open strata, then $H^*(\iota) : H^*(\mathcal{M}) \to H^*(\mathcal{M}^{\rm ss})$ is surjective. This often happens if $V_\alpha^{\rm ss}$ is a semistable subscheme in the sense of Geometric Invariant Theory [44].

We have less justification for (ii), but in the examples the author has in mind, it seems to be related to the ideas of §3.9.1, and considering $H_*(\mathcal{M}^{\rm ss})$ as a subspace of $H_*(\mathcal{M})$ 'supported on indecomposables'.

3.9.3 Other algebraic structures on $\tilde H_*(\mathcal{M})$

Question 3.56. *In the situation of* §3.1, *perhaps under additional assumptions, can we define other interesting algebraic structures on* $\check H_*(\mathcal{M}), \tilde H_*(\mathcal{M})^{t=0}, \ldots$ *by a similar method to the graded Lie brackets of* §3.3–§3.7*? If so, do they satisfy interesting compatibility relations with our vertex algebras and Lie brackets?*

For example:

(i) Under what conditions, or extra structure, on a triangulated category $\mathcal{T}$, can we make the vertex algebra $\check H_*(\mathcal{M})$ in §3.2 into a *vertex operator algebra*, as in §2.2.1? This involves finding a suitable class $\omega \in H_4(\mathcal{M}_0)$.

(ii) Ringel [138] shows some Ringel–Hall algebras $\mathcal{H}$ defined from an abelian category $\mathcal{A}$ have a compatible cocommutative comultiplication making $\mathcal{H}$ into a bialgebra. The author has some ideas on how to define a comultiplication $\tilde H_*(\mathcal{M}^{\rm pl}) \to \tilde H_*(\mathcal{M}^{\rm pl} \times \mathcal{M}^{\rm pl}) \cong \tilde H_*(\mathcal{M}^{\rm pl}) \otimes_R \tilde H_*(\mathcal{M}^{\rm pl})$, under extra assumptions which imply $\mathcal{M}, \mathcal{M}^{\rm pl}$ are smooth Artin $\mathbb{K}$-stacks.

(iii) Many Lie algebras such as Kac–Moody algebras have nondegenerate invariant inner products $(\, , \,)$, and it would be interesting to construct these in our situation, under appropriate assumptions.

3.9.4 Generalizing the (co)homology theories

One could consider replacing ordinary (co)homology $H_*(-), H^*(-)$ in §2.4 by some kind of generalized (co)homology theories, so that parts of Assumption 2.30 might need modification, and investigate whether our theories (with variations) can still be made to work. For example:

(i) We could suppose the category $\mathcal{A}$ or $\mathcal{T}$, and the moduli stack $\mathcal{M}$, carry the action of an algebraic $\mathbb{K}$-group G, such as $\mathbb{G}_m$. Then we could try to replace (co)homology $H_*(\mathcal{M}), H^*(\mathcal{M})$ in §3.1–§3.8 by *equivariant* (co)homology $H_*^G(\mathcal{M}), H^*_G(\mathcal{M})$, which are modules over the R-algebra $H^*_G(*)$. We must consider how G acts on the complexes $\Theta^\bullet_{\alpha,\beta}$ as well as on $\mathcal{A}, \mathcal{T}, \mathcal{M}$.

For example, we could take $\mathcal{A} = \mathrm{coh}(X)$ or $\mathcal{T} = D^b\mathrm{coh}(X)$ for X a toric projective $\mathbb{K}$-scheme, and take G to be the algebraic torus acting on X. Or we could take $\mathcal{A} = \text{mod-}\mathbb{K}Q$ or $\mathcal{T} = D^b\,\text{mod-}\mathbb{K}Q$ for a quiver Q, as in §5.1–§5.4, and $G = \mathbb{G}_m$ acting by rescaling the edge morphisms $\rho_e : X_{t(e)} \to X_{h(e)}$ in each $(X, \rho) \in \text{mod-}\mathbb{K}Q$ by $\lambda : \rho_e \mapsto \lambda\rho_e$ for $\lambda \in \mathbb{G}_m$.

(ii) When $\mathbb{K} = R = \mathbb{C}$, we could consider *mixed Hodge structures* on $H^*(\mathcal{M})$ and $H_*(\mathcal{M})$, giving additional structure on our vertex and Lie algebras.

(iii) We could try to replace (co)homology by *K-theory* $K^*(\mathcal{M}), K_*(\mathcal{M})$.

It would be particularly interesting to realize 'quantum' versions of vertex algebras and Lie algebras this way, just as quantum groups $U_q(\mathfrak{g})$ generalize suitable Lie algebras $\mathfrak{g}$, as in Jantzen [71] and Lusztig [106].

3.9.5 Physical interpretation of our work

Question 3.57. *Do the vertex algebra structures on $\check{H}_*(\mathcal{M})$ in Theorem 3.14 have an interpretation in Conformal Field Theory and String Theory?*

Here the author is thinking especially of our theory applied to categories $\mathcal{A}, \mathcal{T}$ which are already intensively studied in String Theory, such as derived categories $\mathcal{T} = D^b\mathrm{coh}(X)$ for X a complex Calabi–Yau $2n$-fold, which appear in the Homological Mirror Symmetry story and are interpreted as categories of boundary conditions for a Super Conformal Field Theory, and $\mathcal{T} = D^b\,\text{mod-}\mathbb{C}Q$.

3.9.6 Is there a parallel story for the 'odd Calabi–Yau' case?

As in Remark 3.3(A), the most natural way to obtain data satisfying Assumptions 3.1 or 3.2 is to start with a category $\mathcal{A}$ or $\mathcal{T}$ which is $2n$-Calabi–Yau, and take $\Theta^\bullet = (\mathcal{E}xt^\bullet)^\vee$. For more general $\mathcal{A}, \mathcal{T}$, as in Remark 3.3(B) we can make $\mathcal{A}, \mathcal{T}$ act like a $2n$-Calabi–Yau category by taking $\Theta^\bullet = (\mathcal{E}xt^\bullet)^\vee \oplus \sigma^*(\mathcal{E}xt^\bullet)[2n]$. As in Remark 3.17, if $\mathcal{A}, \mathcal{T}$ are $(2n+1)$-Calabi–Yau this gives $[\Theta^\bullet] = 0$ in $K_0(\mathrm{Perf}(\mathcal{M} \times \mathcal{M}))$, and then the vertex algebras $\check{H}_*(\mathcal{M})$ in §3.2 are commutative, and the Lie algebras $\tilde{H}_*(\mathcal{M})^{t=0}, \ldots$ in §3.3–§3.8 are abelian. Thus, in the 'odd Calabi–Yau' case our constructions are basically trivial and boring.

This may be surprising, as there is a lot of very interesting and special geometry for 3-Calabi–Yau categories, but our picture appears not to see it.

Question 3.58. *This whole book is in some sense about the 'even Calabi–Yau' case. Is there a related, but different, story about the 'odd Calabi–Yau' case?*

Kontsevich and Soibelman's Cohomological Hall algebras of 3-Calabi–Yau categories [93, 144] might be a good starting point for thinking about this. Also, there are related, but different, notions of 'orientation data' for even Calabi–Yau categories and for odd Calabi–Yau categories [31, §2–§3], where 'even Calabi–Yau' orientation data is related to the choice of $\epsilon_{\alpha,\beta}$ in Assumption 3.1(d); the 'odd Calabi–Yau' story should involve 'odd Calabi–Yau' orientation data.

4 Proofs of main results in §3

4.1 Proof of Proposition 3.6

Let Assumption 3.2 hold for $\mathcal{T}$. For (a), suppose $\mathcal{T} \not\cong 0$. Then we can choose a nonzero object E in $\mathcal{T}$. Assumption 3.2(iv) gives a morphism $\xi_E : \mathbb{A}^1 \to \mathcal{M}_0$ which on $\mathbb{K}$-points maps $0 \mapsto [E \oplus E[1]]$ and $x \mapsto [0]$ for $0 \neq x \in \mathbb{K}$. Write $f : \mathcal{M}' \hookrightarrow \mathcal{M}$ for the inclusion. Define morphisms

$$\begin{aligned} &g : \mathcal{M} \longrightarrow \mathcal{M}', && F : \mathcal{M}' \times \mathbb{A}^1 \longrightarrow \mathcal{M}', && G : \mathcal{M} \times \mathbb{A}^1 \longrightarrow \mathcal{M}, \text{ by} \\ &g = \Phi|_{\mathcal{M}\times\{[E\oplus E[1]]\}}, && F = \Phi \circ (f \times \xi_E), && G = \Phi \circ (\mathrm{id}_{\mathcal{M}} \times \xi_E). \end{aligned}$$

Note that on $\mathbb{K}$-points g maps $[F] \mapsto [F \oplus E \oplus E[1]]$, and so maps to $\mathcal{M}' \subset \mathcal{M}$ as $[F \oplus E \oplus E[1]] \neq [0]$ for all $[F]$.

Then $g \circ f, f \circ g$ both map $[F] \mapsto [F \oplus E \oplus E[1]]$ on $\mathbb{K}$-points, and F, G both map $([F], 0) \mapsto [F \oplus E \oplus E[1]]$ and $([F], 1) \mapsto [F]$. Thus $F|_{\mathcal{M}'\times\{0\}} = g \circ f$, $F|_{\mathcal{M}'\times\{1\}} = \mathrm{id}_{\mathcal{M}'}$, $G|_{\mathcal{M}\times\{0\}} = f \circ g$, $G|_{\mathcal{M}\times\{1\}} = \mathrm{id}_{\mathcal{M}}$, so f is a homotopy equivalence by Definition 2.37. The same argument works for $\mathcal{M}_0' \hookrightarrow \mathcal{M}_0$.

For (b), by a very similar argument we can show that $\Phi_{\alpha,-\alpha}|_{\mathcal{M}_\alpha\times\{[E[1]]\}} : \mathcal{M}_\alpha \cong \mathcal{M}_\alpha \times \{[E[1]]\} \to \mathcal{M}_0$ is a homotopy inverse for $\Phi_{0,\alpha}|_{\mathcal{M}_0\times\{[E]\}} : \mathcal{M}_0 \to \mathcal{M}_\alpha$. Equation (3.13) then follows from Lemma 2.38.

4.2 Proof of Theorem 3.12

We work in the situation of Theorem 3.12, and we also write $a = \tilde{a} + 2 - \chi(\alpha, \alpha)$, $b = \tilde{b} + 2 - \chi(\beta, \beta)$, $c = \tilde{c} + 2 - \chi(\gamma, \gamma)$, so that $\zeta \in H_a(\mathcal{M}_\alpha)$, $\eta \in H_b(\mathcal{M}_\beta)$, and $\theta \in H_c(\mathcal{M}_\gamma)$ by (3.23).

4.2.1 Proof of equation (3.25)

Consider the restriction $\Theta^\bullet_{\alpha,0}|_{\mathcal{M}_\alpha\times\{[0]\}}$ of $\Theta^\bullet_{\alpha,0}$ to $\mathcal{M}_\alpha \times \{[0]\} \subseteq \mathcal{M}_\alpha \times \mathcal{M}_0$. Since $0 \oplus 0 = 0$, so $\Phi_{0,0}(\mathbb{K}) : ([0], [0]) \mapsto [0]$, and restricting (3.10) with $\beta = \gamma = 0$ to $\mathcal{M}_\alpha \times \{[0]\}$ implies that $\Theta^\bullet_{\alpha,0}|_{\mathcal{M}_\alpha\times\{[0]\}} \cong \Theta^\bullet_{\alpha,0}|_{\mathcal{M}_\alpha\times\{[0]\}} \oplus \Theta^\bullet_{\alpha,0}|_{\mathcal{M}_\alpha\times\{[0]\}}$. Hence $\Theta^\bullet_{\alpha,0}|_{\mathcal{M}_\alpha\times\{[0]\}} = 0$, and similarly $\Theta^\bullet_{0,\alpha}|_{\{[0]\}\times\mathcal{M}_\alpha} = 0$.

Equation (3.22) defines $[\zeta, \mathbb{1}]_n$ and $[\mathbb{1}, \zeta]_n$ for $n \in \mathbb{Z}$. As $\mathbb{1} = H_0([0])(1)$ by Definition 3.10, the terms $(\zeta \boxtimes \mathbb{1}) \cap c_i([\Theta^\bullet_{\alpha,0}])$ and $(\mathbb{1} \boxtimes \zeta) \cap c_i([\Theta^\bullet_{0,\alpha}])$ in (3.22) for $[\zeta, \mathbb{1}]_n, [\mathbb{1}, \zeta]_n$ depend on $c_i(\Theta^\bullet_{\alpha,0}|_{\mathcal{M}_\alpha\times\{[0]\}})$ and $c_i(\Theta^\bullet_{0,\alpha}|_{\{[0]\}\times\mathcal{M}_\alpha})$, and hence are 1 if $i = 0$ and 0 if $i > 0$. Thus (3.22) gives

$$[\zeta, \mathbb{1}]_n = \begin{cases} \epsilon_{\alpha,0} \cdot H_{a-2n-2}(\Xi_{\alpha,0})\bigl(t^{-n-1} \boxtimes \zeta \boxtimes \mathbb{1}\bigr), & n < 0, \\ 0, & n \geqslant 0, \end{cases}$$

$$[\mathbb{1}, \zeta]_n = \begin{cases} \epsilon_{0,\alpha} \cdot H_{a-2n-2}(\Xi_{0,\alpha})\bigl(t^{-n-1} \boxtimes \mathbb{1} \boxtimes \zeta\bigr), & n < 0, \\ 0, & n \geqslant 0. \end{cases}$$

This proves (3.25) when $n \geqslant 0$, so suppose $n < 0$. We have $\epsilon_{\alpha,0} = \epsilon_{0,\alpha} = 1$ by (3.3), so the definition (3.20) of $\Xi_{\alpha,0}, \Xi_{0,\alpha}$ and functoriality of $H_*(-)$ yield

$$\begin{aligned}
[\zeta, \mathbb{1}]_n &= H_{a-2n-2}(\Phi_{\alpha,0})\bigl(H_{a-2n-2}(\Psi_\alpha)(t^{-n-1} \boxtimes \zeta) \boxtimes \mathbb{1}\bigr) \\
&= H_{a-2n-2}(\Phi_{\alpha,0})\bigl((t^{-n-1} \diamond \zeta) \boxtimes \mathbb{1}\bigr) = t^{-n-1} \diamond \zeta, \\
[\mathbb{1}, \zeta]_n &= H_{a-2n-2}(\Phi_{0,\alpha})\bigl(H_{-2n-2}(\Psi_0)(t^{-n-1} \boxtimes \mathbb{1}) \boxtimes \zeta\bigr) \\
&= \begin{cases} H_a(\Phi_{0,\alpha})(\mathbb{1} \boxtimes \zeta) = \zeta, & n = -1, \\ 0, & n < -1. \end{cases}
\end{aligned}$$

Here for the first equation, the second step uses (3.17), and the third that $\Phi_{\alpha,0}|_{\mathcal{M}_\alpha \times \{[0]\}} : \mathcal{M}_\alpha \times * \to \mathcal{M}_\alpha$ is the natural identification, which identifies $\zeta \boxtimes 1 \cong \zeta$ on homology as in Assumption 2.30(a)(iv). The second equation is similar, where for $n < -1$ we note that $H_{-2n-2}(\Psi_0)(t^{-n-1} \boxtimes \mathbb{1})$ factors through $H_{-2n-2}(*) = 0$. This proves (3.25).

4.2.2 Proof of equation (3.26)

For equation (3.26), we have

$$\begin{aligned}
&[t^p \diamond \zeta, \eta]_n = \bigl[H_{a+2p}(\Psi_\alpha)(t^p \boxtimes \zeta), \eta\bigr]_n \\
&= \sum_{\substack{i \geqslant 0:\ 2i \leqslant a+b+2p, \\ i \geqslant n+\chi(\alpha,\beta)+1}} \begin{aligned}[t] &\epsilon_{\alpha,\beta}(-1)^{(a+2p)\chi(\beta,\beta)} \cdot H_{a+b-2n+2p-2\chi(\alpha,\beta)-2}(\Xi_{\alpha,\beta}) \\ &\bigl(t^{i-n-\chi(\alpha,\beta)-1} \boxtimes \bigl[(H_{a+2p}(\Psi_\alpha)(t^p \boxtimes \zeta) \boxtimes \eta) \cap c_i([\Theta^\bullet_{\alpha,\beta}])\bigr]\bigr) \end{aligned} \\
&= \sum_{\substack{i \geqslant 0:\ 2i \leqslant a+b+2p, \\ i \geqslant n+\chi(\alpha,\beta)+1}} \begin{aligned}[t] &\epsilon_{\alpha,\beta}(-1)^{a\chi(\beta,\beta)} \cdot H_{a+b-2n+2p-2\chi(\alpha,\beta)-2}(\Xi_{\alpha,\beta}) \\ &\bigl(t^{i-n-\chi(\alpha,\beta)-1} \boxtimes \bigl[\bigl(H_{a+b+2p}\bigl((\Psi_\alpha \circ (\Pi_1 \times \Pi_2)) \times \Pi_3\bigr) \\ &(t^p \boxtimes \zeta \boxtimes \eta)\bigr) \cap c_i([\Theta^\bullet_{\alpha,\beta}])\bigr]\bigr) \end{aligned} \\
&= \sum_{\substack{i \geqslant 0:\ 2i \leqslant a+b+2p, \\ i \geqslant n+\chi(\alpha,\beta)+1}} \begin{aligned}[t] &\epsilon_{\alpha,\beta}(-1)^{a\chi(\beta,\beta)} \cdot H_{a+b-2n+2p-2\chi(\alpha,\beta)-2}(\Xi_{\alpha,\beta}) \\ &\bigl(t^{i-n-\chi(\alpha,\beta)-1} \boxtimes \bigl[H_{a+b+2p-2i}\bigl((\Psi_\alpha \circ (\Pi_1 \times \Pi_2)) \times \Pi_3\bigr) \\ &\bigl((t^p \boxtimes \zeta \boxtimes \eta) \cap c_i([((\Psi_\alpha \circ (\Pi_1 \times \Pi_2)) \times \Pi_3)^*(\Theta^\bullet_{\alpha,\beta})])\bigr)\bigr]\bigr) \end{aligned} \\
&= \sum_{\substack{i \geqslant 0:\ 2i \leqslant a+b+2p, \\ i \geqslant n+\chi(\alpha,\beta)+1}} \begin{aligned}[t] &\epsilon_{\alpha,\beta}(-1)^{a\chi(\beta,\beta)} \cdot H_{a+b-2n+2p-2\chi(\alpha,\beta)-2}(\Xi_{\alpha,\beta}) \\ &\bigl(t^{i-n-\chi(\alpha,\beta)-1} \boxtimes \bigl[H_{a+b+2p-2i}\bigl((\Psi_\alpha \circ (\Pi_1 \times \Pi_2)) \times \Pi_3\bigr) \\ &\bigl((t^p \boxtimes \zeta \boxtimes \eta) \cap c_i\bigl([\Pi_1^*(E_1) \otimes (\Pi_2 \times \Pi_3)^*(\Theta^\bullet_{\alpha,\beta})]\bigr)\bigr)\bigr]\bigr) \end{aligned} \\
&= \sum_{\substack{i \geqslant 0:\ 2i \leqslant a+b+2p, \\ i \geqslant n+\chi(\alpha,\beta)+1}} \begin{aligned}[t] &\epsilon_{\alpha,\beta}(-1)^{a\chi(\beta,\beta)} \cdot H_{a+b-2n+2p-2\chi(\alpha,\beta)-2}(\Xi_{\alpha,\beta}) \\ &\bigl(t^{i-n-\chi(\alpha,\beta)-1} \boxtimes \bigl[H_{a+b+2p-2i}\bigl((\Psi_\alpha \circ (\Pi_1 \times \Pi_2)) \times \Pi_3\bigr)\bigl((t^p \boxtimes \zeta \boxtimes \eta) \\ &\cap \bigl\{\textstyle\sum_{j=0}^i (-1)^{i-j} \binom{i-\chi(\alpha,\beta)-1}{i-j} c_j([(\Pi_2 \times \Pi_3)^*(\Theta^\bullet_{\alpha,\beta})]) \cup \tau^{i-j}\bigr\}\bigr)\bigr]\bigr) \end{aligned} \\
&= \sum_{\substack{i,j \geqslant 0:\ 2i \leqslant a+b+2p, \\ i \geqslant n+\chi(\alpha,\beta)+1, \\ j \leqslant i,\ i-j \leqslant p}} \begin{aligned}[t] &\epsilon_{\alpha,\beta}(-1)^{i-j+a\chi(\beta,\beta)} \binom{i-\chi(\alpha,\beta)-1}{i-j} \cdot \\ &H_{a+b-2n+2p-2\chi(\alpha,\beta)-2}(\Xi_{\alpha,\beta}) \\ &\bigl(t^{i-n-\chi(\alpha,\beta)-1} \boxtimes \bigl[H_{a+b+2p-2i}\bigl((\Psi_\alpha \circ (\Pi_1 \times \Pi_2)) \times \Pi_3\bigr) \\ &\bigl(t^{p-i+j} \boxtimes ((\zeta \boxtimes \eta) \cap c_j([\Theta^\bullet_{\alpha,\beta}]))\bigr)\bigr]\bigr) \end{aligned}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{\substack{i,j\geqslant 0:\ 2i\leqslant a+b+2p,\\ i\geqslant n+\chi(\alpha,\beta)+1,\\ j\leqslant i,\ i-j\leqslant p}} \epsilon_{\alpha,\beta}(-1)^{i-j+a\chi(\beta,\beta)}\tbinom{i-\chi(\alpha,\beta)-1}{i-j}\cdot \\
&\qquad H_{a+b-2n+2p-2\chi(\alpha,\beta)-2}\bigl(\Xi_{\alpha,\beta}\circ\bigl[\Pi_1\times(\Psi_\alpha\circ(\Pi_2\times\Pi_3))\times\Pi_4)\bigr]\bigr)\\
&\qquad \bigl(t_1^{i-n-\chi(\alpha,\beta)-1}\boxtimes t_2^{p-i+j}\boxtimes((\zeta\boxtimes\eta)\cap c_j([\Theta^\bullet_{\alpha,\beta}]))\bigr)\\
&= \sum_{\substack{i,j\geqslant 0:\ 2i\leqslant a+b+2p,\\ i\geqslant n+\chi(\alpha,\beta)+1,\\ j\leqslant i,\ i-j\leqslant p}} \epsilon_{\alpha,\beta}(-1)^{i-j+a\chi(\beta,\beta)}\tbinom{i-\chi(\alpha,\beta)-1}{i-j}\cdot \\
&\qquad H_{a+b-2n+2p-2\chi(\alpha,\beta)-2}\bigl(\Xi_{\alpha,\beta}\circ\\
&\qquad \bigl[(\Omega\circ(\Pi_1\times\Pi_2))\times\Pi_3\times\Pi_4\bigr]\bigr)\\
&\qquad \bigl(t_1^{i-n-\chi(\alpha,\beta)-1}\boxtimes t_2^{p-i+j}\boxtimes((\zeta\boxtimes\eta)\cap c_j([\Theta^\bullet_{\alpha,\beta}]))\bigr)\\
&= \sum_{\substack{i,j\geqslant 0:\ 2i\leqslant a+b+2p,\\ i\geqslant n+\chi(\alpha,\beta)+1,\\ j\leqslant i,\ i-j\leqslant p}} \epsilon_{\alpha,\beta}(-1)^{i-j+a\chi(\beta,\beta)}\tbinom{i-\chi(\alpha,\beta)-1}{i-j}\tbinom{p+j-n-\chi(\alpha,\beta)-1}{p-i+j}\cdot \\
&\qquad H_{a+b-2n+2p-2\chi(\alpha,\beta)-2}(\Xi_{\alpha,\beta})\\
&\qquad \bigl(t^{p+j-n-\chi(\alpha,\beta)-1}\boxtimes((\zeta\boxtimes\eta)\cap c_j([\Theta^\bullet_{\alpha,\beta}]))\bigr)\\
&= \sum_{\substack{j\geqslant 0:\ 2j\leqslant a+b+2p,\\ j\geqslant n-p+\chi(\alpha,\beta)+1}} \epsilon_{\alpha,\beta}(-1)^{a\chi(\beta,\beta)}\cdot H_{a+b-2n+2p-2\chi(\alpha,\beta)-2}(\Xi_{\alpha,\beta})\\
&\qquad \bigl(t^{j-(n-p)-\chi(\alpha,\beta)-1}\boxtimes((\zeta\boxtimes\eta)\cap c_j([\Theta^\bullet_{\alpha,\beta}]))\bigr)\\
&\qquad \left[\sum_{k=0}^p(-1)^k\tbinom{j+k-\chi(\alpha,\beta)-1}{k}\tbinom{p+j-n-\chi(\alpha,\beta)-1}{p-k}\right]\\
&= (-1)^p\binom{n}{p}\cdot[\zeta,\eta]_{n-p},
\end{aligned}
\tag{4.1}
$$

where the first step uses (3.17) and the second (3.22). In the third we rewrite using the morphism:

$$(\Psi_\alpha\circ(\Pi_1\times\Pi_2))\times\Pi_3:[*/\mathbb{G}_m]\times\mathcal{M}_\alpha\times\mathcal{M}_\beta\longrightarrow\mathcal{M}_\alpha\times\mathcal{M}_\beta,$$

where Π_i means projection to the i^{th} factor of $[*/\mathbb{G}_m]\times\mathcal{M}_\alpha\times\mathcal{M}_\beta$. The fourth step uses Assumption 2.30(a)(ii), (b)(v), the fifth Assumption 3.1(l), the sixth Assumption 2.30(b)(iii), (c)(iii), and the seventh Assumption 2.30(c)(iv).

The eighth step of (4.1) rewrites the expression in terms of homology on $[*/\mathbb{G}_m]^2\times\mathcal{M}_\alpha\times\mathcal{M}_\beta$, where Π_i for $i=1,\ldots,4$ means projection to the i^{th} factor, using functoriality of $H_*(-)$ and compatibility with $\boxtimes$, and t_1^k,t_2^k mean the copies of $t^k\in H_{2k}([*/\mathbb{G}_m])$ associated with the first and second factors of $[*/\mathbb{G}_m]$, respectively. In the ninth step we use the equation in $\mathrm{Ho}(\mathbf{Art}_{\mathbb{K}})$, which follows from (3.7) and (3.20):

$$
\begin{aligned}
\Xi_{\alpha,\beta}\circ\bigl[\Pi_1\times(\Psi_\alpha\circ(\Pi_2\times\Pi_3))\times\Pi_4)\bigr]&=\Xi_{\alpha,\beta}\circ\bigl[(\Omega\circ(\Pi_1\times\Pi_2))\times\Pi_3\times\Pi_4\bigr]:\\
&[*/\mathbb{G}_m]\times[*/\mathbb{G}_m]\times\mathcal{M}_\alpha\times\mathcal{M}_\beta\longrightarrow\mathcal{M}_{\alpha+\beta}.
\end{aligned}
$$

The tenth step uses $H_{2(r+s)}(\Omega)(t_1^r\boxtimes t_2^s)=\binom{r+s}{r}t^{r+s}$ as in (3.15), functoriality of $H_*(-)$, and compatibility with $\boxtimes$. The eleventh step changes variables in the sum from i,j to j and $j=i-j$, and the last step uses (A.8) with $m=p+j-n-\chi(\alpha,\beta)-1$ and (3.22). This proves (3.26).

4.2.3 Proof of equation (3.27)

For equation (3.27), we have

$$
\begin{aligned}
&[\zeta, t^p \diamond \eta]_n = \big[\zeta, H_{b+2p}(\Psi_\beta)(t^p \boxtimes \eta)\big]_n \\
&= \sum_{\substack{i\geqslant 0:\ 2i\leqslant a+b+2p,\\ i\geqslant n+\chi(\alpha,\beta)+1}} \epsilon_{\alpha,\beta}(-1)^{(a+2p)\chi(\beta,\beta)} \cdot H_{a+b-2n+2p-2\chi(\alpha,\beta)-2}(\Xi_{\alpha,\beta}) \\
&\qquad \big(t^{i-n-\chi(\alpha,\beta)-1} \boxtimes \big[(\zeta \boxtimes H_{b+2p}(\Psi_\beta)(t^p\boxtimes\eta)) \cap c_i([\Theta^\bullet_{\alpha,\beta}])\big]\big) \\
&= \sum_{\substack{i\geqslant 0:\ 2i\leqslant a+b+2p,\\ i\geqslant n+\chi(\alpha,\beta)+1}} \epsilon_{\alpha,\beta}(-1)^{a\chi(\beta,\beta)} \cdot H_{a+b-2n+2p-2\chi(\alpha,\beta)-2}(\Xi_{\alpha,\beta}) \\
&\qquad \big(t^{i-n-\chi(\alpha,\beta)-1} \boxtimes \big[\big(H_{a+b+2p}\big(\Pi_2\times(\Psi_\beta\circ(\Pi_1\times\Pi_3))\big) \\
&\qquad (t^p\boxtimes\zeta\boxtimes\eta)\big) \cap c_i([\Theta^\bullet_{\alpha,\beta}])\big]\big) \\
&= \sum_{\substack{i\geqslant 0:\ 2i\leqslant a+b+2p,\\ i\geqslant n+\chi(\alpha,\beta)+1}} \epsilon_{\alpha,\beta}(-1)^{a\chi(\beta,\beta)} \cdot H_{a+b-2n+2p-2\chi(\alpha,\beta)-2}(\Xi_{\alpha,\beta}) \\
&\qquad \big(t^{i-n-\chi(\alpha,\beta)-1} \boxtimes \big[H_{a+b+2p-2i}\big(\Pi_2\times(\Psi_\beta\circ(\Pi_1\times\Pi_3))\big) \\
&\qquad \big((t^p\boxtimes\zeta\boxtimes\eta)\cap c_i([(\Pi_2\times(\Psi_\beta\circ(\Pi_1\times\Pi_3)))^*(\Theta^\bullet_{\alpha,\beta})])\big)\big]\big) \\
&= \sum_{\substack{i\geqslant 0:\ 2i\leqslant a+b+2p,\\ i\geqslant n+\chi(\alpha,\beta)+1}} \epsilon_{\alpha,\beta}(-1)^{a\chi(\beta,\beta)} \cdot H_{a+b-2n+2p-2\chi(\alpha,\beta)-2}(\Xi_{\alpha,\beta}) \\
&\qquad \big(t^{i-n-\chi(\alpha,\beta)-1} \boxtimes \big[H_{a+b+2p-2i}\big(\Pi_2\times(\Psi_\beta\circ(\Pi_1\times\Pi_3)\big) \\
&\qquad \big((t^p\boxtimes\zeta\boxtimes\eta)\cap c_i\big([\Pi_1^*(E_{-1})\otimes(\Pi_2\times\Pi_3)^*(\Theta^\bullet_{\alpha,\beta})]\big)\big)\big]\big) \\
&= \sum_{\substack{i\geqslant 0:\ 2i\leqslant a+b+2p,\\ i\geqslant n+\chi(\alpha,\beta)+1}} \epsilon_{\alpha,\beta}(-1)^{a\chi(\beta,\beta)} \cdot H_{a+b-2n+2p-2\chi(\alpha,\beta)-2}(\Xi_{\alpha,\beta}) \\
&\qquad \big(t^{i-n-\chi(\alpha,\beta)-1}\boxtimes\big[H_{a+b+2p-2i}\big(\Pi_2\times(\Psi_\beta\circ(\Pi_1\times\Pi_3)\big)\big((t^p\boxtimes\zeta\boxtimes\eta) \\
&\qquad \cap\big\{\textstyle\sum_{j=0}^i \binom{i-\chi(\alpha,\beta)-1}{i-j} c_j([(\Pi_2\times\Pi_3)^*(\Theta^\bullet_{\alpha,\beta})])\cup\tau^{i-j}\big\}\big)\big]\big) \\
&= \sum_{\substack{i,j\geqslant 0:\ 2j\leqslant a+b,\\ i\geqslant n+\chi(\alpha,\beta)+1,\\ j\leqslant i,\ i-j\leqslant p}} \epsilon_{\alpha,\beta}(-1)^{a\chi(\beta,\beta)}\tbinom{i-\chi(\alpha,\beta)-1}{i-j} \cdot H_{a+b-2n+2p-2\chi(\alpha,\beta)-2}(\Xi_{\alpha,\beta}) \\
&\qquad \big(t^{i-n-\chi(\alpha,\beta)-1}\boxtimes\big[H_{a+b+2p-2i}\big(\Pi_2\times(\Psi_\beta\circ(\Pi_1\times\Pi_3)\big) \\
&\qquad \big(t^{p-i+j}\boxtimes((\zeta\boxtimes\eta)\cap c_j([\Theta^\bullet_{\alpha,\beta}]))\big)\big]\big) \\
&= \sum_{\substack{i,j\geqslant 0:\ 2j\leqslant a+b,\\ i\geqslant n+\chi(\alpha,\beta)+1,\\ j\leqslant i,\ i-j\leqslant p}} \epsilon_{\alpha,\beta}(-1)^{a\chi(\beta,\beta)}\tbinom{i-\chi(\alpha,\beta)-1}{i-j} \cdot \\
&\qquad H_{a+b-2n+2p-2\chi(\alpha,\beta)-2}\big(\Xi_{\alpha,\beta}\circ\big[\Pi_1\times\Pi_3\times(\Psi_\beta\circ(\Pi_2\times\Pi_4)\big]\big) \\
&\qquad \big(t_1^{i-n-\chi(\alpha,\beta)-1}\boxtimes t_2^{p-i+j}\boxtimes((\zeta\boxtimes\eta)\cap c_j([\Theta^\bullet_{\alpha,\beta}]))\big) \\
&= \sum_{\substack{i,j\geqslant 0:\ 2j\leqslant a+b,\\ i\geqslant n+\chi(\alpha,\beta)+1,\\ j\leqslant i,\ i-j\leqslant p}} \epsilon_{\alpha,\beta}(-1)^{a\chi(\beta,\beta)}\tbinom{i-\chi(\alpha,\beta)-1}{i-j} \cdot \\
&\qquad H_{a+b-2n+2p-2\chi(\alpha,\beta)-2}\big(\Psi_{\alpha+\beta}\circ\big[\Pi_1\times\big(\Xi_{\alpha,\beta}\circ(\Pi_2\times\Pi_3\times\Pi_4)\big)\big]\big) \\
&\qquad \circ H_{a+b-2n+2p-2\chi(\alpha,\beta)-2}\big(\Pi_2\times(\hat\Omega\circ(\Pi_1\times\Pi_2))\times\Pi_3\times\Pi_4\big) \\
&\qquad \big(t_1^{i-n-\chi(\alpha,\beta)-1}\boxtimes t_2^{p-i+j}\boxtimes((\zeta\boxtimes\eta)\cap c_j([\Theta^\bullet_{\alpha,\beta}]))\big) \\
&= \sum_{\substack{i,j,k\geqslant 0:\ 2j\leqslant a+b,\\ j\geqslant k+n-p\\ +\chi(\alpha,\beta)+1,\\ j\leqslant i,\ i-j\leqslant p,\\ k\leqslant p-i+j}} \epsilon_{\alpha,\beta}(-1)^{i+j+k+p+a\chi(\beta,\beta)}\tbinom{i-\chi(\alpha,\beta)-1}{i-j}\tbinom{j-k-n+p-\chi(\alpha,\beta)-1}{p-i+j-k} \cdot \\
&\qquad H_{a+b-2n+2p-2\chi(\alpha,\beta)-2}\big(\Psi_{\alpha+\beta}\circ\big[\Pi_1\times\big(\Xi_{\alpha,\beta}\circ(\Pi_2\times\Pi_3\times\Pi_4)\big)\big]\big) \\
&\qquad \big(t_1^k\boxtimes t_2^{j-k-n+p-\chi(\alpha,\beta)-1}\boxtimes((\zeta\boxtimes\eta)\cap c_j([\Theta^\bullet_{\alpha,\beta}]))\big)
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{\substack{j,k\geqslant 0:\\ 2j\leqslant a+b,\\ j\geqslant k+n-p\\ +\chi(\alpha,\beta)+1}} \begin{aligned}[t]&\epsilon_{\alpha,\beta}(-1)^{p-k+a\chi(\beta,\beta)}\cdot\\
&\Bigl[\textstyle\sum_{i=j}^{p+j-k}(-1)^{i-j}\binom{i-\chi(\alpha,\beta)-1}{i-j}\binom{j-k-n+p-\chi(\alpha,\beta)-1}{p-i+j-k}\Bigr]\cdot\\
&H_{a+b-2n+2p-2\chi(\alpha,\beta)-2}\bigl(\Psi_{\alpha+\beta}\bigr)\\
&\bigl[t^k\boxtimes H_{a+b-2k-2n+2p-2\chi(\alpha,\beta)-2}\bigl(\Xi_{\alpha,\beta}\bigr)\\
&\bigl(t^{j-k-n+p-\chi(\alpha,\beta)-1}\boxtimes((\zeta\boxtimes\eta)\cap c_j([\Theta^\bullet_{\alpha,\beta}]))\bigr)\bigr]\end{aligned}\\
&= \sum_{\substack{j,k\geqslant 0:\\ 2j\leqslant a+b,\\ j\geqslant k+n-p\\ +\chi(\alpha,\beta)+1}} \begin{aligned}[t]&\epsilon_{\alpha,\beta}(-1)^{p-k+a\chi(\beta,\beta)}\Bigl[(-1)^{p-k}\tbinom{n}{p-k}\Bigr]\cdot\\
&H_{a+b-2n+2p-2\chi(\alpha,\beta)-2}\bigl(\Psi_{\alpha+\beta}\bigr)\\
&\bigl[t^k\boxtimes H_{a+b-2k-2n+2p-2\chi(\alpha,\beta)-2}\bigl(\Xi_{\alpha,\beta}\bigr)\\
&\bigl(t^{j-(n+k-p)-\chi(\alpha,\beta)-1}\boxtimes((\zeta\boxtimes\eta)\cap c_j([\Theta^\bullet_{\alpha,\beta}]))\bigr)\bigr]\end{aligned}\\
&= \sum_{k=0}^{p}\binom{n}{p-k}\cdot t^k\diamond[\zeta,\eta]_{n+k-p},
\end{aligned}
\tag{4.2}
$$

where the first eight steps follow the beginning of (4.1). In the ninth step we use functoriality of $H_*(-)$ and the equation in $\mathrm{Ho}(\mathbf{Art}_{\mathbb{K}})$, which follows from (3.6)–(3.7) and (3.20):

$$
\begin{aligned}
&\Xi_{\alpha,\beta}\circ\bigl[\Pi_1\times\Pi_3\times(\Psi_\beta\circ(\Pi_2\times\Pi_4)\bigr]\\
&\quad=\Psi_{\alpha+\beta}\circ\bigl[\Pi_1\times\bigl(\Xi_{\alpha,\beta}\circ(\Pi_2\times\Pi_3\times\Pi_4)\bigr)\bigr]\circ\bigl[\Pi_2\times(\hat\Omega\circ(\Pi_1\times\Pi_2))\times\Pi_3\times\Pi_4\bigr]:\\
&\qquad[*/\mathbb{G}_m]\times[*/\mathbb{G}_m]\times\mathcal{M}_\alpha\times\mathcal{M}_\beta\longrightarrow\mathcal{M}_{\alpha+\beta},
\end{aligned}
\tag{4.3}
$$

where $\hat\Omega:[*/\mathbb{G}_m]\times[*/\mathbb{G}_m]\to[*/\mathbb{G}_m]$ is the stack morphism induced by the group morphism $\mathbb{G}_m^2\to\mathbb{G}_m$ mapping $(\lambda,\mu)\mapsto\lambda\mu^{-1}$.

Here we can explain (4.3) in the language of Assumption 3.1(e),(g),(h), by fixing a $\mathbb{K}$-scheme S and regarding morphisms $S\to\mathcal{M}_\alpha$ as families $E_\alpha\to S$ of objects in $\mathcal{A}$ in class $\alpha\in K(\mathcal{A})$ over the base $\mathbb{K}$-scheme S. Then (4.3) concerns the map, for line bundles $L_1,L_2\to S$ (that is, maps $S\to[*/\mathbb{G}_m]$) and families $E_\alpha,E_\beta\to S$ in $\mathcal{A}$ (that is, morphisms $S\to\mathcal{M}_\alpha$, $S\to\mathcal{M}_\beta$):

$$
\begin{aligned}
(L_1,L_2,E_\alpha,E_\beta)\longmapsto{}&(E_\alpha\otimes L_1)\oplus(E_\beta\otimes L_2)\\
&\cong L_2\otimes\bigl((E_\alpha\otimes(L_1\otimes L_2^{-1}))\oplus E_\beta\bigr),
\end{aligned}
\tag{4.4}
$$

where $\hat\Omega$ corresponds to the map $(L_1,L_2)\mapsto L_1\otimes L_2^{-1}$.

In the tenth step of (4.2), we use that $\Pi_2\times(\hat\Omega\circ(\Pi_1\times\Pi_2)):[*/\mathbb{G}_m]^2\to[*/\mathbb{G}_m]^2$ acts on homology by

$$
H_{2l+2m}\bigl(\Pi_2\times(\hat\Omega\circ(\Pi_1\times\Pi_2))\bigr):t_1^l\boxtimes t_2^m\longmapsto\sum_{k=0}^{m}(-1)^{k+m}\binom{l+m-k}{m-k}t_1^k\boxtimes t_2^{l+m-k},
$$

which we can prove using Assumption 2.30(c). The eleventh step uses functoriality of $H_*(-)$ and compatibility with $\boxtimes$, and rearranges the sum. The twelfth step of (4.2) uses (A.8) with $i-j,j-k-n+p-\chi(\alpha,\beta)-1,n,p-k$ in place of k,m,n,p, and the final step follows from (3.17) and (3.22). This proves (3.27).

4.2.4 Proof of equation (3.28)

For equation (3.28), we have

$$
\begin{aligned}
&\sum_{k\geqslant 0:\ 2k\leqslant \tilde a+\tilde b-2n+2-\chi(\alpha+\beta,\alpha+\beta)} (-1)^{1+\tilde a\tilde b+k+n}\cdot t^k\diamond[\zeta,\eta]_{k+n}\\
&=\sum_{\substack{i,k\geqslant 0:\ 2i\leqslant a+b,\\ i\geqslant n+k+\chi(\alpha,\beta)+1}} (-1)^{1+(a-2+\chi(\alpha,\alpha))(b-2+\chi(\beta,\beta))+k+n}\cdot\epsilon_{\alpha,\beta}(-1)^{a\chi(\beta,\beta)}\cdot\\
&\qquad H_{a+b-2n-2\chi(\alpha,\beta)-2}(\Psi_{\alpha+\beta})(t^k\boxtimes H_{a+b-2k-2n-2\chi(\alpha,\beta)-2}(\Xi_{\alpha,\beta})\\
&\qquad \bigl(t^{i-k-n-\chi(\alpha,\beta)-1}\boxtimes\bigl[(\zeta\boxtimes\eta)\cap c_i([\Theta^\bullet_{\alpha,\beta}])\bigr]\bigr)\\
&=\sum_{\substack{i,k\geqslant 0:\ 2i\leqslant a+b,\\ i\geqslant n+k+\chi(\alpha,\beta)+1}} \epsilon_{\alpha,\beta}(-1)^{1+ab+b\chi(\alpha,\alpha)+\chi(\alpha,\alpha)\chi(\beta,\beta)+k+n}\cdot\\
&\qquad H_{a+b-2n-2\chi(\alpha,\beta)-2}\bigl(\Psi_{\alpha+\beta}\circ(\Pi_1\times(\Xi_{\alpha,\beta}\circ(\Pi_2\times\Pi_3\times\Pi_4)))\bigr)\\
&\qquad \bigl(t_1^k\boxtimes t_2^{i-k-n-\chi(\alpha,\beta)-1}\boxtimes\bigl[(\zeta\boxtimes\eta)\cap c_i([\Theta^\bullet_{\alpha,\beta}])\bigr]\bigr)\\
&=\sum_{\substack{i,k\geqslant 0:\ 2i\leqslant a+b,\\ i\geqslant n+k+\chi(\alpha,\beta)+1}} \epsilon_{\beta,\alpha}(-1)^{1+ab+b\chi(\alpha,\alpha)+\chi(\alpha,\beta)+k+n}\cdot\\
&\qquad H_{a+b-2n-2\chi(\alpha,\beta)-2}\bigl(\Xi_{\beta,\alpha}\circ\bigl[\Pi_1\times\Pi_4\times(\Psi_\alpha\circ(\Pi_2\times\Pi_3))\bigr]\bigr)\\
&\qquad \circ H_{a+b-2n-2\chi(\alpha,\beta)-2}\bigl(\Pi_1\times(\Omega\circ(\Pi_1\times\Pi_2))\times\Pi_3\times\Pi_4\bigr)\\
&\qquad \bigl(t_1^k\boxtimes t_2^{i-k-n-\chi(\alpha,\beta)-1}\boxtimes\bigl[(\zeta\boxtimes\eta)\cap c_i([\Theta^\bullet_{\alpha,\beta}])\bigr]\bigr)\\
&=\sum_{\substack{i,j,k\geqslant 0:\ 2i\leqslant a+b,\\ i\geqslant n+k+\chi(\alpha,\beta)+1,\\ j\leqslant k}} \epsilon_{\beta,\alpha}(-1)^{1+ab+b\chi(\alpha,\alpha)+\chi(\alpha,\beta)+k+n}\tbinom{i+j-k-n-\chi(\alpha,\beta)-1}{j}\cdot\\
&\qquad H_{a+b-2n-2\chi(\alpha,\beta)-2}\bigl(\Xi_{\beta,\alpha}\circ\bigl[\Pi_1\times\Pi_4\times(\Psi_\alpha\circ(\Pi_2\times\Pi_3))\bigr]\bigr)\\
&\qquad \bigl(t_1^{k-j}\boxtimes t_2^{i+j-k-n-\chi(\alpha,\beta)-1}\boxtimes\bigl[(\zeta\boxtimes\eta)\cap c_i([\Theta^\bullet_{\alpha,\beta}])\bigr]\bigr)\\
&=\sum_{\substack{i,l\geqslant 0:\ 2i\leqslant a+b,\\ i\geqslant n+l+\chi(\alpha,\beta)+1}} \epsilon_{\beta,\alpha}(-1)^{ab+b\chi(\alpha,\alpha)+i+l}\Bigl[\textstyle\sum_{j=0}^{l}(-1)^j\binom{l}{j}\Bigr]\cdot\\
&\qquad H_{a+b-2n-2\chi(\alpha,\beta)-2}\bigl(\Xi_{\beta,\alpha}\circ\bigl[\Pi_1\times\Pi_4\times(\Psi_\alpha\circ(\Pi_2\times\Pi_3))\bigr]\bigr)\\
&\qquad \bigl(t_1^{i-l-n-\chi(\alpha,\beta)-1}\boxtimes t_2^l\boxtimes\bigl[(\zeta\boxtimes\eta)\cap c_i([\Theta^\bullet_{\alpha,\beta}])\bigr]\bigr)\\
&=\sum_{\substack{i\geqslant 0:\ 2i\leqslant a+b,\\ i\geqslant n+\chi(\alpha,\beta)+1}} \epsilon_{\beta,\alpha}(-1)^{ab+b\chi(\alpha,\alpha)+i}\cdot H_{a+b-2n-2\chi(\alpha,\beta)-2}(\Xi_{\beta,\alpha})\\
&\qquad \circ H_{a+b-2n-2\chi(\alpha,\beta)-2}\bigl(\Pi_1\times\Pi_4\times(\Psi_\alpha\circ(\Pi_2\times\Pi_3))\bigr)\\
&\qquad \bigl(t_1^{i-n-\chi(\alpha,\beta)-1}\boxtimes t_2^0\boxtimes\bigl[(\zeta\boxtimes\eta)\cap c_i([\Theta^\bullet_{\alpha,\beta}])\bigr]\bigr)\\
&=\sum_{\substack{i\geqslant 0:\ 2i\leqslant a+b,\\ i\geqslant n+\chi(\alpha,\beta)+1}} \epsilon_{\beta,\alpha}(-1)^{b\chi(\alpha,\alpha)+i}\cdot H_{a+b-2n-2\chi(\alpha,\beta)-2}(\Xi_{\beta,\alpha})\\
&\qquad \bigl(t^{i-n-\chi(\alpha,\beta)-1}\boxtimes\bigl[(\eta\boxtimes\zeta)\cap c_i([\sigma^*_{\beta,\alpha}(\Theta^\bullet_{\alpha,\beta})])\bigr]\bigr)\\
&=\sum_{\substack{i\geqslant 0:\ 2i\leqslant a+b,\\ i\geqslant n+\chi(\alpha,\beta)+1}} \epsilon_{\beta,\alpha}(-1)^{b\chi(\alpha,\alpha)}\cdot H_{b+a-2n-2\chi(\beta,\alpha)-2}(\Xi_{\beta,\alpha})\\
&\qquad \bigl(t^{i-n-\chi(\beta,\alpha)-1}\boxtimes\bigl[(\eta\boxtimes\zeta)\cap c_i([\Theta^\bullet_{\beta,\alpha}])\bigr]\bigr)\\
&=[\eta,\zeta]_n,
\end{aligned}
\tag{4.5}
$$

where in the first step we use $\tilde a=a-2+\chi(\alpha,\alpha)$, $\tilde b=b-2+\chi(\beta,\beta)$ and equations (3.17) and (3.22). In the second we rewrite in homology on $[*/\mathbb{G}_m]^2\times\mathcal{M}_\alpha\times\mathcal{M}_\beta$, where Π_i for $i=1,\dots,4$ means projection to the i^{th} factor, using functoriality of $H_*(-)$ and compatibility with $\boxtimes$, and t_1^k,t_2^k mean the $t^k\in H_{2k}([*/\mathbb{G}_m])$

associated with the first and second factors of $[*/\mathbb{G}_m]$, respectively.

In the third step we rewrite the signs by (3.1), and use functoriality of $H_*(-)$ and the equation in $\mathrm{Ho}(\mathbf{Art}_{\mathbb{K}})$, which follows from (3.4), (3.6)–(3.7) and (3.20):

$$\begin{aligned}&\Psi_{\alpha+\beta}\circ(\Pi_1\times(\Xi_{\alpha,\beta}\circ(\Pi_2\times\Pi_3\times\Pi_4)))\\ &=\Xi_{\beta,\alpha}\circ\bigl[\Pi_1\times\Pi_4\times(\Psi_\alpha\circ(\Pi_2\times\Pi_3))\bigr]\circ\bigl[\Pi_1\times(\Omega\circ(\Pi_1\times\Pi_2))\times\Pi_3\times\Pi_4\bigr]:\\ &\qquad[*/\mathbb{G}_m]\times[*/\mathbb{G}_m]\times\mathcal{M}_\alpha\times\mathcal{M}_\beta\longrightarrow\mathcal{M}_{\alpha+\beta}=\mathcal{M}_{\beta+\alpha},\end{aligned}\tag{4.6}$$

where as usual $\Omega:[*/\mathbb{G}_m]\times[*/\mathbb{G}_m]\to[*/\mathbb{G}_m]$ is the stack morphism induced by the group morphism $\mathbb{G}_m^2\to\mathbb{G}_m$ mapping $(\lambda,\mu)\mapsto\lambda\mu$. As for (4.3)–(4.4), we can understand (4.6) in terms of the map of families

$$(L_1,L_2,E_\alpha,E_\beta)\longmapsto L_1\otimes((L_2\otimes E_\alpha)\oplus E_\beta)\cong(L_1\otimes E_\beta)\oplus((L_1\otimes L_2)\otimes E_\alpha).$$

In the fourth step of (4.2), we use that $\Pi_1\times(\Omega\circ(\Pi_1\times\Pi_2)):[*/\mathbb{G}_m]^2\to[*/\mathbb{G}_m]^2$ acts on homology by

$$H_{2k+2l}\bigl(\Pi_1\times(\Omega\circ(\Pi_1\times\Pi_2))\bigr):t_1^k\boxtimes t_2^l\longmapsto\sum_{j=0}^k\binom{j+l}{j}t_1^{k-j}\boxtimes t_2^{j+l},$$

which we can prove using Assumption 2.30(c). In the fifth we change variables from i,j,k to i,j,l with $l=i+j-k-n-\chi(\alpha,\beta)-1$. In the sixth we note that $\sum_{j=0}^l(-1)^j\binom{l}{j}=(1-1)^l$ by the binomial theorem, which is 1 if $l=0$ and 0 otherwise, and use functoriality of $H_*(-)$.

In the seventh step of (4.5) we note that the morphism

$$\Pi_1\times\Pi_4\times(\Psi_\alpha\circ(\Pi_2\times\Pi_3)):[*/\mathbb{G}_m]\times[*/\mathbb{G}_m]\times\mathcal{M}_\alpha\times\mathcal{M}_\beta\longrightarrow[*/\mathbb{G}_m]\times\mathcal{M}_\beta\times\mathcal{M}_\alpha$$

maps $t_1^l\boxtimes t_2^0\boxtimes\bigl[(\zeta\boxtimes\eta)\cap c_i([\Theta^\bullet_{\alpha,\beta}])\bigr]\mapsto(-1)^{ab}t^l\boxtimes\bigl[(\eta\boxtimes\zeta)\cap c_i([\sigma^*_{\beta,\alpha}(\Theta^\bullet_{\alpha,\beta})])\bigr]$ on homology, where $\sigma_{\beta,\alpha}:\mathcal{M}_\beta\times\mathcal{M}_\alpha\to\mathcal{M}_\alpha\times\mathcal{M}_\beta$ exchanges the factors. This is because $H_a(\Psi_\alpha)(t^0\boxtimes\zeta)=\zeta$, and $H_{a+b}(\sigma_{\alpha,\beta})(\zeta\boxtimes\eta)=(-1)^{ab}\eta\boxtimes\zeta$. The eighth step holds as $\sigma^*_{\beta,\alpha}(\Theta^\bullet_{\alpha,\beta})\cong(\Theta^\bullet_{\beta,\alpha})^\vee[2n]$ by Assumption 3.1(j), and $c_i([(\Theta^\bullet_{\beta,\alpha})^\vee[2n]])=c_i([(\Theta^\bullet_{\beta,\alpha})^\vee])=(-1)^ic_i([\Theta^\bullet_{\beta,\alpha}])$ by Assumption 2.30(b)(iv). The last step uses (3.22). This proves (3.28).

4.2.5 Proof of equation (3.29)

Expanding out the first term in (3.29) using (3.17) and (3.22) yields

$$\begin{aligned}&[[\zeta,\eta]_l,\theta]_m=\\ &\sum_{\substack{h,k\geqslant 0:\\ 2h\leqslant a+b+c\\ -2l-2\chi(\alpha,\beta)-2,\\ h\geqslant m+\chi(\alpha+\beta,\gamma)+1,\\ 2k\leqslant a+b,\\ k\geqslant l+\chi(\alpha,\beta)+1}}\begin{aligned}[t]&\epsilon_{\alpha,\beta}(-1)^{a\chi(\beta,\beta)}\epsilon_{\alpha+\beta,\gamma}(-1)^{(a+b)\chi(\gamma,\gamma)}\cdot H_d(\Xi_{\alpha+\beta,\gamma})\\ &\bigl\{t_1^{h-m-\chi(\alpha+\beta,\gamma)-1}\boxtimes\bigl[\bigl(H_{a+b-2l-2\chi(\alpha,\beta)-2}(\Xi_{\alpha,\beta})\\ &\bigl(t_2^{k-l-\chi(\alpha,\beta)-1}\boxtimes\bigl[(\zeta\boxtimes\eta)\cap c_k([\Theta^\bullet_{\alpha,\beta}])\bigr]\bigr)\boxtimes\theta\bigr)\cap c_h([\Theta^\bullet_{\alpha+\beta,\gamma}])\bigr]\bigr\},\end{aligned}\end{aligned}$$

where $d=a+b+c-2l-2m-2\chi(\alpha,\beta)-2\chi(\alpha,\gamma)-2\chi(\beta,\gamma)-4$. Here t_1^i,t_2^i mean $t^i\in H_{2i}([*/\mathbb{G}_m])$ from Assumption 2.30(c) from the first and second copies of $[*/\mathbb{G}_m]$ involved in the formula.

From (3.20) and functoriality of $H_i(-)$ we have

$$H_i(\Xi_{\alpha,\beta})=H_i(\Phi_{\alpha,\beta})\circ H_i\bigl((\Psi_\alpha\circ(\Pi_1\times\Pi_2))\times\Pi_3)\bigr).$$

We substitute this into the previous equation, and then rewrite by combining the pushforwards $H_{\cdots}(\Xi_{\alpha+\beta,\gamma})$ and $H_{\cdots}(\Phi_{\alpha,\beta})$ into one pushforward by

$$\Xi_{\alpha+\beta,\gamma}\circ(\mathrm{id}_{[*/\mathbb{G}_m]}\times\Phi_{\alpha,\beta}\times\mathrm{id}_{\mathcal{M}_\gamma}):[*/\mathbb{G}_m]\times\mathcal{M}_\alpha\times\mathcal{M}_\beta\times\mathcal{M}_\gamma\to\mathcal{M}_{\alpha+\beta+\gamma},$$

using Assumption 2.30(a)(i)–(iii) and (b)(v), and rearrange signs. This yields

$$\begin{aligned}[[\zeta,\eta]_l,\theta]_m=\sum_{\substack{h,k\geqslant 0:\\ 2h\leqslant a+b+c\\ -2l-2\chi(\alpha,\beta)-2,\\ h\geqslant m+\chi(\alpha+\beta,\gamma)+1,\\ 2k\leqslant a+b,\\ k\geqslant l+\chi(\alpha,\beta)+1}}&\epsilon_{\alpha,\beta}\epsilon_{\alpha+\beta,\gamma}(-1)^{a\chi(\beta,\beta)+(a+b)\chi(\gamma,\gamma)}\cdot H_d\bigl(\Xi_{\alpha+\beta,\gamma}\circ\\ &(\mathrm{id}_{[*/\mathbb{G}_m]}\times\Phi_{\alpha,\beta}\times\mathrm{id}_{\mathcal{M}_\gamma})\bigr)\bigl\{t_1^{h-m-\chi(\alpha+\beta,\gamma)-1}\boxtimes\\ &\bigl[\bigl(\bigl\{H_{a+b-2l-2\chi(\alpha,\beta)-2}\bigl((\Psi_\alpha\circ(\Pi_1\times\Pi_2))\times\Pi_3\bigr)\\ &\bigl(t_2^{k-l-\chi(\alpha,\beta)-1}\boxtimes\bigl[(\zeta\boxtimes\eta)\cap c_k([\Theta^\bullet_{\alpha,\beta}])\bigr]\bigr)\bigr\}\boxtimes\theta\bigr)\\ &\cap c_h\bigl([(\Phi_{\alpha,\beta}\times\mathrm{id}_{\mathcal{M}_\gamma})^*(\Theta^\bullet_{\alpha+\beta,\gamma})]\bigr)\bigr]\bigr\}.\end{aligned}$$

We rewrite $(\Phi_{\alpha,\beta}\times\mathrm{id}_{\mathcal{M}_\gamma})^*(\Theta^\bullet_{\alpha+\beta,\gamma})$ as a sum of terms in $\Theta^\bullet_{\alpha,\gamma},\Theta^\bullet_{\beta,\gamma}$ by (3.9), and substitute in $c_h(A+B)=\sum_{h=i+p}c_i(A)\cup c_p(B)$ by (2.33), giving

$$\begin{aligned}[[\zeta,\eta]_l,\theta]_m=\sum_{\substack{i,k,p\geqslant 0:\\ 2k\leqslant a+b,\\ k\geqslant l+\chi(\alpha,\beta)+1,\\ 2(i+p)\leqslant a+b+c\\ -2l-2\chi(\alpha,\beta)-2,\\ i+p\geqslant m+\chi(\alpha+\beta,\gamma)+1}}&\epsilon_{\alpha,\beta}\epsilon_{\alpha+\beta,\gamma}(-1)^{a\chi(\beta,\beta)+(a+b)\chi(\gamma,\gamma)}\cdot H_d\bigl(\Xi_{\alpha+\beta,\gamma}\circ\\ &(\mathrm{id}_{[*/\mathbb{G}_m]}\times\Phi_{\alpha,\beta}\times\mathrm{id}_{\mathcal{M}_\gamma})\bigr)\bigl\{t_1^{i+p-m-\chi(\alpha+\beta,\gamma)-1}\boxtimes\\ &\bigl[\bigl(\bigl\{H_{a+b-2l-2\chi(\alpha,\beta)-2}\bigl((\Psi_\alpha\circ(\Pi_1\times\Pi_2))\times\Pi_3\bigr)\\ &\bigl(t_2^{k-l-\chi(\alpha,\beta)-1}\boxtimes\bigl[(\zeta\boxtimes\eta)\cap c_k([\Theta^\bullet_{\alpha,\beta}])\bigr]\bigr)\bigr\}\boxtimes\theta\bigr)\\ &\cap\bigl(c_i([(\pi_{\mathcal{M}_\beta\times\mathcal{M}_\gamma})^*(\Theta^\bullet_{\beta,\gamma})])\\ &\cup c_p([(\pi_{\mathcal{M}_\alpha\times\mathcal{M}_\gamma})^*(\Theta^\bullet_{\alpha,\gamma})])\bigr)\bigr]\bigr\}.\end{aligned}$$

We combine the two pushforwards $H_{\cdots}(\cdots)$ into one by

$$\begin{aligned}&\Xi_{\alpha+\beta,\gamma}\circ(\Pi_1\times(\Xi_{\alpha,\beta}\circ(\Pi_2\times\Pi_3\times\Pi_4))\times\Pi_5):\\ &[*/\mathbb{G}_m]\times[*/\mathbb{G}_m]\times\mathcal{M}_\alpha\times\mathcal{M}_\beta\times\mathcal{M}_\gamma\longrightarrow\mathcal{M}_{\alpha+\beta+\gamma},\end{aligned}\tag{4.7}$$

using Assumption 2.30(a)(i)–(iii) and (b)(v), where Π_i means projection to the i^{th} factor of $[*/\mathbb{G}_m]\times[*/\mathbb{G}_m]\times\mathcal{M}_\alpha\times\mathcal{M}_\beta\times\mathcal{M}_\gamma$. This involves pulling back $c_i([(\pi_{\mathcal{M}_\beta\times\mathcal{M}_\gamma})^*(\Theta^\bullet_{\beta,\gamma})])\cup c_p([(\pi_{\mathcal{M}_\alpha\times\mathcal{M}_\gamma})^*(\Theta^\bullet_{\alpha,\gamma})])$ by a factor coming from $(\Psi_\alpha\circ(\Pi_1\times\Pi_2))\times\Pi_3$, but the pullback does not affect the $\Theta^\bullet_{\beta,\gamma}$ term, as it is independent of $\mathcal{M}_\alpha$. We obtain:

$$
\begin{aligned}
[[\zeta,\eta]_l,\theta]_m = \sum_{\substack{i,k,p\geqslant 0:\\ 2k\leqslant a+b,\\ k\geqslant l+\chi(\alpha,\beta)+1,\\ 2(i+l)\leqslant a+b+c\\ -2l-2\chi(\alpha,\beta)-2,\\ i+l\geqslant m+\chi(\alpha+\beta,\gamma)+1}} & \epsilon_{\alpha,\beta}\epsilon_{\alpha+\beta,\gamma}(-1)^{a\chi(\beta,\beta)+(a+b)\chi(\gamma,\gamma)}\cdot H_d\bigl(\Xi_{\alpha+\beta,\gamma}\\
&\circ(\Pi_1\times(\Xi_{\alpha,\beta}\circ(\Pi_2\times\Pi_3\times\Pi_4))\times\Pi_5)\bigr)\\
&\bigl\{t_1^{i+p-m-\chi(\alpha+\beta,\gamma)-1}\boxtimes\\
&\bigl[\bigl(t_2^{k-l-\chi(\alpha,\beta)-1}\boxtimes\bigl[(\zeta\boxtimes\eta)\cap c_k([\Theta^\bullet_{\alpha,\beta}])\bigr]\boxtimes\theta\bigr)\\
&\cap\bigl(c_i([(\Pi_4\times\Pi_5)^*(\Theta^\bullet_{\beta,\gamma})])\\
&\cup c_p([((\Psi_\alpha\circ(\Pi_2\times\Pi_3))\times\Pi_5)^*(\Theta^\bullet_{\alpha,\gamma})])\bigr)\bigr]\bigr\}.
\end{aligned}
$$

Next we use equation (3.11) to substitute for $c_p([(\cdots)^*(\Theta^\bullet_{\alpha,\gamma})])$ in the last line. The result is of the form $c_p([(\text{line bundle})\otimes(\cdots)])$, so we can apply (2.36), which we write using a change of variables from p to j,q with $j+q=p$. We use $c_1([E_1])=\tau$ in $H^2([*/\mathbb{G}_m])$, in the notation of Assumption 2.30(c), but we write τ_2 rather than τ to indicate that it corresponds to t_2. We also pull back the $c_k([\Theta^\bullet_{\alpha,\beta}])$ term from $\mathcal{M}_\alpha\times\mathcal{M}_\beta$ to $[*/\mathbb{G}_m]^2\times\mathcal{M}_\alpha\times\mathcal{M}_\beta\times\mathcal{M}_\gamma$. We find:

$$
\begin{aligned}
[[\zeta,\eta]_l,\theta]_m = \sum_{\substack{i,j,k,q\geqslant 0:\\ 2k\leqslant a+b,\\ k\geqslant l+\chi(\alpha,\beta)+1,\\ 2(i+j+q)\leqslant a+b+c\\ -2l-2\chi(\alpha,\beta)-2,\\ i+j+q\geqslant m\\ +\chi(\alpha+\beta,\gamma)+1}} & \epsilon_{\alpha,\beta}\epsilon_{\alpha+\beta,\gamma}(-1)^{a\chi(\beta,\beta)+(a+b)\chi(\gamma,\gamma)}\cdot H_d\bigl(\Xi_{\alpha+\beta,\gamma}\circ\\
&(\Pi_1\times(\Xi_{\alpha,\beta}\circ(\Pi_2\times\Pi_3\times\Pi_4))\times\Pi_5)\bigr)\\
&\bigl\{\bigl(t_1^{i+j-m+q-\chi(\alpha+\beta,\gamma)-1}\boxtimes t_2^{k-l-\chi(\alpha,\beta)-1}\boxtimes\zeta\boxtimes\eta\boxtimes\theta\bigr)\\
&\cap\bigl(H^{2k}(\Pi_3\times\Pi_4)(c_k([\Theta^\bullet_{\alpha,\beta}])\cup H^{2i}(\Pi_4\times\Pi_5)(c_i([\Theta^\bullet_{\beta,\gamma}]))\\
&\cup\bigl[(-1)^q\tbinom{q+j-\chi(\gamma,\alpha)-1}{q}\cdot\\
&\quad H^{2j}(\Pi_3\times\Pi_5)(c_j([\Theta^\bullet_{\alpha,\gamma}]))\cup\tau_2^q\bigr]\bigr]\bigr)\bigr\}.
\end{aligned}
\tag{4.8}
$$

Assumption 2.30(c)(iv) shows that $t_2^p\cap\tau_2^q=t_2^{p-q}$ if $p\geqslant q$ and 0 otherwise. Also the terms in $c_k([\Theta^\bullet_{\alpha,\beta}]),c_i([\Theta^\bullet_{\beta,\gamma}]),c_j([\Theta^\bullet_{\alpha,\gamma}])$ pair only with $\zeta\boxtimes\eta\boxtimes\theta$, not with $t_1^{\cdots}\boxtimes t_2^{\cdots}$. Thus we see that

$$
\begin{aligned}
[[\zeta,\eta]_l,\theta]_m = \sum_{\substack{i,j,k,q\geqslant 0:\\ 2(i+j+k)\leqslant a+b+c,\\ k\geqslant l+q+\chi(\alpha,\beta)+1,\\ i+j+q\geqslant m\\ +\chi(\alpha+\beta,\gamma)+1}} & \epsilon_{\alpha,\beta}\epsilon_{\alpha+\beta,\gamma}(-1)^{q+a\chi(\beta,\beta)+(a+b)\chi(\gamma,\gamma)}\tbinom{q+j-\chi(\gamma,\alpha)-1}{q}\cdot\\
&H_d\bigl(\Xi_{\alpha+\beta,\gamma}\circ(\Pi_1\times(\Xi_{\alpha,\beta}\circ(\Pi_2\times\Pi_3\times\Pi_4)\times\Pi_5))\bigr)\\
&\bigl\{t_1^{i+j-m+q-\chi(\alpha+\beta,\gamma)-1}\boxtimes t_2^{k-l-q-\chi(\alpha,\beta)-1}\boxtimes\\
&\bigl[(\zeta\boxtimes\eta\boxtimes\theta)\cap\bigl(H^{2i}(\pi_{\mathcal{M}_\beta\times\mathcal{M}_\gamma})(c_i([\Theta^\bullet_{\beta,\gamma}]))\\
&\cup H^{2j}(\pi_{\mathcal{M}_\alpha\times\mathcal{M}_\gamma})(c_j([\Theta^\bullet_{\alpha,\gamma}]))\\
&\cup H^{2k}(\pi_{\mathcal{M}_\alpha\times\mathcal{M}_\beta})(c_k([\Theta^\bullet_{\alpha,\beta}]))\bigr)\bigr]\bigr\}.
\end{aligned}
\tag{4.9}
$$

Using (3.6)–(3.7), we can show that the morphism (4.7) factors as $\mathrm{K}_{\alpha,\beta,\gamma}\circ(\Upsilon\times\mathrm{id}_{\mathcal{M}_\alpha\times\mathcal{M}_\beta\times\mathcal{M}_\gamma})$, where

$$
\begin{aligned}
\mathrm{K}_{\alpha,\beta,\gamma}=\;&\Phi_{\alpha+\beta,\gamma}\circ(\Phi_{\alpha,\beta}\times\mathrm{id}_{\mathcal{M}_\gamma})\circ\\
&\bigl((\Psi_\alpha\circ(\Pi_1\times\Pi_3))\times(\Psi_\beta\circ(\Pi_2\times\Pi_4))\times\Pi_5\bigr):\\
&[*/\mathbb{G}_m]\times[*/\mathbb{G}_m]\times\mathcal{M}_\alpha\times\mathcal{M}_\beta\times\mathcal{M}_\gamma\longrightarrow\mathcal{M}_{\alpha+\beta+\gamma},
\end{aligned}
$$

and $\Upsilon : [*/\mathbb{G}_m]^2 \to [*/\mathbb{G}_m]^2$ is induced by the group morphism $\upsilon : \mathbb{G}_m^2 \to \mathbb{G}_m^2$ mapping $\upsilon : (\mu,\nu) \longmapsto (\mu\nu,\mu)$. Thus the pushforward $H_{\cdots}(\cdots)$ in (4.9) factors as $H_{\cdots}(\mathrm{K}_{\alpha,\beta,\gamma}) \circ H_{\cdots}(\Upsilon \times \mathrm{id}_{\mathcal{M}_\alpha\times\mathcal{M}_\beta\times\mathcal{M}_\gamma})$, where we may write $H_{\cdots}(\Upsilon \times \mathrm{id}_{\mathcal{M}_\alpha\times\mathcal{M}_\beta\times\mathcal{M}_\gamma})$ as $H_{\cdots}(\Upsilon)\boxtimes\mathrm{id}$. Hence, from (4.9) we deduce that

$$\begin{aligned}
[[\zeta,\eta]_l,\theta]_m = \sum_{\substack{i,j,k,q\geqslant 0:\\ 2(i+j+k)\leqslant a+b+c,\\ k\geqslant l+q+\chi(\alpha,\beta)+1,\\ i+j+q\geqslant m\\ +\chi(\alpha+\beta,\gamma)+1}} & \epsilon_{\alpha,\beta}\epsilon_{\alpha+\beta,\gamma}(-1)^{q+a\chi(\beta,\beta)+(a+b)\chi(\gamma,\gamma)}\tbinom{q+j-\chi(\gamma,\alpha)-1}{q}\cdot \\
& H_d(\mathrm{K}_{\alpha,\beta,\gamma}) \\
& \bigl\{\bigl[H_{2(i+j+k-l-m-\chi(\alpha,\beta)-\chi(\beta,\gamma)-\chi(\gamma,\alpha)-2)}(\Upsilon) \\
& (t_1^{i+j-m+q-\chi(\alpha+\beta,\gamma)-1}\boxtimes t_2^{k-l-q-\chi(\alpha,\beta)-1})\bigr]\boxtimes \\
& \bigl[(\zeta\boxtimes\eta\boxtimes\theta)\cap\bigl(H^{2i}(\pi_{\mathcal{M}_\beta\times\mathcal{M}_\gamma})(c_i([\Theta^\bullet_{\beta,\gamma}])) \\
& \cup H^{2j}(\pi_{\mathcal{M}_\alpha\times\mathcal{M}_\gamma})(c_j([\Theta^\bullet_{\alpha,\gamma}])) \\
& \cup H^{2k}(\pi_{\mathcal{M}_\alpha\times\mathcal{M}_\beta})(c_k([\Theta^\bullet_{\alpha,\beta}]))\bigr)\bigr]\bigr\}.
\end{aligned}\tag{4.10}$$

Now from the definition of Υ and Assumption 2.30(c) we can show that

$$\begin{aligned}
&H_{\cdots}(\Upsilon)(t_1^{i+j-m+q-\chi(\alpha+\beta,\gamma)-1}\boxtimes t_2^{k-l-q-\chi(\alpha,\beta)-1}) \\
&= \sum_{\substack{r,s\geqslant 0:\ r\geqslant k-l-q-\chi(\alpha,\beta)-1\\ r+s=i+j+k-l-m-\chi(\alpha,\beta)-\chi(\beta,\gamma)-\chi(\gamma,\alpha)-2}} \tbinom{r}{k-l-q-\chi(\alpha,\beta)-1}\cdot t_1^r\boxtimes t_2^s.
\end{aligned}$$

Substituting this into (4.10) and rearranging into two sums yields

$$\begin{aligned}
&[[\zeta,\eta]_l,\theta]_m = \\
&\sum_{\substack{i,j,k,r,s\geqslant 0:\\ 2(i+j+k)\leqslant\\ a+b+c,\\ r+s=i+j+k\\ -l-m-\chi(\alpha,\beta)\\ -\chi(\beta,\gamma)\\ -\chi(\gamma,\alpha)-2}} \begin{aligned}[t] & (-1)^{a\chi(\beta,\beta)+(a+b)\chi(\gamma,\gamma)}\epsilon_{\alpha,\beta}\epsilon_{\alpha+\beta,\gamma}\cdot \\
& H_d(\mathrm{K}_{\alpha,\beta,\gamma})\bigl\{t_1^r\boxtimes t_2^s\boxtimes\bigl[(\zeta\boxtimes\eta\boxtimes\theta)\cap \\
& \bigl(H^{2i}(\pi_{\mathcal{M}_\beta\times\mathcal{M}_\gamma})(c_i([\Theta^\bullet_{\beta,\gamma}])) \\
& \cup H^{2j}(\pi_{\mathcal{M}_\gamma\times\mathcal{M}_\alpha})(c_j([\Theta^\bullet_{\alpha,\gamma}])) \\
& \cup H^{2k}(\pi_{\mathcal{M}_\alpha\times\mathcal{M}_\beta})(c_k([\Theta^\bullet_{\alpha,\beta}]))\bigr)\bigr]\bigr\}\cdot \end{aligned} \\
&\quad\left[\sum_{\substack{q\geqslant 0:\ q\leqslant k-l-\chi(\alpha,\beta)-1,\\ q\geqslant k-l-r-\chi(\alpha,\beta)-1}} (-1)^q\tbinom{q+j-\chi(\gamma,\alpha)-1}{q}\tbinom{r}{k-l-q-\chi(\alpha,\beta)-1}\right].
\end{aligned}$$

The last line $[\cdots]$ can be summed explicitly using (A.8) and (A.4), giving

$$\begin{aligned}
&[[\zeta,\eta]_l,\theta]_m = \\
&\sum_{\substack{i,j,k,r,s\geqslant 0:\\ 2(i+j+k)\leqslant\\ a+b+c,\\ r+s=i+j+k\\ -l-m-\chi(\alpha,\beta)\\ -\chi(\beta,\gamma)\\ -\chi(\gamma,\alpha)-2}} \begin{aligned}[t] & (-1)^{a\chi(\beta,\beta)+(a+b)\chi(\gamma,\gamma)}\epsilon_{\alpha,\beta}\epsilon_{\alpha+\beta,\gamma}\cdot \\
& H_d(\mathrm{K}_{\alpha,\beta,\gamma})\bigl\{t_1^r\boxtimes t_2^s\boxtimes\bigl[(\zeta\boxtimes\eta\boxtimes\theta)\cap \\
& \bigl(H^{2i}(\pi_{\mathcal{M}_\beta\times\mathcal{M}_\gamma})(c_i([\Theta^\bullet_{\beta,\gamma}])) \\
& \cup H^{2j}(\pi_{\mathcal{M}_\gamma\times\mathcal{M}_\alpha})(c_j([\Theta^\bullet_{\alpha,\gamma}])) \\
& \cup H^{2k}(\pi_{\mathcal{M}_\alpha\times\mathcal{M}_\beta})(c_k([\Theta^\bullet_{\alpha,\beta}]))\bigr)\bigr]\bigr\}\cdot \\
& \tbinom{-j+r+\chi(\gamma,\alpha)}{k-l-\chi(\alpha,\beta)-1}. \end{aligned}
\end{aligned}\tag{4.11}$$

By similar proofs we can show that

$$
\begin{aligned}
&[\zeta,[\eta,\theta]_{m+n}]_{l-n}= \\
&\sum_{\substack{i,j,k,r,s\geqslant 0:\\ 2(i+j+k)\leqslant\\ a+b+c,\\ r+s=i+j+k\\ -l-m-\chi(\alpha,\beta)\\ -\chi(\beta,\gamma)\\ -\chi(\gamma,\alpha)-2}}
\begin{aligned}[t]
&(-1)^{a\chi(\beta,\beta)+(a+b)\chi(\gamma,\gamma)}\epsilon_{\beta,\gamma}\epsilon_{\alpha,\beta+\gamma}\cdot\\
&H_d(\mathrm{K}_{\alpha,\beta,\gamma})\bigl\{t_1^r\boxtimes t_2^s\boxtimes\bigl[(\zeta\boxtimes\eta\boxtimes\theta)\cap\\
&\bigl(H^{2i}(\pi_{\mathcal{M}_\beta\times\mathcal{M}_\gamma})(c_i([\Theta^\bullet_{\beta,\gamma}]))\\
&\cup H^{2j}(\pi_{\mathcal{M}_\gamma\times\mathcal{M}_\alpha})(c_j([\Theta^\bullet_{\alpha,\gamma}]))\\
&\cup H^{2k}(\pi_{\mathcal{M}_\alpha\times\mathcal{M}_\beta})(c_k([\Theta^\bullet_{\alpha,\beta}])))\bigr]\bigr\}\cdot\\
&\textstyle\binom{-j+l-n+r+\chi(\gamma,\alpha)}{i-m-n-s-\chi(\beta,\gamma)-1},
\end{aligned}
\end{aligned}
\tag{4.12}
$$

$$
\begin{aligned}
&(-1)^{\tilde a\tilde b}\cdot[\eta,[\zeta,\theta]_n]_{l+m-n}= \\
&\sum_{\substack{i,j,k,r,s\geqslant 0:\\ 2(i+j+k)\leqslant\\ a+b+c,\\ r+s=i+j+k\\ -l-m-\chi(\alpha,\beta)\\ -\chi(\beta,\gamma)\\ -\chi(\gamma,\alpha)-2}}
\begin{aligned}[t]
&(-1)^{a\chi(\beta,\beta)+(a+b)\chi(\gamma,\gamma)}\cdot\\
&(-1)^{\chi(\alpha,\beta)+\chi(\alpha,\alpha)\chi(\beta,\beta)}\epsilon_{\alpha,\gamma}\epsilon_{\beta,\alpha+\gamma}\cdot\\
&H_d(\mathrm{K}_{\alpha,\beta,\gamma})\bigl\{t_1^r\boxtimes t_2^s\boxtimes\bigl[(\zeta\boxtimes\eta\boxtimes\theta)\cap\\
&\bigl(H^{2i}(\pi_{\mathcal{M}_\beta\times\mathcal{M}_\gamma})(c_i([\Theta^\bullet_{\beta,\gamma}]))\\
&\cup H^{2j}(\pi_{\mathcal{M}_\gamma\times\mathcal{M}_\alpha})(c_j([\Theta^\bullet_{\alpha,\gamma}]))\\
&\cup H^{2k}(\pi_{\mathcal{M}_\alpha\times\mathcal{M}_\beta})(c_k([\Theta^\bullet_{\alpha,\beta}])))\bigr]\bigr\}\cdot\\
&(-1)^{k+\chi(\alpha,\beta)}\textstyle\binom{-i+l-m-n+s+\chi(\beta,\gamma)}{j-n-r-\chi(\gamma,\alpha)-1}.
\end{aligned}
\end{aligned}
\tag{4.13}
$$

Here the differences with the proof of (4.11) are that, firstly, for (4.12)–(4.13) we use (3.12) rather than (3.11), and $c_1([E_{-1}])=-\tau$, so τ_2^q in (4.8) is replaced by $(-\tau_2)^q$. Secondly, for (4.13), under the isomorphism $H_*(\mathcal{M}_\beta\times\mathcal{M}_\alpha)\cong H_*(\mathcal{M}_\alpha\times\mathcal{M}_\beta)$ we have $\eta\boxtimes\zeta\cong(-1)^{ab}\zeta\boxtimes\eta$. Thirdly, for (4.13) we use Assumptions 2.30(b)(iv) and 3.1(j) to convert $c_k([\Theta^\bullet_{\beta,\alpha}])$ to $(-1)^k c_k([\Theta^\bullet_{\alpha,\beta}])$.

Using (A.4), (A.8) and $r+s=i+j+k-l-m-\chi(\alpha,\beta)-\chi(\beta,\gamma)-\chi(\gamma,\alpha)-2$, we find that

$$
\begin{aligned}
\sum_{n\geqslant 0:\ n\leqslant i-m-s-\chi(\beta,\gamma)-1}&(-1)^n\textstyle\binom{l}{n}\cdot\binom{-j+l-n+r+\chi(\gamma,\alpha)}{i-m-n-s-\chi(\beta,\gamma)-1}\\
&=(-1)^{i-m-s-\chi(\beta,\gamma)-1}\textstyle\binom{-k+l+\chi(\alpha,\beta)}{i-m-s-\chi(\beta,\gamma)-1},
\end{aligned}
\tag{4.14}
$$

$$
\begin{aligned}
\sum_{n\geqslant 0:\ n\leqslant j-r-\chi(\gamma,\alpha)-1}&(-1)^{n+l}\textstyle\binom{l}{n}\cdot(-1)^{k+\chi(\alpha,\beta)}\binom{-i+l-m-n+s+\chi(\beta,\gamma)}{j-n-r-\chi(\gamma,\alpha)-1}\\
&=(-1)^{k-l-\chi(\alpha,\beta)}\textstyle\binom{-i-m+s+\chi(\beta,\gamma)}{j-r-\chi(\gamma,\alpha)-1}.
\end{aligned}
\tag{4.15}
$$

The next equation follows from (3.1)–(3.2):

$$
\epsilon_{\alpha,\beta}\epsilon_{\alpha+\beta,\gamma}=\epsilon_{\beta,\gamma}\epsilon_{\alpha,\beta+\gamma}=(-1)^{\chi(\alpha,\beta)+\chi(\alpha,\alpha)\chi(\beta,\beta)}\epsilon_{\alpha,\gamma}\epsilon_{\beta,\alpha+\gamma}.
\tag{4.16}
$$

Combining (4.11)–(4.16) yields

$$
\begin{aligned}
&[[\zeta,\eta]_l,\theta]_m-\sum_{\substack{n\geqslant 0:\ 2m+2n\leqslant\tilde b+\tilde c\\ +2-\chi(\beta+\gamma,\beta+\gamma)}}(-1)^n\binom{l}{n}\cdot[\zeta,[\eta,\theta]_{m+n}]_{l-n}\\
&+\sum_{\substack{n\geqslant 0:\ 2n\leqslant\tilde a+\tilde c\\ +2-\chi(\alpha+\gamma,\alpha+\gamma)}}(-1)^{n+l+\tilde a\tilde b}\binom{l}{n}[\eta,[\zeta,\theta]_n]_{l+m-n}=\\
&\sum_{\substack{i,j,k,r,s\geqslant 0:\\ 2(i+j+k)\leqslant\\ a+b+c,\\ r+s=i+j+k\\ -l-m-\chi(\alpha,\beta)\\ -\chi(\beta,\gamma)\\ -\chi(\gamma,\alpha)-2}}
\begin{aligned}[t]
&(-1)^{a\chi(\beta,\beta)+(a+b)\chi(\gamma,\gamma)}\epsilon_{\alpha,\beta}\epsilon_{\alpha+\beta,\gamma}\cdot\\
&H_d(\mathrm{K}_{\alpha,\beta,\gamma})\bigl\{t_1^r\boxtimes t_2^s\boxtimes\bigl[(\zeta\boxtimes\eta\boxtimes\theta)\cap\\
&\bigl(H^{2i}(\pi_{\mathcal{M}_\beta\times\mathcal{M}_\gamma})(c_i([\Theta^\bullet_{\beta,\gamma}]))\\
&\cup H^{2j}(\pi_{\mathcal{M}_\gamma\times\mathcal{M}_\alpha})(c_j([\Theta^\bullet_{\alpha,\gamma}]))\\
&\cup H^{2k}(\pi_{\mathcal{M}_\alpha\times\mathcal{M}_\beta})(c_k([\Theta^\bullet_{\alpha,\beta}]))\bigr)\bigr]\bigr\}\cdot\\
&\Bigl[\tbinom{-j+r+\chi(\gamma,\alpha)}{k-l-\chi(\alpha,\beta)-1}-(-1)^{i-m-s-\chi(\beta,\gamma)-1}\tbinom{-k+l+\chi(\alpha,\beta)}{i-m-s-\chi(\beta,\gamma)-1}\\
&+(-1)^{k-l-\chi(\alpha,\beta)}\tbinom{-i-m+s+\chi(\beta,\gamma)}{j-r-\chi(\gamma,\alpha)-1}\Bigr]=0,
\end{aligned}
\end{aligned}
$$

where the second step follows from (A.9). This proves (3.29).

4.3 Proof of Proposition 3.18

We work in the situation of Proposition 3.18, and write $a=\tilde a+2-\chi(\alpha,\alpha)$, $b=\tilde b+2-\chi(\beta,\beta)$, $c=\tilde c+2-\chi(\gamma,\gamma)$, and $d=\tilde a+\tilde b+\tilde c+2-\chi(\alpha+\beta+\gamma,\alpha+\beta+\gamma)$, so that $\zeta\in H_a(\mathcal{M}_\alpha)$, $\eta\in H_b(\mathcal{M}_\beta)$, and $\theta\in H_c(\mathcal{M}_\gamma)$ by (3.23). Apply $t^{m+n-l}\diamond -$ to (3.29) with m,n,p in place of l,m,n, multiply by $(-1)^{\tilde c\tilde a+m+n}\frac{(m+n-l)!}{m!n!}x^m(x+y)^n$, and sum over $m,n\geqslant 0$ with $m+n\geqslant l$. This yields

$$
\begin{aligned}
0=&\sum_{m,n\geqslant 0:\ m+n\geqslant l,\ 2(m+n)\leqslant d}(-1)^{\tilde c\tilde a+m+n}\tfrac{(m+n-l)!}{m!n!}x^m(x+y)^n\cdot t^{m+n-l}\diamond[[\zeta,\eta]_m,\theta]_n\\
-&\sum_{m,n,p\geqslant 0:\ m+n\geqslant l,\ 2(m+n)\leqslant d,\ p\leqslant m}(-1)^{\tilde c\tilde a+m+n+p}\tfrac{(m+n-l)!}{m!n!}\tbinom{m}{p}x^m(x+y)^n\cdot t^{m+n-l}\diamond[\zeta,[\eta,\theta]_{n+p}]_{m-p}\\
+&\sum_{m,n,p\geqslant 0:\ m+n\geqslant l,\ 2(m+n)\leqslant d,\ p\leqslant m}(-1)^{\tilde a\tilde b+\tilde c\tilde a+n+p}\tfrac{(m+n-l)!}{m!n!}\tbinom{m}{p}x^m(x+y)^n\cdot t^{m+n-l}\diamond[\eta,[\zeta,\theta]_p]_{m+n-p}.
\end{aligned}
$$

We use (3.28) three times to rewrite the second and third terms:

$$
\begin{aligned}
0 = &\sum_{m,n\geqslant 0:\ m+n\geqslant l,\ 2(m+n)\leqslant d} (-1)^{\tilde c\tilde a+m+n} \tfrac{(m+n-l)!}{m!n!} x^m (x+y)^n \cdot t^{m+n-l} \diamond [[\zeta,\eta]_m,\theta]_n \\
- &\sum_{\substack{m,n,p,q\geqslant 0:\\ 2(m+n+q)\leqslant d,\\ p\leqslant m}} (-1)^{\tilde c\tilde a+m+n+p} \tfrac{(m+n-l)!}{m!n!} \tbinom{m}{p} x^m (x+y)^n \cdot \\
& \qquad t^{m+n-l} \diamond \bigl((-1)^{1+\tilde a(\tilde b+\tilde c)+m-p+q} \cdot t^q \diamond [[\eta,\theta]_{n+p},\zeta]_{m-p+q}\bigr) \\
+ &\sum_{\substack{m,n,p,q,r\geqslant 0:\\ 2(m+n+q)\leqslant d,\\ p\leqslant m,\ 2r\leqslant \tilde a+\tilde c-2p\\ +2-\chi(\alpha+\gamma,\alpha+\gamma)}} (-1)^{\tilde a\tilde b+\tilde c\tilde a+n+p} \tfrac{(m+n-l)!}{m!n!} \tbinom{m}{p} x^m (x+y)^n \cdot \\
& \qquad t^{m+n-l} \diamond \bigl((-1)^{1+\tilde b(\tilde a+\tilde c)+m+n-p+q} \cdot t^q \diamond \\
& \qquad [(-1)^{1+\tilde c\tilde a+p+r} \cdot t^r \diamond [\theta,\zeta]_{p+r},\eta]_{m+n-p+q}\bigr).
\end{aligned}
$$

We substitute (3.26) in the third term, use equations (3.15) and (3.18) to combine terms $t^r \diamond (t^s \diamond -)$, and simplify signs, giving

$$
\begin{aligned}
0 = &\sum_{m,n\geqslant 0:\ 2(m+n)\leqslant d} (-1)^{\tilde c\tilde a+m+n} \tfrac{(m+n-l)!}{m!n!} x^m (x+y)^n \cdot t^{m+n-l} \diamond [[\zeta,\eta]_m,\theta]_n \\
+ &\sum_{\substack{m,n,p,q\geqslant 0:\\ 2(m+n+q)\leqslant d,\\ p\leqslant m}} (-1)^{\tilde a\tilde b+n+q} \tfrac{(m+n-l)!}{m!n!} \tbinom{m}{p} \tbinom{m+n-l+q}{m+n-l} x^m (x+y)^n \cdot \\
& \qquad t^{m+n-l+q} \diamond [[\eta,\theta]_{n+p},\zeta]_{m-p+q} \\
+ &\sum_{\substack{m,n,p,q,r\geqslant 0:\\ 2(m+n+q)\leqslant d,\\ p+r\leqslant m+n+q}} (-1)^{\tilde b\tilde c+m+p+q} \tfrac{(m+n-l)!}{m!n!} \tbinom{m}{p} \tbinom{m+n-p+q}{r} \tbinom{m+n-l+q}{m+n-l} x^m (x+y)^n \cdot \\
& \qquad t^{m+n-l+q} \diamond [[\theta,\zeta]_{p+r},\eta]_{m+n-p+q-r}.
\end{aligned}
$$

Next we change variables in the second sum to $m' = n+p$ and $n' = m-p+q$, eliminating m, n, and in the third sum to $m' = p+r$ and $n' = m+n-p+q-r$, eliminating m and p, giving

$$
\begin{aligned}
0 = &\sum_{m,n\geqslant 0:\ m+n\geqslant l,\ 2(m+n)\leqslant d} (-1)^{\tilde c\tilde a+m+n} \tfrac{(m+n-l)!}{m!n!} x^m (x+y)^n \cdot t^{m+n-l} \diamond [[\zeta,\eta]_m,\theta]_n \\
+ &\sum_{\substack{m',n'\geqslant 0:\\ 2(m'+n')\leqslant d}} (-1)^{\tilde a\tilde b+m'+n'} \tfrac{(m'+n'-l)!}{m'!n'!} \cdot t^{m'+n'-l} \diamond [[\eta,\theta]_{m'},\zeta]_{n'} \cdot \\
& \qquad \Bigl[\sum_{p,q\geqslant 0:\ p\leqslant m', q\leqslant n'} (-1)^{n'+p-q} \tbinom{m'}{p} \tbinom{n'}{q} x^{n'+p-q} (x+y)^{m'-p}\Bigr] \\
+ &\sum_{\substack{m',n'\geqslant 0:\\ 2(m'+n')\leqslant d}} (-1)^{\tilde b\tilde c+m'+n'} \tfrac{(m'+n'-l)!}{m'!n'!} \cdot t^{m'+n'-l} \diamond [[\theta,\zeta]_{m'},\eta]_{n'} \cdot \\
& \qquad \Bigl[\sum_{n,q,r\geqslant 0:\ r\leqslant m',\ n+q\leqslant n'+r} (-1)^{m'+n+r} \tbinom{m'}{r} \tfrac{(n'+r)!}{(n'+r-n-q)!n!q!} x^{m'+n'-n-q} (x+y)^n\Bigr].
\end{aligned}
\tag{4.17}
$$

Now by the binomial theorem and $x+y+z=1$ we see that

$$
\begin{aligned}
\sum_{p,q\geqslant 0:\ p\leqslant m', q\leqslant n'} (-1)^{n'+p-q} \tbinom{m'}{p} \tbinom{n'}{q} x^{n'+p-q} (x+y)^{m'-p} &= (-x+(x+y))^{m'} (1-x)^{n'} \\
&= y^{m'} (y+z)^{n'},
\end{aligned}
\tag{4.18}
$$

$$
\begin{aligned}
&\sum_{n,q,r\geqslant 0:\, r\leqslant m',\, n+q\leqslant n'+r}(-1)^{m'+n+r}\tbinom{m'}{r}\tfrac{(n'+r)!}{n!(n'+r-n-q)!q!}x^{m'+n'-n-q}(x+y)^n\\
&\qquad=\sum_{r=0}^{m'}\binom{m'}{r}(-x)^{m'-r}(-(x+y)+1+x)^{n'+r}\\
&\qquad=\sum_{r=0}^{m'}\binom{m'}{r}(-x)^{m'-r}(z+x)^{n'+r}=z^{m'}(z+x)^{n'}.
\end{aligned}
\tag{4.19}
$$

Combining (4.17)–(4.19) proves (3.35), and Proposition 3.18.

4.4 Proof of Proposition 3.24

For part (a), as $\Pi_\alpha^{\mathrm{pl}}:\mathcal{M}'_\alpha\to\mathcal{M}_\alpha^{\mathrm{pl}}$ is rationally trivial, by Definition 2.26 there exists a morphism $f:T\to\mathcal{M}_\alpha^{\mathrm{pl}}$ which is surjective and a $[*/\mathbb{Z}_n]$-fibration over each connected component of $\mathcal{M}_\alpha^{\mathrm{pl}}$, such that the pullback principal $[*/\mathbb{G}_m]$-bundle is trivial. Consider the diagram in $\mathrm{Ho}(\mathbf{Art}_{\mathbb{K}}^{\mathrm{lft}})$ and $\mathbf{Art}_{\mathbb{K}}^{\mathrm{lft}}$:

$$
\begin{tikzcd}
{[*/\mathbb{G}_m]^2\times T} \arrow[r, "\Omega\times\mathrm{id}_T"'] \arrow[d, "\mathrm{id}_{[*/\mathbb{G}_m]}\times f'"] & {[*/\mathbb{G}_m]\times T} \arrow[r, "\pi_T"'] \arrow[d, "f'"'] & T \arrow[d, "f"'] \\
{[*/\mathbb{G}_m]\times\mathcal{M}'_\alpha} \arrow[r, "\Psi'_\alpha"] & \mathcal{M}'_\alpha \arrow[r, "\Pi_\alpha^{\mathrm{pl}}"] & \mathcal{M}_\alpha^{\mathrm{pl}},
\end{tikzcd}
\tag{4.20}
$$

where $\Omega:[*/\mathbb{G}_m]^2\to[*/\mathbb{G}_m]$ is as in Assumption 3.1(h). Here the squares are 2-Cartesian in $\mathbf{Art}_{\mathbb{K}}^{\mathrm{lft}}$, and so commute in $\mathrm{Ho}(\mathbf{Art}_{\mathbb{K}}^{\mathrm{lft}})$, as they give the pullback principal $[*/\mathbb{G}_m]$-bundle by $f:T\to\mathcal{M}_\alpha^{\mathrm{pl}}$ with its $[*/\mathbb{G}_m]$-action, where we use the fact that the pullback is trivial to write $\mathcal{M}'_\alpha\times_{\Pi_\alpha^{\mathrm{pl}},\mathcal{M}_\alpha^{\mathrm{pl}},f}T=[*/\mathbb{G}_m]\times T$. Since f is a surjective local $[*/\mathbb{Z}_n]$-fibration, and the squares are 2-Cartesian, the other vertical morphisms $f',\mathrm{id}_{[*/\mathbb{G}_m]}\times f'$ are also local $[*/\mathbb{Z}_n]$-fibrations.

Applying $H_*(-)$ to (4.20) gives a commutative diagram:

$$
\begin{tikzcd}
{R[t]\otimes_R R[t]\otimes_R H_*(T)} \arrow[r, "H_*(\Omega\times\mathrm{id}_T)"'] \arrow[d, "\cong"', "H_*(\mathrm{id}_{[*/\mathbb{G}_m]}\times f')"] & {R[t]\otimes_R H_*(T)} \arrow[r, "H_*(\pi_T)"'] \arrow[d, "\cong"', "H_*(f')"] & H_*(T) \arrow[d, "H_*(f)"', "\cong"] \\
{R[t]\otimes_R H_*(\mathcal{M}'_\alpha)} \arrow[r, "H_*(\Psi'_\alpha)"] & H_*(\mathcal{M}'_\alpha) \arrow[r, "H_*(\Pi_\alpha^{\mathrm{pl}})"] & H_*(\mathcal{M}_\alpha^{\mathrm{pl}}).
\end{tikzcd}
\tag{4.21}
$$

Here we have used Assumption 2.31(b) and $H_*([*/\mathbb{G}_m])=R[t]$ to rewrite the homology of the $[*/\mathbb{G}_m]$-factors. The columns in (4.21) are isomorphisms by Assumption 2.31(a), as the columns in (4.20) are local $[*/\mathbb{Z}_n]$-fibrations.

Three of the horizontal morphisms in (4.21) may be written explicitly as

$$
H_*(\Omega\times\mathrm{id}_T):t^m\otimes t^n\otimes\zeta\longmapsto(t^m\star t^n)\otimes\zeta=\tbinom{m+n}{n}t^{m+n}\otimes\zeta,\tag{4.22}
$$

$$
H_*(\Psi'_\alpha):t^m\otimes\eta\longmapsto t^m\diamond\eta,\tag{4.23}
$$

$$
H_*(\pi_T):t^n\otimes\zeta\longmapsto\begin{cases}\zeta, & n=0,\\ 0, & n>0,\end{cases}\tag{4.24}
$$

using (3.14)–(3.15) in (4.22), equation (3.17) in (4.23), and that $\pi : [*/\mathbb{G}_m] \to *$ acts on homology by $t^0 \mapsto 1$ and $t^n \mapsto 0$ for $n > 0$ in (4.24).

Now the left hand square of (4.21) and (4.22)–(4.23) show that the middle column of (4.21) identifies the $R[t]$-actions on $R[t] \otimes_R H_*(T)$ and $H_*(\mathcal{M}'_\alpha)$. Therefore the right hand square of (4.21), with columns isomorphisms, and (4.24), imply that (3.49) is exact, and hence $\Pi^{\rm pl}_{t=0} : H_*(\mathcal{M}'_\alpha)^{t=0} \to H_*(\mathcal{M}^{\rm pl}_\alpha)$ in (3.50) is an isomorphism, as we want.

For (b), let $\mathbb{K}$ be algebraically closed, and suppose $0 \neq \alpha, \beta \in K(\mathcal{A})$, with $\chi(\alpha,\beta) \neq 0$ and $\mathcal{M}_\beta \neq \emptyset$. Choose $[E] \in \mathcal{M}_\beta(\mathbb{K})$. Then the restriction (pullback) of $\Theta^\bullet_{\alpha,\beta}$ to $\mathcal{M}'_\alpha \cong \mathcal{M}'_\alpha \times \{[E]\} \subset \mathcal{M}_\alpha \times \mathcal{M}_\beta$ is a perfect complex $\Theta^\bullet_{\alpha,\beta}|_{\mathcal{M}'_\alpha \times \{[E]\}}$ on $\mathcal{M}'_\alpha$, of rank $\chi(\alpha,\beta) \neq 0$, and Assumption 3.1(l) implies that $\Theta^\bullet_{\alpha,\beta}|_{\mathcal{M}'_\alpha \times \{[E]\}}$ has a weight one $[*/\mathbb{G}_m]$-action compatible with the $[*/\mathbb{G}_m]$-action Ψ'_α on $\mathcal{M}'_\alpha$, in the sense of §2.3.8. Thus Proposition 2.29 (which requires $\mathbb{K}$ to be algebraically closed) says that $\Pi^{\rm pl}_\alpha : \mathcal{M}'_\alpha \to \mathcal{M}^{\rm pl}_\alpha$ is rationally trivial, as this is the principal $[*/\mathbb{G}_m]$-bundle with $[*/\mathbb{G}_m]$-action Ψ'_α by Definition 3.22.

For (c), suppose $K(\mathcal{A})$ is free abelian, and $\chi : K(\mathcal{A}) \times K(\mathcal{A}) \to \mathbb{Z}$ is nondegenerate, and $\mathcal{M}_0(\mathbb{K}) = \{[0]\}$. Let $\alpha \in K(\mathcal{A})$ with $\alpha \neq 0$. Then there exists $\beta \in K(\mathcal{A})$ with $\chi(\alpha,\beta) \neq 0$, as χ is nondegenerate. By Assumption 3.1(b) the projection $K_0(\mathcal{A}) \to K(\mathcal{A})$ is surjective, so $K_0(\mathcal{A})$ cannot map into the kernel of $\chi(\alpha,-)$, and thus there exists $\tilde\beta \in K(\mathcal{A})$ with $\chi(\alpha,\tilde\beta) \neq 0$ and $\mathcal{M}_{\tilde\beta} \neq \emptyset$. So (b) shows that $\Pi^{\rm pl}_\alpha : \mathcal{M}'_\alpha \to \mathcal{M}^{\rm pl}_\alpha$ is rationally trivial, whenever $\alpha \neq 0$. When $\alpha = 0$ we have $\mathcal{M}'_0 = \mathcal{M}^{\rm pl}_0 = \emptyset$ as $\mathcal{M}_0(\mathbb{K}) = \{[0]\}$. Hence $\Pi^{\rm pl} : \mathcal{M}' \to \mathcal{M}^{\rm pl}$ is rationally trivial. The rest is immediate.

4.5 Proof of Proposition 3.26

Work in the situation of Definition 3.22, in either the abelian or triangulated category case, over $\mathbb{K} = \mathbb{C}$, with the homology theories of (higher) Artin $\mathbb{C}$-stacks over R described in Example 2.35. Then $\Pi^{\rm pl} : \mathcal{M}' \to \mathcal{M}^{\rm pl}$ is a principal $[*/\mathbb{G}_m]$-bundle, so passing to classifying spaces or topological realizations as in Example 2.35(a),(c) gives a map of topological spaces $F^{\mathbf{Top}_\infty}_{\mathbf{HSt}_\mathbb{C}}(\Pi^{\rm pl}) : F^{\mathbf{Top}_\infty}_{\mathbf{HSt}_\mathbb{C}}(\mathcal{M}') \to F^{\mathbf{Top}_\infty}_{\mathbf{HSt}_\mathbb{C}}(\mathcal{M}^{\rm pl})$ which is (at least up to homotopy) a topological fibration with fibre $B\mathbb{G}_m$. Applying homology $H_*(-)$ gives $H_*(\Pi^{\rm pl}) : H_*(\mathcal{M}') \to H_*(\mathcal{M}^{\rm pl})$.

Now as in McCleary [111, §5], given a topological fibration $\pi : E \to B$ with fibre F, the *homology Leray–Serre spectral sequence* is a first quadrant spectral sequence with second page $E^2_{p,q} \cong H_p(B, \mathcal{H}_q(F,R))$, which converges to $H_{p+q}(E,R)$. In our case, as $\Pi^{\rm pl}$ is a principal bundle, $\mathcal{H}_q(F,R)$ is the constant sheaf with fibre $H_q(F,R)$ for $F = B\mathbb{G}_m$, and as $H_q(F,R)$ is finitely generated and free over R we have $H_p(B, \mathcal{H}_q(F,R)) \cong H_p(B,R) \otimes_R H_q(F,R)$. Thus

$$E^2_{p,q} = H_p(\mathcal{M}^{\rm pl}) \otimes_R H_q([*/\mathbb{G}_m]) \cong \begin{cases} H_p(\mathcal{M}^{\rm pl}), & q = 2k,\ k = 0,1,2,\ldots, \\ 0, & \text{otherwise.} \end{cases}$$

We can now use the theory of spectral sequences, which involve taking homology of complexes to compute the subsequent pages $(E^r_{p,q})_{p,q\geqslant 0}$ for $r = 3,4,\ldots$.

Here $E^r_{p,q}$ becomes independent of r for $r > \max(p+1,q)$, and then we write $E^\infty_{p,q} = E^r_{p,q}$. Finally $H_{p+q}(E,R) = H_{p+q}(\mathcal{M}')$ has a filtration $F_q H_{p+q}(\mathcal{M}')$ with $E^\infty_{p,q} = F_q H_{p+q}(\mathcal{M}')/F_{q+1}H_{p+q}(\mathcal{M}')$. In degrees 0,1 we have

$$E^\infty_{0,0} \cong E^2_{0,0} = H_0(\mathcal{M}^{\mathrm{pl}}), \qquad E^\infty_{1,0} \cong E^2_{1,0} = H_1(\mathcal{M}^{\mathrm{pl}}), \qquad E^\infty_{0,1} = 0,$$

so that as $E^\infty_{0,1} = 0$ we have

$$H_0(\mathcal{M}') \cong E^\infty_{0,0} \cong H_0(\mathcal{M}^{\mathrm{pl}}), \qquad H_1(\mathcal{M}') \cong E^\infty_{1,0} \cong H_1(\mathcal{M}^{\mathrm{pl}}). \tag{4.25}$$

As t has degree 2 we have $H_k(\mathcal{M}')^{t=0} = H_k(\mathcal{M}')$ for $k=0,1$, and the morphisms $\Pi^{\mathrm{pl}}_{t=0} : H_k(\mathcal{M}'_0)^{t=0} \to H_k(\mathcal{M}^{\mathrm{pl}}_0)$ for $k=0,1$ are the isomorphisms (4.25), proving the cases $k=0,1$ of the proposition.

In degree 2 we have $d_2 : E^2_{3,0} = H_3(\mathcal{M}^{\mathrm{pl}}) \to E^2_{0,2} = H_0(\mathcal{M}^{\mathrm{pl}})$, and then

$$E^\infty_{2,0} \cong E^2_{2,0} = H_2(\mathcal{M}^{\mathrm{pl}}), \;\; E^\infty_{1,1} = 0, \;\; E^\infty_{0,2} \cong H_0(\mathcal{M}^{\mathrm{pl}})/\operatorname{Im} d_2 \cong H_0(\mathcal{M}')/\operatorname{Im} d_2.$$

As $E^\infty_{1,1} = 0$ we have an exact sequence

$$0 \longrightarrow E^\infty_{0,2} \cong H_0(\mathcal{M}')/\operatorname{Im} d_2 \xrightarrow{\iota} H_2(\mathcal{M}') \xrightarrow{H_2(\Pi^{\mathrm{pl}})} H_2(\mathcal{M}^{\mathrm{pl}}) \cong E^\infty_{2,0} \longrightarrow 0. \tag{4.26}$$

One can show using [111, §5] that we have a commutative diagram

$$\begin{tikzcd} H_0(\mathcal{M}') \arrow[r, "\text{projection}"'] \arrow[rr, bend left, "t\diamond-"] & H_0(\mathcal{M}')/\operatorname{Im} d_2 \arrow[r, "\iota \text{ in (4.26)}"'] & H_2(\mathcal{M}'). \end{tikzcd} \tag{4.27}$$

Comparing this with (3.36), (3.49), (3.50) and (4.26) shows $\Pi^{\mathrm{pl}}_{t=0} : H_2(\mathcal{M}')^{t=0} \to H_2(\mathcal{M}^{\mathrm{pl}})$ is an isomorphism, as we have to prove.

4.6 Proof of Theorem 3.29

Work in the situation of Definition 3.28. For part (a) we must show $[\,,\,]^{\mathrm{pl}}$ satisfies graded antisymmetry (3.45) and the graded Jacobi identity (3.46).

To prove graded antisymmetry we must introduce some new notation. Let $\alpha,\beta \in K(\mathcal{A})$. As for $\check\Pi^{\mathrm{pl}}_{\alpha,\beta}, \Phi^{\mathrm{pl}}_{\alpha,\beta}, \Psi^{\mathrm{pl}}_{\alpha,\beta}$ in (3.53), using the 2-co-Cartesian property of (3.52), we can construct a natural morphism $\sigma^{\mathrm{pl}}_{\alpha,\beta} : (\mathcal{M}_\alpha \times \mathcal{M}_\beta)^{\mathrm{pl}} \to (\mathcal{M}_\beta \times \mathcal{M}_\alpha)^{\mathrm{pl}}$, the analogue of $\sigma_{\alpha,\beta}$ in Assumption 3.1(g), which exchanges the factors $\mathcal{M}_\alpha, \mathcal{M}_\beta$. It is an isomorphism in $\mathrm{Ho}(\mathbf{Art}^{\mathrm{lft}}_{\mathbb{K}})$ with inverse $\sigma^{\mathrm{pl}}_{\beta,\alpha}$. Also write $\sigma^{\mathrm{pl,pl}}_{\alpha,\beta} : \mathcal{M}^{\mathrm{pl}}_\alpha \times \mathcal{M}^{\mathrm{pl}}_\beta \to \mathcal{M}^{\mathrm{pl}}_\beta \times \mathcal{M}^{\mathrm{pl}}_\alpha$ for the exchange of factors. Then we have commutative diagrams in $\mathrm{Ho}(\mathbf{Art}^{\mathrm{lft}}_{\mathbb{K}})$:

$$\begin{tikzcd} \mathcal{M}'_\alpha \times \mathcal{M}'_\beta \arrow[r, "\sigma'_{\alpha,\beta}"'] \arrow[d, "\Pi^{\mathrm{pl}}_{\alpha,\beta}"] & \mathcal{M}'_\beta \times \mathcal{M}'_\alpha \arrow[d, "\Pi^{\mathrm{pl}}_{\beta,\alpha}"'] \\ (\mathcal{M}_\alpha \times \mathcal{M}_\beta)^{\mathrm{pl}} \arrow[r, "\sigma^{\mathrm{pl}}_{\alpha,\beta}"] \arrow[d, "\check\Pi^{\mathrm{pl}}_{\alpha,\beta}"] & (\mathcal{M}_\beta \times \mathcal{M}_\alpha)^{\mathrm{pl}} \arrow[d, "\check\Pi^{\mathrm{pl}}_{\beta,\alpha}"'] \\ \mathcal{M}^{\mathrm{pl}}_\alpha \times \mathcal{M}^{\mathrm{pl}}_\beta \arrow[r, "\sigma^{\mathrm{pl,pl}}_{\alpha,\beta}"] & \mathcal{M}^{\mathrm{pl}}_\beta \times \mathcal{M}^{\mathrm{pl}}_\alpha, \end{tikzcd} \tag{4.28}$$

$$
\begin{tikzcd}
(\mathcal{M}_\alpha\times\mathcal{M}_\beta)^{\mathrm{pl}} \arrow[rr, "\sigma^{\mathrm{pl}}_{\alpha,\beta}"'] \arrow[drr, "\Phi^{\mathrm{pl}}_{\alpha,\beta}"'] & & (\mathcal{M}_\beta\times\mathcal{M}_\alpha)^{\mathrm{pl}} \arrow[d, "\Phi^{\mathrm{pl}}_{\beta,\alpha}"'] \\
& & \mathcal{M}^{\mathrm{pl}}_{\alpha+\beta}.
\end{tikzcd}
\tag{4.29}
$$

As $\check\Theta^\bullet_{\alpha,\beta}$ is defined so that $\Theta^\bullet_{\alpha,\beta}|_{\mathcal{M}'_\alpha\times\mathcal{M}'_\beta}\cong(\Pi^{\mathrm{pl}}_{\alpha,\beta})^*(\check\Theta^\bullet_{\alpha,\beta})$, we see from Assumption 3.1(j) and the top square of (4.28) which is a pullback square of principal $[*/\mathbb{G}_m]$-bundles that there is a natural isomorphism

$$
(\sigma^{\mathrm{pl}}_{\alpha,\beta})^*(\check\Theta^\bullet_{\beta,\alpha})\cong(\check\Theta^\bullet_{\alpha,\beta})^\vee[2n], \tag{4.30}
$$

which is equivariant under the $[*/\mathbb{G}_m]$-actions.

Let $\zeta\in H_a(\mathcal{M}^{\mathrm{pl}}_\alpha)$ and $\eta\in H_b(\mathcal{M}^{\mathrm{pl}}_\beta)$ for $a,b\geqslant 0$, and write $\tilde a=a+\chi(\alpha,\alpha)-2$ and $\tilde b=b+\chi(\beta,\beta)-2$, so that $\zeta\in\tilde H_{\tilde a}(\mathcal{M}^{\mathrm{pl}}_\alpha)$ and $\eta\in\tilde H_{\tilde b}(\mathcal{M}^{\mathrm{pl}}_\beta)$, and set $d=a+b-2\chi(\alpha,\beta)-2$. Then we have

$$
\begin{aligned}
&[\eta,\zeta]^{\mathrm{pl}}=\epsilon_{\beta,\alpha}(-1)^{b\chi(\alpha,\alpha)}\cdot H_d(\Phi^{\mathrm{pl}}_{\beta,\alpha})\circ\mathrm{PE}([\check\Theta^\bullet_{\beta,\alpha}])[\eta\boxtimes\zeta]\\
&=\epsilon_{\beta,\alpha}(-1)^{b\chi(\alpha,\alpha)}\cdot H_d(\Phi^{\mathrm{pl}}_{\beta,\alpha})\circ\mathrm{PE}([\check\Theta^\bullet_{\beta,\alpha}])\bigl[(-1)^{ab}H_{a+b}(\sigma^{\mathrm{pl,pl}}_{\alpha,\beta})(\zeta\boxtimes\eta)\bigr]\\
&=\epsilon_{\beta,\alpha}(-1)^{ab+b\chi(\alpha,\alpha)}\cdot H_d(\Phi^{\mathrm{pl}}_{\beta,\alpha})\circ H_d(\sigma^{\mathrm{pl}}_{\alpha,\beta})\circ\mathrm{PE}\bigl([(\sigma^{\mathrm{pl}}_{\alpha,\beta})^*(\check\Theta^\bullet_{\beta,\alpha})]\bigr)[\theta\boxtimes\eta]\\
&=\epsilon_{\beta,\alpha}(-1)^{ab+b\chi(\alpha,\alpha)}\cdot H_d(\Phi^{\mathrm{pl}}_{\alpha,\beta})\circ\mathrm{PE}\bigl([(\check\Theta^\bullet_{\alpha,\beta})^\vee[2n]]\bigr)[\theta\boxtimes\eta]\\
&=\epsilon_{\beta,\alpha}(-1)^{ab+b\chi(\alpha,\alpha)}\cdot(-1)^{\chi(\alpha,\beta)+1}H_d(\Phi^{\mathrm{pl}}_{\alpha,\beta})\circ\mathrm{PE}\bigl([\check\Theta^\bullet_{\alpha,\beta}]\bigr)[\theta\boxtimes\eta]\\
&=\epsilon_{\beta,\alpha}(-1)^{ab+b\chi(\alpha,\alpha)}\cdot(-1)^{\chi(\beta,\alpha)+1}\cdot\epsilon_{\alpha,\beta}(-1)^{a\chi(\beta,\beta)}\cdot[\zeta,\eta]^{\mathrm{pl}}\\
&=(-1)^{\tilde a\tilde b+1}[\zeta,\eta]^{\mathrm{pl}}.
\end{aligned}
$$

Here we use (3.55) in the first and fifth steps, supercommutativity of $\boxtimes$ in the second, Assumption 2.39(b) applied to the bottom square of (4.28) which is a pullback square of principal $[*/\mathbb{G}_m]$-bundles in the third, equation (4.29) and functoriality of $H_*(-)$ and (4.30) in the fourth, Assumption 2.39(c) and $\operatorname{rank}\check\Theta^\bullet_{\beta,\alpha}=\chi(\alpha,\beta)$ in the fifth, and (3.1) and $\tilde a=a+\chi(\alpha,\alpha)-2$, $\tilde b=b+\chi(\beta,\beta)-2$ in the sixth. Thus $[\,,\,]^{\mathrm{pl}}$ satisfies graded antisymmetry.

To prove the graded Jacobi identity, let $\alpha,\beta,\gamma\in K(\mathcal{A})$. Then we have a principal $[*/\mathbb{G}_m]^3$-bundle

$$
\Pi^{\mathrm{pl}}_\alpha\times\Pi^{\mathrm{pl}}_\beta\times\Pi^{\mathrm{pl}}_\gamma:\mathcal{M}'_\alpha\times\mathcal{M}'_\beta\times\mathcal{M}'_\gamma\longrightarrow\mathcal{M}^{\mathrm{pl}}_\alpha\times\mathcal{M}^{\mathrm{pl}}_\beta\times\mathcal{M}^{\mathrm{pl}}_\gamma,
$$

with $[*/\mathbb{G}_m]^3$-action defined using $\Psi'_\alpha\times\Psi'_\beta\times\Psi'_\gamma$. We will apply Assumption

2.39(e) to this. In place of (2.52) we have the commutative diagram:

$$
\begin{tikzcd}
 & & (\mathcal{M}_\alpha\times\mathcal{M}_\beta)^{\mathrm{pl}}\times\mathcal{M}_\gamma^{\mathrm{pl}} \arrow[dr, bend left, "\Pi^{\mathrm{pl}}_{\alpha,\beta}\times\mathrm{id}_{\mathcal{M}^{\mathrm{pl}}_\gamma}"] & \\
\mathcal{M}'_\alpha\times\mathcal{M}'_\beta\times\mathcal{M}'_\gamma \arrow[r, "\Pi^{\mathrm{pl}}_{\alpha,\beta,\gamma}"] & (\mathcal{M}_\alpha\times\mathcal{M}_\beta\times\mathcal{M}_\gamma)^{\mathrm{pl}} \arrow[ur, bend left, "\tau_{\alpha\beta}"] \arrow[r, "\tau_{\beta\gamma}"'] \arrow[dr, bend right, "\tau_{\gamma\alpha}"'] & (\mathcal{M}_\beta\times\mathcal{M}_\gamma)^{\mathrm{pl}}\times\mathcal{M}_\alpha^{\mathrm{pl}} \arrow[r, "\Pi^{\mathrm{pl}}_{\beta,\gamma}\times\mathrm{id}_{\mathcal{M}^{\mathrm{pl}}_\alpha}"'] & \mathcal{M}^{\mathrm{pl}}_\alpha\times\mathcal{M}^{\mathrm{pl}}_\beta\times\mathcal{M}^{\mathrm{pl}}_\gamma. \\
 & & (\mathcal{M}_\gamma\times\mathcal{M}_\alpha)^{\mathrm{pl}}\times\mathcal{M}_\beta^{\mathrm{pl}} \arrow[ur, bend right, "\Pi^{\mathrm{pl}}_{\gamma,\alpha}\times\mathrm{id}_{\mathcal{M}^{\mathrm{pl}}_\beta}"'] &
\end{tikzcd}
\tag{4.31}
$$

Here and below, for notational simplicity we implicitly identify $\mathcal{M}'_\alpha\times\mathcal{M}'_\beta\times\mathcal{M}'_\gamma$ with its cyclic permutations $\mathcal{M}'_\beta\times\mathcal{M}'_\gamma\times\mathcal{M}'_\alpha$ and $\mathcal{M}'_\gamma\times\mathcal{M}'_\alpha\times\mathcal{M}'_\beta$, omitting the permutation isomorphisms from our notation. We do the same for $(\mathcal{M}_\alpha\times\mathcal{M}_\beta\times\mathcal{M}_\gamma)^{\mathrm{pl}}$ and $\mathcal{M}^{\mathrm{pl}}_\alpha\times\mathcal{M}^{\mathrm{pl}}_\beta\times\mathcal{M}^{\mathrm{pl}}_\gamma$.

As for the definition of $(\mathcal{M}_\alpha\times\mathcal{M}_\beta)^{\mathrm{pl}}$ in Definition 3.28, we define $\Pi^{\mathrm{pl}}_{\alpha,\beta,\gamma}:\mathcal{M}'_\alpha\times\mathcal{M}'_\beta\times\mathcal{M}'_\gamma\to(\mathcal{M}_\alpha\times\mathcal{M}_\beta\times\mathcal{M}_\gamma)^{\mathrm{pl}}$ in (4.31) to be the principal $[*/\mathbb{G}_m]$-bundle with $[*/\mathbb{G}_m]$-action given by Proposition 2.25(a) associated to the free $[*/\mathbb{G}_m]$-action on $\mathcal{M}'_\alpha\times\mathcal{M}'_\beta\times\mathcal{M}'_\gamma$:

$$
\begin{gathered}
\bigl((\Psi'_\alpha\circ\Pi_{[*/\mathbb{G}_m]\times\mathcal{M}'_\alpha}),(\Psi'_\beta\circ\Pi_{[*/\mathbb{G}_m]\times\mathcal{M}'_\beta}),(\Psi'_\gamma\circ\Pi_{[*/\mathbb{G}_m]\times\mathcal{M}'_\gamma})\bigr):\\
[*/\mathbb{G}_m]\times\mathcal{M}'_\alpha\times\mathcal{M}'_\beta\times\mathcal{M}'_\gamma\longrightarrow\mathcal{M}'_\alpha\times\mathcal{M}'_\beta\times\mathcal{M}'_\gamma,
\end{gathered}
$$

the diagonal action of the $[*/\mathbb{G}_m]$-actions $\Psi'_\alpha,\Psi'_\beta,\Psi'_\gamma$ on $\mathcal{M}'_\alpha,\mathcal{M}'_\beta,\mathcal{M}'_\gamma$.

As for the morphisms (3.53) we can construct a natural morphism

$$
\Phi^{\mathrm{pl}}_{\alpha,\beta,\gamma}:(\mathcal{M}_\alpha\times\mathcal{M}_\beta\times\mathcal{M}_\gamma)^{\mathrm{pl}}\longrightarrow(\mathcal{M}_{\alpha+\beta}\times\mathcal{M}_\gamma)^{\mathrm{pl}}
$$

in a commutative diagram

$$
\begin{tikzcd}
\mathcal{M}'_\alpha\times\mathcal{M}'_\beta\times\mathcal{M}'_\gamma \arrow[rr, "\Phi'_{\alpha,\beta}\times\mathrm{id}_{\mathcal{M}'_\gamma}"'] \arrow[d, "\Pi^{\mathrm{pl}}_{\alpha,\beta,\gamma}"] & & \mathcal{M}'_{\alpha+\beta}\times\mathcal{M}'_\gamma \arrow[d, "\Pi^{\mathrm{pl}}_{\alpha+\beta,\gamma}"'] \\
(\mathcal{M}_\alpha\times\mathcal{M}_\beta\times\mathcal{M}_\gamma)^{\mathrm{pl}} \arrow[rr, "\Phi^{\mathrm{pl}}_{\alpha,\beta,\gamma}"] \arrow[d, "\tau_{\alpha\beta}"] & & (\mathcal{M}_{\alpha+\beta}\times\mathcal{M}_\gamma)^{\mathrm{pl}} \arrow[d, "\Pi^{\mathrm{pl}}_{\alpha+\beta,\gamma}"'] \\
(\mathcal{M}_\alpha\times\mathcal{M}_\beta)^{\mathrm{pl}}\times\mathcal{M}^{\mathrm{pl}}_\gamma \arrow[rr, "\Phi^{\mathrm{pl}}_{\alpha,\beta}\times\mathrm{id}_{\mathcal{M}^{\mathrm{pl}}_\gamma}"] & & \mathcal{M}^{\mathrm{pl}}_{\alpha+\beta}\times\mathcal{M}^{\mathrm{pl}}_\gamma.
\end{tikzcd}
\tag{4.32}
$$

Using the implicit identifications $(\mathcal{M}_\alpha\times\mathcal{M}_\beta\times\mathcal{M}_\gamma)^{\mathrm{pl}}\cong(\mathcal{M}_\beta\times\mathcal{M}_\gamma\times\mathcal{M}_\alpha)^{\mathrm{pl}}\cong(\mathcal{M}_\gamma\times\mathcal{M}_\alpha\times\mathcal{M}_\beta)^{\mathrm{pl}}$, using (3.4)–(3.5) we can show that

$$
\begin{gathered}
\Phi^{\mathrm{pl}}_{\alpha+\beta,\gamma}\circ\Phi^{\mathrm{pl}}_{\alpha,\beta,\gamma}=\Phi^{\mathrm{pl}}_{\beta+\gamma,\alpha}\circ\Phi^{\mathrm{pl}}_{\beta,\gamma,\alpha}=\Phi^{\mathrm{pl}}_{\gamma+\alpha,\beta}\circ\Phi^{\mathrm{pl}}_{\gamma,\alpha,\beta}:\\
(\mathcal{M}_\alpha\times\mathcal{M}_\beta\times\mathcal{M}_\gamma)^{\mathrm{pl}}\longrightarrow\mathcal{M}^{\mathrm{pl}}_{\alpha+\beta+\gamma}.
\end{gathered}
\tag{4.33}
$$

Let $\zeta\in H_a(\mathcal{M}^{\mathrm{pl}}_\alpha)$, $\eta\in H_b(\mathcal{M}^{\mathrm{pl}}_\beta)$ and $\theta\in H_c(\mathcal{M}^{\mathrm{pl}}_\gamma)$, so that $\zeta\boxtimes\eta\boxtimes\theta\in H_{a+b+c}(\mathcal{M}^{\mathrm{pl}}_\alpha\times\mathcal{M}^{\mathrm{pl}}_\beta\times\mathcal{M}^{\mathrm{pl}}_\gamma)$. Apply Assumption 2.39(e) to (4.31) with

$\Pi^*_{(\mathcal{M}_\alpha\times\mathcal{M}_\beta)^{\mathrm{pl}}}(\check{\Theta}^\bullet_{\alpha,\beta})$, $\Pi^*_{(\mathcal{M}_\beta\times\mathcal{M}_\gamma)^{\mathrm{pl}}}(\check{\Theta}^\bullet_{\beta,\gamma})$, $\Pi^*_{(\mathcal{M}_\gamma\times\mathcal{M}_\alpha)^{\mathrm{pl}}}(\check{\Theta}^\bullet_{\gamma,\alpha})$ in place of $\mathcal{E}^\bullet_{12}, \mathcal{E}^\bullet_{23}$, $\mathcal{E}^\bullet_{31}$, respectively. Evaluating (2.53) on $\zeta\boxtimes\eta\boxtimes\theta$ shows that

$$\begin{aligned}
&(-1)^{\chi(\gamma,\alpha)}\,\mathrm{PE}\bigl([\tau_{23}^*\circ\Pi^*_{(\mathcal{M}_\beta\times\mathcal{M}_\gamma)^{\mathrm{pl}}}(\check{\Theta}^\bullet_{\beta,\gamma})\oplus\tau_{31}^*\circ\Pi^*_{(\mathcal{M}_\gamma\times\mathcal{M}_\alpha)^{\mathrm{pl}}}(\check{\Theta}^\bullet_{\gamma,\alpha})^\vee]\bigr)\\
&\qquad\circ\mathrm{PE}\bigl([\Pi^*_{(\mathcal{M}_\alpha\times\mathcal{M}_\beta)^{\mathrm{pl}}}(\check{\Theta}^\bullet_{\alpha,\beta})]\bigr)(\zeta\boxtimes\eta\boxtimes\theta)+\\
&(-1)^{\chi(\alpha,\beta)}\,\mathrm{PE}\bigl([\tau_{31}^*\circ\Pi^*_{(\mathcal{M}_\gamma\times\mathcal{M}_\alpha)^{\mathrm{pl}}}(\check{\Theta}^\bullet_{\gamma,\alpha})\oplus\tau_{12}^*\circ\Pi^*_{(\mathcal{M}_\alpha\times\mathcal{M}_\beta)^{\mathrm{pl}}}(\check{\Theta}^\bullet_{\alpha,\beta})^\vee]\bigr)\\
&\qquad\circ\mathrm{PE}\bigl([\Pi^*_{(\mathcal{M}_\beta\times\mathcal{M}_\gamma)^{\mathrm{pl}}}(\check{\Theta}^\bullet_{\beta,\gamma})]\bigr)(\zeta\boxtimes\eta\boxtimes\theta)+\\
&(-1)^{\chi(\beta,\gamma)}\,\mathrm{PE}\bigl([\tau_{12}^*\circ\Pi^*_{(\mathcal{M}_\alpha\times\mathcal{M}_\beta)^{\mathrm{pl}}}(\check{\Theta}^\bullet_{\alpha,\beta})\oplus\tau_{23}^*\circ\Pi^*_{(\mathcal{M}_\beta\times\mathcal{M}_\gamma)^{\mathrm{pl}}}(\check{\Theta}^\bullet_{\beta,\gamma})^\vee]\bigr)\\
&\qquad\circ\mathrm{PE}\bigl([\Pi^*_{(\mathcal{M}_\gamma\times\mathcal{M}_\alpha)^{\mathrm{pl}}}(\check{\Theta}^\bullet_{\gamma,\alpha})]\bigr)(\zeta\boxtimes\eta\boxtimes\theta)=0.
\end{aligned}\tag{4.34}$$

By supercommutativity of $\boxtimes$, the natural isomorphisms

$$\mathcal{M}^{\mathrm{pl}}_\alpha\times\mathcal{M}^{\mathrm{pl}}_\beta\times\mathcal{M}^{\mathrm{pl}}_\gamma\cong\mathcal{M}^{\mathrm{pl}}_\beta\times\mathcal{M}^{\mathrm{pl}}_\gamma\times\mathcal{M}^{\mathrm{pl}}_\alpha\cong\mathcal{M}^{\mathrm{pl}}_\gamma\times\mathcal{M}^{\mathrm{pl}}_\alpha\times\mathcal{M}^{\mathrm{pl}}_\beta$$

induce identifications in homology

$$(-1)^{ca}\zeta\boxtimes\eta\boxtimes\theta\cong(-1)^{ab}\eta\boxtimes\theta\boxtimes\zeta\cong(-1)^{bc}\theta\boxtimes\zeta\boxtimes\eta.$$

Multiplying (4.34) by $(-1)^{ca}$ and using these identifications and (4.30) yields

$$\begin{aligned}
&(-1)^{ca+\chi(\gamma,\alpha)}\,\mathrm{PE}\bigl([\tau_{23}^*\circ\Pi^*_{(\mathcal{M}_\beta\times\mathcal{M}_\gamma)^{\mathrm{pl}}}(\check{\Theta}^\bullet_{\beta,\gamma})\oplus\tau_{31}^*\circ\Pi^*_{(\mathcal{M}_\gamma\times\mathcal{M}_\alpha)^{\mathrm{pl}}}\\
&\qquad\circ(\sigma^{\mathrm{pl}}_{\gamma,\alpha})^*(\check{\Theta}^\bullet_{\alpha,\gamma})[-2n]]\bigr)\circ\mathrm{PE}\bigl([\Pi^*_{(\mathcal{M}_\alpha\times\mathcal{M}_\beta)^{\mathrm{pl}}}(\check{\Theta}^\bullet_{\alpha,\beta})]\bigr)(\zeta\boxtimes\eta\boxtimes\theta)+\\
&(-1)^{ab+\chi(\alpha,\beta)}\,\mathrm{PE}\bigl([\tau_{31}^*\circ\Pi^*_{(\mathcal{M}_\gamma\times\mathcal{M}_\alpha)^{\mathrm{pl}}}(\check{\Theta}^\bullet_{\gamma,\alpha})\oplus\tau_{12}^*\circ\Pi^*_{(\mathcal{M}_\alpha\times\mathcal{M}_\beta)^{\mathrm{pl}}}\\
&\qquad\circ(\sigma^{\mathrm{pl}}_{\alpha,\beta})^*(\check{\Theta}^\bullet_{\beta,\alpha})[-2n]]\bigr)\circ\mathrm{PE}\bigl([\Pi^*_{(\mathcal{M}_\beta\times\mathcal{M}_\gamma)^{\mathrm{pl}}}(\check{\Theta}^\bullet_{\beta,\gamma})]\bigr)(\eta\boxtimes\theta\boxtimes\zeta)+\\
&(-1)^{bc+\chi(\beta,\gamma)}\,\mathrm{PE}\bigl([\tau_{12}^*\circ\Pi^*_{(\mathcal{M}_\alpha\times\mathcal{M}_\beta)^{\mathrm{pl}}}(\check{\Theta}^\bullet_{\alpha,\beta})\oplus\tau_{23}^*\circ\Pi^*_{(\mathcal{M}_\beta\times\mathcal{M}_\gamma)^{\mathrm{pl}}}\\
&\qquad\circ(\sigma^{\mathrm{pl}}_{\beta,\gamma})^*(\check{\Theta}^\bullet_{\gamma,\beta})[-2n]]\bigr)\circ\mathrm{PE}\bigl([\Pi^*_{(\mathcal{M}_\gamma\times\mathcal{M}_\alpha)^{\mathrm{pl}}}(\check{\Theta}^\bullet_{\gamma,\alpha})]\bigr)(\theta\boxtimes\zeta\boxtimes\eta)=0.
\end{aligned}$$

Using Assumption 2.39(d), and dropping the shifts $[-2n]$ as they do not affect K-theory classes, we see that

$$\begin{aligned}
&(-1)^{ca+\chi(\gamma,\alpha)}\,\mathrm{PE}\bigl([\tau_{23}^*\circ\Pi^*_{(\mathcal{M}_\beta\times\mathcal{M}_\gamma)^{\mathrm{pl}}}(\check{\Theta}^\bullet_{\beta,\gamma})\oplus\tau_{31}^*\circ\Pi^*_{(\mathcal{M}_\gamma\times\mathcal{M}_\alpha)^{\mathrm{pl}}}\circ(\sigma^{\mathrm{pl}}_{\gamma,\alpha})^*(\check{\Theta}^\bullet_{\alpha,\gamma})]\bigr)\\
&\qquad\circ\bigl((\mathrm{PE}([\check{\Theta}^\bullet_{\alpha,\beta}])(\zeta\boxtimes\eta))\boxtimes\theta\bigr)+\\
&(-1)^{ab+\chi(\alpha,\beta)}\,\mathrm{PE}\bigl([\tau_{31}^*\circ\Pi^*_{(\mathcal{M}_\gamma\times\mathcal{M}_\alpha)^{\mathrm{pl}}}(\check{\Theta}^\bullet_{\gamma,\alpha})\oplus\tau_{12}^*\circ\Pi^*_{(\mathcal{M}_\alpha\times\mathcal{M}_\beta)^{\mathrm{pl}}}\circ(\sigma^{\mathrm{pl}}_{\alpha,\beta})^*(\check{\Theta}^\bullet_{\beta,\alpha})]\bigr)\\
&\qquad\circ\bigl((\mathrm{PE}([\check{\Theta}^\bullet_{\beta,\gamma}])(\eta\boxtimes\theta))\boxtimes\zeta\bigr)+\\
&(-1)^{bc+\chi(\beta,\gamma)}\,\mathrm{PE}\bigl([\tau_{12}^*\circ\Pi^*_{(\mathcal{M}_\alpha\times\mathcal{M}_\beta)^{\mathrm{pl}}}(\check{\Theta}^\bullet_{\alpha,\beta})\oplus\tau_{23}^*\circ\Pi^*_{(\mathcal{M}_\beta\times\mathcal{M}_\gamma)^{\mathrm{pl}}}\circ(\sigma^{\mathrm{pl}}_{\beta,\gamma})^*(\check{\Theta}^\bullet_{\gamma,\beta})]\bigr)\\
&\qquad\circ\bigl((\mathrm{PE}([\check{\Theta}^\bullet_{\gamma,\alpha}])(\theta\boxtimes\zeta))\boxtimes\eta\bigr)=0.
\end{aligned}\tag{4.35}$$

By pushing (3.9) down along the $[*/\mathbb{G}_m]$-bundles $\Pi^{\mathrm{pl}}_{\alpha,\beta,\gamma},\Pi^{\mathrm{pl}}_{\alpha+\beta,\gamma}$ in the top square of (4.32), we obtain an isomorphism

$$\begin{aligned}
&\tau_{23}^*\circ\Pi^*_{(\mathcal{M}_\beta\times\mathcal{M}_\gamma)^{\mathrm{pl}}}(\check{\Theta}^\bullet_{\beta,\gamma})\oplus\tau_{31}^*\circ\Pi^*_{(\mathcal{M}_\gamma\times\mathcal{M}_\alpha)^{\mathrm{pl}}}\circ(\sigma^{\mathrm{pl}}_{\gamma,\alpha})^*(\check{\Theta}^\bullet_{\alpha,\gamma})\\
&\qquad\cong(\Phi^{\mathrm{pl}}_{\alpha,\beta,\gamma})^*(\check{\Theta}^\bullet_{\alpha+\beta,\gamma}).
\end{aligned}$$

Substituting this and its cyclic permutations into (4.35), and pushing forward to $H_*(\mathcal{M}^{\mathrm{pl}}_{\alpha+\beta+\gamma})$ by the morphisms (4.33) using functoriality of $H_*(-)$ yields

$$
\begin{aligned}
&(-1)^{ca+\chi(\gamma,\alpha)}H_d(\Phi^{\mathrm{pl}}_{\alpha+\beta,\gamma})\circ H_d(\Phi^{\mathrm{pl}}_{\alpha,\beta,\gamma})\circ \mathrm{PE}\bigl([(\Phi^{\mathrm{pl}}_{\alpha,\beta,\gamma})^*(\check\Theta^\bullet_{\alpha+\beta,\gamma})]\bigr)\\
&\qquad\circ\bigl((\mathrm{PE}([\check\Theta^\bullet_{\alpha,\beta}])(\zeta\boxtimes\eta))\boxtimes\theta\bigr)+\\
&(-1)^{ab+\chi(\alpha,\beta)}H_d(\Phi^{\mathrm{pl}}_{\beta+\gamma,\alpha})\circ H_d(\Phi^{\mathrm{pl}}_{\beta,\gamma,\alpha})\circ \mathrm{PE}\bigl([(\Phi^{\mathrm{pl}}_{\beta,\gamma,\alpha})^*(\check\Theta^\bullet_{\beta+\gamma,\alpha})]\bigr)\\
&\qquad\circ\bigl((\mathrm{PE}([\check\Theta^\bullet_{\beta,\gamma}])(\eta\boxtimes\theta))\boxtimes\zeta\bigr)+\\
&(-1)^{bc+\chi(\beta,\gamma)}H_d(\Phi^{\mathrm{pl}}_{\gamma+\alpha,\beta})\circ H_d(\Phi^{\mathrm{pl}}_{\gamma,\alpha,\beta})\circ \mathrm{PE}\bigl([(\Phi^{\mathrm{pl}}_{\gamma,\alpha,\beta})^*(\check\Theta^\bullet_{\gamma+\alpha,\beta})]\bigr)\\
&\qquad\circ\bigl((\mathrm{PE}([\check\Theta^\bullet_{\gamma,\alpha}])(\theta\boxtimes\zeta))\boxtimes\eta\bigr)=0,
\end{aligned}
\tag{4.36}
$$

where $d=a+b+c-2(\chi(\alpha,\beta)+\chi(\beta,\gamma)+\chi(\gamma,\alpha))-4$.

Applying Assumption 2.39(b) to the bottom square of (4.32), which is a pullback square of principal $[*/\mathbb{G}_m]$-bundles, gives

$$
\begin{aligned}
&H_d(\Phi^{\mathrm{pl}}_{\alpha,\beta,\gamma})\circ \mathrm{PE}\bigl([(\Phi^{\mathrm{pl}}_{\alpha,\beta,\gamma})^*(\check\Theta^\bullet_{\alpha+\beta,\gamma})]\bigr)\\
&\qquad=\mathrm{PE}([\check\Theta^\bullet_{\alpha+\beta,\gamma}])\circ H_{a+b+c-2\chi(\alpha,\beta)-2}(\Phi^{\mathrm{pl}}_{\alpha,\beta}\times\mathrm{id}_{\mathcal{M}^{\mathrm{pl}}_\gamma}).
\end{aligned}
$$

Substituting this and its cyclic permutations into (4.36) shows that

$$
\begin{aligned}
&(-1)^{ca+\chi(\gamma,\alpha)}H_d(\Phi^{\mathrm{pl}}_{\alpha+\beta,\gamma})\circ \mathrm{PE}([\check\Theta^\bullet_{\alpha+\beta,\gamma}])\circ H_{a+b+c-2\chi(\alpha,\beta)-2}(\Phi^{\mathrm{pl}}_{\alpha,\beta}\times\mathrm{id}_{\mathcal{M}^{\mathrm{pl}}_\gamma})\\
&\qquad\circ\bigl((\mathrm{PE}([\check\Theta^\bullet_{\alpha,\beta}])(\zeta\boxtimes\eta))\boxtimes\theta\bigr)+\\
&(-1)^{ab+\chi(\alpha,\beta)}H_d(\Phi^{\mathrm{pl}}_{\beta+\gamma,\alpha})\circ \mathrm{PE}([\check\Theta^\bullet_{\beta+\gamma,\alpha}])\circ H_{b+c+a-2\chi(\beta,\gamma)-2}(\Phi^{\mathrm{pl}}_{\beta,\gamma}\times\mathrm{id}_{\mathcal{M}^{\mathrm{pl}}_\alpha})\\
&\qquad\circ\bigl((\mathrm{PE}([\check\Theta^\bullet_{\beta,\gamma}])(\eta\boxtimes\theta))\boxtimes\zeta\bigr)+\\
&(-1)^{bc+\chi(\beta,\gamma)}H_d(\Phi^{\mathrm{pl}}_{\gamma+\alpha,\beta})\circ \mathrm{PE}([\check\Theta^\bullet_{\gamma+\alpha,\beta}])\circ H_{c+a+b-2\chi(\gamma,\alpha)-2}(\Phi^{\mathrm{pl}}_{\gamma,\alpha}\times\mathrm{id}_{\mathcal{M}^{\mathrm{pl}}_\beta})\\
&\qquad\circ\bigl((\mathrm{PE}([\check\Theta^\bullet_{\gamma,\alpha}])(\theta\boxtimes\zeta))\boxtimes\eta\bigr)=0.
\end{aligned}
\tag{4.37}
$$

Using (3.1)–(3.2) we find that:

$$
\begin{aligned}
&(-1)^{\chi(\gamma,\alpha)+\chi(\gamma,\gamma)\chi(\alpha,\alpha)}\epsilon_{\alpha,\beta}\epsilon_{\alpha+\beta,\gamma}\\
&\quad=(-1)^{\chi(\alpha,\beta)+\chi(\alpha,\alpha)\chi(\beta,\beta)}\epsilon_{\beta,\gamma}\epsilon_{\beta+\gamma,\alpha}\\
&\quad=(-1)^{\chi(\beta,\gamma)+\chi(\beta,\beta)\chi(\gamma,\gamma)}\epsilon_{\gamma,\alpha}\epsilon_{\gamma+\alpha,\beta}.
\end{aligned}
$$

Multiplying (4.37) by these signs and by $(-1)^{a\chi(\beta,\beta)+b\chi(\gamma,\gamma)+c\chi(\alpha,\alpha)}$ and using compatibility of $H_*(-)$ and $\boxtimes$, we deduce that

$$
\begin{aligned}
&(-1)^{(c+\chi(\gamma,\gamma)-2)(a+\chi(\alpha,\alpha)-2)}\cdot\epsilon_{\alpha+\beta,\gamma}(-1)^{(a+b)\chi(\gamma,\gamma)}\cdot H_d(\Phi^{\mathrm{pl}}_{\alpha+\beta,\gamma})\circ\mathrm{PE}([\check\Theta^\bullet_{\alpha+\beta,\gamma}])\\
&\quad\circ\bigl\{\bigl(\epsilon_{\alpha,\beta}(-1)^{a\chi(\beta,\beta)}\cdot H_{a+b-2\chi(\alpha,\beta)-2}(\Phi^{\mathrm{pl}}_{\alpha,\beta})\circ\mathrm{PE}([\check\Theta^\bullet_{\alpha,\beta}])(\zeta\boxtimes\eta)\bigr)\boxtimes\theta\bigr\}+\\
&(-1)^{(a+\chi(\alpha,\alpha)-2)(b+\chi(\beta,\beta)-2)}\cdot\epsilon_{\beta+\gamma,\alpha}(-1)^{(b+c)\chi(\alpha,\alpha)}\cdot H_d(\Phi^{\mathrm{pl}}_{\beta+\gamma,\alpha})\circ\mathrm{PE}([\check\Theta^\bullet_{\beta+\gamma,\alpha}])\\
&\quad\circ\bigl\{\bigl(\epsilon_{\beta,\gamma}(-1)^{b\chi(\gamma,\gamma)}\cdot H_{b+c-2\chi(\beta,\gamma)-2}(\Phi^{\mathrm{pl}}_{\beta,\gamma})\circ\mathrm{PE}([\check\Theta^\bullet_{\beta,\gamma}])(\eta\boxtimes\theta)\bigr)\boxtimes\zeta\bigr\}+\\
&(-1)^{(b+\chi(\beta,\beta)-2)(c+\chi(\gamma,\gamma)-2)}\cdot\epsilon_{\gamma+\alpha,\beta}(-1)^{(c+a)\chi(\beta,\beta)}\cdot H_d(\Phi^{\mathrm{pl}}_{\gamma+\alpha,\beta})\circ\mathrm{PE}([\check\Theta^\bullet_{\gamma+\alpha,\beta}])\\
&\quad\circ\bigl\{\bigl(\epsilon_{\gamma,\alpha}(-1)^{c\chi(\alpha,\alpha)}\cdot H_{c+a-2\chi(\gamma,\alpha)-2}(\Phi^{\mathrm{pl}}_{\gamma,\alpha})\circ\mathrm{PE}([\check\Theta^\bullet_{\gamma,\alpha}])(\theta\boxtimes\zeta)\bigr)\boxtimes\eta\bigr\}=0.
\end{aligned}
$$

By (3.55) and $\tilde a=a+\chi(\alpha,\alpha)-2$, $\tilde b=b+\chi(\beta,\beta)-2$ and $\tilde c=c+\chi(\gamma,\gamma)-2$, this is equivalent to

$$(-1)^{\tilde c\tilde a}[[\zeta,\eta]^{\rm pl},\theta]^{\rm pl}+(-1)^{\tilde a\tilde b}[[\eta,\theta]^{\rm pl},\zeta]^{\rm pl}+(-1)^{\tilde b\tilde c}[[\theta,\zeta]^{\rm pl},\eta]^{\rm pl}=0.$$

This proves the graded Jacobi identity for $[\,,\,]^{\rm pl}$, completing part (a).

For part (b), let $\alpha,\beta\in K(\mathcal{A})$. Consider the diagram

$$\begin{tikzcd}
{\begin{subarray}{l}[*/\mathbb{G}_m]\times[*/\mathbb{G}_m]\\ \times\mathcal{M}'_\alpha\times\mathcal{M}'_\beta\end{subarray}} \arrow[r, "\Omega\times\mathrm{id}_{\mathcal{M}'_\alpha\times\mathcal{M}'_\beta}"'] \arrow[d, "\mathrm{id}_{[*/\mathbb{G}_m]^2}\times\Pi^{\rm pl}_{\alpha,\beta}"] & {\begin{subarray}{l}[*/\mathbb{G}_m]\times\\ \mathcal{M}'_\alpha\times\mathcal{M}'_\beta\end{subarray}} \arrow[r, "\Pi_{\mathcal{M}'_\alpha\times\mathcal{M}'_\beta}"'] \arrow[d, "\mathrm{id}_{[*/\mathbb{G}_m]}\times\Pi^{\rm pl}_{\alpha,\beta}"] & \mathcal{M}'_\alpha\times\mathcal{M}'_\beta \arrow[d, "\Pi^{\rm pl}_{\alpha,\beta}"'] \\
{\begin{subarray}{l}[*/\mathbb{G}_m]\times[*/\mathbb{G}_m]\\ \times(\mathcal{M}_\alpha\times\mathcal{M}_\beta)^{\rm pl}\end{subarray}} \arrow[r, "\Omega\times\mathrm{id}_{(\mathcal{M}_\alpha\times\mathcal{M}_\beta)^{\rm pl}}"] \arrow[d, "\mathrm{id}_{[*/\mathbb{G}_m]}\times\Psi^{\rm pl}_{\alpha,\beta}"] & {\begin{subarray}{l}[*/\mathbb{G}_m]\times\\ (\mathcal{M}_\alpha\times\mathcal{M}_\beta)^{\rm pl}\end{subarray}} \arrow[r, "\Pi_{(\mathcal{M}_\alpha\times\mathcal{M}_\beta)^{\rm pl}}"] \arrow[d, "\Psi^{\rm pl}_{\alpha,\beta}"] & (\mathcal{M}_\alpha\times\mathcal{M}_\beta)^{\rm pl} \arrow[d, "\check\Pi^{\rm pl}_{\alpha,\beta}"'] \\
{\begin{subarray}{l}[*/\mathbb{G}_m]\times\\ (\mathcal{M}_\alpha\times\mathcal{M}_\beta)^{\rm pl}\end{subarray}} \arrow[r, "\Psi^{\rm pl}_{\alpha,\beta}"] & (\mathcal{M}_\alpha\times\mathcal{M}_\beta)^{\rm pl} \arrow[r, "\check\Pi^{\rm pl}_{\alpha,\beta}"] & \mathcal{M}^{\rm pl}_\alpha\times\mathcal{M}^{\rm pl}_\beta.
\end{tikzcd} \tag{4.38}$$

Here the top squares obviously commute, the bottom left square commutes by (2.20) for the $[*/\mathbb{G}_m]$-action $\Psi^{\rm pl}_{\alpha,\beta}$, and the bottom right square commutes as $\check\Pi^{\rm pl}_{\alpha,\beta}$ is a principal $[*/\mathbb{G}_m]$-bundle with $[*/\mathbb{G}_m]$-action $\Psi^{\rm pl}_{\alpha,\beta}$.

The three right hand horizontal morphisms in (4.38) are principal $[*/\mathbb{G}_m]$-bundles, with the top two trivial, and the left hand morphisms are the corresponding $[*/\mathbb{G}_m]$-actions. The right hand squares are pullback squares of principal $[*/\mathbb{G}_m]$-bundles, and the left hand squares show these pullbacks are compatible with the given $[*/\mathbb{G}_m]$-actions.

We have a perfect complex $\check\Theta^\bullet_{\alpha,\beta}$ on $(\mathcal{M}_\alpha\times\mathcal{M}_\beta)^{\rm pl}$, the bottom middle object in (4.38), which is weight 1 for the $[*/\mathbb{G}_m]$-action. Pullback by the middle column in (4.38) yields

$$\begin{aligned}
&(\Psi^{\rm pl}_{\alpha,\beta}\circ(\mathrm{id}_{[*/\mathbb{G}_m]^2}\times\Pi^{\rm pl}_{\alpha,\beta}))^*(\check\Theta^\bullet_{\alpha,\beta})\\
&\quad\cong(\mathrm{id}_{[*/\mathbb{G}_m]^2}\times\Pi^{\rm pl}_{\alpha,\beta})^*(\pi^*_{[*/\mathbb{G}_m]}(E_1)\otimes\pi^*_{(\mathcal{M}_\alpha\times\mathcal{M}_\beta)^{\rm pl}}(\check\Theta^\bullet_{\alpha,\beta}))\\
&\quad\cong\pi^*_{[*/\mathbb{G}_m]}(E_1)\otimes\pi^*_{\mathcal{M}'_\alpha\times\mathcal{M}'_\beta}(\Theta^\bullet_{\alpha,\beta}),
\end{aligned}\tag{4.39}$$

where the first step holds by (2.28) as $\check\Theta^\bullet_{\alpha,\beta}$ is weight 1 for $\Psi^{\rm pl}_{\alpha,\beta}$, and the second as $(\Pi^{\rm pl}_{\alpha,\beta})^*(\check\Theta^\bullet_{\alpha,\beta})\cong\Theta^\bullet_{\alpha,\beta}|_{\mathcal{M}'_\alpha\times\mathcal{M}'_\beta}$. Let $\zeta\in H_a(\mathcal{M}'_\alpha)$ and $\eta\in H_b(\mathcal{M}'_\beta)$, and set

$d=a+b-2\chi(\alpha,\beta)-2$. Then we have

$$
\begin{aligned}
&\Pi^{\rm pl}_{t=0}\bigl([\Pi(\zeta),\Pi(\eta)]^{t=0}\bigr)=\Pi^{\rm pl}_{t=0}\circ\Pi\bigl([\zeta,\eta]_0\bigr)=H_d(\Pi^{\rm pl}_{\alpha+\beta})\bigl([\zeta,\eta]_0\bigr)\\
&=H_d(\Pi^{\rm pl}_{\alpha+\beta})\left[\sum_{\substack{i\geqslant 0:\,2i\leqslant a+b,\\ i\geqslant\chi(\alpha,\beta)+1}}\begin{array}{l}\epsilon_{\alpha,\beta}(-1)^{a\chi(\beta,\beta)}\cdot H_d(\Xi_{\alpha,\beta})\\ \bigl(t^{i-\chi(\alpha,\beta)-1}\boxtimes\bigl[(\zeta\boxtimes\eta)\cap c_i([\Theta^\bullet_{\alpha,\beta}])\bigr]\bigr)\end{array}\right]\\
&=\epsilon_{\alpha,\beta}(-1)^{a\chi(\beta,\beta)}\cdot H_d(\Pi^{\rm pl}_{\alpha+\beta}\circ\Xi_{\alpha,\beta})\\
&\qquad\circ\,\mathrm{PE}\bigl([\pi^*_{[*/\mathbb{G}_m]}(E_1)\otimes\pi^*_{\mathcal{M}'_\alpha\times\mathcal{M}'_\beta}(\Theta^\bullet_{\alpha,\beta})]\bigr)(\zeta\boxtimes\eta)\\
&=\epsilon_{\alpha,\beta}(-1)^{a\chi(\beta,\beta)}\cdot H_d\bigl(\Phi^{\rm pl}_{\alpha,\beta}\circ\Psi^{\rm pl}_{\alpha,\beta}\circ(\mathrm{id}_{[*/\mathbb{G}_m]^2}\times\Pi^{\rm pl}_{\alpha,\beta})\bigr)\\
&\qquad\circ\,\mathrm{PE}\bigl([(\Psi^{\rm pl}_{\alpha,\beta}\circ(\mathrm{id}_{[*/\mathbb{G}_m]^2}\times\Pi^{\rm pl}_{\alpha,\beta}))^*(\check\Theta^\bullet_{\alpha,\beta})]\bigr)(\zeta\boxtimes\eta)\\
&=\epsilon_{\alpha,\beta}(-1)^{a\chi(\beta,\beta)}\cdot H_d(\Phi^{\rm pl}_{\alpha,\beta})\circ\mathrm{PE}([\check\Theta^\bullet_{\alpha,\beta}])\circ H_{a+b}(\check\Pi^{\rm pl}_{\alpha,\beta}\circ\Pi^{\rm pl}_{\alpha,\beta})(\zeta\boxtimes\eta)\\
&=\epsilon_{\alpha,\beta}(-1)^{a\chi(\beta,\beta)}\cdot H_d(\Phi^{\rm pl}_{\alpha,\beta})\circ\mathrm{PE}([\check\Theta^\bullet_{\alpha,\beta}])\bigl(\Pi^{\rm pl}_\alpha(\zeta)\boxtimes\Pi^{\rm pl}_\beta(\eta)\bigr)\\
&=\bigl[H_a(\Pi^{\rm pl}_\alpha)(\zeta),H_b(\Pi^{\rm pl}_\beta)(\eta)\bigr]^{\rm pl}=\bigl[\Pi^{\rm pl}_{t=0}\circ\Pi(\zeta),\Pi^{\rm pl}_{t=0}\circ\Pi(\eta)\bigr]^{\rm pl}.
\end{aligned}
\tag{4.40}
$$

Here $\Pi(\zeta)$ involves the projection $\Pi:H_*(\mathcal{M}'_\alpha)\to H_*(\mathcal{M}'_\alpha)^{t=0}$ from Definition 3.19, and $\Pi^{\rm pl}_{t=0}$ is as in (3.50). The first step of (4.40) uses $[\Pi(\zeta),\Pi(\eta)]^{t=0}=\Pi([\zeta,\eta]_0)$ by (3.42). The second and ninth steps follow from $\Pi^{\rm pl}_{t=0}\circ\Pi=H_*(\Pi^{\rm pl}_\gamma)$ as in Definition 3.22, for $\gamma=\alpha,\beta,\alpha+\beta$. We use (3.22) in the third step, Assumption 2.39(a) and functoriality of $H_*(-)$ in the fourth, (4.39) and $\Pi^{\rm pl}_{\alpha+\beta}\circ\Xi_{\alpha,\beta}=\Phi^{\rm pl}_{\alpha,\beta}\circ\Psi^{\rm pl}_{\alpha,\beta}\circ(\mathrm{id}_{[*/\mathbb{G}_m]^2}\times\Pi^{\rm pl}_{\alpha,\beta})$ in the fifth, Assumption 2.39(b) for the right hand rectangle in (4.38) in the sixth, $\check\Pi^{\rm pl}_{\alpha,\beta}\circ\Pi^{\rm pl}_{\alpha,\beta}=\Pi^{\rm pl}_\alpha\times\Pi^{\rm pl}_\beta$ and compatibility of $\boxtimes$ with $H_*(-)$ in the seventh, and (3.55) in the eighth.

Since $\Pi:H_*(\mathcal{M}')\to H_*(\mathcal{M}')^{t=0}$ is surjective, equation (4.40) implies that $\Pi^{\rm pl}_{t=0}:H_*(\mathcal{M}')^{t=0}\to H_*(\mathcal{M}^{\rm pl})$ is a morphism of graded Lie algebras over R. This completes the proof of Theorem 3.29.

4.7 Proof of Theorem 3.33

Work in the situation of Definition 3.32. We must show $[\,,\,]^{\mathrm{rk}>0}$ satisfies graded antisymmetry (3.45) and the graded Jacobi identity (3.46). Let $\zeta\in H_a(\mathcal{M}_\alpha)$, $\eta\in H_b(\mathcal{M}_\beta)$, and $\theta\in H_c(\mathcal{M}_\gamma)$ for $\alpha,\beta,\gamma\in K(\mathcal{A})$ with $\mathrm{rk}\,\alpha,\mathrm{rk}\,\beta,\mathrm{rk}\,\gamma>0$ and $a,b,c\geqslant 0$, and write $a=\tilde a+2-\chi(\alpha,\alpha)$, $b=\tilde b+2-\chi(\beta,\beta)$, $c=\tilde c+2-\chi(\gamma,\gamma)$, so that $\zeta \in \tilde{H}_{\tilde{a}}(\mathcal{M}_\alpha)$, and so on. Then for antisymmetry we have

$$
\begin{aligned}
[\eta,\zeta]^{\mathrm{rk}>0} &= \sum_{l\geqslant 0: 2l\leqslant a+b-2\chi(\alpha,\beta)-2} \Bigl(\frac{-\operatorname{rk}\beta}{\operatorname{rk}(\alpha+\beta)}\Bigr)^l \cdot t^l \diamond [\eta,\zeta]_l \\
&= \sum_{l,m\geqslant 0:\ 2(l+m)\leqslant a+b-2\chi(\alpha,\beta)-2} (-1)^{1+\tilde{a}\tilde{b}+l+m} \Bigl(\frac{-\operatorname{rk}\beta}{\operatorname{rk}(\alpha+\beta)}\Bigr)^l \cdot t^l \diamond \bigl(t^m \diamond [\zeta,\eta]_{l+m}\bigr) \\
&= \sum_{l,m\geqslant 0:\ 2(l+m)\leqslant a+b-2\chi(\alpha,\beta)-2} (-1)^{1+\tilde{a}\tilde{b}+l+m} \Bigl(\frac{-\operatorname{rk}\beta}{\operatorname{rk}(\alpha+\beta)}\Bigr)^l \cdot \bigl(t^l \star t^m\bigr) \diamond [\zeta,\eta]_{l+m} \\
&= \sum_{l,m\geqslant 0:\ 2(l+m)\leqslant a+b-2\chi(\alpha,\beta)-2} (-1)^{1+\tilde{a}\tilde{b}+l+m} \binom{l+m}{l} \Bigl(\frac{-\operatorname{rk}\beta}{\operatorname{rk}(\alpha+\beta)}\Bigr)^l \cdot t^{l+m} \diamond [\zeta,\eta]_{l+m} \\
&= \sum_{n\geqslant 0:\ 2n\leqslant a+b-2\chi(\alpha,\beta)-2} (-1)^{1+\tilde{a}\tilde{b}+n} \Bigl(1+\frac{-\operatorname{rk}\beta}{\operatorname{rk}(\alpha+\beta)}\Bigr)^n \cdot t^n \diamond [\zeta,\eta]_n \\
&= (-1)^{\tilde{a}\tilde{b}+1} \sum_{n\geqslant 0:\ 2n\leqslant a+b-2\chi(\alpha,\beta)-2} \Bigl(\frac{-\operatorname{rk}\alpha}{\operatorname{rk}(\alpha+\beta)}\Bigr)^n \cdot t^n \diamond [\zeta,\eta]_n \\
&= (-1)^{\tilde{a}\tilde{b}+1}[\zeta,\eta]^{\mathrm{rk}>0},
\end{aligned}
$$

using (3.59) in the first and last steps, (3.28) in the second, (3.18) in the third, (3.15) in the fourth, and changing variables to $n=l+m$ and using the binomial theorem in the fifth.

For the Jacobi identity, with $d=\tilde{a}+\tilde{b}+\tilde{c}-\chi(\alpha+\beta+\gamma,\alpha+\beta+\gamma)+2$ we have

$$
\begin{aligned}
&(-1)^{\tilde{c}\tilde{a}}[[\zeta,\eta]^{\mathrm{rk}>0},\theta]^{\mathrm{rk}>0}+(-1)^{\tilde{a}\tilde{b}}[[\eta,\theta]^{\mathrm{rk}>0},\zeta]^{\mathrm{rk}>0}+(-1)^{\tilde{b}\tilde{c}}[[\theta,\zeta]^{\mathrm{rk}>0},\eta]^{\mathrm{rk}>0} \\
&= \sum_{l,m\geqslant 0:\ 2l\leqslant d,\ 2m\leqslant a+b-2\chi(\alpha,\beta)-2} (-1)^{\tilde{c}\tilde{a}} \bigl(\tfrac{-\operatorname{rk}\alpha}{\operatorname{rk}(\alpha+\beta)}\bigr)^m \bigl(\tfrac{-\operatorname{rk}(\alpha+\beta)}{\operatorname{rk}(\alpha+\beta+\gamma)}\bigr)^l \cdot t^l \diamond [t^m \diamond [\zeta,\eta]_m,\theta]_l \\
&+ \sum_{l,m\geqslant 0:\ 2l\leqslant d,\ 2m\leqslant b+c-2\chi(\beta,\gamma)-2} (-1)^{\tilde{a}\tilde{b}} \bigl(\tfrac{-\operatorname{rk}\beta}{\operatorname{rk}(\beta+\gamma)}\bigr)^m \bigl(\tfrac{-\operatorname{rk}(\beta+\gamma)}{\operatorname{rk}(\alpha+\beta+\gamma)}\bigr)^l \cdot t^l \diamond [t^m \diamond [\eta,\theta]_m,\zeta]_l \\
&+ \sum_{l,m\geqslant 0:\ 2l\leqslant d,\ 2m\leqslant c+a-2\chi(\gamma,\alpha)-2} (-1)^{\tilde{b}\tilde{c}} \bigl(\tfrac{-\operatorname{rk}\gamma}{\operatorname{rk}(\gamma+\alpha)}\bigr)^m \bigl(\tfrac{-\operatorname{rk}(\gamma+\alpha)}{\operatorname{rk}(\alpha+\beta+\gamma)}\bigr)^l \cdot t^l \diamond [t^m \diamond [\theta,\zeta]_m,\eta]_l \\
&= \sum_{l,m\geqslant 0:\ 2l\leqslant d,\ m\leqslant l} (-1)^{\tilde{c}\tilde{a}}(-1)^l \tbinom{l}{m} \tfrac{(\operatorname{rk}\alpha)^m(\operatorname{rk}(\alpha+\beta))^{l-m}}{(\operatorname{rk}(\alpha+\beta+\gamma))^{m+l-m}} \cdot t^l \diamond [[\zeta,\eta]_m,\theta]_{l-m} \\
&+ \sum_{l,m\geqslant 0:\ 2l\leqslant d,\ m\leqslant l} (-1)^{\tilde{a}\tilde{b}}(-1)^l \tbinom{l}{m} \tfrac{(\operatorname{rk}\beta)^m(\operatorname{rk}(\beta+\gamma))^{l-m}}{(\operatorname{rk}(\alpha+\beta+\gamma))^{m+l-m}} \cdot t^l \diamond [[\eta,\theta]_m,\zeta]_{l-m} \\
&+ \sum_{l,m\geqslant 0:\ 2l\leqslant d,\ m\leqslant l} (-1)^{\tilde{b}\tilde{c}}(-1)^l \tbinom{l}{m} \tfrac{(\operatorname{rk}\gamma)^m(\operatorname{rk}(\gamma+\alpha))^{l-m}}{(\operatorname{rk}(\alpha+\beta+\gamma))^{m+l-m}} \cdot t^l \diamond [[\theta,\zeta]_m,\eta]_{l-m}
\end{aligned}
\tag{4.41}
$$

$$
\begin{aligned}
&= \sum_{m,n\geqslant 0:\ 2(m+n)\leqslant d} (-1)^{\tilde c\tilde a+m+n}\tbinom{m+n}{n} x^m(x+y)^n\cdot t^{m+n}\diamond[[\zeta,\eta]_m,\theta]_n\\
&+ \sum_{m,n\geqslant 0:\ 2(m+n)\leqslant d} (-1)^{\tilde a\tilde b+m+n}\tbinom{m+n}{n} y^m(y+z)^n\cdot t^{m+n}\diamond[[\eta,\theta]_m,\zeta]_n\\
&+ \sum_{m,n\geqslant 0:\ 2(m+n)\leqslant d} (-1)^{\tilde b\tilde c+m+n}\tbinom{m+n}{n} z^m(z+x)^n\cdot t^{m+n}\diamond[[\theta,\zeta]_m,\eta]_n=0,
\end{aligned}
$$

using (3.59) in the first step, (3.26) in the second, changing variables from l to $n=l-m$ and substituting $x=\frac{\mathrm{rk}\,\alpha}{\mathrm{rk}(\alpha+\beta+\gamma)}$, $y=\frac{\mathrm{rk}\,\beta}{\mathrm{rk}(\alpha+\beta+\gamma)}$, $z=\frac{\mathrm{rk}\,\gamma}{\mathrm{rk}(\alpha+\beta+\gamma)}$ so that $x+y+z=1$ in the third, and using (3.35) with $l=0$ in the fourth.

4.8 Proof of Proposition 3.35

Let $\alpha,\beta\in K(\mathcal{A})$ with $\mathrm{rk}\,\alpha,\mathrm{rk}\,\beta>0$ and $\zeta\in H_a(\mathcal{M}_\alpha)$, $\eta\in H_b(\mathcal{M}_\beta)$ for $a,b\geqslant 0$. Then setting $d=a+b-2\chi(\alpha,\beta)-2$, we have

$$
\begin{aligned}
&[\zeta,\eta]^{\mathrm{rk}>0}=\sum_{\substack{i,n\geqslant 0:\ 2i\leqslant a+b,\\ i\geqslant n+\chi(\alpha,\beta)+1}} \epsilon_{\alpha,\beta}(-1)^{a\chi(\beta,\beta)}\bigl(\tfrac{-\mathrm{rk}\,\alpha}{\mathrm{rk}(\alpha+\beta)}\bigr)^n\cdot H_d(\Psi_{\alpha+\beta})\bigl(t^n\boxtimes\\
&\qquad\bigl\{H_{d-2n}(\Xi_{\alpha,\beta})\bigl(t^{i-n-\chi(\alpha,\beta)-1}\boxtimes\bigl[(\zeta\boxtimes\eta)\cap c_i([\Theta^\bullet_{\alpha,\beta}])\bigr]\bigr)\bigr\}\bigr)\\
&=\sum_{\substack{i,n\geqslant 0:\ 2i\leqslant a+b,\\ i\geqslant n+\chi(\alpha,\beta)+1}} \epsilon_{\alpha,\beta}(-1)^{a\chi(\beta,\beta)}\bigl(\tfrac{-\mathrm{rk}\,\alpha}{\mathrm{rk}(\alpha+\beta)}\bigr)^n\cdot H_d\bigl(\Psi_{\alpha+\beta}\circ(\mathrm{id}_{\mathbb{G}_m}\times\Xi_{\alpha,\beta})\bigr)\\
&\qquad\bigl(t_1^n\boxtimes t_2^{i-n-\chi(\alpha,\beta)-1}\boxtimes\bigl[(\zeta\boxtimes\eta)\cap c_i([\Theta^\bullet_{\alpha,\beta}])\bigr]\bigr)\\
&=\sum_{\substack{i,n\geqslant 0:\ 2i\leqslant a+b,\\ i\geqslant n+\chi(\alpha,\beta)+1}} \epsilon_{\alpha,\beta}(-1)^{a\chi(\beta,\beta)}\bigl(\tfrac{-\mathrm{rk}\,\alpha}{\mathrm{rk}(\alpha+\beta)}\bigr)^n\cdot H_d(\mathrm{X}_{\alpha,\beta})\circ H_d(\Upsilon\times\mathrm{id}_{\mathcal{M}_\alpha\times\mathcal{M}_\beta})\\
&\qquad\bigl(t_1^n\boxtimes t_2^{i-n-\chi(\alpha,\beta)-1}\boxtimes\bigl[(\zeta\boxtimes\eta)\cap c_i([\Theta^\bullet_{\alpha,\beta}])\bigr]\bigr)\\
&=\sum_{\substack{i,n,p,q\geqslant 0:\ 2i\leqslant a+b,\ n\leqslant p,\\ p+q=i-\chi(\alpha,\beta)-1}} \epsilon_{\alpha,\beta}(-1)^{a\chi(\beta,\beta)}\tbinom{p}{n-q}\bigl(\tfrac{-\mathrm{rk}\,\alpha}{\mathrm{rk}(\alpha+\beta)}\bigr)^{n-q}\bigl(\tfrac{-\mathrm{rk}\,\alpha}{\mathrm{rk}(\alpha+\beta)}\bigr)^q\cdot\\
&\qquad H_d(\mathrm{X}_{\alpha,\beta})\bigl(t_1^p\boxtimes t_2^q\boxtimes\bigl[(\zeta\boxtimes\eta)\cap c_i([\Theta^\bullet_{\alpha,\beta}])\bigr]\bigr)\\
&=\sum_{\substack{i,p,q\geqslant 0:\ 2i\leqslant a+b,\\ i=p+q+\chi(\alpha,\beta)+1}} \epsilon_{\alpha,\beta}(-1)^{q+a\chi(\beta,\beta)}\tfrac{(\mathrm{rk}\,\beta)^p(\mathrm{rk}\,\alpha)^q}{(\mathrm{rk}(\alpha+\beta))^{p+q}}\cdot\\
&\qquad H_d(\mathrm{X}_{\alpha,\beta})\bigl(t_1^p\boxtimes t_2^q\boxtimes\bigl[(\zeta\boxtimes\eta)\cap c_i([\Theta^\bullet_{\alpha,\beta}])\bigr]\bigr),
\end{aligned}
\tag{4.42}
$$

using (3.17), (3.22) and (3.59) in the first step, functoriality of $H_*(-)$ and compatibility with $\boxtimes$ in the second, and the next equation which follows from (3.6)–(3.7), (3.20), and (3.62), and functoriality of $H_*(-)$, in the third:

$$
\begin{aligned}
\Psi_{\alpha+\beta}\circ(\mathrm{id}_{\mathbb{G}_m}\times\Xi_{\alpha,\beta})=\mathrm{X}_{\alpha,\beta}\circ(\Upsilon\times\mathrm{id}_{\mathcal{M}_\alpha\times\mathcal{M}_\beta}):&\\
[*/\mathbb{G}_m]\times[*/\mathbb{G}_m]\times\mathcal{M}_\alpha\times\mathcal{M}_\beta\longrightarrow\mathcal{M}_{\alpha+\beta},&
\end{aligned}
$$

where $\Upsilon:[*/\mathbb{G}_m]^2\to[*/\mathbb{G}_m]^2$ is the stack morphism induced by the group morphism $\upsilon:\mathbb{G}_m^2\to\mathbb{G}_m^2$ mapping $\upsilon:(\lambda,\mu)\mapsto(\lambda\mu,\lambda)$

The fourth step of (4.42) substitutes in the formula for $H_*(\Upsilon)$:

$$
H_{2n+2m}(\Upsilon):t_1^n\boxtimes t_2^m\longmapsto\sum_{p,q\geqslant 0:\ p+q=n+m,\ q\leqslant n}\tbinom{p}{n-q}\cdot t_1^p\boxtimes t_2^q,
$$

which can be proved from Assumption 2.30(c). The last step uses

$$\textstyle\sum_{n=q}^{p+q}\binom{p}{n-q}\bigl(\frac{-\operatorname{rk}\alpha}{\operatorname{rk}(\alpha+\beta)}\bigr)^{n-q}=\bigl(1-\frac{\operatorname{rk}\alpha}{\operatorname{rk}(\alpha+\beta)}\bigr)^p=\bigl(\frac{\operatorname{rk}\beta}{\operatorname{rk}(\alpha+\beta)}\bigr)^p,$$

by the binomial theorem. Equation (4.42) proves (3.63), and Proposition 3.35.

4.9 Proof of Proposition 3.37

In the situation of Proposition 3.37, we have

$$\begin{aligned}
&[s^m\heartsuit\zeta,s^n\heartsuit\eta]^{\operatorname{rk}>0}=m!n!(\operatorname{rk}\alpha)^{-m}(\operatorname{rk}\beta)^{-n}\cdot[t^m\diamond\zeta,t^n\diamond\eta]^{\operatorname{rk}>0}\\
&=m!n!(\operatorname{rk}\alpha)^{-m}(\operatorname{rk}\beta)^{-n}\cdot\sum_{l\geqslant 0:\ 2(l-m-n)\leqslant a+b-2\chi(\alpha,\beta)-2}\bigl(\tfrac{-\operatorname{rk}\alpha}{\operatorname{rk}(\alpha+\beta)}\bigr)^l\cdot t^l\diamond[t^m\diamond\zeta,t^n\diamond\eta]_l\\
&=\sum_{\substack{k,l\geqslant 0:\ k\leqslant n,\\ 2(k+l-m-n)\leqslant\\ a+b-2\chi(\alpha,\beta)-2}}\begin{aligned}[t]&m!n!(\operatorname{rk}\alpha)^{-m}(\operatorname{rk}\beta)^{-n}\bigl(\tfrac{-\operatorname{rk}\alpha}{\operatorname{rk}(\alpha+\beta)}\bigr)^l\cdot\\&(-1)^m\tbinom{l}{m}\tbinom{l-m}{n-k}\cdot t^l\diamond\bigl(t^k\diamond[\zeta,\eta]_{k+l-m-n}\bigr)\end{aligned}\\
&=\sum_{\substack{k,l\geqslant 0:\ k\leqslant n,\\ 2(k+l-m-n)\leqslant\\ a+b-2\chi(\alpha,\beta)-2}}\begin{aligned}[t]&m!n!(\operatorname{rk}\alpha)^{-m}(\operatorname{rk}\beta)^{-n}\bigl(\tfrac{-\operatorname{rk}\alpha}{\operatorname{rk}(\alpha+\beta)}\bigr)^l\cdot\\&(-1)^m\tbinom{l}{m}\tbinom{l-m}{n-k}\tbinom{k+l}{k}\cdot t^{k+l}\diamond[\zeta,\eta]_{k+l-m-n}\end{aligned}\\
&=\sum_{\substack{j,k\geqslant 0:\ k\leqslant n,\\ 2j\leqslant a+b\\ -2\chi(\alpha,\beta)-2}}\begin{aligned}[t]&m!n!(\operatorname{rk}\alpha)^{-m}(\operatorname{rk}\beta)^{-n}\bigl(\tfrac{-\operatorname{rk}\alpha}{\operatorname{rk}(\alpha+\beta)}\bigr)^{j-k+m+n}\cdot\\&(-1)^m\tfrac{(j-k+m+n)!}{m!(j-k+n)!}\tfrac{(j-k+n)!}{(n-k)!j!}\tfrac{(j+m+n)!}{k!(j-k+m+n)!}\cdot t^{j+m+n}\diamond[\zeta,\eta]_j\end{aligned}\\
&=\sum_{\substack{j\geqslant 0:\ 2j\leqslant a+b\\ -2\chi(\alpha,\beta)-2}}\begin{aligned}[t]&(\operatorname{rk}\alpha)^{-m}(\operatorname{rk}\beta)^{-n}\bigl(\tfrac{-\operatorname{rk}\alpha}{\operatorname{rk}(\alpha+\beta)}\bigr)^{j+m}\bigl[\textstyle\sum_{k=0}^n\binom{n}{k}\bigl(\frac{-\operatorname{rk}\alpha}{\operatorname{rk}(\alpha+\beta)}\bigr)^{n-k}\bigr]\cdot\\&(-1)^m(m+n)!\cdot t^{m+n}\diamond\bigl(t^j\diamond[\zeta,\eta]_j\bigr)\end{aligned}\\
&=(m+n)!(\operatorname{rk}(\alpha+\beta))^{-m-n}\cdot t^{m+n}\diamond\sum_{j\geqslant 0:\ 2j\leqslant a+b-2\chi(\alpha,\beta)-2}\bigl(\tfrac{-\operatorname{rk}\alpha}{\operatorname{rk}(\alpha+\beta)}\bigr)^j\cdot t^j\diamond[\zeta,\eta]_j\\
&=s^{m+n}\heartsuit[\zeta,\eta]^{\operatorname{rk}>0},
\end{aligned}$$

using (3.64) in the first step, (3.59) in the second, (3.26)–(3.27) in the third, (3.15) and (3.18) in the fourth, changing variables from l to $j=k+l-m-n$ in the fifth, using (3.26)–(3.27) and rearranging in the sixth, substituting in $\sum_{k=0}^n\binom{n}{k}\bigl(\frac{-\operatorname{rk}\alpha}{\operatorname{rk}(\alpha+\beta)}\bigr)^{n-k}=\bigl(\frac{\operatorname{rk}\beta}{\operatorname{rk}(\alpha+\beta)}\bigr)^n$ by the binomial theorem in the seventh, and (3.59) and (3.64) in the eighth. This proves (3.65), and the proposition.

4.10 Proof of Theorem 3.40

We work in the situation of Definition 3.39.

4.10.1 Proof of graded antisymmetry of $[\,,\,]^{\rm mix}$

First we prove graded antisymmetry of $[\,,\,]^{\rm mix}$, that if $\alpha,\beta\in K(\mathcal{A})$ and $\bar\zeta\in H_a(\mathcal{M}_\alpha)^{\rm mix}$, $\bar\eta\in H_b(\mathcal{M}_\beta)^{\rm mix}$, and $\tilde a=a+\chi(\alpha,\alpha)-2$, $\tilde b=b+\chi(\beta,\beta)-2$, then

$$[\bar\eta,\bar\zeta]^{\rm mix}=(-1)^{\tilde a\tilde b+1}[\bar\zeta,\bar\eta]^{\rm mix}. \tag{4.43}$$

We do this following the cases of Definition 3.39(a)–(e).

In case (a) ($\operatorname{rk}\alpha, \operatorname{rk}\beta, \operatorname{rk}(\alpha+\beta) \neq 0$), equation (4.43) follows from the proof of Theorem 3.33 in §4.7, and in case (b) ($\operatorname{rk}\alpha = \operatorname{rk}\beta = 0$) it follows from Theorem 3.20. For case (c), with $\operatorname{rk}\alpha = 0$ and $\operatorname{rk}\beta \neq 0$, let $s^m \otimes (\zeta + I_t \diamond H_*(\mathcal{M}_\alpha)) \in H_a(\mathcal{M}_\alpha)^{\mathrm{mix}}$ for $\zeta \in H_a(\mathcal{M}_\alpha)$, and $\eta \in H_b(\mathcal{M}_\beta)^{\mathrm{mix}} = H_b(\mathcal{M}_\beta)$. Then

$$\begin{aligned}
&\bigl[\eta, s^m \otimes (\zeta + I_t \diamond H_*(\mathcal{M}_\alpha))\bigr]^{\mathrm{mix}} = s^m \heartsuit \Bigl(\sum_{n \geqslant 0:\ 2n \leqslant a+b-2\chi(\alpha,\beta)-2} (-1)^n t^n \diamond [\eta, \zeta]_n \Bigr) \\
&= s^m \heartsuit \Bigl(\sum_{k,n \geqslant 0:\ 2(k+n) \leqslant a+b-2\chi(\alpha,\beta)-2} (-1)^{1+\tilde{a}\tilde{b}+k} \cdot t^n \diamond \bigl(t^k \diamond [\zeta,\eta]_{k+n}\bigr) \Bigr) \\
&= s^m \heartsuit \Bigl(\sum_{k,n \geqslant 0:\ 2(k+n) \leqslant a+b-2\chi(\alpha,\beta)-2} (-1)^{1+\tilde{a}\tilde{b}+k} \tbinom{k+n}{k} \cdot t^{k+n} \diamond [\zeta,\eta]_{k+n} \Bigr) \\
&= s^m \heartsuit \Bigl(\sum_{j \geqslant 0:\ 2j \leqslant a+b-2\chi(\alpha,\beta)-2} (-1)^{1+\tilde{a}\tilde{b}} \bigl[\textstyle\sum_{k=0}^{j} (-1)^k \binom{j}{k}\bigr] \cdot t^j \diamond [\zeta,\eta]_j \Bigr) \\
&= (-1)^{1+\tilde{a}\tilde{b}} \cdot s^m \heartsuit [\zeta,\eta]_0 = (-1)^{1+\tilde{a}\tilde{b}} \bigl[s^m \otimes (\zeta + I_t \diamond H_*(\mathcal{M}_\alpha)), \eta\bigr]^{\mathrm{mix}},
\end{aligned}$$

using (3.69) in the first step, (3.28) and (3.64) in the second, and (3.15) and (3.18) in the third, changing variables from n to $j = k+n$ in the fourth, substituting $\sum_{k=0}^{j}(-1)^k \binom{j}{k} = (1-1)^j$ by the binomial theorem which is 1 if $j = 0$ and 0 otherwise in the fifth, and using (3.68) in the sixth. This proves (4.43) in case (c). Case (d) follows from case (c) by exchanging ζ, η.

For case (e) ($\operatorname{rk}\alpha = -\operatorname{rk}\beta \neq 0$), we have

$$\begin{aligned}
[\eta,\zeta]^{\mathrm{mix}} &= \sum_{n \geqslant 0:\ 2n \leqslant a+b-2\chi(\alpha,\beta)-2} \tfrac{(\operatorname{rk}\alpha)^n}{n!} \cdot s^n \otimes \bigl([\eta,\zeta]_n + I_t \diamond H_*(\mathcal{M}_0)\bigr) \\
&= \sum_{n \geqslant 0:\ 2n \leqslant a+b-2\chi(\alpha,\beta)-2} \tfrac{(-\operatorname{rk}\beta)^n}{n!} \cdot s^n \otimes \bigl((-1)^{1+\tilde{a}\tilde{b}+n}[\zeta,\eta]_n + I_t \diamond H_*(\mathcal{M}_0)\bigr) = (-1)^{\tilde{a}\tilde{b}+1}[\zeta,\eta]^{\mathrm{mix}},
\end{aligned}$$

using (3.70) in the first and third steps, and (3.28) modulo I_t and $\operatorname{rk}\alpha = -\operatorname{rk}\beta$ in the second. This proves (4.43).

4.10.2 Proof of equation (3.72)

Let $\alpha, \beta \in K(\mathcal{A})$, $\zeta \in H_*(\mathcal{M}_\alpha)^{\mathrm{mix}}$, $\eta \in H_*(\mathcal{M}_\beta)^{\mathrm{mix}}$ and $m, n \geqslant 0$. We will prove (3.72) following the cases of Definition 3.39(a)–(e). Case (a) ($\operatorname{rk}\alpha, \operatorname{rk}\beta, \operatorname{rk}(\alpha+\beta) \neq 0$) follows from Proposition 3.37. Case (b) ($\operatorname{rk}\alpha = \operatorname{rk}\beta = 0$) is obvious from $s^m \heartsuit (s^n \otimes \bar{\zeta}) = s^{m+n} \otimes \bar{\zeta}$ and (3.67). For case (c), with $\operatorname{rk}\alpha = 0$ and $\operatorname{rk}\beta \neq 0$, let $s^p \otimes (\zeta + I_t \diamond H_*(\mathcal{M}_\alpha)) \in H_a(\mathcal{M}_\alpha)^{\mathrm{mix}}$ for $\zeta \in H_a(\mathcal{M}_\alpha)$, and

$\eta \in H_b(\mathcal{M}_\beta)^{\mathrm{mix}} = H_b(\mathcal{M}_\beta)$. Then

$$\begin{aligned}
&\bigl[s^m \heartsuit (s^p \otimes (\zeta + I_t \diamond H_*(\mathcal{M}_\alpha))), s^n \heartsuit \eta\bigr]^{\mathrm{mix}} \\
&= \bigl[s^{m+p} \otimes (\zeta + I_t \diamond H_*(\mathcal{M}_\alpha))), n!(\mathrm{rk}\,\beta)^{-n} \cdot t^n \diamond \eta\bigr]^{\mathrm{mix}} \\
&= n!(\mathrm{rk}\,\beta)^{-n} \cdot s^{m+p} \heartsuit [\zeta, t^n \diamond \eta]_0 = s^{m+p} \heartsuit \bigl(n!(\mathrm{rk}\,\beta)^{-n} \cdot t^n \diamond [\zeta,\eta]_0\bigr) \\
&= s^{m+p} \heartsuit \bigl(s^n \heartsuit [\zeta,\eta]_0\bigr) = s^{m+n} \heartsuit \bigl(s^p \heartsuit [\zeta,\eta]_0\bigr) \\
&= s^{m+n} \heartsuit \bigl[s^p \otimes (\zeta + I_t \diamond H_*(\mathcal{M}_\alpha)), \eta\bigr]^{\mathrm{mix}},
\end{aligned}$$

using $s^m \heartsuit (s^p \otimes \bar\zeta) = s^{m+p} \otimes \bar\zeta$ and (3.64) in the first step, (3.68) in the second, (3.27) in the third, (3.64) in the fourth, that $\heartsuit$ is an $R[s]$-action in the fifth, and (3.68) in the sixth. Case (d) follows from (c) and (4.43). For case (e) ($\mathrm{rk}\,\alpha = -\mathrm{rk}\,\beta \neq 0$), if $\zeta \in H_a(\mathcal{M}_\alpha)$, $\eta \in H_b(\mathcal{M}_\beta)$ and $m, n \geqslant 0$ we have

$$\begin{aligned}
&\bigl[s^m \heartsuit \zeta, s^n \heartsuit \eta\bigr]^{\mathrm{mix}} \\
&= \sum_{p \geqslant 0:\ 2p \leqslant a+b+2m+2n-2\chi(\alpha,\beta)-2} m!(\mathrm{rk}\,\alpha)^{-m} n!(\mathrm{rk}\,\beta)^{-n} \frac{(\mathrm{rk}\,\beta)^p}{p!} \cdot s^p \otimes \bigl([t^m \diamond \zeta, t^n \eta]_p + I_t \diamond H_*(\mathcal{M}_0)\bigr) \\
&= \sum_{p \geqslant 0:\ 2p \leqslant a+b+2m+2n-2\chi(\alpha,\beta)-2} \tfrac{m!n!}{p!} (\mathrm{rk}\,\beta)^{p-m-n} \tbinom{p}{m} \tbinom{p-m}{n} \cdot s^p \otimes \bigl([\zeta,\eta]_{p-m-n} + I_t \diamond H_*(\mathcal{M}_0)\bigr) \\
&= \sum_{q \geqslant 0:\ 2q \leqslant a+b-2\chi(\alpha,\beta)-2} \tfrac{(\mathrm{rk}\,\beta)^q}{q!} \cdot s^{m+n+q} \otimes \bigl([\zeta,\eta]_q + I_t \diamond H_*(\mathcal{M}_0)\bigr) = s^{m+n} \heartsuit [\zeta,\eta]^{\mathrm{mix}},
\end{aligned}$$

using (3.64) and (3.70) in the first step, (3.26)–(3.27) modulo I_t and $\mathrm{rk}\,\alpha = -\mathrm{rk}\,\beta$ in the second, changing variables to $q = p - m - n$ and rearranging binomial coefficients in the third, and using (3.70) and $s^{m+n} \heartsuit (s^q \otimes \bar\theta) = s^{m+n+q} \otimes \bar\theta$ in the fourth. This proves (3.72).

4.10.3 Proof of the graded Jacobi identity for $[\,,\,]^{\mathrm{mix}}$

Finally we prove the graded Jacobi identity, that if $\alpha, \beta, \gamma \in K(\mathcal{A})$ and $\bar\zeta \in H_a(\mathcal{M}_\alpha)^{\mathrm{mix}}$, $\bar\eta \in H_b(\mathcal{M}_\beta)^{\mathrm{mix}}$, $\bar\theta \in H_c(\mathcal{M}_\gamma)^{\mathrm{mix}}$, and $\tilde a = a + \chi(\alpha,\alpha) - 2$, $\tilde b = b + \chi(\beta,\beta) - 2$, $\tilde c = c + \chi(\gamma,\gamma) - 2$, then

$$(-1)^{\tilde c \tilde a} [[\bar\zeta, \bar\eta]^{\mathrm{mix}}, \bar\theta]^{\mathrm{mix}} + (-1)^{\tilde a \tilde b} [[\bar\eta, \bar\theta]^{\mathrm{mix}}, \bar\zeta]^{\mathrm{mix}} + (-1)^{\tilde b \tilde c} [[\bar\theta, \bar\zeta]^{\mathrm{mix}}, \bar\eta]^{\mathrm{mix}} = 0. \quad (4.44)$$

We must split into cases according to whether each of $\mathrm{rk}\,\alpha, \mathrm{rk}\,\beta, \mathrm{rk}\,\gamma, \mathrm{rk}(\alpha+\beta)$, $\mathrm{rk}(\beta+\gamma), \mathrm{rk}(\gamma+\alpha), \mathrm{rk}(\alpha+\beta+\gamma)$ are zero or nonzero. This gives 18 possible cases, but as (4.44) is invariant under cyclic permutations of (α,β,γ), $(\tilde a, \tilde b, \tilde c)$, $(\bar\zeta, \bar\eta, \bar\theta)$, up to cyclic permutations we reduce to the following eight cases:

(a) all $\mathrm{rk}\,\alpha, \ldots, \mathrm{rk}(\alpha+\beta+\gamma)$ nonzero; (b) $\mathrm{rk}\,\alpha = \mathrm{rk}\,\beta = \mathrm{rk}\,\gamma = 0$;
(c) $\mathrm{rk}\,\alpha = \mathrm{rk}\,\beta = 0$, $\mathrm{rk}\,\gamma \neq 0$; (d) $\mathrm{rk}\,\alpha = 0$, all others nonzero;
(e) $\mathrm{rk}\,\alpha = \mathrm{rk}(\beta+\gamma) = 0$, $\mathrm{rk}\,\beta \neq 0$; (f) $\mathrm{rk}(\alpha+\beta) = 0$, all others nonzero;
(g) $\mathrm{rk}(\alpha+\beta) = \mathrm{rk}(\alpha+\gamma) = 0$, $\mathrm{rk}\,\alpha \neq 0$; (h) $\mathrm{rk}(\alpha+\beta+\gamma) = 0$, others nonzero.

We also make the following simplification: as (3.72) holds (proved in §4.10.2), if $\operatorname{rk}\alpha=0$ it is enough to verify (4.44) for elements $\bar\zeta=s^0\otimes(\zeta+I_t\diamond H_*(\mathcal{M}_\alpha))$ in $H_a(\mathcal{M}_\alpha)^{\rm mix}$, as $s^m\heartsuit\bigl(s^0\otimes(\zeta+I_t\diamond H_*(\mathcal{M}_\alpha))\bigr)=s^m\otimes(\zeta+I_t\diamond H_*(\mathcal{M}_\alpha))$, and similarly if $\operatorname{rk}\beta,\operatorname{rk}\gamma=0$. We set $d=a+b+c-2\chi(\alpha,\beta)-2\chi(\alpha,\gamma)-2\chi(\beta,\gamma)-4$.

Case (a) follows from Theorem 3.33, and case (b) from (3.67) and Theorem 3.20. For case (c), let $\operatorname{rk}\alpha=\operatorname{rk}\beta=0$, $\operatorname{rk}\gamma\ne 0$ and consider $\bar\zeta=s^0\otimes(\zeta+I_t\diamond H_*(\mathcal{M}_\alpha))$, $\bar\eta=s^0\otimes(\eta+I_t\diamond H_*(\mathcal{M}_\beta))$ and $\bar\theta=\theta$ for $\zeta\in H_a(\mathcal{M}_\alpha)$, $\eta\in H_b(\mathcal{M}_\beta)$, and $\theta\in H_c(\mathcal{M}_\gamma)$. Then

$$
\begin{aligned}
&(-1)^{\tilde c\tilde a}[[\bar\zeta,\bar\eta]^{\rm mix},\bar\theta]^{\rm mix}+(-1)^{\tilde a\tilde b}[[\bar\eta,\bar\theta]^{\rm mix},\bar\zeta]^{\rm mix}+(-1)^{\tilde b\tilde c}[[\bar\theta,\bar\zeta]^{\rm mix},\bar\eta]^{\rm mix}\\
&=(-1)^{\tilde c\tilde a}s^0\heartsuit[[\zeta,\eta]_0,\theta]_0+(-1)^{\tilde a\tilde b}s^0\heartsuit\Bigl(\sum_{m\geqslant 0:\ 2m\leqslant a+b+c-2\chi(\beta+\gamma,\alpha)-2}(-1)^m t^m\diamond[[\eta,\theta]_0,\zeta]_m\Bigr)\\
&+(-1)^{\tilde b\tilde c}s^0\heartsuit\Bigl(\sum_{\substack{m,n\geqslant 0:\ 2n\leqslant c+a-2\chi(\gamma,\alpha)-2,\\ 2m\leqslant a+b+c-2\chi(\gamma+\alpha,\beta)-2}}(-1)^{m+n}t^m\diamond[t^n\diamond[\theta,\zeta]_n,\eta]_m\Bigr)\\
&=s^0\heartsuit\Bigl((-1)^{\tilde c\tilde a}[[\zeta,\eta]_0,\theta]_0+(-1)^{\tilde a\tilde b}\sum_{m\geqslant 0:\ 2m\leqslant a+b+c-2\chi(\beta+\gamma,\alpha)-2}(-1)^m t^m\diamond[[\eta,\theta]_0,\zeta]_m\\
&+(-1)^{\tilde b\tilde c}\sum_{\substack{m,n\geqslant 0:\ n\leqslant m,\\ 2m\leqslant a+b+c-2\chi(\gamma+\alpha,\beta)-2}}(-1)^m\binom{m}{n}t^m\diamond[[\theta,\zeta]_n,\eta]_{m-n}\Bigr)=0,
\end{aligned}
$$

using (3.67)–(3.69) in the first step, (3.26) in the second, and (3.35) with $l=0$, $x=y=0$ and $z=1$ in the third.

In case (d), if $\bar\zeta=s^0\otimes(\zeta+I_t\diamond H_*(\mathcal{M}_\alpha))$, $\bar\eta=\eta$ and $\bar\theta=\theta$ for $\zeta\in H_a(\mathcal{M}_\alpha)$, $\eta\in H_b(\mathcal{M}_\beta)$, and $\theta\in H_c(\mathcal{M}_\gamma)$, then (4.44) follows from (4.41) with $\operatorname{rk}\alpha=0$, since (3.68)–(3.69) come from (3.59) with $\operatorname{rk}\alpha=0$, and $\operatorname{rk}\beta=0$, respectively.

In case (e), if $\bar\zeta=s^0\otimes(\zeta+I_t\diamond H_*(\mathcal{M}_\alpha))$, $\bar\eta=\eta$, $\bar\theta=\theta$ for $\zeta\in H_a(\mathcal{M}_\alpha)$, $\eta\in H_b(\mathcal{M}_\beta)$, $\theta\in H_c(\mathcal{M}_\gamma)$, then

$$
\begin{aligned}
&(-1)^{\tilde c\tilde a}[[\bar\zeta,\bar\eta]^{\rm mix},\bar\theta]^{\rm mix}+(-1)^{\tilde a\tilde b}[[\bar\eta,\bar\theta]^{\rm mix},\bar\zeta]^{\rm mix}+(-1)^{\tilde b\tilde c}[[\bar\theta,\bar\zeta]^{\rm mix},\bar\eta]^{\rm mix}\\
&=\sum_{m\geqslant 0:\ 2m\leqslant d}(-1)^{\tilde c\tilde a}\tfrac{(\operatorname{rk}\gamma)^m}{m!}\cdot s^m\otimes\bigl([[\zeta,\eta]_0,\theta]_m+I_t\diamond H_*(\mathcal{M}_0)\bigr)\\
&+\sum_{m\geqslant 0:\ 2m\leqslant b+c-2\chi(\beta,\gamma)-2}(-1)^{\tilde a\tilde b}\tfrac{(\operatorname{rk}\gamma)^m}{m!}\cdot s^m\otimes\bigl([[\eta,\theta]_m,\zeta]_0+I_t\diamond H_*(\mathcal{M}_0)\bigr)\\
&+\sum_{m,n\geqslant 0:\ 2m\leqslant d,\ 2n\leqslant c+a-2\chi(\gamma,\alpha)-2}(-1)^{\tilde b\tilde c+n}\tfrac{(\operatorname{rk}\beta)^m}{m!}\cdot s^m\otimes\bigl([t^n\diamond[\theta,\zeta]_n,\eta]_m+I_t\diamond H_*(\mathcal{M}_0)\bigr)\\
&=\sum_{\substack{m\geqslant 0:\\ 2m\leqslant d}}\tfrac{(\operatorname{rk}\gamma)^m}{m!}\cdot s^m\otimes\Bigl((-1)^{\tilde c\tilde a}[[\zeta,\eta]_0,\theta]_m+(-1)^{\tilde a\tilde b}[[\eta,\theta]_m,\zeta]_0\\
&\qquad+\sum_{n\geqslant 0:\ 2n\leqslant c+a-2\chi(\gamma,\alpha)-2}(-1)^{\tilde b\tilde c+m}\tbinom{m}{n}[[\theta,\zeta]_n,\eta]_{m-n}+I_t\diamond H_*(\mathcal{M}_0)\Bigr)
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{\substack{m\geqslant 0:\\ 2m\leqslant d}} \tfrac{(\operatorname{rk}\gamma)^m}{m!}\cdot s^m\otimes\Bigl((-1)^{\tilde c\tilde a}\,[[\zeta,\eta]_0,\theta]_m-(-1)^{\tilde c\tilde a}\,[\zeta,[\eta,\theta]_m]_0\\
&\qquad\qquad +\sum_{k,n\geqslant 0:\ 2k+2n\leqslant c+a-2\chi(\gamma,\alpha)-2}(-1)^k\tbinom{m}{n}\,[\eta,t^k\diamond[\zeta,\theta]_{k+n}]_{m-n}+I_t\diamond H_*(\mathcal{M}_0)\Bigr)\\
&= \sum_{\substack{m\geqslant 0:\\ 2m\leqslant d}} \tfrac{(\operatorname{rk}\gamma)^m}{m!}\cdot s^m\otimes(-1)^{\tilde c\tilde a}\Bigl([[\zeta,\eta]_0,\theta]_m-[\zeta,[\eta,\theta]_m]_0\\
&\qquad\qquad +\sum_{k,n\geqslant 0:\ 2k+2n\leqslant c+a-2\chi(\gamma,\alpha)-2}(-1)^{\tilde c\tilde a+k}\tbinom{m}{n}\tbinom{m-n}{k}\,[\eta,[\zeta,\theta]_{k+n}]_{m-n-k}+I_t\diamond H_*(\mathcal{M}_0)\Bigr)\\
&= \sum_{\substack{m\geqslant 0:\\ 2m\leqslant d}} \tfrac{(\operatorname{rk}\gamma)^m}{m!}\cdot s^m\otimes(-1)^{\tilde c\tilde a}\Bigl([[\zeta,\eta]_0,\theta]_m-[\zeta,[\eta,\theta]_m]_0\\
&\qquad\qquad +\sum_{k,p\geqslant 0:\ k\leqslant p,\ 2p\leqslant c+a-2\chi(\gamma,\alpha)-2}(-1)^{\tilde c\tilde a+k}\tbinom{m}{p}\tbinom{p}{k}\,[\eta,[\zeta,\theta]_p]_{m-p}+I_t\diamond H_*(\mathcal{M}_0)\Bigr)\\
&= \sum_{\substack{m\geqslant 0:\\ 2m\leqslant d}} \tfrac{(\operatorname{rk}\gamma)^m}{m!}\cdot s^m\otimes(-1)^{\tilde c\tilde a}\Bigl([[\zeta,\eta]_0,\theta]_m-[\zeta,[\eta,\theta]_m]_0\\
&\qquad\qquad +(-1)^{\tilde c\tilde a}[\eta,[\zeta,\theta]_0]_m+I_t\diamond H_*(\mathcal{M}_0)\Bigr)=0,
\end{aligned}
$$

using (3.68)–(3.70) in the first step, (3.26) and $\operatorname{rk}\beta=-\operatorname{rk}\gamma$ in the second, (3.28) in the third, (3.27) in the fourth, changing variables from n to $p=k+n$ in the fifth, using $\sum_{k=0}^p(-1)^k\binom{p}{k}=1$ if $p=0$ and 0 if $p>0$ in the sixth, and using (3.29) with $l=0$ in the seventh.

In case (f), if $\bar\zeta=\zeta$, $\bar\eta=\eta$, $\bar\theta=\theta$ for $\zeta\in H_a(\mathcal{M}_\alpha)$, $\eta\in H_b(\mathcal{M}_\beta)$, and $\theta\in H_c(\mathcal{M}_\gamma)$, we have

$$
\begin{aligned}
&(-1)^{\tilde c\tilde a}[[\bar\zeta,\bar\eta]^{\rm mix},\bar\theta]^{\rm mix}+(-1)^{\tilde a\tilde b}[[\bar\eta,\bar\theta]^{\rm mix},\bar\zeta]^{\rm mix}+(-1)^{\tilde b\tilde c}[[\bar\theta,\bar\zeta]^{\rm mix},\bar\eta]^{\rm mix}\\
&=\sum_{m\geqslant 0:\ 2m\leqslant a+b-2\chi(\alpha,\beta)-2}(-1)^{\tilde c\tilde a}\tfrac{(\operatorname{rk}\beta)^m}{m!}\cdot s^m\heartsuit[[\zeta,\eta]_m,\theta]_0\\
&+\sum_{l,m\geqslant 0:\ 2l\leqslant d,\ 2m\leqslant b+c-2\chi(\beta,\gamma)-2}(-1)^{\tilde a\tilde b}\bigl(\tfrac{-\operatorname{rk}\beta}{\operatorname{rk}(\beta+\gamma)}\bigr)^m\bigl(\tfrac{-\operatorname{rk}(\beta+\gamma)}{\operatorname{rk}\gamma}\bigr)^l\cdot t^l\diamond[t^m\diamond[\eta,\theta]_m,\zeta]_l\\
&+\sum_{l,m\geqslant 0:\ 2l\leqslant d,\ 2m\leqslant c+a-2\chi(\gamma,\alpha)-2}(-1)^{\tilde b\tilde c}\bigl(\tfrac{-\operatorname{rk}\gamma}{\operatorname{rk}(\gamma+\alpha)}\bigr)^m\bigl(\tfrac{-\operatorname{rk}(\gamma+\alpha)}{\operatorname{rk}\gamma}\bigr)^l\cdot t^l\diamond[t^m\diamond[\theta,\zeta]_m,\eta]_l\\
&=\sum_{m\geqslant 0:\ 2m\leqslant a+b-2\chi(\alpha,\beta)-2}(-1)^{\tilde c\tilde a}\bigl(\tfrac{\operatorname{rk}\beta}{\operatorname{rk}\gamma}\bigr)^m\cdot t^m\diamond[[\zeta,\eta]_m,\theta]_0\\
&+\sum_{l,m\geqslant 0:\ 2l\leqslant d,\ m\leqslant l}(-1)^{\tilde a\tilde b}(-1)^l\tbinom{l}{m}\tfrac{(\operatorname{rk}\beta)^m(\operatorname{rk}(\beta+\gamma))^{l-m}}{(\operatorname{rk}\gamma)^{m+l-m}}\cdot t^l\diamond[[\eta,\theta]_m,\zeta]_{l-m}\\
&+\sum_{l,m\geqslant 0:\ 2l\leqslant d,\ m\leqslant l}(-1)^{\tilde b\tilde c}(-1)^l\tbinom{l}{m}\tfrac{(\operatorname{rk}\gamma)^m(\operatorname{rk}(\gamma+\alpha))^{l-m}}{(\operatorname{rk}\gamma)^{m+l-m}}\cdot t^l\diamond[[\theta,\zeta]_m,\eta]_{l-m}
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{m,n\geqslant 0:\ 2(m+n)\leqslant d} (-1)^{\tilde c\tilde a+m+n}\tbinom{m+n}{n} x^m(x+y)^n\cdot t^{m+n}\diamond[[\zeta,\eta]_m,\theta]_n\\
&+ \sum_{m,n\geqslant 0:\ 2(m+n)\leqslant d} (-1)^{\tilde a\tilde b+m+n}\tbinom{m+n}{n} y^m(y+z)^n\cdot t^{m+n}\diamond[[\eta,\theta]_m,\zeta]_n\\
&+ \sum_{m,n\geqslant 0:\ 2(m+n)\leqslant d} (-1)^{\tilde b\tilde c+m+n}\tbinom{m+n}{n} z^m(z+x)^n\cdot t^{m+n}\diamond[[\theta,\zeta]_m,\eta]_n=0,
\end{aligned}
$$

using (3.59), (3.68) and (3.70) in the first step, (3.26) and (3.64) in the second, changing variables from l to $n=l-m$ and substituting $x=-\frac{\mathrm{rk}\,\beta}{\mathrm{rk}\,\gamma}$, $y=\frac{\mathrm{rk}\,\beta}{\mathrm{rk}\,\gamma}$, $z=1$ so that $x+y+z=1$ in the third, noting that $x+y=0$ so only $n=0$ contributes in the first sum, and using (3.35) with $l=0$ in the fourth.

Similarly, in case (g), if $\bar\zeta=\zeta$, $\bar\eta=\eta$, $\bar\theta=\theta$ for $\zeta\in H_a(\mathcal{M}_\alpha)$, $\eta\in H_b(\mathcal{M}_\beta)$, and $\theta\in H_c(\mathcal{M}_\gamma)$, we have

$$
\begin{aligned}
&(-1)^{\tilde c\tilde a}[[\bar\zeta,\bar\eta]^{\mathrm{mix}},\bar\theta]^{\mathrm{mix}}+(-1)^{\tilde a\tilde b}[[\bar\eta,\bar\theta]^{\mathrm{mix}},\bar\zeta]^{\mathrm{mix}}+(-1)^{\tilde b\tilde c}[[\bar\theta,\bar\zeta]^{\mathrm{mix}},\bar\eta]^{\mathrm{mix}}\\
&= \sum_{m\geqslant 0:\ 2m\leqslant a+b-2\chi(\alpha,\beta)-2} (-1)^{\tilde c\tilde a}\tfrac{(\mathrm{rk}\,\beta)^m}{m!}\cdot s^m\heartsuit[[\zeta,\eta]_m,\theta]_0\\
&+ \sum_{l,m\geqslant 0:\ 2l\leqslant d,\ 2m\leqslant b+c-2\chi(\beta,\gamma)-2} (-1)^{\tilde a\tilde b}\Bigl(\tfrac{-\mathrm{rk}\,\beta}{\mathrm{rk}(\beta+\gamma)}\Bigr)^m\Bigl(\tfrac{-\mathrm{rk}(\beta+\gamma)}{\mathrm{rk}\,\gamma}\Bigr)^l\cdot t^l\diamond[t^m\diamond[\eta,\theta]_m,\zeta]_l\\
&+ \sum_{m\geqslant 0:\ 2m\leqslant c+a-2\chi(\gamma,\alpha)-2} (-1)^{\tilde b\tilde c}\tfrac{(\mathrm{rk}\,\alpha)^m}{m!}\cdot s^m\heartsuit[[\theta,\zeta]_m,\eta]_0\\
&= \sum_{m\geqslant 0:\ 2m\leqslant a+b-2\chi(\alpha,\beta)-2} (-1)^{\tilde c\tilde a}\Bigl(\tfrac{\mathrm{rk}\,\beta}{\mathrm{rk}\,\gamma}\Bigr)^m\cdot t^m\diamond[[\zeta,\eta]_m,\theta]_0\\
&+ \sum_{l,m\geqslant 0:\ 2l\leqslant d,\ m\leqslant l} (-1)^{\tilde a\tilde b}(-1)^l\tbinom{l}{m}\tfrac{(\mathrm{rk}\,\beta)^m(\mathrm{rk}(\beta+\gamma))^{l-m}}{(\mathrm{rk}\,\gamma)^{m+l-m}}\cdot t^l\diamond[[\eta,\theta]_m,\zeta]_{l-m}\\
&+ \sum_{m\geqslant 0:\ 2m\leqslant c+a-2\chi(\gamma,\alpha)-2} (-1)^{\tilde b\tilde c}\Bigl(\tfrac{\mathrm{rk}\,\alpha}{\mathrm{rk}\,\beta}\Bigr)^m\cdot t^m\diamond[[\theta,\zeta]_m,\eta]_0\\
&= \sum_{m,n\geqslant 0:\ 2(m+n)\leqslant d} (-1)^{\tilde c\tilde a+m+n}\tbinom{m+n}{n} x^m(x+y)^n\cdot t^{m+n}\diamond[[\zeta,\eta]_m,\theta]_n\\
&+ \sum_{m,n\geqslant 0:\ 2(m+n)\leqslant d} (-1)^{\tilde a\tilde b+m+n}\tbinom{m+n}{n} y^m(y+z)^n\cdot t^{m+n}\diamond[[\eta,\theta]_m,\zeta]_n\\
&+ \sum_{m,n\geqslant 0:\ 2(m+n)\leqslant d} (-1)^{\tilde b\tilde c+m+n}\tbinom{m+n}{n} z^m(z+x)^n\cdot t^{m+n}\diamond[[\theta,\zeta]_m,\eta]_n=0,
\end{aligned}
$$

using (3.59), (3.68) and (3.70) in the first step, (3.26) and (3.64) in the second, changing variables from l to $n=l-m$ and substituting $x=-1$, $y=z=1$ in the third, noting that $x+y=z+x=0$ so only $n=0$ contributes in the first and third sums of the third step, and using (3.35) with $l=0$ in the fourth.

In case (h), if $\bar\zeta=\zeta$, $\bar\eta=\eta$, $\bar\theta=\theta$ for $\zeta\in H_a(\mathcal{M}_\alpha)$, $\eta\in H_b(\mathcal{M}_\beta)$, and

$\theta \in H_c(\mathcal{M}_\gamma)$, we have

$$
\begin{aligned}
&(-1)^{\tilde c\tilde a}[[\bar\zeta,\bar\eta]^{\rm mix},\bar\theta]^{\rm mix}+(-1)^{\tilde a\tilde b}[[\bar\eta,\bar\theta]^{\rm mix},\bar\zeta]^{\rm mix}+(-1)^{\tilde b\tilde c}[[\bar\theta,\bar\zeta]^{\rm mix},\bar\eta]^{\rm mix}\\
&=\sum_{l,m\geqslant 0:\ 2l\leqslant d,\ 2m\leqslant a+b-2\chi(\alpha,\beta)-2}(-1)^{\tilde c\tilde a}\bigl(\tfrac{-\operatorname{rk}\alpha}{\operatorname{rk}(\alpha+\beta)}\bigr)^m\tfrac{(\operatorname{rk}\gamma)^l}{l!}\cdot s^l\otimes\bigl([t^m\diamond[\zeta,\eta]_m,\theta]_l+I_t\diamond H_*(\mathcal{M}_0)\bigr)\\
&+\sum_{l,m\geqslant 0:\ 2l\leqslant d,\ 2m\leqslant b+c-2\chi(\beta,\gamma)-2}(-1)^{\tilde a\tilde b}\bigl(\tfrac{-\operatorname{rk}\beta}{\operatorname{rk}(\beta+\gamma)}\bigr)^m\tfrac{(\operatorname{rk}\alpha)^l}{l!}\cdot s^l\otimes\bigl([t^m\diamond[\eta,\theta]_m,\zeta]_l+I_t\diamond H_*(\mathcal{M}_0)\bigr)\\
&+\sum_{l,m\geqslant 0:\ 2l\leqslant d,\ 2m\leqslant c+a-2\chi(\gamma,\alpha)-2}(-1)^{\tilde b\tilde c}\bigl(\tfrac{-\operatorname{rk}\gamma}{\operatorname{rk}(\gamma+\alpha)}\bigr)^m\tfrac{(\operatorname{rk}\beta)^l}{l!}\cdot s^l\otimes\bigl([t^m\diamond[\theta,\zeta]_m,\eta]_l+I_t\diamond H_*(\mathcal{M}_0)\bigr)\\
&=\sum_{l,m\geqslant 0:\ 2l\leqslant d,\ m\leqslant l}(-1)^{\tilde c\tilde a+m}\tbinom{l}{m}\bigl(\tfrac{\operatorname{rk}\alpha}{\operatorname{rk}\gamma}\bigr)^m\tfrac{(\operatorname{rk}\gamma)^l}{l!}\cdot s^l\otimes\bigl([[\zeta,\eta]_m,\theta]_{l-m}+I_t\diamond H_*(\mathcal{M}_0)\bigr)\\
&+\sum_{l,m\geqslant 0:\ 2l\leqslant d,\ m\leqslant l}(-1)^{\tilde a\tilde b+m}\tbinom{l}{m}\bigl(\tfrac{\operatorname{rk}\beta}{\operatorname{rk}\alpha}\bigr)^m\tfrac{(\operatorname{rk}\alpha)^l}{l!}\cdot s^l\otimes\bigl([[\eta,\theta]_m,\zeta]_{l-m}+I_t\diamond H_*(\mathcal{M}_0)\bigr)\\
&+\sum_{l,m\geqslant 0:\ 2l\leqslant d,\ m\leqslant l}(-1)^{\tilde b\tilde c+m}\tbinom{l}{m}\bigl(\tfrac{\operatorname{rk}\gamma}{\operatorname{rk}\beta}\bigr)^m\tfrac{(\operatorname{rk}\beta)^l}{l!}\cdot s^l\otimes\bigl([[\theta,\zeta]_m,\eta]_{l-m}+I_t\diamond H_*(\mathcal{M}_0)\bigr)\\
&=\sum_{l\geqslant 0:\ 2l\leqslant d}s^l\otimes\Bigl(\sum_{\substack{m,n\geqslant 0:\\ m+n=l}}\begin{aligned}[t]&(-1)^{\tilde c\tilde a+m}\tfrac{(\operatorname{rk}\alpha)^m(\operatorname{rk}\gamma)^n}{m!n!}[[\zeta,\eta]_m,\theta]_n+\\&(-1)^{\tilde a\tilde b+m}\tfrac{(\operatorname{rk}\beta)^m(\operatorname{rk}\alpha)^n}{m!n!}[[\eta,\theta]_m,\zeta]_n+\\&(-1)^{\tilde b\tilde c+m}\tfrac{(\operatorname{rk}\gamma)^m(\operatorname{rk}\beta)^n}{m!n!}[[\theta,\zeta]_m,\eta]_n+I_t\diamond H_*(\mathcal{M}_0)\Bigr)=0,\end{aligned}
\end{aligned}
\tag{4.45}
$$

using (3.59) and (3.70) in the first step, (3.26) and $\operatorname{rk}(\alpha+\beta+\gamma)=0$ in the second, setting $n=l-m$ and rewriting in the third.

In the fourth step we note that reducing (3.35) modulo I_t yields

$$
\begin{aligned}
&\sum_{m,n\geqslant 0:\ m+n=l}(-1)^{\tilde c\tilde a+l}\frac{1}{m!n!}x^m(x+y)^n\cdot[[\zeta,\eta]_m,\theta]_n+\\
&\sum_{m,n\geqslant 0:\ m+n=l}(-1)^{\tilde a\tilde b+l}\frac{1}{m!n!}y^m(y+z)^n\cdot t^{m+n-l}\diamond[[\eta,\theta]_m,\zeta]_n+\\
&\sum_{m,n\geqslant 0:\ m+n=l}(-1)^{\tilde b\tilde c+l}\frac{1}{m!n!}z^m(z+x)^n\cdot t^{m+n-l}\diamond[[\theta,\zeta]_m,\eta]_n=0\qquad\mod I_t.
\end{aligned}
$$

This holds initially for x,y,z with $x+y+z=1$, but as it is homogeneous of degree l in x,y,z it also holds for x,y,z with $x+y+z\neq 0$, and then taking limits shows it holds for all x,y,z. Putting $x=\operatorname{rk}\alpha$, $y=\operatorname{rk}\beta$ and $z=\operatorname{rk}\gamma$ and noting that $\operatorname{rk}\alpha+\operatorname{rk}\beta+\operatorname{rk}\gamma=0$ gives the fourth step of (4.45). This proves (4.44), and Theorem 3.40.

4.11 Proof of Theorem 3.47

Work in the situation of Definition 3.46. For (a), let $\alpha,\beta\in K(\mathcal{A})$, and consider the diagram in $\mathrm{Ho}(\mathbf{Art}^{\mathrm{lft}}_{\mathbb{K}})$:

$$\begin{tikzcd}
\mathcal{M}_\alpha^{\mathrm{fpd}}\times\mathcal{M}_\beta^{\mathrm{fpd}} \arrow[rr, "\pi"'] \arrow[d, "\pi_{\mathcal{M}_\alpha}\times\pi_{\mathcal{M}_\beta}"] & & {*} \arrow[dl, "{(L_\alpha,L_\beta)}"'] \arrow[dd, "L_{\alpha+\beta}"] \\
\mathcal{M}_\alpha\times\mathcal{M}_\beta \arrow[r, "(\hat\Pi^{\mathrm{pl}}\circ\det_\alpha)\times(\hat\Pi^{\mathrm{pl}}\circ\det_\beta)"] \arrow[d, "\Phi_{\alpha,\beta}"] & \mathcal{P}^{\mathrm{pl}}\times\mathcal{P}^{\mathrm{pl}} \arrow[dr, "\hat\Phi"] & \\
\mathcal{M}_{\alpha+\beta} \arrow[rr, "\hat\Pi^{\mathrm{pl}}\circ\det_{\alpha+\beta}"] & & \mathcal{P}^{\mathrm{pl}}.
\end{tikzcd} \tag{4.46}$$

Here the top quadrilateral commutes by (3.80), the bottom by (3.73) and (3.78), and the right hand triangle by Assumption 3.43(f). Thus (4.46) commutes.

Lifting to $\mathbf{Art}^{\mathrm{lft}}_{\mathbb{K}}$ gives a 2-morphism $\eta':(\hat\Pi^{\mathrm{pl}}\circ\det_{\alpha+\beta})\circ(\Phi_{\alpha,\beta}\circ(\pi_{\mathcal{M}_\alpha}\times\pi_{\mathcal{M}_\beta}))\Rightarrow L_{\alpha+\beta}\circ\pi$. By the 2-Cartesian property in Definition 2.20 of the second diagram in (3.80) for $\alpha+\beta$, there exist a 1-morphism

$$b:\mathcal{M}_\alpha^{\mathrm{fpd}}\times\mathcal{M}_\beta^{\mathrm{fpd}}\longrightarrow\mathcal{M}_{\alpha+\beta}^{\mathrm{fpd}}$$

in $\mathbf{Art}^{\mathrm{lft}}_{\mathbb{K}}$ and 2-isomorphisms

$$\zeta:\pi_{\mathcal{M}_{\alpha+\beta}}\circ b\Longrightarrow\Phi_{\alpha,\beta}\circ(\pi_{\mathcal{M}_\alpha}\times\pi_{\mathcal{M}_\beta}),\qquad \theta:\pi\circ b\Rightarrow\pi.$$

Now as in §2.3.2, $\mathcal{M}_\alpha,\mathcal{M}_\beta,\mathcal{M}_{\alpha+\beta}$ are Artin $\mathbb{K}$-stacks, which are categories with functors $p_{\mathcal{M}_\alpha}:\mathcal{M}_\alpha\to\mathbf{Sch}_{\mathbb{K}},\ldots,p_{\mathcal{M}_{\alpha+\beta}}:\mathcal{M}_{\alpha+\beta}\to\mathbf{Sch}_{\mathbb{K}}$, and $\mathcal{M}_\alpha^{\mathrm{fpd}}\subset\mathcal{M}_\alpha$, $\mathcal{M}_\beta^{\mathrm{fpd}}\subset\mathcal{M}_\beta$, $\mathcal{M}_{\alpha+\beta}^{\mathrm{fpd}}\subset\mathcal{M}_{\alpha+\beta}$ are substacks, which as in Definition 2.19 are subcategories closed under isomorphisms, and $\pi_{\mathcal{M}_\alpha}:\mathcal{M}_\alpha^{\mathrm{fpd}}\to\mathcal{M}_\alpha,\ldots,\pi_{\mathcal{M}_{\alpha+\beta}}:\mathcal{M}_{\alpha+\beta}^{\mathrm{fpd}}\to\mathcal{M}_{\alpha+\beta}$ are the inclusions of subcategories.

Let $A\in\mathcal{M}_\alpha^{\mathrm{fpd}}$, $B\in\mathcal{M}_\alpha^{\mathrm{fpd}}$ be objects in these subcategories. Then (A,B) is an object in $\mathcal{M}_\alpha^{\mathrm{fpd}}\times\mathcal{M}_\beta^{\mathrm{fpd}}$, and evaluating ζ at (A,B) gives an isomorphism

$$\zeta(A,B):b(A,B)=\pi_{\mathcal{M}_{\alpha+\beta}}\circ b(A,B)\to\Phi_{\alpha,\beta}\circ(\pi_{\mathcal{M}_\alpha}\times\pi_{\mathcal{M}_\beta})(A,B)=\Phi_{\alpha,\beta}(A,B)$$

in the category $\mathcal{M}_{\alpha+\beta}$. As $b(A,B)$ is an object in $\mathcal{M}_{\alpha+\beta}^{\mathrm{fpd}}\subset\mathcal{M}_{\alpha+\beta}$, which is closed under isomorphisms in $\mathcal{M}_{\alpha+\beta}$, we see that $\Phi_{\alpha,\beta}(A,B)$ is an object in $\mathcal{M}_{\alpha+\beta}^{\mathrm{fpd}}$. A similar argument shows $\Phi_{\alpha,\beta}$ maps morphisms in $\mathcal{M}_\alpha^{\mathrm{fpd}}\times\mathcal{M}_\beta^{\mathrm{fpd}}$ to $\mathcal{M}_{\alpha+\beta}^{\mathrm{fpd}}$. Hence $\Phi_{\alpha,\beta}$ restricts to a unique $\Phi_{\alpha,\beta}^{\mathrm{fpd}}$ in (3.83), as we have to prove.

For the restriction $\Psi_\alpha^{\mathrm{fpd}}$ in (3.84), we use a very similar argument, but replacing (4.46) by the diagram

$$\begin{tikzcd}
{[*/\mathbb{G}_m]}\times\mathcal{M}_\alpha^{\mathrm{fpd}} \arrow[rrr, "\pi"'] \arrow[d, "\mathrm{id}\times\pi_{\mathcal{M}_\alpha}"] & & & {*} \arrow[dd, "L_\alpha"] \\
{[*/\mathbb{G}_m]}\times\mathcal{M}_\alpha \arrow[r, "\Upsilon_{\mathrm{rk}\,\alpha}\times\det_\alpha"] \arrow[d, "\Psi_\alpha"] & {[*/\mathbb{G}_m]}\times\mathcal{P} \arrow[r, "\pi_{\mathcal{P}}"] \arrow[d, "\hat\Psi"] & \mathcal{P} \arrow[dr, "\hat\Pi^{\mathrm{pl}}"] & \\
\mathcal{M}_\alpha \arrow[r, "\det_\alpha"] & \mathcal{P} \arrow[rr, "\hat\Pi^{\mathrm{pl}}"] & & \mathcal{P}^{\mathrm{pl}}.
\end{tikzcd}$$

Here the top hexagon commutes by (3.80), the bottom left quadrilateral by (3.74), and the bottom right by Assumption 3.43(h).

By restricting to substacks, we see that $\Phi_{\alpha,\beta}^{\mathrm{fpd}}, \Psi_\alpha^{\mathrm{fpd}}$ satisfy the analogues of (3.4)–(3.7). So $\Psi_\alpha^{\mathrm{fpd}}$ is a $[*/\mathbb{G}_m]$-action on $\mathcal{M}_\alpha^{\mathrm{fpd}}$, and $\Psi_\alpha^{\prime\mathrm{fpd}}$ a free $[*/\mathbb{G}_m]$-action on $\mathcal{M}_\alpha^{\prime\mathrm{fpd}}$. We noted in Definition 3.46 that (3.82) is 2-Cartesian, with columns principal $[*/\mathbb{G}_m]$-fibrations. The rows of (3.82) are also inclusions of substacks, so the $[*/\mathbb{G}_m]$-action for $\Pi_{\alpha,\mathrm{fpd}}^{\mathrm{pfd}}$ must be the restriction of the $[*/\mathbb{G}_m]$-action Ψ'_α for Π_α^{pl}. That is, $\Pi_{\alpha,\mathrm{fpd}}^{\mathrm{pfd}}$ has $[*/\mathbb{G}_m]$-action $\Psi_\alpha^{\prime\mathrm{fpd}}$. This proves (a).

For (b), let $\mathbb{K}$ be algebraically closed, and recall from Definition 2.23 that a $[*/\mathbb{G}_m]$-principal bundle $\rho : S \to T$ lies in a commutative square (2.23) which locally over T is equivalent to (2.24). Consider the diagram (really two diagrams A, B combined)

$$
\begin{tikzcd}
\begin{array}{c}\mathcal{M}_\alpha^{\prime\mathrm{fd}} \approx \\ {[*/\mathbb{Z}_n]} \times \mathcal{M}_\alpha^{\mathrm{pfd}}\end{array} \arrow[rrr, "\pi"'] \arrow[dr, "\Pi_{\alpha,\mathrm{fd}}^{\mathrm{pfd}} \approx \pi_{\mathcal{M}_\alpha^{\mathrm{pfd}}}"'] \arrow[dd, "\begin{array}{c}\pi_{\mathcal{M}'_\alpha} \approx \\ \mathrm{inc} \times \pi_{\mathcal{M}_\alpha^{\mathrm{pl}}}\end{array}"] & & & * \arrow[dr, "\mathrm{id}"] \arrow[dd, "L_\alpha \approx (*, L_\alpha)"', near start] & \\
& \mathcal{M}_\alpha^{\mathrm{pfd}} \arrow[rrr, "\pi"', crossing over] \arrow[dd, "\pi_{\mathcal{M}_\alpha^{\mathrm{pl}}}", near start] & & & * \arrow[dd, "L_\alpha"'] \\
\begin{array}{c}\mathcal{M}'_\alpha \approx \\ {[*/\mathbb{G}_m]} \times \mathcal{M}_\alpha^{\mathrm{pl}}\end{array} \arrow[rrr, "\begin{array}{c}\det'_\alpha \approx \\ \Upsilon_{\mathrm{rk}\,\alpha} \times \det_\alpha^{\mathrm{pl}}\end{array}"', near end] \arrow[dr, "\Pi_\alpha^{\mathrm{pl}} \approx \pi_{\mathcal{M}_\alpha^{\mathrm{pl}}}"'] & & & \begin{array}{c}\mathcal{P} \approx \\ {[*/\mathbb{G}_m]} \times \mathcal{P}^{\mathrm{pl}}\end{array} \arrow[dr, "\hat{\Pi}^{\mathrm{pl}} \approx \pi_{\mathcal{P}^{\mathrm{pl}}}"'] & \\
& \mathcal{M}_\alpha^{\mathrm{pl}} \arrow[rrr, "\det_\alpha^{\mathrm{pl}}"] & & & \mathcal{P}^{\mathrm{pl}}
\end{tikzcd}
\tag{4.47}
$$

in $\mathrm{Ho}(\mathbf{Art}_{\mathbb{K}}^{\mathrm{lft}})$. Here when we write '$A \approx B$' for an object or morphism, we mean that B is the local approximation of A over $\mathcal{M}_\alpha^{\mathrm{pl}}$ and $\mathcal{P}^{\mathrm{pl}}$, as in (2.23)–(2.24). If we just write a single object or morphism A, then $A = B$, that is, A is equal to its local approximation.

For the 'A' part of the '$A \approx B$' in (4.47), the top left and bottom right rectangles are the 2-Cartesian squares (3.80) defining $\mathcal{M}_\alpha^{\mathrm{fd}}$ and $\mathcal{M}_\alpha^{\mathrm{pfd}}$. The diagonal morphisms are those used to define $\Pi_{\alpha,\mathrm{fd}}^{\mathrm{pfd}}$ in (3.81), so the 'A' diagram commutes.

For the 'B' part of the '$A \approx B$' in (4.47), we have $\mathcal{M}'_\alpha \approx [*/\mathbb{G}_m] \times \mathcal{M}_\alpha^{\mathrm{pl}}$ locally over $\mathcal{M}_\alpha^{\mathrm{pl}}$ and $\mathcal{P} \approx [*/\mathbb{G}_m] \times \mathcal{P}^{\mathrm{pl}}$ locally over $\mathcal{P}^{\mathrm{pl}}$ by applying the local approximation (2.23)–(2.24) to (3.48) and (3.76), and these local approximations identify $\Pi_\alpha^{\mathrm{pl}}, \hat{\Pi}^{\mathrm{pl}}$ with the projections to $\mathcal{M}_\alpha^{\mathrm{pl}}, \mathcal{P}^{\mathrm{pl}}$. We see that $\det'_\alpha \approx \Upsilon_{\mathrm{rk}\,\alpha} \times \det_\alpha^{\mathrm{pl}}$ from (3.48), (3.74), and (3.76).

The approximation $\mathcal{M}_\alpha^{\mathrm{fd}} \approx [*/\mathbb{Z}_n] \times \mathcal{M}_\alpha^{\mathrm{pfd}}$ now follows by completing the top left 2-Cartesian square of 'B' morphisms, using the 2-Cartesian square

$$
\begin{tikzcd}
{[*/\mathbb{Z}_n]} \arrow[rr, "\pi"'] \arrow[d, "\mathrm{inc}"] & \arrow[d, phantom, "\Uparrow"] & * \arrow[d, "*"'] \\
{[*/\mathbb{G}_m]} \arrow[rr, "\Upsilon_{\mathrm{rk}\,\alpha}"] & \; & {[*/\mathbb{G}_m]}
\end{tikzcd}
$$

in $\mathbf{Art}_{\mathbb{K}}^{\mathrm{lft}}$, where $n = |\operatorname{rk}\alpha|$, and $\mathrm{inc} : [*/\mathbb{Z}_n] \to [*/\mathbb{G}_m]$ is induced by the inclusion $\mathbb{Z}_n \hookrightarrow \mathbb{G}_m$ as the group of n^{th} roots of unity in $\mathbb{G}_m \subset \mathbb{K}$, noting that we assume $\mathbb{K}$ is algebraically closed.

This shows that $\mathcal{M}_\alpha^{\mathrm{fd}} \approx [*/\mathbb{Z}_n] \times \mathcal{M}_\alpha^{\mathrm{pfd}}$ locally over $\mathcal{M}_\alpha^{\mathrm{pl}}$ and $\mathcal{P}^{\mathrm{pl}}$, and $\Pi_{\alpha,\mathrm{fd}}^{\mathrm{pfd}} : \mathcal{M}_\alpha^{\mathrm{fd}} \to \mathcal{M}_\alpha^{\mathrm{pfd}}$ is locally equivalent to the projection $[*/\mathbb{Z}_n] \times \mathcal{M}_\alpha^{\mathrm{pfd}} \to \mathcal{M}_\alpha^{\mathrm{pfd}}$. That is, $\Pi_{\alpha,\mathrm{fd}}^{\mathrm{pfd}}$ is a locally trivial $[*/\mathbb{Z}_n]$-fibration, as we have to prove. The rest of (b) is immediate.

For (c), by (3.85) there exists a 2-morphism $\eta' : \det_\alpha^{\mathrm{pl}} \Rightarrow L_\alpha \circ \pi$ in $\mathbf{Art}_{\mathbb{K}}^{\mathrm{lft}}$. Thus by the universal property in Definition 2.20 of the 2-Cartesian square (3.80) defining $\mathcal{M}_\alpha^{\mathrm{pfd}}$, there exists a 1-morphism $b : \mathcal{M}_\alpha^{\mathrm{pl}} \to \mathcal{M}_\alpha^{\mathrm{pfd}}$ and a 2-morphism $\zeta : \pi_{\mathcal{M}_\alpha^{\mathrm{pl}}} \circ b \Rightarrow \mathrm{id}_{\mathcal{M}_\alpha^{\mathrm{pl}}}$. As $\pi_{\mathcal{M}_\alpha^{\mathrm{pl}}} : \mathcal{M}_\alpha^{\mathrm{pfd}} \hookrightarrow \mathcal{M}_\alpha^{\mathrm{pl}}$ is the inclusion of a substack, this implies that $\mathcal{M}_\alpha^{\mathrm{pfd}} = \mathcal{M}_\alpha^{\mathrm{pl}}$. To show that $\mathcal{M}_\alpha^{\mathrm{fpd}} = \mathcal{M}_\alpha$, we note that (3.77) and (3.84) imply that $\hat\Pi^{\mathrm{pl}} \circ \det_\alpha = L_\alpha \circ \pi : \mathcal{M}_\alpha \to \mathcal{P}^{\mathrm{pl}}$ in $\mathrm{Ho}(\mathbf{Art}_{\mathbb{K}}^{\mathrm{lft}})$, and then use the same argument.

Generalizing to the triangulated category case needs only the obvious modifications. This completes the proof of Theorem 3.47.

Part II
Applications in Algebraic Geometry

5 Lie algebras from quiver representations

We will now apply the constructions of Part I to the abelian category $\mathcal{A} = \text{mod-}\mathbb{C}Q$ and the triangulated category $\mathcal{T} = D^b\,\text{mod-}\mathbb{C}Q$ for a quiver Q, and relate the resulting vertex algebras and Lie algebras to lattice vertex algebras and Kac–Moody algebras. For simplicity we restrict to the field $\mathbb{K} = \mathbb{C}$, and to (co)homology theories $H_*(-), H^*(-)$ of Artin $\mathbb{C}$-stacks over a field R of characteristic zero, defined using ordinary (co)homology as in Example 2.35. But the analogues should also work for other fields $\mathbb{K}$ for which the results of §5.2 hold.

5.1 Background on quivers and Ringel–Hall algebras

5.1.1 Quivers and quiver representations

Here are the basic definitions in quiver theory, following Assem at al. [4, §II].

Definition 5.1. A *quiver* Q is a finite directed graph. That is, Q is a quadruple (Q_0, Q_1, h, t), where Q_0 is a finite set of *vertices*, Q_1 is a finite set of *edges*, and $h, t : Q_1 \to Q_0$ are maps giving the *head* and *tail* of each edge.

We call Q *acyclic* if it contains no directed cycles of edges.

We say that Q *has no vertex loops* if there are no edges starting and finishing at the same vertex.

The *underlying graph* of Q is the undirected graph obtained by forgetting the orientations of the edges of Q.

The *path algebra* $\mathbb{K}Q$ is an associative algebra over the field $\mathbb{K}$ with basis all *paths of length* $k \geqslant 0$, that is, sequences of the form

$$v_0 \xrightarrow{e_1} v_1 \to \cdots \to v_{k-1} \xrightarrow{e_k} v_k, \tag{5.1}$$

where $v_0, \ldots, v_k \in Q_0$, $e_1, \ldots, e_k \in Q_1$, $t(a_i) = v_{i-1}$ and $h(a_i) = v_i$. Multiplication is given by composition of paths in reverse order, or zero if the paths do not compose. Then $\mathbb{K}Q$ is finite-dimensional if and only if Q is acyclic.

We define representations of quivers.

Definition 5.2. Let $Q = (Q_0, Q_1, h, t)$ be a quiver. A *representation* of Q consists of finite-dimensional $\mathbb{K}$-vector spaces X_v for each $v \in Q_0$, and linear maps $\rho_e : X_{t(e)} \to X_{h(e)}$ for each $e \in Q_1$. Representations of Q are in 1-1 correspondence with *finite-dimensional left* $\mathbb{K}Q$*-modules* (X, ρ), as follows.

Given X_v, ρ_e, define $X = \bigoplus_{v \in Q_0} X_v$, and a linear $\rho : \mathbb{K}Q \to \text{End}(X)$ taking (5.1) to the linear map $X \to X$ acting as $\rho_{e_k} \circ \rho_{e_{k-1}} \circ \cdots \circ \rho_{e_1}$ on X_{v_0}, and 0 on X_v for $v \neq v_0$. Then (X, ρ) is a left $\mathbb{K}Q$-module. Conversely, any such (X, ρ) comes from a unique representation of Q.

A *morphism of representations* $\phi : (X, \rho) \to (Y, \sigma)$ is a linear map $\phi : X \to Y$ with $\phi \circ \rho(\gamma) = \sigma(\gamma) \circ \phi$ for all $\gamma \in \mathbb{K}Q$. Equivalently, ϕ defines linear maps $\phi_v : X_v \to Y_v$ for all $v \in Q_0$ with $\phi_{h(e)} \circ \rho_e = \sigma_e \circ \phi_{t(e)}$ for all $e \in Q_1$. Write mod-$\mathbb{K}Q$ for the categories of representations of Q. It is a $\mathbb{K}$-linear abelian category. We will be interested in taking $\mathcal{A} = \text{mod-}\mathbb{K}Q$ and $\mathcal{T} = D^b \,\text{mod-}\mathbb{K}Q$ in the constructions of Part I.

If (X, ρ) is a representation of Q, define the *dimension vector* $\mathbf{dim}(X, \rho)$ in $\mathbb{N}^{Q_0} \subset \mathbb{Z}^{Q_0}$ of (X, ρ) by $\mathbf{dim}(X, \rho) : v \mapsto \dim_{\mathbb{K}} X_v$. This induces a surjective morphism $\mathbf{dim} : K_0(\text{mod-}\mathbb{K}Q) \to \mathbb{Z}^{Q_0}$, which is an isomorphism if Q is acyclic. When $\mathcal{A} = \text{mod-}\mathbb{K}Q$ we will always take the quotient group $K(\mathcal{A})$ of $K_0(\mathcal{A})$ in Assumption 3.1(b) to be $\mathbb{Z}^{Q_0}$, using this morphism $\mathbf{dim}$.

If Q is a quiver, the moduli stack $\mathcal{M}^Q$ of objects (X, ρ) in mod-$\mathbb{K}Q$ is a smooth Artin $\mathbb{K}$-stack, locally of finite type. For $\boldsymbol{d} \in \mathbb{N}^{Q_0}$, the open and closed substack $\mathcal{M}^Q_{\boldsymbol{d}}$ of (X, ρ) with $\mathbf{dim}(X, \rho) = \boldsymbol{d}$ is of finite type, and has a very explicit description: as a quotient $\mathbb{K}$-stack we have

$$\mathcal{M}^Q_{\boldsymbol{d}} \cong \bigl[\textstyle\prod_{e \in Q_1} \mathrm{Hom}(\mathbb{K}^{\boldsymbol{d}(t(e))}, \mathbb{K}^{\boldsymbol{d}(h(e))}) / \prod_{v \in Q_0} \mathrm{GL}(\boldsymbol{d}(v), \mathbb{K})\bigr]. \tag{5.2}$$

Let $Q = (Q_0, Q_1, h, t)$ be a quiver. It is well known that $\mathrm{Ext}^i(D, E) = 0$ for all $D, E \in \text{mod-}\mathbb{K}Q$ and $i > 1$, and

$$\dim_{\mathbb{K}} \mathrm{Hom}(D, E) - \dim_{\mathbb{K}} \mathrm{Ext}^1(D, E) = \chi_Q(\mathbf{dim}\, D, \mathbf{dim}\, E), \tag{5.3}$$

where $\chi_Q : \mathbb{Z}^{Q_0} \times \mathbb{Z}^{Q_0} \to \mathbb{Z}$ is the *Euler form* of mod-$\mathbb{K}Q$, given by

$$\chi_Q(\boldsymbol{d}, \boldsymbol{e}) = \textstyle\sum_{v \in Q_0} \boldsymbol{d}(v)\boldsymbol{e}(v) - \sum_{e \in Q_1} \boldsymbol{d}(t(e))\boldsymbol{e}(h(e)).$$

We write $\chi_Q^{\mathrm{sym}} : \mathbb{Z}^{Q_0} \times \mathbb{Z}^{Q_0} \to \mathbb{Z}$ for the *symmetrized Euler form* $\chi_Q^{\mathrm{sym}}(\boldsymbol{d}, \boldsymbol{e}) = \chi_Q(\boldsymbol{d}, \boldsymbol{e}) + \chi_Q(\boldsymbol{e}, \boldsymbol{d})$. It is independent of the orientation of the edges of Q.

Write n_{vw} for the number of edges $\overset{v}{\bullet} \to \overset{w}{\bullet}$ in Q, and write $a_{vw} = 2\delta_{vw} - n_{vw} - n_{wv}$, for all $v, w \in Q_0$. Then for all $\boldsymbol{d}, \boldsymbol{e} \in \mathbb{Z}^{Q_0}$ we have

$$\chi_Q(\boldsymbol{d}, \boldsymbol{e}) = \sum_{v,w \in Q_0} (\delta_{vw} - n_{vw})\boldsymbol{d}(v)\boldsymbol{e}(w), \quad \chi_Q^{\mathrm{sym}}(\boldsymbol{d}, \boldsymbol{e}) = \sum_{v,w \in Q_0} a_{vw}\boldsymbol{d}(v)\boldsymbol{e}(w). \tag{5.4}$$

That is, the matrices of χ_Q and χ_Q^{sym} are $(\delta_{vw} - n_{vw})_{v,w \in Q_0}$ and $A = (a_{vw})_{v,w \in Q_0}$. If Q has no vertex loops then $n_{vv} = 0$, so $a_{vv} = 2$, and $a_{vw} \leqslant 0$ for $v \neq w$. This condition was important in the theory of Kac–Moody algebras in §2.1.2, as then $(a_{vw})_{v,w \in Q_0}$ is a generalized Cartan matrix.

5.1.2 Gabriel's Theorem and Kac's Theorem

A quiver Q is called *of finite type* if there are only finitely many isomorphism classes of indecomposable objects in mod-$\mathbb{K}Q$. Gabriel's Theorem [4, §VII.5] classifies quivers of finite type over algebraically closed fields.

Theorem 5.3 (Gabriel's Theorem)**.** *Let $\mathbb{K}$ be an algebraically closed field. A quiver Q is of finite type over $\mathbb{K}$ if and only if the underlying graph is a finite* *disjoint union of Dynkin diagrams of type A, D or E. If Q is of finite type then the map $[E] \mapsto \mathbf{dim}[E]$ induces a* 1-1 *correspondence between isomorphism classes $[E]$ of indecomposable objects E in* mod-$\mathbb{K}Q$ *and the set of positive roots in $\mathbb{Z}^{Q_0}$ of the corresponding Dynkin diagram. Note that these do not depend on the orientation of the arrows in Q.*

Kac [82, 83] proves the following generalization:

Theorem 5.4. *For any quiver Q without vertex loops and any algebraically closed field $\mathbb{K}$, the set of dimension vectors $\boldsymbol{d} = \mathbf{dim}[E]$ in $\mathbb{Z}^{Q_0}$ of indecomposable objects E in* mod-$\mathbb{K}Q$ *coincides with the set of positive roots Δ_+ in $\mathbb{Z}^{Q_0}$ of the derived Kac–Moody algebra $\mathfrak{g}'(A)$ from* §2.1.2 *associated to the generalized Cartan matrix $A = (a_{vw})_{v,w \in Q_0}$ of Q in Definition* 5.2. *If $\boldsymbol{d}$ is a real root then there is a unique isomorphism class $[E]$ of indecomposables $E \in$* mod-$\mathbb{K}Q$ *with $\mathbf{dim}[E] = \boldsymbol{d}$. If $\boldsymbol{d}$ is an imaginary root there are many such $[E]$.*

In fact Kac does not require Q to have no vertex loops, and so works with generalized Kac–Moody algebras as in Remark 2.5(ii).

5.1.3 Ringel–Hall algebras

Theorems 5.3 and 5.4 suggest that there should be a connection between categories of quiver representations mod-$\mathbb{K}Q$ and the theory of Lie algebras, and that one might be able to reconstruct the Lie algebra $\mathfrak{g}'(A)$ corresponding to Q from the abelian category mod-$\mathbb{K}Q$ (or perhaps the derived category D^b mod-$\mathbb{K}Q$).

Investigating this connection led to the idea of *Ringel–Hall algebra*, originally due to Ringel [136, 137]. Schiffmann [141] gives a good survey. The basic idea is that given a suitable abelian category $\mathcal{A}$, one defines an associative algebra $H_{\mathcal{A}}$. Ringel–Hall type algebras are defined in four main contexts:

- *Counting subobjects over finite fields* $\mathbb{F}_q$, as in Ringel [136, 137].
- *Constructible functions on moduli spaces* are used by Lusztig [105, §10.18–§10.19], Nakajima [118, §10], Frenkel, Malkin and Vybornov [49], Riedtmann [134] and others.
- *Perverse sheaves on moduli spaces* are used by Lusztig [105].
- *Homology of moduli spaces*, as in Nakajima [118].

We will explain Ringel's finite field version [136, 137]:

Definition 5.5. Let $\mathbb{F}_q$ be a finite field with q elements, and $\mathcal{A}$ be a small $\mathbb{F}_q$-linear abelian category with $\dim_{\mathbb{F}_q} \mathrm{Hom}_{\mathcal{A}}(E,F) < \infty$, $\dim_{\mathbb{F}_q} \mathrm{Ext}^1_{\mathcal{A}}(E,F) < \infty$ for all $E, F \in \mathcal{A}$. Write $\mathcal{M}(\mathbb{F}_q)$ for the set of isomorphism classes $[E]$ of objects E in $\mathcal{A}$, which is the set of $\mathbb{F}_q$-points of the moduli stack $\mathcal{M}$ of objects in $\mathcal{A}$.

Define $H_{\mathcal{A}}$ to be the $\mathbb{C}$-vector space of functions $f : \mathcal{M}(\mathbb{F}_q) \to \mathbb{C}$ with finite support. Then $\mathcal{M}(\mathbb{F}_q)$ has basis $\delta_{[E]}$ for $[E] \in \mathcal{M}(\mathbb{F}_q)$, where $\delta_{[E]}([F])$ is 1 if $[E] = [F]$ and 0 otherwise. For all $E, F, G \in \mathcal{A}$, write $N_{E,F,G}$ for the (finite) number of subobjects $U \subset F$ in $\mathcal{A}$ with $F/U \cong E$ and $U \cong G$. Define a $\mathbb{C}$-bilinear multiplication $* : H_{\mathcal{A}} \times H_{\mathcal{A}} \to H_{\mathcal{A}}$ by

$$\delta_{[E]} * \delta_{[G]} = \sum_{[F] \in \mathcal{M}(\mathbb{K})} N_{E,F,G} \cdot \delta_{[F]}.$$

Then as in [136, Prop. 1], $H_{\mathcal{A}}$ is an associative $\mathbb{C}$-algebra with identity $\delta_{[0]}$.

Suppose that for all $E, F \in \mathcal{A}$ we have $\dim_{\mathbb{F}_q} \mathrm{Ext}^i_{\mathcal{A}}(E,F) < \infty$ for all $i \geqslant 0$ with $\mathrm{Ext}^i_{\mathcal{A}}(E,F) = 0$ for $i \gg 0$, so that the Euler form $\chi_{\mathcal{A}}([E],[F]) = \sum_{i \geqslant 0} (-1)^i \dim_{\mathbb{F}_q} \mathrm{Ext}^i_{\mathcal{A}}(E,F)$ is defined. Define the *twisted Hall algebra* $H^{\mathrm{tw}}_{\mathcal{A}}$ to be $H^{\mathrm{tw}}_{\mathcal{A}} = H_{\mathcal{A}}$, but with the twisted multiplication

$$\delta_{[E]} *^{\mathrm{tw}} \delta_{[G]} = \sqrt{q}^{\,\chi([E],[G])} \sum_{[F] \in \mathcal{M}(\mathbb{K})} N_{E,F,G} \cdot \delta_{[F]}.$$

Again, $H^{\mathrm{tw}}_{\mathcal{A}}$ is an associative $\mathbb{C}$-algebra with identity $\delta_{[0]}$.

When $\mathcal{A} = \text{mod-}\mathbb{F}_q Q$ for a quiver Q with no oriented cycles, Ringel [136,137] and Green [56] (see Schiffmann [141, §3.3] for a good explanation) describe the twisted Hall algebra $H^{\mathrm{tw}}_{\text{mod-}\mathbb{F}_q Q}$ in terms of quantum groups:

Theorem 5.6. *Let Q be a quiver with no oriented cycles, and $\mathbb{F}_q$ be a finite field with q elements. Then there is a unique, injective $\mathbb{C}$-algebra morphism $\Upsilon : U_{q^{1/2}}(\mathfrak{n}_+) \hookrightarrow H^{\mathrm{tw}}_{\text{mod-}\mathbb{F}_q Q}$ with $\Upsilon(S_v) = \delta_{[E_v]}$ for all $v \in Q_0$, where S_v is the v^{th} generator of $U_{q^{1/2}}(\mathfrak{n}_+)$, and $E_v \in \text{mod-}\mathbb{F}_q Q$ has $\mathbf{dim}\, E_v = \delta_v \in \mathbb{Z}^{Q_0}$. Here $U_{q^{1/2}}(\mathfrak{n}_+)$ is the quantum group $U_v(\mathfrak{n}_+)$ of the positive part $\mathfrak{n}_+$ of the Kac–Moody algebra $\mathfrak{g}'(A) = \mathfrak{h} \oplus \mathfrak{n}_+ \oplus \mathfrak{n}_-$ associated to Q, specialized at $v = q^{1/2}$. Furthermore, Υ is an isomorphism if and only if Q is of finite type, i.e. its underlying graph is a disjoint union of Dynkin diagrams of type A, D or E.*

Here as in Jantzen [71] and Lusztig [106], the *quantum group* $U_v(\mathfrak{g})$ of $\mathfrak{g} = \mathfrak{g}'(A)$ is a family of associative $\mathbb{C}$-algebras (actually, Hopf algebras) depending on a parameter v, and equal to the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$ when $v = 1$. Quantum groups have many applications in mathematics and physics.

It was a long standing problem [137, p. 583] to reconstruct the full Lie algebra $\mathfrak{g}$, or its universal enveloping algebra $U(\mathfrak{g})$, or the full quantum group $U_v(\mathfrak{g})$, from some version of the derived category $D^b \,\text{mod-}\mathbb{F}_q Q$. Progress on this was made by Peng and Xiao [129, 130], who used a somewhat ad hoc construction to recover $\mathfrak{g}$ from the 2-periodic derived category $(D^b \,\text{mod-}\mathbb{F}_q Q)/T^2$ for $T : D^b \,\text{mod-}\mathbb{F}_q Q \to D^b \,\text{mod-}\mathbb{F}_q Q$ the translation functor, by Toën [148] and Xiao and Fu [154], who defined an associative 'derived Hall algebra' for derived categories over $\mathbb{F}_q$ such as $D^b \,\text{mod-}\mathbb{F}_q Q$ (but which does not give $U_{q^{1/2}}(\mathfrak{g})$), and by Bridgeland [25], who defined a localization $(H^{\mathrm{tw}}_{\mathcal{C}_{\mathbb{Z}_2}(\text{mod-}\mathbb{F}_q Q)})^{\mathrm{loc}}$ of a twisted Hall algebra of 2-periodic complexes in $\text{mod-}\mathbb{F}_q Q$ with an embedding $U_{q^{1/2}}(\mathfrak{g}) \hookrightarrow (H^{\mathrm{tw}}_{\mathcal{C}_{\mathbb{Z}_2}(\text{mod-}\mathbb{F}_q Q)})^{\mathrm{loc}}$, which is an isomorphism if Q is of finite type.

5.2 The (co)homology of $[*/\operatorname{GL}(r,\mathbb{C})]$ and $\mathbf{Perf}_{\mathbb{C}}^r$

For use in §5.3, we now describe the (co)homology of the stacks $[*/\operatorname{GL}(r,\mathbb{C})]$ and $\mathbf{Perf}_{\mathbb{C}}^r$, and compute how morphisms $\Phi_{r,s},\Psi_r,\bar\Phi_{r,s},\bar\Psi_r$ act on them.

5.2.1 Cohomology of $[*/\operatorname{GL}(r,\mathbb{C})],\mathbf{Perf}_{\mathbb{C}}^r$ using Chern classes

The next proposition is well known:

Proposition 5.7. *Let R be any commutative ring, and let $H_*(-),H^*(-)$ be the cohomology theories of Artin $\mathbb{C}$-stacks over R described in Example* 2.35. *Then for any $r\geqslant 0$ there is a canonical isomorphism of graded R-algebras*

$$H^*([*/\operatorname{GL}(r,\mathbb{C})])\cong R[\gamma_1,\gamma_2,\ldots,\gamma_r], \tag{5.5}$$

where γ_i is a formal variable of degree $2i$. That is, $H^{2k}([*/\operatorname{GL}(r,\mathbb{C})])$ is the free R-module with basis the monomials $\gamma_1^{a_1}\cdots\gamma_r^{a_r}$ for all $a_1,\ldots,a_r$ in $\mathbb{N}$ with $a_1+2a_2+\cdots+ra_r=k$, and $H^{2k+1}([*/\operatorname{GL}(r,\mathbb{C})])=0$.

The γ_i may be interpreted in terms of Chern classes. Let $E_r\to[*/\operatorname{GL}(r,\mathbb{C})]$ be the rank r vector bundle associated to the obvious representation of $\operatorname{GL}(r,\mathbb{C})$ on $\mathbb{C}^r$. Then $c_i(E_r)\cong\gamma_i$ under (5.5) for $i=1,\ldots,r$. If S is an Artin $\mathbb{C}$-stack and $E\to S$ is a rank r vector bundle, there is a unique morphism $\phi:S\to[*/\operatorname{GL}(r,\mathbb{C})]$ in $\operatorname{Ho}(\mathbf{Art}_{\mathbb{C}})$ with $E\cong\phi^*(E_r)$, and $c_i(E)=H^{2i}(\phi)(\gamma_i)$ in $H^{2i}(S)$ under (5.5) for $i=1,\ldots,r$.

Proof. As in Example 2.35(a), $H^*([*/\operatorname{GL}(r,\mathbb{C})])$ is the homology of a classifying space for the topological stack $F_{\mathbf{Art}_{\mathbb{C}}}^{\mathbf{TopSta}}([*/\operatorname{GL}(r,\mathbb{C})])$. This is a classifying space $B\operatorname{GL}(r,\mathbb{C})$ for $\operatorname{GL}(r,\mathbb{C})$ in the usual sense, as in May [109, §16.5, §23], and is also a classifying space $B\mathrm{U}(r)$ for $\mathrm{U}(r)$ as the group morphism $\mathrm{U}(r)\hookrightarrow\operatorname{GL}(r,\mathbb{C})$ is a homotopy equivalence. The proposition then follows from the computation of $H^*(B\mathrm{U}(r),\mathbb{Z})$ by Milnor and Stasheff [114, Th. 14.5], and the Universal Coefficient Theorem in Spanier [145, Th. 5.5.10]. □

Definition 5.8. As in Example 2.35(c), write $\mathbf{Perf}_{\mathbb{C}}$ for the moduli stack of perfect complexes on the point $*=\operatorname{Spec}\mathbb{C}$. Write $\mathbf{Perf}_{\mathbb{C}}=\coprod_{r\in\mathbb{Z}}\mathbf{Perf}_{\mathbb{C}}^r$, where $\mathbf{Perf}_{\mathbb{C}}^r$ is the moduli stack of rank r perfect complexes on $*$. Then $\mathbf{Perf}_{\mathbb{C}},\mathbf{Perf}_{\mathbb{C}}^r$ are higher $\mathbb{C}$-stacks. If S is a $\mathbb{C}$-scheme then $\mathbf{Hom}(S,\mathbf{Perf}_{\mathbb{C}}^r)$ is the ∞-groupoid of rank r perfect complexes on S, up to quasi-isomorphism.

For $r\geqslant 0$, there is a natural inclusion $\iota_r:[*/\operatorname{GL}(r,\mathbb{C})]\hookrightarrow\mathbf{Perf}_{\mathbb{C}}^r$ as an open substack, by regarding $[*/\operatorname{GL}(r,\mathbb{C})]$ as the moduli stack of rank r vector bundles on $*$, and considering vector bundles as examples of perfect complexes.

The following proposition is also well known, at least to experts.

Proposition 5.9. *Let R be any commutative ring, and let $H_*(-),H^*(-)$ be the cohomology theories of higher Artin $\mathbb{C}$-stacks over R described in Example* 2.35. *Then for any $r\in\mathbb{Z}$ there is a canonical isomorphism of graded R-algebras*

$$H^*(\mathbf{Perf}_{\mathbb{C}}^r)\cong R[\gamma_1,\gamma_2,\ldots], \tag{5.6}$$

where γ_i is a formal variable of degree $2i$. That is, $H^{2k}(\mathbf{Perf}_{\mathbb{C}}^r)$ is the free R-module with basis the monomials $\gamma_1^{a_1}\cdots\gamma_k^{a_k}$ for all $a_1,\ldots,a_k$ in $\mathbb{N}$ with $a_1+2a_2+\cdots+ka_k=k$, and $H^{2k+1}(\mathbf{Perf}_{\mathbb{C}}^r)=0$.

The γ_i may be interpreted in terms of Chern classes. Let $\mathcal{E}_r^\bullet\to\mathbf{Perf}_{\mathbb{C}}^r$ be the universal rank r perfect complex on $\mathbf{Perf}_{\mathbb{C}}^r$. Then $c_i(\mathcal{E}_r^\bullet)\cong\gamma_i$ under (5.6) *for all $i\geqslant 1$. If S is an Artin $\mathbb{C}$-stack and $\mathcal{E}^\bullet\to S$ is a rank r perfect complex, there is a unique morphism $\phi:S\to\mathbf{Perf}_{\mathbb{C}}^r$ in $\mathrm{Ho}(\mathbf{HSt}_{\mathbb{C}})$ with $\mathcal{E}^\bullet\cong\phi^*(\mathcal{E}_r^\bullet)$, and $c_i(\mathcal{E}^\bullet)=H^{2i}(\phi)(\gamma_i)$ in $H^{2i}(S)$ under* (5.6) *for all $i\geqslant 1$.*

If $r\geqslant 0$, so we have an open inclusion $\iota_r:[*/\mathrm{GL}(r,\mathbb{C})]\hookrightarrow\mathbf{Perf}_{\mathbb{C}}^r$, then under the identifications (5.5)–(5.6) *we have*

$$H^{2i}(\iota_r):\gamma_i\longmapsto\begin{cases}\gamma_i, & i=1,\ldots,r,\\ 0, & i>r.\end{cases}\tag{5.7}$$

Proof. As in Example 2.35(c), we define $H^*(\mathbf{Perf}_{\mathbb{C}}^r)$ to be the ordinary cohomology $H^*\bigl(F_{\mathbf{HSt}_{\mathbb{C}}}^{\mathbf{Top}_\infty}(\mathbf{Perf}_{\mathbb{C}}^r),R\bigr)$, where $F_{\mathbf{HSt}_{\mathbb{C}}}^{\mathbf{Top}_\infty}:\mathbf{HSt}_{\mathbb{C}}^{\mathrm{lft}}\to\mathbf{Top}_\infty$ is the 'topological realization' ∞-functor. It follows from Blanc [16, Th.s 4.5 & 4.21] that (as a connective symmetric spectrum) $F_{\mathbf{HSt}_{\mathbb{C}}}^{\mathbf{Top}_\infty}(\mathbf{Perf}_{\mathbb{C}})$ is homotopy equivalent to the spectrum $K\mathrm{U}=\mathbb{Z}\times B\mathrm{U}$ of complex topological K-theory, and $F_{\mathbf{HSt}_{\mathbb{C}}}^{\mathbf{Top}_\infty}(\mathbf{Perf}_{\mathbb{C}}^r)$ is homotopy equivalent to $B\mathrm{U}=\lim_{n\to\infty}B\mathrm{U}(n)$ for each $r\in\mathbb{Z}$. So the cohomology of $F_{\mathbf{HSt}_{\mathbb{C}}}^{\mathbf{Top}_\infty}(\mathbf{Perf}_{\mathbb{C}}^r)$ is the limit as $r\to\infty$ in (5.5), giving (5.6). Equation (5.7) holds as $\iota_r^*(\mathcal{E}_r^\bullet)\cong E_r$, so $H^{2i}(\iota_r)$ maps $c_i(\mathcal{E}_r^\bullet)\mapsto c_i(E_r)$, where $c_i(\mathcal{E}_r^\bullet)\cong\gamma_i$ for all i, and $c_i(E_r)\cong\gamma_i$ for $i=1,\ldots,r$, and $c_i(E_r)=0$ for $i>r$. $\square$

5.2.2 (Co)homology of $[*/\mathrm{GL}(r,\mathbb{C})],\mathbf{Perf}_{\mathbb{C}}^r$ using Chern characters

As in §2.4.2, we prefer to work with *Chern characters* rather than Chern classes, so we can use the formulae (2.47) for direct sums and tensor products. Thus we will use alternative presentations for $H^*([*/\mathrm{GL}(r,\mathbb{C})])$ and $H^*(\mathbf{Perf}_{\mathbb{C}}^r)$:

Definition 5.10. Let R be a field of characteristic zero, such as $\mathbb{Q},\mathbb{R}$ or $\mathbb{C}$, and let $H_*(-),H^*(-)$ be the cohomology theories of (higher) Artin $\mathbb{C}$-stacks over R described in Example 2.35. Propositions 5.7 and 5.9 give descriptions (5.5) for $H^*([*/\mathrm{GL}(r,\mathbb{C})])$ when $r\geqslant 0$, and (5.6) for $H^*(\mathbf{Perf}_{\mathbb{C}}^r)$ when $r\in\mathbb{Z}$. For all $i=1,2,\ldots$, define $\beta_i\in H^{2i}([*/\mathrm{GL}(r,\mathbb{C})])$ to be identified with $\mathrm{Ch}_i(\gamma_1,\ldots,\gamma_i)$ under (5.5), where $\gamma_j=0$ for $j>r$, and define $\beta_i\in H^{2i}(\mathbf{Perf}_{\mathbb{C}}^r)$ to be identified with $\mathrm{Ch}_i(\gamma_1,\ldots,\gamma_i)$ under (5.6), where $\mathrm{Ch}_1,\mathrm{Ch}_2,\ldots$ are the universal polynomials defined in (2.48)–(2.49). Also set $\beta_0=r\cdot 1$ in $H^0([*/\mathrm{GL}(r,\mathbb{C})])$ and $H^0(\mathbf{Perf}_{\mathbb{C}}^r)$.

Then by (2.47) and Propositions 5.7 and 5.9, we have $\beta_i=\mathrm{ch}_i(E_r)$ in $H^{2i}([*/\mathrm{GL}(r,\mathbb{C})])$ and $\beta_i=\mathrm{ch}_i(\mathcal{E}_r^\bullet)$ in $H^{2i}(\mathbf{Perf}_{\mathbb{C}}^r)$. If S is an Artin $\mathbb{C}$-stack and $E\to S$ is a rank r vector bundle, there is $\phi:S\to[*/\mathrm{GL}(r,\mathbb{C})]$ with $E\cong\phi^*(E_r)$, and $\mathrm{ch}_i(E)=H^{2i}(\phi)(\beta_i)$ in $H^{2i}(S)$ for all i. Similarly if $\mathcal{E}^\bullet\to S$ is a rank r perfect complex, there is $\phi:S\to\mathbf{Perf}_{\mathbb{C}}^r$ with $\mathcal{E}^\bullet\cong\phi^*(\mathcal{E}_r^\bullet)$, and $\mathrm{ch}_i(\mathcal{E}^\bullet)=H^{2i}(\phi)(\beta_i)$ in $H^{2i}(S)$ for all i.

As the coefficient of γ_k in $\mathrm{Ch}_k(\gamma_1,\ldots,\gamma_k)$ is $(-1)^{k-1}/(k-1)!$, which is nonzero, the elements $\beta_1,\ldots,\beta_i$ freely generate $R[\gamma_1,\ldots,\gamma_i]$, giving an isomorphism

$$H^*([*/\mathrm{GL}(r,\mathbb{C})])\cong R[\beta_1,\beta_2,\ldots,\beta_r] \tag{5.8}$$

for $r\geqslant 0$, where β_i is a formal variable of degree $2i$. Note that (5.5) and (5.8) are *different* isomorphisms. Similarly, for $r\in\mathbb{Z}$ we have

$$H^*(\mathbf{Perf}^r_{\mathbb{C}})\cong R[\beta_1,\beta_2,\beta_3,\ldots]. \tag{5.9}$$

Now we have also defined elements $\beta_{r+1},\beta_{r+2},\ldots$ in $H^*([*/\mathrm{GL}(r,\mathbb{C})])$, which are nonzero, but are not really taken into account by the description (5.8). As in §2.4.2, under (5.5) we can write γ_i in terms of the β_j by $\gamma_i=\mathrm{C}_i(\beta_1,\ldots,\beta_i)$, where $\mathrm{C}_1,\mathrm{C}_2,\ldots$ are the universal polynomials defined in (2.51)–(2.52), since the C_i's are the inverse polynomials to the Ch_i's. But $\gamma_i=0$ in $H^*([*/\mathrm{GL}(r,\mathbb{C})])$ if $i>r$ by definition, so the elements $\beta_1,\beta_2,\ldots$ in $H^*([*/\mathrm{GL}(r,\mathbb{C})])$ satisfy $\mathrm{C}_i(\beta_1,\ldots,\beta_i)=0$ for all $i>r$. Therefore we may also write

$$\begin{aligned}H^*([*/\mathrm{GL}(r,\mathbb{C})])&\cong R[\beta_1,\beta_2,\beta_3,\ldots]/(\mathrm{C}_i(\beta_1,\ldots,\beta_i)=0\ \ \forall i>r)\\&=R[\beta_1,\beta_2,\beta_3,\ldots]/I_r,\end{aligned} \tag{5.10}$$

where β_i is a formal variable of degree $2i$, and $I_r:=(\mathrm{C}_i(\beta_1,\ldots,\beta_i)=0\ \forall i>r)$ is the ideal in $R[\beta_1,\beta_2,\ldots]$ generated by $\mathrm{C}_i(\beta_1,\ldots,\beta_i)$ for $i=r+1,r+2,\ldots$. Under (5.9)–(5.10), the morphism $H^*(\iota_r):H^*(\mathbf{Perf}^r_{\mathbb{C}})\to H^*([*/\mathrm{GL}(r,\mathbb{C})])$ is identified with the projection $R[\beta_1,\beta_2,\beta_3,\ldots]\to R[\beta_1,\beta_2,\beta_3,\ldots]/I_r$.

Consider the graded R-vector space $R[b_1,b_2,\ldots]$, where b_i is a formal variable of degree $2i$. Define an R-bilinear pairing $\cdot:R[b_1,b_2,\ldots]\times R[\beta_1,\beta_2,\ldots]\to R$ by

$$\bigl(b_1^{m_1}b_2^{m_2}\cdots b_N^{m_N}\bigr)\cdot\bigl(\beta_1^{n_1}\beta_2^{n_2}\cdots\beta_N^{n_N}\bigr)=\begin{cases}\dfrac{\prod_{i=1}^N m_i!}{\prod_{i=1}^N((i-1)!)^{m_i}}, & m_i=n_i,\ \text{all } i,\\ 0, & \text{otherwise.}\end{cases} \tag{5.11}$$

Here we can write any two monomials in $R[b_1,b_2,\ldots]$ and $R[\beta_1,\beta_2,\ldots]$ as $b_1^{m_1}\cdots b_N^{m_N}$ and $\beta_1^{n_1}\cdots\beta_N^{n_N}$ for $N\gg 0$ and $m_i,n_i\in\mathbb{N}$, by allowing $m_i=n_i=0$ for $i\gg 0$. The peculiar normalization $\prod_i[m_i!/((i-1)!)^{m_i}]$ in (5.11) makes Theorem 5.19 below work. Equation (5.11) induces isomorphisms for each $k\geqslant 0$

$$R[b_1,b_2,\ldots]_k\cong\bigl(R[\beta_1,\beta_2,\ldots]_k\bigr)^*, \tag{5.12}$$

where $R[b_1,b_2,\ldots]_k,R[\beta_1,\beta_2,\ldots]_k$ are the degree k subspaces of $R[b_1,b_2,\ldots]$ and $R[\beta_1,\beta_2,\ldots]$. Hence by (5.9), (5.10) and (5.12) we have isomorphisms

$$H_k(\mathbf{Perf}^r_{\mathbb{C}})\cong H^k(\mathbf{Perf}^r_{\mathbb{C}})^*\cong\bigl(R[\beta_1,\beta_2,\beta_3,\ldots]_k\bigr)^*\cong R[b_1,b_2,\ldots]_k, \tag{5.13}$$

$$\begin{aligned}H_k([*/\mathrm{GL}(r,\mathbb{C})])&\cong H^k([*/\mathrm{GL}(r,\mathbb{C})])^*\cong\bigl(R[\beta_1,\beta_2,\beta_3,\ldots]_k/I_{r,k}\bigr)^*\\&\cong I^\circ_{r,k}:=\bigl\{b\in R[b_1,b_2,\ldots]_k:b\cdot\beta=0\ \ \forall\beta\in I_{r,k}\bigr\},\end{aligned} \tag{5.14}$$

where $I_{r,k} = I_r \cap R[\beta_1, \beta_2, \ldots]_k$, and $I^\circ_{r,k}$ is its annihilator in $R[b_1, b_2, \ldots]_k$. Here the first steps hold as $H^k(\mathbf{Perf}^r_{\mathbb{C}})^*, H^k([*/\operatorname{GL}(r,\mathbb{C})])$ are finite-dimensional over a field R. Thus we have isomorphisms

$$H_*(\mathbf{Perf}^r_{\mathbb{C}}) \cong R[b_1, b_2, \ldots], \tag{5.15}$$

$$H_*([*/\operatorname{GL}(r,\mathbb{C})]) \cong I^\circ_r \subset R[b_1, b_2, \ldots], \tag{5.16}$$

where I°_r is the annihilator of the ideal I_r in $R[\beta_1, \beta_2, \ldots]$. Under (5.15)–(5.16), the morphism $H_*(\iota_r) : H_*([*/\operatorname{GL}(r,\mathbb{C})]) \to H_*(\mathbf{Perf}^r_{\mathbb{C}})$ is identified with the inclusion $I^\circ_r \hookrightarrow R[b_1, b_2, \ldots]$.

We can also write the cap products on the (co)homology of $\mathbf{Perf}^r_{\mathbb{C}}$ and $[*/\operatorname{GL}(r,\mathbb{C})]$ using (5.9)–(5.10) and (5.13)–(5.14). For $0 \leqslant l \leqslant k$, define an R-bilinear map $\cap : R[b_1, b_2, \ldots]_k \times R[\beta_1, \beta_2, \ldots]_l \to R[b_1, b_2, \ldots]_{k-l}$ by

$$\begin{aligned}&\bigl(b_1^{m_1} \cdots b_N^{m_N}\bigr) \cap \bigl(\beta_1^{n_1} \cdots \beta_N^{n_N}\bigr)\\ &\quad = \begin{cases} \frac{\prod_{i=1}^N m_i!}{\prod_{i=1}^N (m_i - n_i)!((i-1)!)^{n_i}} b_1^{m_1-n_1} \cdots b_N^{m_N-n_N}, & m_i \geqslant n_i, \text{ all } i, \\ 0, & \text{otherwise.} \end{cases}\end{aligned} \tag{5.17}$$

Then $-\cap\beta_i$ acts on $R[b_1, b_2, \ldots]$ as $\frac{1}{(i-1)!}\frac{\mathrm{d}}{\mathrm{d}b_i}$ for each $i = 1, 2, \ldots$. Equation (5.17) is dual to multiplication $R[\beta_1, \beta_2, \ldots]_l \times R[\beta_1, \beta_2, \ldots]_{k-l} \to R[\beta_1, \beta_2, \ldots]_k$ under (5.11). Since (5.9) identifies the cup product on $H^*(\mathbf{Perf}^r_{\mathbb{C}})$ with multiplication in $R[\beta_1, \beta_2, \ldots]$, and the cap product is dual to the cup product as R is a field, we see that (5.17) is identified by (5.9) and (5.13) with the cap product

$$\cap : H_k(\mathbf{Perf}^r_{\mathbb{C}}) \times H^l(\mathbf{Perf}^r_{\mathbb{C}}) \longrightarrow H_{k-l}(\mathbf{Perf}^r_{\mathbb{C}}).$$

Similarly, (5.17) restricts to zero on $I^\circ_{r,k} \times I_{r,l}$ and maps $I^\circ_{r,k} \times R[\beta_1, \beta_2, \ldots]_l \to I^\circ_{r,k-l}$, and so descends to an R-bilinear map

$$\cap : I^\circ_{r,k} \times \bigl(R[\beta_1, \beta_2, \ldots]_l / I_{r,l}\bigr) \longrightarrow I^\circ_{r,k-l},$$

which is identified by (5.10) and (5.14) with the cap product

$$\cap : H_k([*/\operatorname{GL}(r,\mathbb{C})]) \times H^l([*/\operatorname{GL}(r,\mathbb{C})]) \longrightarrow H_{k-l}([*/\operatorname{GL}(r,\mathbb{C})]).$$

5.2.3 Morphisms $\Phi_{r,s}, \Psi_r, \bar\Phi_{r,s}, \bar\Psi_r$, and their action on (co)homology

We define morphisms $\Phi_{r,s}, \Psi_r, \bar\Phi_{r,s}, \bar\Psi_r$ related to the morphisms Φ, Ψ in Assumption 3.1, and compute their action on (co)homology.

Definition 5.11. For $r, s \geqslant 0$, define morphisms of algebraic $\mathbb{C}$-groups

$$\begin{aligned} \phi_{r,s} &: \operatorname{GL}(r,\mathbb{C}) \times \operatorname{GL}(s,\mathbb{C}) \longrightarrow \operatorname{GL}(r+s,\mathbb{C}), & \phi_{r,s} &: (A, B) \longmapsto \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}, \\ \psi_r &: \mathbb{G}_m \times \operatorname{GL}(r,\mathbb{C}) \longrightarrow \operatorname{GL}(r,\mathbb{C}), & \psi_r &: (\lambda, A) \longmapsto \lambda A, \end{aligned} \tag{5.18}$$

where A, B are $r \times r$ and $s \times s$ complex matrices and $\lambda \in \mathbb{G}_m$, and write

$$\begin{aligned} \Phi_{r,s} &: [*/\operatorname{GL}(r,\mathbb{C})] \times [*/\operatorname{GL}(s,\mathbb{C})] \longrightarrow [*/\operatorname{GL}(r+s,\mathbb{C})], \\ \Psi_r &: [*/\mathbb{G}_m] \times [*/\operatorname{GL}(r,\mathbb{C})] \longrightarrow [*/\operatorname{GL}(r,\mathbb{C})], \end{aligned}$$

for the morphisms of Artin $\mathbb{C}$-stacks induced by $\phi_{r,s}, \psi_r$.

Writing $E_r \to [*/\operatorname{GL}(r,\mathbb{C})]$, $E_s \to [*/\operatorname{GL}(s,\mathbb{C})]$, $E_{r+s} \to [*/\operatorname{GL}(r+s,\mathbb{C})]$ for the natural vector bundles from Proposition 5.7, and $E_1 \to [*/\mathbb{G}_m]$ for the line bundle from Assumption 2.30(c), by (5.18) we have isomorphisms

$$\begin{aligned} \Phi_{r,s}^*(E_{r+s}) &\cong \pi^*_{[*/\operatorname{GL}(r,\mathbb{C})]}(E_r) \oplus \pi^*_{[*/\operatorname{GL}(s,\mathbb{C})]}(E_s), \\ \Psi_r^*(E_r) &\cong \pi^*_{[*/\mathbb{G}_m]}(E_1) \otimes \pi^*_{[*/\operatorname{GL}(r,\mathbb{C})]}(E_r). \end{aligned} \tag{5.19}$$

Similarly, for $r, s \in \mathbb{Z}$ define morphisms of higher Artin $\mathbb{C}$-stacks

$$\begin{aligned} \bar\Phi_{r,s} : \mathbf{Perf}_{\mathbb{C}}^r \times \mathbf{Perf}_{\mathbb{C}}^s \longrightarrow \mathbf{Perf}_{\mathbb{C}}^{r+s} \quad &\text{by} \quad \bar\Phi_{r,s}(\mathcal{F}_r^\bullet, \mathcal{F}_s^\bullet) = \mathcal{F}_r^\bullet \oplus \mathcal{F}_s^\bullet, \\ \bar\Psi_r : [*/\mathbb{G}_m] \times \mathbf{Perf}_{\mathbb{C}}^r \longrightarrow \mathbf{Perf}_{\mathbb{C}}^r \quad &\text{by} \quad \bar\Psi_r(L, \mathcal{F}_r^\bullet) = L \otimes \mathcal{F}_r^\bullet. \end{aligned} \tag{5.20}$$

That is, if S is a $\mathbb{C}$-scheme then $\mathbf{Perf}_{\mathbb{C}}^r(S) = \mathbf{Hom}(S, \mathbf{Perf}_{\mathbb{C}}^r)$ is the ∞-groupoid of rank r perfect complexes $\mathcal{F}_r^\bullet$ on S, and $\bar\Phi_{r,s}(S) : \mathbf{Perf}_{\mathbb{C}}^r(S) \times \mathbf{Perf}_{\mathbb{C}}^s(S) \to \mathbf{Perf}_{\mathbb{C}}^{r+s}(S)$ maps $(\mathcal{F}_r^\bullet, \mathcal{F}_s^\bullet) \mapsto \mathcal{F}_r^\bullet \oplus \mathcal{F}_s^\bullet$. Similarly, $[*/\mathbb{G}_m](S)$ is the groupoid of line bundles $L \to S$, and $\bar\Psi_r(S) : [*/\mathbb{G}_m](S) \times \mathbf{Perf}_{\mathbb{C}}^r(S) \to \mathbf{Perf}_{\mathbb{C}}^r(S)$ maps $(L, \mathcal{F}_r^\bullet) \mapsto L \otimes \mathcal{F}_r^\bullet$. The analogue of (5.19) is

$$\begin{aligned} \bar\Phi_{r,s}^*(\mathcal{E}_{r+s}^\bullet) &\cong \pi^*_{\mathbf{Perf}_{\mathbb{C}}^r}(\mathcal{E}_r^\bullet) \oplus \pi^*_{\mathbf{Perf}_{\mathbb{C}}^s}(\mathcal{E}_s^\bullet), \\ \bar\Psi_r^*(\mathcal{E}_r^\bullet) &\cong \pi^*_{[*/\mathbb{G}_m]}(E_1) \otimes \pi^*_{\mathbf{Perf}_{\mathbb{C}}^r}(\mathcal{E}_r^\bullet). \end{aligned} \tag{5.21}$$

The following commute in $\operatorname{Ho}(\mathbf{HSt}_{\mathbb{C}}^{\mathrm{lft}})$:

$$\begin{tikzcd} {[*/\operatorname{GL}(r,\mathbb{C})] \times [*/\operatorname{GL}(s,\mathbb{C})]} \arrow[r, "\Phi_{r,s}"'] \arrow[d, "\iota_r \times \iota_s"] & {[*/\operatorname{GL}(r+s,\mathbb{C})]} \arrow[d, "\iota_{r+s}"'] \\ \mathbf{Perf}_{\mathbb{C}}^r \times \mathbf{Perf}_{\mathbb{C}}^s \arrow[r, "\bar\Phi_{r,s}"] & \mathbf{Perf}_{\mathbb{C}}^{r+s}, \end{tikzcd} \tag{5.22}$$

$$\begin{tikzcd} {[*/\mathbb{G}_m] \times [*/\operatorname{GL}(r,\mathbb{C})]} \arrow[r, "\Psi_r"'] \arrow[d, "\mathrm{id}_{[*/\mathbb{G}_m]} \times \iota_r"] & {[*/\operatorname{GL}(r,\mathbb{C})]} \arrow[d, "\iota_r"'] \\ {[*/\mathbb{G}_m] \times \mathbf{Perf}_{\mathbb{C}}^r} \arrow[r, "\bar\Psi_r"] & \mathbf{Perf}_{\mathbb{C}}^r. \end{tikzcd} \tag{5.23}$$

Applying Chern characters to (5.21) and using (2.47) yields

$$\begin{aligned} \operatorname{ch}_i(\bar\Phi_{r,s}^*(\mathcal{E}_{r+s}^\bullet)) &= \operatorname{ch}_i(\pi^*_{\mathbf{Perf}_{\mathbb{C}}^r}(E_r)) + \operatorname{ch}_i(\pi^*_{\mathbf{Perf}_{\mathbb{C}}^s}(\mathcal{E}_s^\bullet)), \\ \operatorname{ch}_i(\bar\Psi_r^*(\mathcal{E}_r^\bullet)) &= \sum_{j,k \geqslant 0: i=j+k} \operatorname{ch}_j(\pi^*_{[*/\mathbb{G}_m]}(E_1)) \cup \operatorname{ch}_k(\pi^*_{\mathbf{Perf}_{\mathbb{C}}^r}(\mathcal{E}_r^\bullet)). \end{aligned}$$

Equation (5.19) implies the analogues for $\Psi_{r,s}, \Psi_r$. Under the identification (5.9) we have $\operatorname{ch}_0(\mathcal{E}_r^\bullet) = r \cdot 1$ and $\operatorname{ch}_i(\mathcal{E}_r^\bullet) = \beta_i$ for $i > 0$, and similarly for $s, r+s$, and under the identification $H^*([*/\mathbb{G}_m])\cong R[\tau]$ from Assumption 2.30(c) we have $\mathrm{ch}_j(E_1)=\frac{1}{j!}\tau^j$. This yields

$$H^{2i}(\bar\Phi_{r,s})(\beta_i)=\beta_i\boxtimes 1+1\boxtimes\beta_i,\qquad H^{2i}(\bar\Psi_r)(\beta_i)=\sum_{j=0}^i\frac{1}{j!}\tau^j\boxtimes\beta_{i-j},$$

where $\beta_0=r\cdot 1$. As $H^*(\bar\Phi_{r,s})$ and $H^*(\bar\Psi_r)$ are algebra morphisms, we deduce the action on the full algebras $R[\beta_1,\beta_2,\beta_3,\ldots]$:

$$\begin{aligned}&H^*(\bar\Phi_{r,s})(\beta_1^{n_1}\cdots\beta_N^{n_N})\\&\quad=\sum_{\substack{0\leqslant m_i\leqslant n_i,\\ i=1,\ldots,N}}\prod_{i=1}^N\binom{n_i}{m_i}\cdot(\beta_1^{m_1}\cdots\beta_N^{m_N})\boxtimes(\beta_1^{n_1-m_1}\cdots\beta_N^{n_N-m_N}),\end{aligned}\tag{5.24}$$

$$\begin{aligned}&H^*(\bar\Psi_r)(\beta_{k_1}\cdots\beta_{k_N})_{N\geqslant 0,\ k_1,\ldots,k_N>0}\\&\quad=\sum_{\substack{0\leqslant j_i\leqslant k_i,\\ i=1,\ldots,N}}\frac{1}{j_1!\cdots j_N!}\tau^{j_1+\cdots+j_N}\boxtimes(\beta_{k_1-j_1}\cdots\beta_{k_N-j_N}).\end{aligned}\tag{5.25}$$

Here $H^*(\bar\Phi_{r,s})$ is independent of r,s, but $H^*(\bar\Psi_r)$ depends on r as it involves $\beta_0=r\cdot 1$. Using the identification (5.10), the actions of $H^*(\Phi_{r,s}),H^*(\Psi_r)$ are given by the same formulae (5.24)–(5.25), modulo the ideals I_r,I_s,I_{r+s}.

Under the isomorphism (5.15) and $H_*([*/\mathbb{G}_m])\cong R[t]$ from Assumption 2.30(c), the homology actions $H_*(\bar\Phi_{r,s}),H_*(\bar\Psi_r)$ are the dual R-linear maps to $H^*(\bar\Phi_{r,s}),H^*(\bar\Psi_r)$ in (5.24)–(5.25) under the dual pairing (5.11), and the dual pairing between $R[\tau]$ and $R[t]$ in Assumption 2.30(c). Calculation gives

$$H_*(\bar\Phi_{r,s})\bigl[(b_1^{m_1}\cdots b_N^{m_N})\boxtimes(b_1^{n_1}\cdots b_N^{n_N})\bigr]=b_1^{m_1+n_1}\cdots b_N^{m_N+n_N},\tag{5.26}$$

$$\begin{aligned}&H_*(\bar\Psi_r)\bigl[t^k\boxtimes(b_{j_1}\cdots b_{j_M})\bigr]_{M\geqslant 0,\ j_1,\ldots,j_M>0}\\&=\sum_{\substack{k_1,\ldots,k_M\geqslant 0,\ N\geqslant 0,\ l_1,\ldots,l_N>0:\\ k=k_1+\cdots+k_M+l_1+\cdots+l_N}}\tfrac{r^N}{N!l_1\cdots l_N}\textstyle\prod_{i=1}^N\binom{j_i+k_i-1}{j_i-1}\cdot(b_{j_1+k_1}\cdots b_{j_M+k_M}b_{l_1}\cdots b_{l_N}).\end{aligned}\tag{5.27}$$

Here in (5.27), the terms in $l_1,\ldots,l_N$ correspond to those j_i in (5.25) with $j_i=k_i$, giving a term $\beta_{k_i-j_i}=\beta_0=r\cdot 1$ in (5.25), and the terms in $k_1,\ldots,k_M$ correspond to those j_i in (5.25) with $j_i<k_i$. When $k=1$ in (5.27) we get

$$\begin{aligned}&H_*(\bar\Psi_r)\bigl[t\boxtimes(b_{j_1}\cdots b_{j_N})\bigr]_{M\geqslant 0,\ j_1,\ldots,j_M>0}\\&=r\cdot(b_1b_{j_1}\cdots b_{j_M})+\sum_{i=1}^M j_i\cdot b_{j_1}\cdots b_{j_{i-1}}b_{j_i+1}b_{j_{i+1}}\cdots b_{j_M}.\end{aligned}\tag{5.28}$$

By (5.22)–(5.23), under the isomorphisms (5.14) and $H_*([*/\mathbb{G}_m])\cong R[t]$, the actions $H_*(\Phi_{r,s}),H_*(\Psi_r)$ of $\Phi_{r,s}$ and Ψ_r on homology are given by

$$\begin{aligned}H_*(\Phi_{r,s})&\cong H_*(\bar\Phi_{r,s})|_{I_r^\circ\boxtimes I_s^\circ}:I_r^\circ\boxtimes I_s^\circ\longrightarrow I_{r+s}^\circ,\\ H_*(\Psi_r)&\cong H_*(\bar\Psi_r)|_{R[t]\boxtimes I_r^\circ}:R[t]\boxtimes I_r^\circ\longrightarrow I_r^\circ.\end{aligned}$$

5.3 Lie algebras from $\mathcal{A} = \text{mod-}\mathbb{C}Q$ and $\mathcal{T} = D^b\,\text{mod-}\mathbb{C}Q$

5.3.1 Set up of the problem

The next definition describes the situation we will study in the rest of §5.3–§5.4.

Definition 5.12. Let Q be a quiver, and use the notation of Definitions 5.1–5.2. We will apply the constructions of Part I to the abelian category $\mathcal{A} = \text{mod-}\mathbb{C}Q$, and to the triangulated category $\mathcal{T} = D^b\,\text{mod-}\mathbb{C}Q$, over the field $\mathbb{K} = \mathbb{C}$. We cover the abelian and triangulated cases simultaneously. A bar accent '$^-$' will denote objects in the triangulated case, so for example we write $\mathcal{M}_{\boldsymbol{d}}, \Phi_{\boldsymbol{d},\boldsymbol{e}}, \Psi_{\boldsymbol{d}}, \ldots$ in the abelian case, but $\bar{\mathcal{M}}_{\boldsymbol{d}}, \bar{\Phi}_{\boldsymbol{d},\boldsymbol{e}}, \bar{\Psi}_{\boldsymbol{d}}, \ldots$ in the triangulated case.

We must specify the data in Assumption 3.1 for $\mathcal{A} = \text{mod-}\mathbb{C}Q$, and in Assumption 3.2 for $\mathcal{T} = D^b\,\text{mod-}\mathbb{C}Q$. In Assumption 3.1(b) we take $K(\mathcal{A}) = K(\mathcal{T}) = \mathbb{Z}^{Q_0}$ to be the lattice of dimension vectors of Q, and write elements of $\mathbb{Z}^{Q_0}$ as $\boldsymbol{d}, \boldsymbol{e}, \ldots$, regarded as maps $\boldsymbol{d} : Q_0 \to \mathbb{Z}$. In Assumption 3.1(c) we take χ to be the symmetrized Euler form χ_Q^{sym} from Definition 5.2. As in (5.4) this is

$$\chi(\boldsymbol{d}, \boldsymbol{e}) = \sum_{v,w \in Q_0} a_{vw} \boldsymbol{d}(v) \boldsymbol{e}(w), \tag{5.29}$$

where $A = (a_{vw})_{v,w \in Q_0}$ with $a_{vw} = 2\delta_{vw} - n_{vw} - n_{wv}$, for n_{vw} the number of edges $\overset{v}{\bullet} \to \overset{w}{\bullet}$ in Q. If Q has no vertex loops then A is a generalized Cartan matrix, and has an associated Kac–Moody algebra $\mathfrak{g}'(A)$ as in §2.1.2.

In Assumption 3.1(d) we define signs $\epsilon_{\boldsymbol{d},\boldsymbol{e}}$ for all $\boldsymbol{d}, \boldsymbol{e} \in \mathbb{Z}^{Q_0}$ by

$$\epsilon_{\boldsymbol{d},\boldsymbol{e}} = (-1)^{\sum_{v,w \in Q_0} n_{vw} \boldsymbol{d}(v) \boldsymbol{e}(w)}.$$

Then (5.29) and $a_{vw} = 2\delta_{vw} - n_{vw} - n_{wv}$ imply that (3.1) holds, and the map $(\boldsymbol{d}, \boldsymbol{e}) \mapsto \epsilon_{\boldsymbol{d},\boldsymbol{e}}$ is biadditive, so (3.2)–(3.3) hold.

We write $\mathcal{M}$ for the moduli stack of objects in $\mathcal{A} = \text{mod-}\mathbb{C}Q$, and $\bar{\mathcal{M}}$ for the moduli stack of objects in $\mathcal{T} = D^b\,\text{mod-}\mathbb{C}Q$. Then $\mathcal{M}$ is an Artin $\mathbb{C}$-stack , and $\bar{\mathcal{M}}$ a higher Artin $\mathbb{C}$-stack, both locally of finite type. We have decompositions $\mathcal{M} = \coprod_{\boldsymbol{d} \in \mathbb{N}^{Q_0}} \mathcal{M}_{\boldsymbol{d}}$ and $\bar{\mathcal{M}} = \coprod_{\boldsymbol{d} \in \mathbb{Z}^{Q_0}} \bar{\mathcal{M}}_{\boldsymbol{d}}$, for $\mathcal{M}_{\boldsymbol{d}}, \bar{\mathcal{M}}_{\boldsymbol{d}}$ the open and closed substacks of $\mathcal{M}, \bar{\mathcal{M}}$ of objects with dimension vector $\boldsymbol{d}$. As in (5.2) we have

$$\mathcal{M}_{\boldsymbol{d}} \cong \bigl[\textstyle\prod_{e \in Q_1} \text{Hom}(\mathbb{C}^{\boldsymbol{d}(t(e))}, \mathbb{C}^{\boldsymbol{d}(h(e))}) / \prod_{v \in Q_0} \text{GL}(\boldsymbol{d}(v), \mathbb{C})\bigr].$$

There is a natural morphism $\iota : \mathcal{M} \hookrightarrow \bar{\mathcal{M}}$ restricting to $\iota_{\boldsymbol{d}} : \mathcal{M}_{\boldsymbol{d}} \hookrightarrow \bar{\mathcal{M}}_{\boldsymbol{d}}$ for all $\boldsymbol{d} \in \mathbb{N}^{Q_0}$, from the degree 0 inclusion $\text{mod-}\mathbb{C}Q \hookrightarrow D^b\,\text{mod-}\mathbb{C}Q$.

As in Assumption 3.1(g),(h) we have morphisms $\Phi : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$, $\Phi_{\boldsymbol{d},\boldsymbol{e}} : \mathcal{M}_{\boldsymbol{d}} \times \mathcal{M}_{\boldsymbol{e}} \to \mathcal{M}_{\boldsymbol{d}+\boldsymbol{e}}$, $\Psi : [*/\mathbb{G}_m] \times \mathcal{M} \to \mathcal{M}$, $\Psi_{\boldsymbol{d}} : [*/\mathbb{G}_m] \times \mathcal{M}_{\boldsymbol{d}} \to \mathcal{M}_{\boldsymbol{d}}$ in the abelian case, and $\bar{\Phi} : \bar{\mathcal{M}} \times \bar{\mathcal{M}} \to \bar{\mathcal{M}}$, $\bar{\Phi}_{\boldsymbol{d},\boldsymbol{e}} : \bar{\mathcal{M}}_{\boldsymbol{d}} \times \bar{\mathcal{M}}_{\boldsymbol{e}} \to \bar{\mathcal{M}}_{\boldsymbol{d}+\boldsymbol{e}}$, $\bar{\Psi} : [*/\mathbb{G}_m] \times \bar{\mathcal{M}} \to \bar{\mathcal{M}}$, $\bar{\Psi}_{\boldsymbol{d}} : [*/\mathbb{G}_m] \times \bar{\mathcal{M}}_{\boldsymbol{d}} \to \bar{\mathcal{M}}_{\boldsymbol{d}}$ in the triangulated case.

As in Remark 3.3 there are natural perfect complexes $\mathcal{E}xt^\bullet$ in $\text{Perf}(\mathcal{M} \times \mathcal{M})$ and $\bar{\mathcal{E}}xt^\bullet$ in $\text{Perf}(\bar{\mathcal{M}} \times \bar{\mathcal{M}})$ whose cohomology at each $\mathbb{C}$-point in $\mathcal{M} \times \mathcal{M}$ and

$\bar{\mathcal{M}} \times \bar{\mathcal{M}}$ computes the Ext groups in $\mathcal{A}$ and $\mathcal{T}$. Writing $\mathcal{E}xt^\bullet_{\boldsymbol{d},\boldsymbol{e}}$, $\bar{\mathcal{E}}xt^\bullet_{\boldsymbol{d},\boldsymbol{e}}$ for their restrictions to $\mathcal{M}_{\boldsymbol{d}} \times \mathcal{M}_{\boldsymbol{e}}$, $\bar{\mathcal{M}}_{\boldsymbol{d}} \times \bar{\mathcal{M}}_{\boldsymbol{e}}$, by (5.3) and (5.4) we have

$$\operatorname{rank} \mathcal{E}xt^\bullet_{\boldsymbol{d},\boldsymbol{e}} = \operatorname{rank} \bar{\mathcal{E}}xt^\bullet_{\boldsymbol{d},\boldsymbol{e}} = \chi_Q(\boldsymbol{d},\boldsymbol{e}) = \sum_{v,w\in Q_0} (\delta_{vw} - n_{vw})\boldsymbol{d}(v)\boldsymbol{e}(w). \tag{5.30}$$

As in Remark 3.3(B) and Assumption 3.1(i) we set $\Theta^\bullet = (\mathcal{E}xt^\bullet)^\vee \oplus \sigma^*(\mathcal{E}xt^\bullet)$ in $\operatorname{Perf}(\mathcal{M} \times \mathcal{M})$, and $\bar{\Theta}^\bullet = (\bar{\mathcal{E}}xt^\bullet)^\vee \oplus \bar{\sigma}^*(\bar{\mathcal{E}}xt^\bullet)$ in $\operatorname{Perf}(\bar{\mathcal{M}} \times \bar{\mathcal{M}})$, where $\sigma : \mathcal{M} \times \mathcal{M} \to \mathcal{M} \times \mathcal{M}$ and $\bar{\sigma} : \bar{\mathcal{M}} \times \bar{\mathcal{M}} \to \bar{\mathcal{M}} \times \bar{\mathcal{M}}$ are exchange of factors. Then (5.30) and $\chi = \chi_Q^{\rm sym}$ imply that $\operatorname{rank} \Theta^\bullet_{\boldsymbol{d},\boldsymbol{e}} = \operatorname{rank} \bar{\Theta}^\bullet_{\boldsymbol{d},\boldsymbol{e}} = \chi(\boldsymbol{d},\boldsymbol{e})$, as in Assumption 3.1(i). The isomorphisms (3.8)–(3.12) satisfying Assumption 3.1(j)–(l) follow easily from properties of $\mathcal{E}xt^\bullet, \bar{\mathcal{E}}xt^\bullet$.

This defines all the data satisfying Assumption 3.1 for $\mathcal{A} = \text{mod-}\mathbb{C}Q$, and Assumption 3.2 for $\mathcal{T} = D^b \text{mod-}\mathbb{C}Q$. Let R be a field of characteristic zero, such as $\mathbb{Q}, \mathbb{R}$ or $\mathbb{C}$, and let $H_*(-), H^*(-)$ be the cohomology theories of (higher) Artin $\mathbb{C}$-stacks over R described in Example 2.35. We can now apply the constructions of §3.1–§3.8 in these two situations. In particular, we will relate the vertex algebras $\check{H}_*(\bar{\mathcal{M}})$ in §3.2 to lattice vertex algebras, and the '$t = 0$' Lie algebras $\tilde{H}_0(\mathcal{M})^{t=0}$ and $\tilde{H}_0(\bar{\mathcal{M}})^{t=0}$ in §3.3 to Kac–Moody algebras.

In the triangulated case, the shift functor $[1] : D^b \text{mod-}\mathbb{C}Q \to D^b \text{mod-}\mathbb{C}Q$ induces an isomorphism $\bar{\Sigma} : \bar{\mathcal{M}} \to \bar{\mathcal{M}}$ in $\operatorname{Ho}(\mathbf{HSt}^{\rm lft}_{\mathbb{C}})$, restricting to an isomorphism $\bar{\Sigma}_{\boldsymbol{d}} : \bar{\mathcal{M}}_{\boldsymbol{d}} \to \bar{\mathcal{M}}_{-\boldsymbol{d}}$ for all $\boldsymbol{d} \in \mathbb{Z}^{Q_0}$. This $\bar{\Sigma}$ preserves all the structures above, including $(\bar{\Sigma}\times\bar{\Sigma})^*(\bar{\Theta}^\bullet) \cong \bar{\Theta}^\bullet$. Thus, $H_*(\bar{\Sigma}) : H_*(\bar{\mathcal{M}}) \to H_*(\bar{\mathcal{M}})$ descends to an isomorphism of graded Lie algebras $H_*(\bar{\Sigma})^{t=0} : H_*(\bar{\mathcal{M}})^{t=0} \to H_*(\bar{\mathcal{M}})^{t=0}$. We will see later that $H_*(\bar{\Sigma})^2 = \mathrm{id}$ and $(H_*(\bar{\Sigma})^{t=0})^2 = \mathrm{id}$, although $\bar{\Sigma}^2 \neq \mathrm{id}_{\bar{\mathcal{M}}}$.

5.3.2 The (co)homology of $\mathcal{M}_{\boldsymbol{d}}, \bar{\mathcal{M}}_{\boldsymbol{d}}$

Work in the situation of Definition 5.12.

Proposition 5.13. *For each $v \in Q_0$, define morphisms $\bar{\Pi}^v_{\boldsymbol{d}} : \bar{\mathcal{M}}_{\boldsymbol{d}} \to \mathbf{Perf}^{\boldsymbol{d}(v)}_{\mathbb{C}}$, $\Pi^v_{\boldsymbol{d}} : \mathcal{M}_{\boldsymbol{d}} \to [*/\operatorname{GL}(\boldsymbol{d}(v),\mathbb{C})]$, which map a (complex of) Q-representations $(X_{v':v'\in Q_0}, \rho_{e:e\in Q_1})$ to the vector space (or complex) X_v at vertex v. Then the following are homotopy equivalences of (higher) stacks, as in Definition* 2.37:

$$\begin{aligned} &\textstyle\prod_{v\in Q_0} \bar{\Pi}^v_{\boldsymbol{d}} : \bar{\mathcal{M}}_{\boldsymbol{d}} \longrightarrow \prod_{v\in Q_0} \mathbf{Perf}^{\boldsymbol{d}(v)}_{\mathbb{C}} && \text{for all } \boldsymbol{d} \in \mathbb{Z}^{Q_0}, \\ &\textstyle\prod_{v\in Q_0} \Pi^v_{\boldsymbol{d}} : \mathcal{M}_{\boldsymbol{d}} \longrightarrow \prod_{v\in Q_0} [*/\operatorname{GL}(\boldsymbol{d}(v),\mathbb{C})] && \text{for all } \boldsymbol{d} \in \mathbb{N}^{Q_0}. \end{aligned} \tag{5.31}$$

Hence by Lemma 2.38, *for all $\boldsymbol{d}$ we have natural isomorphisms*

$$\begin{aligned} H^*(\bar{\mathcal{M}}_{\boldsymbol{d}}) &\cong \textstyle\bigotimes_{v\in Q_0} H^*(\mathbf{Perf}^{\boldsymbol{d}(v)}_{\mathbb{C}}), \\ H_*(\bar{\mathcal{M}}_{\boldsymbol{d}}) &\cong \textstyle\bigotimes_{v\in Q_0} H_*(\mathbf{Perf}^{\boldsymbol{d}(v)}_{\mathbb{C}}), \\ H^*(\mathcal{M}_{\boldsymbol{d}}) &\cong \textstyle\bigotimes_{v\in Q_0} H^*([*/\operatorname{GL}(\boldsymbol{d}(v),\mathbb{C})]), \\ H_*(\mathcal{M}_{\boldsymbol{d}}) &\cong \textstyle\bigotimes_{v\in Q_0} H_*([*/\operatorname{GL}(\boldsymbol{d}(v),\mathbb{C})]). \end{aligned} \tag{5.32}$$

Proof. For all $\boldsymbol{d}$, define morphisms

$$\bar{g}_{\boldsymbol{d}} : \prod_{v\in Q_0} \mathbf{Perf}_{\mathbb{C}}^{\boldsymbol{d}(v)} \longrightarrow \bar{\mathcal{M}}_{\boldsymbol{d}}, \quad g_{\boldsymbol{d}} : \prod_{v\in Q_0}[*/\mathrm{GL}(\boldsymbol{d}(v),\mathbb{C})] \longrightarrow \mathcal{M}_{\boldsymbol{d}},$$

to map a collection $(X_v : v \in Q_0)$ of rank $\boldsymbol{d}(v)$ vector spaces or complexes, to the corresponding (complex of) Q-representations $(X_{v':v'\in Q_0}, 0_{e\in Q_1})$ for which the edge maps $\rho_e : X_{h(e)} \to X_{t(e)}$ are zero for all $e \in Q_1$. Then $(\prod_{v\in Q_0} \bar{\Pi}^v_{\boldsymbol{d}}) \circ \bar{g}_{\boldsymbol{d}}$ and $(\prod_{v\in Q_0} \Pi^v_{\boldsymbol{d}}) \circ g_{\boldsymbol{d}}$ are the identity. The compositions $\bar{g}_{\boldsymbol{d}} \circ (\prod_{v\in Q_0} \bar{\Pi}^v_{\boldsymbol{d}})$ and $g_{\boldsymbol{d}} \circ (\prod_{v\in Q_0} \Pi^v_{\boldsymbol{d}})$ are not the identity. However, there are natural morphisms $\bar{F} : \mathbb{C} \times \bar{\mathcal{M}}_{\boldsymbol{d}} \longrightarrow \bar{\mathcal{M}}_{\boldsymbol{d}}$ and $F : \mathbb{C} \times \mathcal{M}_{\boldsymbol{d}} \longrightarrow \mathcal{M}_{\boldsymbol{d}}$ mapping $\big(t, (X_{v:v\in Q_0}, \rho_{e:e\in Q_1})\big) \mapsto (X_{v:v\in Q_0}, t\rho_{e:e\in Q_1})$ on $\mathbb{C}$-points, scaling the edge maps ρ_e by $t \in \mathbb{C}$, and

$$\begin{aligned} \bar{F}|_{\{0\}\times\bar{\mathcal{M}}_{\boldsymbol{d}}} &= \bar{g}_{\boldsymbol{d}} \circ (\textstyle\prod_{v\in Q_0} \bar{\Pi}^v_{\boldsymbol{d}}), & \bar{F}|_{\{1\}\times\bar{\mathcal{M}}_{\boldsymbol{d}}} &= \mathrm{id}, \\ F|_{\{0\}\times\mathcal{M}_{\boldsymbol{d}}} &= g_{\boldsymbol{d}} \circ (\textstyle\prod_{v\in Q_0} \Pi^v_{\boldsymbol{d}}), & F|_{\{1\}\times\mathcal{M}_{\boldsymbol{d}}} &= \mathrm{id}. \end{aligned}$$

Hence (5.31) are homotopy equivalences, and (5.32) follows. $\square$

Combining isomorphisms (5.9)–(5.10), (5.13)–(5.14), (5.32) and the Künneth Theorem gives explicit descriptions of the (co)homology of $\bar{\mathcal{M}}_{\boldsymbol{d}}, \mathcal{M}_{\boldsymbol{d}}$. Explicitly, for each $\boldsymbol{d} \in \mathbb{Z}^{Q_0}$, by (5.9) and (5.32) we write

$$H^*(\bar{\mathcal{M}}_{\boldsymbol{d}}) \cong R[\beta_{\boldsymbol{d},v,i} : v \in Q_0,\ i = 1, 2, \ldots], \tag{5.33}$$

where $\beta_{\boldsymbol{d},v,i}$ is a formal variable of degree $2i$, and for each $v \in Q_0$, the factor of $H^*(\mathbf{Perf}_{\mathbb{C}}^{\boldsymbol{d}(v)})$ in (5.32) is identified with $R[\beta_{\boldsymbol{d},v,i} : i \geqslant 1]$ by (5.9). By (5.13) and (5.32) we write

$$H_*(\bar{\mathcal{M}}_{\boldsymbol{d}}) \cong R[b_{\boldsymbol{d},v,i} : v \in Q_0,\ i = 1, 2, \ldots], \tag{5.34}$$

where $b_{\boldsymbol{d},v,i}$ is a formal variable of degree $2i$. Here as in (5.11), the pairing between $R[\beta_{\boldsymbol{d},v,i} : v \in Q_0,\ i \geqslant 1]$ and $R[b_{\boldsymbol{d},v,i} : v \in Q_0,\ i \geqslant 1]$ corresponding to the R-bilinear dual pairing between $H_*(\bar{\mathcal{M}}_{\boldsymbol{d}})$ and $H^*(\bar{\mathcal{M}}_{\boldsymbol{d}})$ is

$$\begin{aligned} &\big(\textstyle\prod_{v\in Q_0,\ i\geqslant 1} b_{\boldsymbol{d},v,i}^{m_{v,i}}\big) \cdot \big(\textstyle\prod_{v\in Q_0,\ i\geqslant 1} \beta_{\boldsymbol{d},v,i}^{n_{v,i}}\big) \\ &= \begin{cases} \dfrac{\prod_{v\in Q_0,\ i\geqslant 1} m_{v,i}!}{\prod_{v\in Q_0,\ i\geqslant 1}((i-1)!)^{m_{v,i}}}, & m_{v,i} = n_{v,i}, \text{ all } v, i, \\ 0, & \text{otherwise,} \end{cases} \end{aligned} \tag{5.35}$$

where $m_{v,i}, n_{v,i} \in \mathbb{N}$ with only finitely many $m_{v,i}$ and $n_{v,i}$ nonzero. The peculiar normalization $\prod_{v,i}[m_{v,i}!/((i-1)!)^{m_{v,i}}]$ in (5.35) is chosen to give the isomorphism we want between $\check{H}_*(\bar{\mathcal{M}})$ and a lattice vertex algebra from §2.2.4 in Theorem 5.19 below. Also as in (5.17), the R-bilinear cap product $\cap$ on $H_*(\bar{\mathcal{M}}_{\boldsymbol{d}})$ and $H^*(\bar{\mathcal{M}}_{\boldsymbol{d}})$ is identified by (5.33)–(5.34) with

$$\begin{aligned} &\big(\textstyle\prod_{v\in Q_0,\ i\geqslant 1} b_{\boldsymbol{d},v,i}^{m_{v,i}}\big) \cap \big(\textstyle\prod_{v\in Q_0,\ i\geqslant 1} \beta_{\boldsymbol{d},v,i}^{n_{v,i}}\big) = \\ &\begin{cases} \prod\limits_{v\in Q_0,\ i\geqslant 1} \frac{m_{v,i}!}{(m_{v,i}-n_{v,i})!((i-1)!)^{n_{v,i}}} \cdot \prod\limits_{v\in Q_0,\ i\geqslant 1} b_{\boldsymbol{d},v,i}^{m_{v,i}-n_{v,i}}, & m_{v,i} \geqslant n_{v,i}, \text{ all } v, i, \\ 0, & \text{otherwise.} \end{cases} \end{aligned} \tag{5.36}$$

Similarly, for each $\boldsymbol{d}\in\mathbb{N}^{Q_0}$, by (5.10) and (5.32) we write

$$\begin{aligned}H^*(\mathcal{M}_{\boldsymbol{d}})\cong R[\beta_{\boldsymbol{d},v,i}:v\in Q_0,\ i=1,2,\ldots]/I_{\boldsymbol{d}},\ \text{ where } I_{\boldsymbol{d}} \text{ is the ideal}\\ I_{\boldsymbol{d}}=\bigl(\mathrm{C}_i(\beta_{\boldsymbol{d},v,1},\ldots,\beta_{\boldsymbol{d},v,i})=0\ \text{ for all } v\in Q_0 \text{ and } i>\boldsymbol{d}(v)\bigr),\end{aligned}\tag{5.37}$$

and by (5.16) and (5.32) we write

$$H_*(\mathcal{M}_{\boldsymbol{d}})\cong I^\circ_{\boldsymbol{d}}\subset R[b_{\boldsymbol{d},v,i}:v\in Q_0,\ i=1,2,\ldots],\tag{5.38}$$

where $I^\circ_{\boldsymbol{d}}$ is the annihilator of $I_{\boldsymbol{d}}$ in (5.37) under the dual pairing (5.35). The dual pairing and cap product for $H_*(\mathcal{M}_{\boldsymbol{d}}),H^*(\mathcal{M}_{\boldsymbol{d}})$ are identified with those induced on $I^\circ_{\boldsymbol{d}}$ and $R[\beta_{\boldsymbol{d},v,i}:v\in Q_0,\ i\geqslant 1]/I_{\boldsymbol{d}}$ by (5.35)–(5.36).

For the rest of §5.3–§5.4 we make the identifications (5.33)–(5.34) and (5.37)–(5.38), so we just write $H^*(\bar{\mathcal{M}}_{\boldsymbol{d}})=R[\beta_{\boldsymbol{d},v,i}:v\in Q_0,\ i\geqslant 1]$, and so on. We also write $R[\beta_{\boldsymbol{d},v,i}:v\in Q_0,\ i\geqslant 1]_k$, $R[b_{\boldsymbol{d},v,i}:v\in Q_0,\ i\geqslant 1]_k$, $I_{\boldsymbol{d},k}$, $I^\circ_{\boldsymbol{d},k}$ for the degree k graded subspaces of $R[\beta_{\boldsymbol{d},v,i}:v\in Q_0,\ i\geqslant 1],\ldots,I^\circ_{\boldsymbol{d}}$. We write $1_{\boldsymbol{d}}$ for the identities in $R[\beta_{\boldsymbol{d},v,i}:v\in Q_0,\ i\geqslant 1]$ and $R[b_{\boldsymbol{d},v,i}:v\in Q_0,\ i\geqslant 1]$, the generators of $H^0(\bar{\mathcal{M}}_{\boldsymbol{d}}),H^0(\mathcal{M}_{\boldsymbol{d}}),H_0(\bar{\mathcal{M}}_{\boldsymbol{d}})$ and $H_0(\mathcal{M}_{\boldsymbol{d}})$, and we write $\beta_{\boldsymbol{d},v,0}=\boldsymbol{d}(v)\cdot 1_{\boldsymbol{d}}$ for $v\in Q_0$.

We can also use (5.8)–(5.9) and (5.32) to compute Betti numbers of $\bar{\mathcal{M}}_{\boldsymbol{d}}$ and $\bar{\mathcal{M}}_{\boldsymbol{d}}$. In generating function form we have

$$\sum_{k\geqslant 0}\dim H^k(\bar{\mathcal{M}}_{\boldsymbol{d}})q^k=\sum_{k\geqslant 0}\dim H_k(\bar{\mathcal{M}}_{\boldsymbol{d}})q^k=\prod_{i=1}^\infty(1-q^{2i})^{-|Q_0|},\tag{5.39}$$

$$\sum_{k\geqslant 0}\dim H^k(\mathcal{M}_{\boldsymbol{d}})q^k=\sum_{k\geqslant 0}\dim H_k(\mathcal{M}_{\boldsymbol{d}})q^k=\prod_{v\in Q_0}\prod_{i=1}^{\boldsymbol{d}(v)}(1-q^{2i})^{-1}.\tag{5.40}$$

As in Definition 5.12, translation $[1]:D^b\operatorname{mod-}\mathbb{C}Q\to D^b\operatorname{mod-}\mathbb{C}Q$ induces an isomorphism $\bar\Sigma:\bar{\mathcal{M}}\to\bar{\mathcal{M}}$, restricting to $\bar\Sigma_{\boldsymbol{d}}:\bar{\mathcal{M}}_{\boldsymbol{d}}\to\bar{\mathcal{M}}_{-\boldsymbol{d}}$. It fits into a commutative diagram

$$\begin{tikzcd}[column sep=huge]\bar{\mathcal{M}}_{\boldsymbol{d}} \arrow[r, "\bar\Sigma_{\boldsymbol{d}}"'] \arrow[d, "\prod_{v\in Q_0}\bar\Pi^v_{\boldsymbol{d}}"] & \bar{\mathcal{M}}_{-\boldsymbol{d}} \arrow[d, "\prod_{v\in Q_0}\bar\Pi^v_{-\boldsymbol{d}}"'] \\ \prod_{v\in Q_0}\mathbf{Perf}_{\mathbb{C}}^{\boldsymbol{d}(v)} \arrow[r, "\prod_{v\in Q_0}\Sigma_{\boldsymbol{d}(v)}"] & \prod_{v\in Q_0}\mathbf{Perf}_{\mathbb{C}}^{-\boldsymbol{d}(v)},\end{tikzcd}\tag{5.41}$$

where $\Sigma_r:\mathbf{Perf}^r_{\mathbb{C}}\to\mathbf{Perf}^{-r}_{\mathbb{C}}$ is induced by $[1]:D^b\operatorname{Vect}_{\mathbb{C}}\to D^b\operatorname{Vect}_{\mathbb{C}}$.

Now $\Sigma_r^*(\mathcal{E}^\bullet_{-r})\cong\mathcal{E}^\bullet_r[-1]$, where $\mathcal{E}^\bullet_r$ is the complex on $\mathbf{Perf}^r_{\mathbb{C}}$ from Proposition 5.9. Thus $H^{2i}(\Sigma_r)(\mathrm{ch}_i(\mathcal{E}^\bullet_{-r}))=-\mathrm{ch}_i(\mathcal{E}^\bullet_r)$, as $\mathrm{ch}_i(\mathcal{F}^\bullet[-1])=-\mathrm{ch}_i(\mathcal{F}^\bullet)$. Since $H^{2i}(\bar\Pi^v_{\boldsymbol{d}})(\mathrm{ch}_i(\mathcal{E}^\bullet_{\boldsymbol{d}(v)}))=\beta_{\boldsymbol{d},v,i}$, we deduce from (5.41) that

$$H^{2i}(\bar\Sigma_{\boldsymbol{d}})(\beta_{-\boldsymbol{d},v,i})=-\beta_{\boldsymbol{d},v,i}.$$

As $H^*(\bar\Sigma_{\boldsymbol{d}})$ is an algebra morphism, it follows that

$$H^*(\bar\Sigma_{\boldsymbol{d}})\bigl(\textstyle\prod_{v\in Q_0,\ i\geqslant 1}\beta_{-\boldsymbol{d},v,i}^{m_{v,i}}\bigr)=(-1)^{\sum_{v,i}m_{v,i}}\textstyle\prod_{v\in Q_0,\ i\geqslant 1}\beta_{\boldsymbol{d},v,i}^{m_{v,i}},$$

for $m_{v,i}\in\mathbb{N}$ with only finitely many nonzero. The dual action on homology is

$$H_*(\bar\Sigma_{\boldsymbol d})\bigl(\textstyle\prod_{v\in Q_0,\,i\geqslant 1}b_{\boldsymbol d,v,i}^{m_{v,i}}\bigr)=(-1)^{\sum_{v,i}m_{v,i}}\textstyle\prod_{v\in Q_0,\,i\geqslant 1}b_{-\boldsymbol d,v,i}^{m_{v,i}}. \tag{5.42}$$

Note that $H^*(\bar\Sigma)^2$ and $H_*(\bar\Sigma)^2$ are the identities, although $\bar\Sigma^2\neq\mathrm{id}_{\overline{\mathcal M}}$.

5.3.3 The actions of $\bar\Phi_{\boldsymbol d,\boldsymbol e},\bar\Psi_{\boldsymbol d},\Phi_{\boldsymbol d,\boldsymbol e},\Psi_{\boldsymbol d}$ on homology

For all $\boldsymbol d,\boldsymbol e\in\mathbb{Z}^{Q_0}$ we have a commutative diagram

$$\begin{tikzcd}
\overline{\mathcal M}_{\boldsymbol d}\times\overline{\mathcal M}_{\boldsymbol e} \arrow[rr, "\bar\Phi_{\boldsymbol d,\boldsymbol e}"] \arrow[d, "\prod_{v\in Q_0}\bar\Pi^v_{\boldsymbol d}\times\prod_{v\in Q_0}\bar\Pi^v_{\boldsymbol e}"] && \overline{\mathcal M}_{\boldsymbol d+\boldsymbol e} \arrow[d, "\prod_{v\in Q_0}\bar\Pi^v_{\boldsymbol d+\boldsymbol e}"] \\
\bigl(\prod_{v\in Q_0}\mathbf{Perf}_{\mathbb C}^{\boldsymbol d(v)}\bigr)\times\bigl(\prod_{v\in Q_0}\mathbf{Perf}_{\mathbb C}^{\boldsymbol e(v)}\bigr) \arrow[rr, "\prod_{v\in Q_0}\bar\Phi_{\boldsymbol d(v),\boldsymbol e(v)}"] && \prod_{v\in Q_0}\mathbf{Perf}_{\mathbb C}^{(\boldsymbol d+\boldsymbol e)(v)},
\end{tikzcd} \tag{5.43}$$

where $\bar\Phi_{\boldsymbol d,\boldsymbol e}$ is as in Assumption 3.1(g) and Definition 5.12, and $\prod_{v\in Q_0}\bar\Pi^v_{\boldsymbol d}$ is as in (5.31), and $\bar\Phi_{\boldsymbol d(v),\boldsymbol e(v)}$ is as in (5.20). Applying $H_*(-)$ to (5.43) gives a commutative diagram on homology, where the columns are the isomorphisms (5.32). Thus by (5.15), (5.26), and (5.34), the action of $\bar\Phi_{\boldsymbol d,\boldsymbol e}$ on homology is

$$H_*(\bar\Phi_{\boldsymbol d,\boldsymbol e})\bigl[\bigl(\prod_{v\in Q_0,\,i\geqslant 1}b_{\boldsymbol d,v,i}^{m_{v,i}}\bigr)\boxtimes\bigl(\prod_{v\in Q_0,\,i\geqslant 1}b_{\boldsymbol e,v,i}^{n_{v,i}}\bigr)\bigr]=\prod_{v\in Q_0,\,i\geqslant 1}b_{\boldsymbol d+\boldsymbol e,v,i}^{m_{v,i}+n_{v,i}}, \tag{5.44}$$

where $m_{v,i},n_{v,i}\in\mathbb{N}$ with only finitely many $m_{v,i}$ and $n_{v,i}$ nonzero. By (5.38), $H_*(\Phi_{\boldsymbol d,\boldsymbol e})$ is the restriction of (5.44) to a map $I^\circ_{\boldsymbol d}\boxtimes I^\circ_{\boldsymbol e}\to I^\circ_{\boldsymbol d+\boldsymbol e}$.

Similarly, for all $\boldsymbol d\in\mathbb{Z}^{Q_0}$ we have a commutative diagram

$$\begin{tikzcd}
{[*/\mathbb{G}_m]}\times\overline{\mathcal M}_{\boldsymbol d} \arrow[rr, "\bar\Psi_{\boldsymbol d}"] \arrow[d, "\mathrm{id}_{[*/\mathbb{G}_m]}\times\prod_{v\in Q_0}\bar\Pi^v_{\boldsymbol d}"] && \overline{\mathcal M}_{\boldsymbol d} \arrow[d, "\prod_{v\in Q_0}\bar\Pi^v_{\boldsymbol d}"] \\
{[*/\mathbb{G}_m]}\times\bigl(\prod_{v\in Q_0}\mathbf{Perf}_{\mathbb C}^{\boldsymbol d(v)}\bigr) \arrow[rr, "\prod_{v\in Q_0}\bar\Psi_{\boldsymbol d(v)}\circ\Pi_{[*/\mathbb{G}_m]\times\mathbf{Perf}_{\mathbb C}^{\boldsymbol d(v)}}"] && \prod_{v\in Q_0}\mathbf{Perf}_{\mathbb C}^{\boldsymbol d(v)}.
\end{tikzcd} \tag{5.45}$$

We can write the bottom morphism in (5.45) as a composition

$$[*/\mathbb{G}_m]\times\bigl(\prod_{v\in Q_0}\mathbf{Perf}_{\mathbb C}^{\boldsymbol d(v)}\bigr)\xrightarrow{\Delta_{Q_0}\times\mathrm{id}}\prod_{v\in Q_0}\bigl([*/\mathbb{G}_m]\times\mathbf{Perf}_{\mathbb C}^{\boldsymbol d(v)}\bigr)\xrightarrow{\prod_{v\in Q_0}\bar\Psi_{\boldsymbol d(v)}}\prod_{v\in Q_0}\mathbf{Perf}_{\mathbb C}^{\boldsymbol d(v)},$$

where $\Delta_{Q_0}:[*/\mathbb{G}_m]\to\prod_{v\in Q_0}[*/\mathbb{G}_m]$ is the diagonal map. Using the isomorphism $H_*([*/\mathbb{G}_m])\cong R[t]$ from Assumption 2.30(c), we find $H_*(\Delta_{Q_0})$ acts by

$$H_{2k}(\Delta_{Q_0})(t^k)=\sum_{\substack{k_v\geqslant 0,\ v\in Q_0:\\ k=\sum_{v\in Q_0}k_v}}\ \prod_{v\in Q_0}t_v^{k_v},$$

writing $H_*(\prod_{v\in Q_0}[*/\mathbb{G}_m])=R[t_v:v\in Q_0]$ in the obvious way.

Combining this with (5.45) and the action (5.27) of $H_*(\bar\Psi_{\boldsymbol{d}(v)})$ shows that the action of $\bar\Psi_{\boldsymbol{d}}$ on homology is

$$
\begin{aligned}
&H_*(\bar\Psi_{\boldsymbol{d}})\bigl[t^k\boxtimes(b_{\boldsymbol{d},v_1,j_1}\cdots b_{\boldsymbol{d},v_M,j_M})\bigr]_{M\geqslant 0,\; v_1,\ldots,v_M\in Q_0,\; j_1,\ldots,j_M>0}\\
&=\sum_{\substack{k_1,\ldots,k_M\geqslant 0,\; N\geqslant 0,\\ w_1,\ldots,w_N\in Q_0,\; l_1,\ldots,l_N>0:\\ k=k_1+\cdots+k_M+l_1+\cdots+l_N}} \frac{\prod_{i=1}^N \boldsymbol{d}(w_i)}{N!l_1\cdots l_N}\prod_{i=1}^M\binom{j_i+k_i-1}{j_i-1}\cdot\\
&\qquad (b_{\boldsymbol{d},v_1,j_1+k_1}\cdots b_{\boldsymbol{d},v_M,j_M+k_M}b_{\boldsymbol{d},w_1,l_1}\cdots b_{\boldsymbol{d},w_N,l_N}).
\end{aligned}
\tag{5.46}
$$

By (5.38), $H_*(\Psi_{\boldsymbol{d}})$ is the restriction of (5.46) to a map $R[t]\boxtimes I^\circ_{\boldsymbol{d}}\to I^\circ_{\boldsymbol{d}}$. As for (5.28), when $k=1$, equation (5.46) simplifies to

$$
\begin{aligned}
&H_*(\bar\Psi_{\boldsymbol{d}})\bigl[t\boxtimes(b_{\boldsymbol{d},v_1,j_1}\cdots b_{\boldsymbol{d},v_M,j_M})\bigr]_{M\geqslant 0,\; v_1,\ldots,v_M\in Q_0,\; j_1,\ldots,j_M>0}\\
&=\sum_{w\in Q_0}\boldsymbol{d}(w)\cdot(b_{\boldsymbol{d},w,1}b_{\boldsymbol{d},v_1,j_1}\cdots b_{\boldsymbol{d},v_M,j_M})\\
&+\sum_{i=1}^M j_i\cdot b_{\boldsymbol{d},v_1,j_1}\cdots b_{\boldsymbol{d},v_{i-1},j_{i-1}}b_{\boldsymbol{d},v_i,j_i+1}b_{\boldsymbol{d},v_{i+1},j_{i+1}}\cdots b_{\boldsymbol{d},v_M,j_M}.
\end{aligned}
\tag{5.47}
$$

Now Definition 3.9 defined an action $\diamond$ of $R[t]$ on $H_*(\bar{\mathcal{M}}_{\boldsymbol{d}})$ using $H_*(\bar\Psi_{\boldsymbol{d}})$, so (5.47) yields

$$
\begin{aligned}
&t\diamond(b_{\boldsymbol{d},v_1,j_1}\cdots b_{\boldsymbol{d},v_M,j_M})=\sum_{w\in Q_0}\boldsymbol{d}(w)\cdot(b_{\boldsymbol{d},w,1}b_{\boldsymbol{d},v_1,j_1}\cdots b_{\boldsymbol{d},v_M,j_M})\\
&+\sum_{i=1}^M j_i\cdot b_{\boldsymbol{d},v_1,j_1}\cdots b_{\boldsymbol{d},v_{i-1},j_{i-1}}b_{\boldsymbol{d},v_i,j_i+1}b_{\boldsymbol{d},v_{i+1},j_{i+1}}\cdots b_{\boldsymbol{d},v_M,j_M}.
\end{aligned}
\tag{5.48}
$$

Lemma 5.14. *The map $t\diamond-:H_k(\bar{\mathcal{M}}_{\boldsymbol{d}})\to H_{k+2}(\bar{\mathcal{M}}_{\boldsymbol{d}})$ is injective for all $\boldsymbol{d}\in\mathbb{Z}^{Q_0}$ and $k\in\mathbb{N}$, except when $\boldsymbol{d}=k=0$, when it has kernel $H_0(\bar{\mathcal{M}}_0)=R$. The same holds for the map $t\diamond-:H_k(\mathcal{M}_{\boldsymbol{d}})\to H_{k+2}(\mathcal{M}_{\boldsymbol{d}})$.*

Proof. By (5.34), $H_*(\bar{\mathcal{M}}_{\boldsymbol{d}})$ has basis the monomials $\mu=\prod_{v\in Q_0,\;j\geqslant 1}b_{\boldsymbol{d},v,j}^{m_{v,j}}$ where $m_{v,j}\in\mathbb{N}$ with only finitely many $m_{v,j}$ nonzero. To each such monomial μ, let us associate the number $M(\mu):=\max_{v,j:m_{v,j}>0}j$ which is the largest j with $m_{v,j}>0$ for some $v\in Q_0$, and write $M(1)=0$ for the monomial $\mu=1$ for which this is undefined. Suppose $t\diamond(\sum_{\text{monomials }\mu}a_\mu\cdot\mu)=0$ in $H_*(\bar{\mathcal{M}}_{\boldsymbol{d}})$, for coefficients $a_\mu\in R$ with only finitely many a_μ nonzero, but not all a_μ zero.

Let M be the maximum of the $M(\mu)$ with $a_\mu\neq 0$. If $M>0$, then by considering the coefficients of monomials μ' with $M(\mu')=M+1$ in the equation $t\diamond(\sum_\mu a_\mu\cdot\mu)=0$ we derive a contradiction, as such terms μ' come only from those μ with $a_\mu\neq 0$ and $M(\mu)=M$, and are injective on such terms μ. Hence $M=0$, and the only possibility with $a_\mu\neq 0$ is $\mu=1$. But $t\diamond 1=\sum_{w\in Q_0}\boldsymbol{d}(w)\cdot b_{\boldsymbol{d},w,1}$ by (5.48), so $t\diamond 1=0$ if and only if $\boldsymbol{d}=0$. This proves the lemma for $\bar{\mathcal{M}}_{\boldsymbol{d}}$, and the analogue for $\mathcal{M}_{\boldsymbol{d}}$ follows by restriction. $\square$

In §3.3 we defined the '$t=0$' homology $H_*(\bar{\mathcal{M}}_{\boldsymbol{d}})^{t=0}$. As R is a field of characteristic zero, this is $H_k(\bar{\mathcal{M}}_{\boldsymbol{d}})^{t=0}=H_k(\bar{\mathcal{M}}_{\boldsymbol{d}})/(t\diamond H_{k-2}(\bar{\mathcal{M}}_{\boldsymbol{d}}))$. By Lemma

5.14, this has dimension

$$\dim H_k(\bar{\mathcal{M}}_{\boldsymbol{d}})^{t=0} = \begin{cases} \dim H_k(\bar{\mathcal{M}}_{\boldsymbol{d}}) - \dim H_{k-2}(\bar{\mathcal{M}}_{\boldsymbol{d}}), & (\boldsymbol{d},k) \neq (0,2), \\ \dim H_2(\bar{\mathcal{M}}_0), & (\boldsymbol{d},k) = (0,2), \end{cases}$$

and similarly for $\dim H_k(\mathcal{M}_{\boldsymbol{d}})^{t=0}$. Hence by (5.39)–(5.40) we have

$$\sum_{k\geqslant 0} \dim H_k(\bar{\mathcal{M}}_{\boldsymbol{d}})^{t=0} q^k = q^2\delta_{\boldsymbol{d}0} + (1-q^2)\prod_{i=1}^{\infty}(1-q^{2i})^{-|Q_0|},$$

$$\sum_{k\geqslant 0} \dim H_k(\mathcal{M}_{\boldsymbol{d}})^{t=0} q^k = q^2\delta_{\boldsymbol{d}0} + (1-q^2)\prod_{v\in Q_0}\prod_{i=1}^{\boldsymbol{d}(v)}(1-q^{2i})^{-1},$$

where $\delta_{\boldsymbol{d}0} = 1$ if $\boldsymbol{d} = 0$ and $\delta_{\boldsymbol{d}0} = 0$ otherwise. With the alternative grading $\tilde{H}_*(-)$ in (3.43), this gives

$$\sum_{k\geqslant \chi(\boldsymbol{d},\boldsymbol{d})-2} \dim \tilde{H}_k(\bar{\mathcal{M}}_{\boldsymbol{d}})^{t=0} q^k = \delta_{\boldsymbol{d}0} + q^{\chi(\boldsymbol{d},\boldsymbol{d})-2}(1-q^2)\prod_{i=1}^{\infty}(1-q^{2i})^{-|Q_0|}, \tag{5.49}$$

$$\sum_{k\geqslant \chi(\boldsymbol{d},\boldsymbol{d})-2} \dim \tilde{H}_k(\mathcal{M}_{\boldsymbol{d}})^{t=0} q^k = \delta_{\boldsymbol{d}0} + q^{\chi(\boldsymbol{d},\boldsymbol{d})-2}(1-q^2)\prod_{v\in Q_0}\prod_{i=1}^{\boldsymbol{d}(v)}(1-q^{2i})^{-1}. \tag{5.50}$$

5.3.4 The Chern characters and Chern classes of $\bar{\Theta}^\bullet_{\boldsymbol{d},\boldsymbol{e}}, \Theta^\bullet_{\boldsymbol{d},\boldsymbol{e}}$

Definition 5.12 defined perfect complexes $\bar{\Theta}^\bullet_{\boldsymbol{d},\boldsymbol{e}}, \Theta^\bullet_{\boldsymbol{d},\boldsymbol{e}}$ on $\bar{\mathcal{M}}_{\boldsymbol{d}}\times\bar{\mathcal{M}}_{\boldsymbol{e}}$ and $\mathcal{M}_{\boldsymbol{d}}\times\mathcal{M}_{\boldsymbol{e}}$. We now compute their Chern characters and Chern classes.

Proposition 5.15. *For all $\boldsymbol{d},\boldsymbol{e}\in\mathbb{Z}^{Q_0}$ and $i\geqslant 0$, in $H^{2i}(\bar{\mathcal{M}}_{\boldsymbol{d}}\times\bar{\mathcal{M}}_{\boldsymbol{e}})$ we have*

$$\mathrm{ch}_i([\bar{\Theta}^\bullet_{\boldsymbol{d},\boldsymbol{e}}]) = \sum_{v,w\in Q_0}\sum_{j,k\geqslant 0: j+k=i}(-1)^k a_{vw}\cdot\beta_{\boldsymbol{d},v,j}\boxtimes\beta_{\boldsymbol{e},w,k}, \tag{5.51}$$

where $A=(a_{vw})_{v,w\in Q_0}$ is as in Definition 5.12, and $\beta_{\boldsymbol{d},v,0}=\boldsymbol{d}(v)\cdot 1_{\boldsymbol{d}}$. Hence as in §2.4.2, using the polynomials C_i in (2.51)–(2.52) we have

$$c_i([\bar{\Theta}^\bullet_{\boldsymbol{d},\boldsymbol{e}}]) = \mathrm{C}_i\bigl(\mathrm{ch}_1([\bar{\Theta}^\bullet_{\boldsymbol{d},\boldsymbol{e}}]),\ldots,\mathrm{ch}_i([\bar{\Theta}^\bullet_{\boldsymbol{d},\boldsymbol{e}}])\bigr). \tag{5.52}$$

Also $\mathrm{ch}_i(\Theta^\bullet_{\boldsymbol{d},\boldsymbol{e}}), c_i(\Theta^\bullet_{\boldsymbol{d},\boldsymbol{e}})$ are the images of (5.51)–(5.52) in $H^{2i}(\mathcal{M}_{\boldsymbol{d}}\times\mathcal{M}_{\boldsymbol{e}})$.

Proof. Let $\bar{g}_{\boldsymbol{d}}:\prod_{v\in Q_0}\mathbf{Perf}_{\mathbb{C}}^{\boldsymbol{d}(v)}\to\bar{\mathcal{M}}_{\boldsymbol{d}}$ be as in the proof of Proposition 5.13. Then on $\bigl(\prod_{v\in Q_0}\mathbf{Perf}_{\mathbb{C}}^{\boldsymbol{d}(v)}\bigr)\times\bigl(\prod_{v\in Q_0}\mathbf{Perf}_{\mathbb{C}}^{\boldsymbol{e}(v)}\bigr)$ for $\boldsymbol{d},\boldsymbol{e}\in\mathbb{Z}^{Q_0}$ we have

$$\begin{aligned}(\bar{g}_{\boldsymbol{d}}\times\bar{g}_{\boldsymbol{e}})^*(\bar{\mathcal{E}}xt^\bullet_{\boldsymbol{d},\boldsymbol{e}}) &\cong \textstyle\bigoplus_{v\in Q_0}\pi^*_{\mathbf{Perf}_{\mathbb{C}}^{\boldsymbol{d}(v)}}\bigl((\mathcal{E}^\bullet_{\boldsymbol{d}(v)})^\vee\bigr)\otimes\pi^*_{\mathbf{Perf}_{\mathbb{C}}^{\boldsymbol{e}(v)}}\bigl(\mathcal{E}^\bullet_{\boldsymbol{e}(v)}\bigr)\\ &\oplus\textstyle\bigoplus_{e\in Q_1}\bigl(\pi^*_{\mathbf{Perf}_{\mathbb{C}}^{\boldsymbol{d}(h(e))}}\bigl((\mathcal{E}^\bullet_{\boldsymbol{d}(h(e))})^\vee\bigr)\otimes\pi^*_{\mathbf{Perf}_{\mathbb{C}}^{\boldsymbol{e}(t(e))}}\bigl(\mathcal{E}^\bullet_{\boldsymbol{e}(t(e))}\bigr)\bigr)[-1],\end{aligned} \tag{5.53}$$

where $\bar{\mathcal{E}}xt^\bullet_{\boldsymbol{d},\boldsymbol{e}}$ is as in Definition 5.12, and the perfect complex $\mathcal{E}^\bullet_r$ on $\mathbf{Perf}^r_{\mathbb{C}}$ is as in Proposition 5.9.

Equation (5.53) holds as $\bar{\mathcal{E}}xt^\bullet_{\boldsymbol{d},\boldsymbol{e}}$ is the derived Hom of complexes of Q-representations on $\bar{\mathcal{M}}_{\boldsymbol{d}}\times\bar{\mathcal{M}}_{\boldsymbol{e}}$, from the pullback of the universal representation on $\bar{\mathcal{M}}_{\boldsymbol{d}}$, to the pullback of the universal representation on $\bar{\mathcal{M}}_{\boldsymbol{e}}$. When we pullback by $\bar{g}_{\boldsymbol{d}}\times\bar{g}_{\boldsymbol{e}}$, we again get a derived Hom of complexes of Q-representations, but now from $\bigoplus_{v\in Q_0}\pi^*_{\mathbf{Perf}^{\boldsymbol{d}(v)}_{\mathbb{C}}}(\mathcal{E}^\bullet_{\boldsymbol{d}(v)})$ regarded as a Q-representation with edge morphisms zero, to $\bigoplus_{v\in Q_0}\pi^*_{\mathbf{Perf}^{\boldsymbol{e}(v)}_{\mathbb{C}}}(\mathcal{E}^\bullet_{\boldsymbol{e}(v)})$ as a Q-representation with edge morphisms zero, and we can show this is equivalent to (5.53).

Taking K-theory classes of (5.53) gives an equation

$$\begin{aligned}&K_0(\bar{g}_{\boldsymbol{d}}\times\bar{g}_{\boldsymbol{e}})([\bar{\mathcal{E}}xt^\bullet_{\boldsymbol{d},\boldsymbol{e}}])\\&=\sum_{v,w\in Q_0}(\delta_{vw}-n_{vw})K_0(\pi_{\mathbf{Perf}^{\boldsymbol{d}(v)}_{\mathbb{C}}})\bigl([\mathcal{E}^\bullet_{\boldsymbol{d}(v)}]^\vee\bigr)\otimes K_0(\pi_{\mathbf{Perf}^{\boldsymbol{e}(w)}_{\mathbb{C}}})\bigl([\mathcal{E}^\bullet_{\boldsymbol{e}(w)}]\bigr)\end{aligned}\tag{5.54}$$

in $K_0\bigl(\mathrm{Perf}\bigl((\prod_v\mathbf{Perf}^{\boldsymbol{d}(v)}_{\mathbb{C}})\times(\prod_v\mathbf{Perf}^{\boldsymbol{e}(v)}_{\mathbb{C}})\bigr)\bigr)$, where n_{vw} is the number of edges $\overset{v}{\bullet}\to\overset{w}{\bullet}$ in Q, as in Definition 5.12. Now $\bar{\Theta}^\bullet_{\boldsymbol{d},\boldsymbol{e}}=(\bar{\mathcal{E}}xt^\bullet_{\boldsymbol{d},\boldsymbol{e}})^\vee\oplus\bar{\sigma}^*_{\boldsymbol{d},\boldsymbol{e}}(\bar{\mathcal{E}}xt^\bullet_{\boldsymbol{e},\boldsymbol{d}})$ by definition, so adding the dual of (5.54) to (5.54) with $\boldsymbol{d},\boldsymbol{e}$ exchanged yields

$$\begin{aligned}&K_0(\bar{g}_{\boldsymbol{d}}\times\bar{g}_{\boldsymbol{e}})([\bar{\Theta}^\bullet_{\boldsymbol{d},\boldsymbol{e}}])\\&=\sum_{v,w\in Q_0}a_{vw}K_0(\pi_{\mathbf{Perf}^{\boldsymbol{d}(v)}_{\mathbb{C}}})\bigl([\mathcal{E}^\bullet_{\boldsymbol{d}(v)}]\bigr)\otimes K_0(\pi_{\mathbf{Perf}^{\boldsymbol{e}(w)}_{\mathbb{C}}})\bigl([\mathcal{E}^\bullet_{\boldsymbol{e}(w)}]^\vee\bigr)\end{aligned}\tag{5.55}$$

where $a_{vw}=2\delta_{vw}-n_{vw}-n_{wv}$, as in Definition 5.12.

Applying Chern characters ch_i to (5.55) gives an equation in the cohomology of $(\prod_v\mathbf{Perf}^{\boldsymbol{d}(v)}_{\mathbb{C}})\times(\prod_v\mathbf{Perf}^{\boldsymbol{e}(v)}_{\mathbb{C}})$. But (5.32) identifies this with $H^*(\bar{\mathcal{M}}_{\boldsymbol{d}}\times\bar{\mathcal{M}}_{\boldsymbol{e}})$, and under this identification $H^*(\bar{g}_{\boldsymbol{d}}\times\bar{g}_{\boldsymbol{e}})$ is the identity, so we can omit it. Thus

$$\begin{aligned}\mathrm{ch}_i([\bar{\Theta}^\bullet_{\boldsymbol{d},\boldsymbol{e}}])&=\sum_{v,w\in Q_0}a_{vw}\sum_{j,k\geqslant 0:j+k=i}\mathrm{ch}_j([\mathcal{E}^\bullet_{\boldsymbol{d}(v)}])\boxtimes\mathrm{ch}_k([(\mathcal{E}^\bullet_{\boldsymbol{e}(v)})^\vee])\\&=\sum_{v,w\in Q_0}\sum_{j,k\geqslant 0:j+k=i}(-1)^k a_{vw}\cdot\beta_{\boldsymbol{d},v,j}\boxtimes\beta_{\boldsymbol{e},w,k},\end{aligned}$$

where the first step uses (2.47), and the second uses $\mathrm{ch}_k((\mathcal{E}^\bullet)^\vee)=(-1)^k\,\mathrm{ch}_k(\mathcal{E}^\bullet)$ and $\mathrm{ch}_j(\mathcal{E}^\bullet_{\boldsymbol{d}(v)})=\beta_{\boldsymbol{d},v,j}$, $\mathrm{ch}_k(\mathcal{E}^\bullet_{\boldsymbol{e}(v)})=\beta_{\boldsymbol{e},v,k}$. This proves (5.51), and the rest of the proposition is immediate. □

Note that (5.51)–(5.52) depend only on the underlying graph of Q.

5.3.5 Explicit computation of $[\zeta,\eta]_n$

Combining (5.51) with the generating function expression (3.34) for $[\zeta,\eta]_n$ in terms of Chern characters $\mathrm{ch}_i([\bar{\Theta}^\bullet_{\boldsymbol{d},\boldsymbol{e}}])$ yields:

Corollary 5.16. *For all $\boldsymbol{d},\boldsymbol{e}\in\mathbb{Z}^{Q_0}$ and $\zeta\in\tilde H_*(\mathcal{M}_{\boldsymbol{d}})$, $\eta\in\tilde H_*(\mathcal{M}_{\boldsymbol{e}})$, we have*

$$
\begin{aligned}
&Y(\zeta,z)\eta:=\sum_{n\in\mathbb{Z}}[\zeta,\eta]_n z^{-n-1}=\epsilon_{\boldsymbol{d},\boldsymbol{e}}z^{\chi(\boldsymbol{d},\boldsymbol{e})}\cdot H_*(\bar\Phi_{\boldsymbol{d},\boldsymbol{e}})\circ H_*(\bar\Psi_{\boldsymbol{d}}\times\mathrm{id}_{\bar{\mathcal{M}}_{\boldsymbol{e}}}) \qquad (5.56)\\
&\Bigl\{\Bigl(\sum_{i\geqslant 0}z^it^i\Bigr)\boxtimes\Bigl((\zeta\boxtimes\eta)\cap\exp\Bigl[\sum_{v,w\in Q_0}a_{vw}\sum_{\substack{j,k\geqslant 0:\\ j+k\geqslant 1}}\begin{array}{l}(-1)^{j-1}(j+k-1)!z^{-j-k}\cdot\\ \beta_{\boldsymbol{d},v,j}\boxtimes\beta_{\boldsymbol{e},w,k}\end{array}\Bigr]\Bigr)\Bigr\}
\end{aligned}
$$

in $\tilde H_*(\bar{\mathcal{M}}_{\boldsymbol{d}+\boldsymbol{e}})[[z,z^{-1}]]$, where z is a formal variable.

Here we omit the factor $(-1)^{a\chi(\beta,\beta)}$ in (3.34) as $\bar{\mathcal{M}}_{\boldsymbol{e}}$ has only even homology, and we have used (3.20) to expand $H_*(\Xi_{\boldsymbol{d},\boldsymbol{e}})$. Note that $\cap$, $H_*(\bar\Phi_{\boldsymbol{d},\boldsymbol{e}})$ and $H_*(\bar\Psi_{\boldsymbol{d}})$ in (5.56) are given explicitly by equations (5.36), (5.44) and (5.46).

The next two propositions simplify (5.56) in the case when $\zeta=1_{\boldsymbol{d}}$, and in the case when $\boldsymbol{d}=0$ and $\zeta=b_{0,v,m}$.

Proposition 5.17. *For all $\boldsymbol{d},\boldsymbol{e}\in\mathbb{Z}^{Q_0}$ and $\eta\in\tilde H_*(\mathcal{M}_{\boldsymbol{e}})$, we have*

$$
\begin{aligned}
&Y(1_{\boldsymbol{d}},z)\eta:=\sum_{n\in\mathbb{Z}}[1_{\boldsymbol{d}},\eta]_n z^{-n-1}=\epsilon_{\boldsymbol{d},\boldsymbol{e}}z^{\chi(\boldsymbol{d},\boldsymbol{e})}\cdot H_*(\bar\Phi_{\boldsymbol{d},\boldsymbol{e}}) \qquad (5.57)\\
&\Bigl\{\exp\Bigl[\sum_{v\in Q_0,\,j\geqslant 1}\boldsymbol{d}(v)z^j\tfrac{1}{j}b_{\boldsymbol{d},v,j}\Bigr]\boxtimes\Bigl(\eta\cap\exp\Bigl[-\sum_{v,w\in Q_0,\,k\geqslant 1}a_{vw}\boldsymbol{d}(v)(k-1)!z^{-k}\beta_{\boldsymbol{e},w,k}\Bigr]\Bigr)\Bigr\}.
\end{aligned}
$$

Proof. Take $\zeta=1_{\boldsymbol{d}}$ in (5.56). Then in the cap product $(1_{\boldsymbol{d}}\boxtimes\eta)\cap\exp[\cdots]$, all terms in $\beta_{\boldsymbol{d},v,j}$ for $j>0$ give zero as $1_{\boldsymbol{d}}\cap\beta_{\boldsymbol{d},v,j}=0$, so we can restrict to $j=0$, when $\beta_{\boldsymbol{d},v,0}=\boldsymbol{d}(v)$. Then $-\cap\exp[\cdots]$ only affects the factor η in $1_{\boldsymbol{d}}\boxtimes\eta$, so we can rewrite it as $1_{\boldsymbol{d}}\boxtimes(\eta\cap\exp[\cdots])$. But the $H_*(\bar\Psi_{\boldsymbol{d}})$ only affects the $t^i\boxtimes 1_{\boldsymbol{d}}$ factor, not the $(\eta\cap\exp[\cdots])$ factor. Therefore we can simplify the sum as

$$
\begin{aligned}
&Y(1_{\boldsymbol{d}},z)\eta=\epsilon_{\boldsymbol{d},\boldsymbol{e}}z^{\chi(\boldsymbol{d},\boldsymbol{e})}\cdot H_*(\bar\Phi_{\boldsymbol{d},\boldsymbol{e}}) \qquad (5.58)\\
&\Bigl\{H_*(\bar\Psi_{\boldsymbol{d}})\Bigl(\sum_{i\geqslant 0}z^it^i\boxtimes 1_{\boldsymbol{d}}\Bigr)\boxtimes\Bigl(\eta\cap\exp\Bigl[-\sum_{v,w\in Q_0}a_{vw}\boldsymbol{d}(v)\sum_{k\geqslant 1}(k-1)!z^{-k}\beta_{\boldsymbol{e},w,k}\Bigr]\Bigr)\Bigr\}.
\end{aligned}
$$

From (5.46) we can show that

$$
H_*(\bar\Psi_{\boldsymbol{d}})\bigl(\textstyle\sum_{i\geqslant 0}z^it^i\boxtimes 1_{\boldsymbol{d}}\bigr)=\exp\bigl[\textstyle\sum_{v\in Q_0,\,j\geqslant 1}\boldsymbol{d}(v)z^j\frac{1}{j}b_{\boldsymbol{d},v,j}\bigr]. \qquad (5.59)
$$

Substituting this into (5.58) yields (5.57). □

Proposition 5.18. *For all $v\in Q_0$, $m\geqslant 1$, $\boldsymbol{e}\in\mathbb{Z}^{Q_0}$ and $\eta\in\tilde H_*(\mathcal{M}_{\boldsymbol{e}})$, we have*

$$
\begin{aligned}
Y(b_{0,v,m},z)\eta:=\sum_{n\in\mathbb{Z}}[b_{0,v,m},\eta]_n z^{-n-1}=\sum_{i\geqslant 0}z^i\tbinom{m+i-1}{m-1}\cdot b_{\boldsymbol{e},v,m+i}\,\eta+{}\\
\sum_{v,w\in Q_0,\,k\geqslant 0}a_{vw}\tfrac{(-1)^{m-1}(m+k-1)!}{(m-1)!}z^{-m-k}\cdot(\eta\cap\beta_{\boldsymbol{e},w,k}). \qquad (5.60)
\end{aligned}
$$

Here $b_{\boldsymbol{e},v,m+i}\,\eta$ means multiplication in the R-algebra $R[b_{\boldsymbol{e},v,i}]_{v,i}$ in (5.34).

Proof. Take $\boldsymbol{d}=0$ and $\zeta=b_{0,v,m}$ in (5.56). Then in $(b_{0,v,i}\boxtimes\eta)\cap\exp[\cdots]$, all terms in $\beta_{0,v,j}$ for $j\neq m$ in $\exp[\cdots]$ give zero as $b_{0,v,m}\cap\beta_{0,v,j}=0$ for $j\neq i$, noting that when $j=0$ we have $\beta_{0,v,j}=0(v)1_0=0$ as $\boldsymbol{d}=0$. Also, although the $\exp[\cdots]$ involves terms of order l in $[\cdots]$, only $l=0$ and $l=1$ terms contribute, as $b_{0,v,m}\cap\beta_{0,v,m}^2=0$. Hence (5.56) becomes

$$
\begin{aligned}
Y(b_{0,v,m},z)\eta &:= \sum_{n\in\mathbb{Z}}[b_{0,v,m},\eta]_n z^{-n-1} = H_*(\bar\Phi_{0,\boldsymbol{e}})\circ H_*(\bar\Psi_0\times\mathrm{id}_{\bar{\mathcal{M}}_{\boldsymbol{e}}}) \qquad (5.61)\\
&\Bigl\{\Bigl(\sum_{i\geqslant 0}z^it^i\Bigr)\boxtimes\Bigl((b_{0,v,m}\boxtimes\eta)\cap\Bigl[1+\sum_{v,w\in Q_0}a_{vw}\sum_{k\geqslant 0}(-1)^{m-1}(m+k-1)!z^{-m-k}\cdot\beta_{0,v,m}\boxtimes\beta_{\boldsymbol{e},w,k}\Bigr]\Bigr)\Bigr\}\\
&= H_*(\bar\Phi_{0,\boldsymbol{e}})\circ H_*(\bar\Psi_0\times\mathrm{id}_{\bar{\mathcal{M}}_{\boldsymbol{e}}})\\
&\Bigl\{\Bigl(\sum_{i\geqslant 0}z^it^i\Bigr)\boxtimes\Bigl((b_{0,v,m}\boxtimes\eta)+\sum_{v,w\in Q_0,\,k\geqslant 0}a_{vw}\tfrac{(-1)^{m-1}(m+k-1)!}{(m-1)!}z^{-m-k}\cdot 1_0\boxtimes(\eta\cap\beta_{\boldsymbol{e},w,k})\Bigr)\Bigr\}\\
&= H_*(\bar\Phi_{0,\boldsymbol{e}})\Bigl\{H_*(\bar\Psi_0)\Bigl(\sum_{i\geqslant 0}z^it^i\boxtimes b_{0,v,m}\Bigr)\boxtimes\eta+\\
&H_*(\bar\Psi_0)\Bigl(\sum_{i\geqslant 0}z^it^i\boxtimes 1_0\Bigr)\boxtimes\Bigl(\sum_{v,w\in Q_0,\,k\geqslant 0}a_{vw}\tfrac{(-1)^{m-1}(m+k-1)!}{(m-1)!}z^{-m-k}\cdot(\eta\cap\beta_{\boldsymbol{e},w,k})]\Bigr)\Bigr\},
\end{aligned}
$$

where in the first step we use $\epsilon_{0,\boldsymbol{e}}=1$ and $\chi(0,\boldsymbol{e})=0$, and in the second $b_{0,v,m}\cap\beta_{0,v,m}=1/(m-1)!$ by (5.36). Now from (5.46) we can show that

$$
H_*(\bar\Psi_0)\bigl(\textstyle\sum_{i\geqslant 0}z^it^i\boxtimes b_{0,v,m}\bigr)=\textstyle\sum_{i\geqslant 0}z^i\binom{m+i-1}{m-1}b_{0,v,m+i},
$$

and (5.59) gives $H_*(\bar\Psi_0)\bigl(\sum_{i\geqslant 0}z^it^i\boxtimes 1_0\bigr)=1_0$. Substituting into (5.61) yields

$$
\begin{aligned}
Y(b_{0,v,m},z)\eta = H_*(\bar\Phi_{0,\boldsymbol{e}})\Bigl\{&\sum_{i\geqslant 0}z^i\tbinom{m+i-1}{m-1}b_{0,v,m+i}\boxtimes\eta+\\
&\sum_{v,w\in Q_0,\,k\geqslant 0}a_{vw}\tfrac{(-1)^{m-1}(m+k-1)!}{(m-1)!}z^{-m-k}\cdot 1_0\boxtimes(\eta\cap\beta_{\boldsymbol{e},w,k})]\Bigr\}.
\end{aligned}
$$

Equation (5.60) now follows from (5.44). $\square$

5.3.6 $\check H_*(\bar{\mathcal{M}})$ is a lattice vertex algebra

In the first main result of this section, we show that $\check H_*(\bar{\mathcal{M}})$ is isomorphic to a lattice vertex algebra from §2.2.4.

Theorem 5.19. *Let Q be any quiver, and $\mathbb{K}=\mathbb{C}$, and R be a field of characteristic zero, and let $\check H_*(\bar{\mathcal{M}})$ be the graded vertex algebra over R constructed in Theorem* 3.14 *from the triangulated category $\mathcal{T}=D^b\,\mathrm{mod}\text{-}\mathbb{C}Q$ with additional data as in Definition* 5.12*. Use the notation of* §5.3.1–§5.3.2.

Then there is an isomorphism of graded vertex algebras over R between $\check H_*(\bar{\mathcal{M}})$ and the lattice vertex algebra V_* defined in Definition 2.15 *and Theorem* 2.16 *using the lattice $\mathbb{Z}^{Q_0}$ and intersection form χ, identifying $\prod_{v\in Q_0,\,i\geqslant 1}b_{\boldsymbol{d},v,i}^{n_{v,i}}$ in $\check H_*(\bar{\mathcal{M}}_{\boldsymbol{d}})$ with $e^{\boldsymbol{d}}\otimes\prod_{v\in Q_0,\,i\geqslant 1}b_{v,i}^{n_{v,i}}$ in V_* for all $\boldsymbol{d}\in\mathbb{Z}^{Q_0}$ and $n_{v,i}\in\mathbb{N}$ with only finitely many $n_{v,i}$ nonzero.*

Hence by Theorem 2.16, *if χ is nondegenerate, then writing $C=(c_{vw})_{v,w\in Q_0}$ for the inverse matrix of $A=(a_{vw})_{v,w\in Q_0}$ over $\mathbb{Q}$, then $\check H_*(\bar{\mathcal{M}})$ is a simple graded vertex algebra, and it is a graded vertex operator algebra, with central charge $c=|Q_0|$ and conformal vector*

$$\omega=\tfrac{1}{2}\sum_{v,w\in Q_0} c_{vw}\, b_{0,v,1}b_{0,w,1}\in H_4(\bar{\mathcal{M}}_0).$$

Proof. By (3.31) and (5.34), $\check H_*(\bar{\mathcal{M}})$ has a basis of elements $\prod_{v\in Q_0,\, i\geqslant 1} b_{\boldsymbol{d},v,i}^{n_{v,i}}$ graded of degree $-\frac{1}{2}\chi(\boldsymbol{d},\boldsymbol{d})+\sum_{v,i} n_{v,i}$ for $\boldsymbol{d}\in\mathbb{Z}^{Q_0}$ and $n_{v,i}\in\mathbb{N}$ with finitely many $n_{v,i}\neq 0$. As in (2.13)–(2.14), V_* has a basis of elements $e^{\boldsymbol{d}}\otimes\prod_{v,i} b_{v,i}^{n_{v,i}}$ graded of degree $-\frac{1}{2}\chi(\boldsymbol{d},\boldsymbol{d})+\sum_{v,i} n_{v,i}$ for the same $\boldsymbol{d}, n_{v,i}$. Thus, there is a unique isomorphism of $\frac{1}{2}\mathbb{Z}$-graded R-vector spaces $\iota:\check H_*(\bar{\mathcal{M}})\to V_*$ identifying $\prod_{v\in Q_0,\, i\geqslant 1} b_{\boldsymbol{d},v,i}^{n_{v,i}}$ in $\check H_*(\bar{\mathcal{M}}_{\boldsymbol{d}})$ with $e^{\boldsymbol{d}}\otimes\prod_{v\in Q_0,\, i\geqslant 1} b_{v,i}^{n_{v,i}}$ in V_* for all $\boldsymbol{d}, n_{v,i}$.

We claim that ι identifies the vertex algebra structure on $\check H_*(\bar{\mathcal{M}})$ with the unique vertex algebra structure on V_* given by Theorem 2.16. To see this, note that ι maps $\mathbb{1}=1_0\mapsto\mathbb{1}=e^0\otimes 1$.

If $\alpha\in\mathbb{Z}^{Q_0}$ and $n\in\mathbb{Z}$ then Definition 2.15 and (2.15) determine $(e^0\otimes\sum_{v\in Q_0}\alpha(v)b_{v,1})_n : V_*\to V_*$, and Proposition 5.18 with $m=1$ determines $(\sum_{v\in Q_0}\alpha(v)\, b_{0,v,1})_n:\check H_*(\bar{\mathcal{M}})\to\check H_*(\bar{\mathcal{M}})$. Comparing we see that ι identifies these two actions, noting that for $n>0$, $\alpha_n:V_*\to V_*$ acts as $n\sum_{v\in Q_0}\alpha(v)\frac{\mathrm{d}}{\mathrm{d}b_{v,n}}$ by Definition 2.15(iii), and $-\cap(\sum_{v\in Q_0}\alpha(v)\, n!\beta_{\boldsymbol{d},v,n}):\check H_*(\bar{\mathcal{M}}_{\boldsymbol{d}})\to\check H_*(\bar{\mathcal{M}}_{\boldsymbol{d}})$ is given by (5.36) and acts by $n\sum_{v\in Q_0}\alpha(v)\,\frac{\mathrm{d}}{\mathrm{d}b_{\boldsymbol{d},v,n}}$.

If $\alpha\in\mathbb{Z}^{Q_0}$ then (2.16) determines $Y(e^\alpha\otimes 1,z):V_*\to V_*[[z,z^{-1}]]$, and Proposition 5.17 determines $Y(1_\alpha,z):\check H_*(\bar{\mathcal{M}})\to\check H_*(\bar{\mathcal{M}})[[z,z^{-1}]]$. Comparing we see that ι identifies these two actions, as the factors $\epsilon_{\alpha,\beta}z^{\chi(\alpha,\beta)}$ in (2.16) and $\epsilon_{\boldsymbol{d},\boldsymbol{e}}z^{\chi(\boldsymbol{d},\boldsymbol{e})}$ in (5.57) correspond, and the factors $\exp\bigl[-\sum_{n<0}\frac{1}{n}z^{-n}\alpha_n\bigr]$ in (2.16) and $\exp\bigl[\sum_{v,j}\boldsymbol{d}(v)z^j\frac{1}{j}b_{\boldsymbol{d},v,j}\bigr]$ in (5.57) correspond by Definition 2.15(iii), and the factors $\exp\bigl[-\sum_{n>0}\frac{1}{n}z^{-n}\alpha_n\bigr]$ in (2.16) and $-\cap\exp\bigl[-\sum_{v,w,k}a_{vw}\boldsymbol{d}(v)(k-1)!z^{-k}\beta_{\boldsymbol{e},w,k}\bigr]$ correspond by Definition 2.15(i) and (5.36). Thus the first part of Theorem 2.16 shows that ι identifies the vertex algebra structures on $\check H_*(\bar{\mathcal{M}})$ and V_*, proving the first part of the theorem. The second part is immediate. $\square$

Remark 5.20. (i) In examples, such as the lattice vertex algebras in §2.2.4, (graded) vertex algebra structures are often constructed on the underlying vector space V_* of a (graded) commutative algebra, although the algebra structure on V_* is *not* part of the vertex algebra structure (and the grading is different).

In our situation $H_*(\mathcal{M})$ is naturally a graded commutative algebra, with multiplication $\zeta\cdot\eta=H_*(\Phi)(\zeta\boxtimes\eta)$ for $\Phi:\mathcal{M}\times\mathcal{M}\to\mathcal{M}$ as in Assumption 3.1(g), although $H_*(\Phi)$ is not part of the algebraic structures we usually consider on $H_*(\mathcal{M})$ (e.g. $H_*(\Phi)$ does not respect the primary grading $\tilde H_*(\mathcal{M})$). Thus, every graded vertex algebra $\check H_*(\mathcal{M})$ arising from Theorem 3.14 is also the underlying vector space of a graded commutative algebra (though with a different grading).

Equation (5.44) implies that the isomorphism between $\check H_*(\mathcal{M})$ and the lattice vertex algebra in Theorem 5.19 identifies the graded commutative algebra structure from $H_*(\Phi)$ on $\check H_*(\mathcal{M})$ with the algebra structure used to build the lattice vertex algebra. This was not obvious in advance, it is a consequence of the weird formula (5.35) which determined the isomorphism.

Our construction provides a geometric motivation for the existence of these underlying (graded) algebra structures on (graded) vertex algebras.

(ii) Theorem 5.4 gives a graded vertex algebra $\check H_*(\mathcal{M})$ from $\mathcal{T} = D^b \operatorname{mod-}\mathbb{C}Q$ over any commutative ring R, not just for R a field of characteristic zero, though the explicit computations in §5.3.2–§5.3.5 need R to be a $\mathbb{Q}$-algebra so we can use Chern characters. As in Remark 2.17(iii) one can define lattice vertex algebras over a general commutative ring R, and the author expects the first part of Theorem 5.19 also holds in this case.

5.3.7 Relating $\tilde H_0(\bar{\mathcal{M}})^{t=0}$ and $\tilde H_0(\mathcal{M})^{t=0}$ to Kac–Moody algebras

When Q has no vertex loops, we now relate the '$t=0$' Lie algebras $\tilde H_0(\bar{\mathcal{M}})^{t=0}$ and $\tilde H_0(\mathcal{M})^{t=0}$ to Kac–Moody algebras. Much of the next theorem follows from Theorem 5.19 and known facts about the relation between lattice vertex algebras and Kac–Moody algebras, but we provide an independent proof. The theorem should be compared to Theorem 5.4.

Theorem 5.21. *Suppose Q is a quiver without vertex loops. Then:*

(a) *Define $\delta_v \in \mathbb{Z}^{Q_0}$ by $\delta_v(w) = 1$ if $v = w$ and $\delta_v(w) = 0$ otherwise. Then for all $v \in Q_0$ we have*

$$\begin{aligned} \tilde H_0(\bar{\mathcal{M}}_{\delta_v})^{t=0} &= \tilde H_0(\mathcal{M}_{\delta_v})^{t=0} = H_0(\bar{\mathcal{M}}_{\delta_v}) = \langle 1_{\delta_v}\rangle_R, \\ \tilde H_0(\bar{\mathcal{M}}_{-\delta_v})^{t=0} &= H_0(\bar{\mathcal{M}}_{-\delta_v}) = \langle 1_{-\delta_v}\rangle_R, \\ \tilde H_0(\bar{\mathcal{M}}_0)^{t=0} &= H_2(\bar{\mathcal{M}}_0) = \langle b_{0,v,1} : v \in Q_0\rangle_R. \end{aligned} \tag{5.62}$$

Define $E_v = 1_{\delta_v}$ in $\tilde H_0(\mathcal{M}_{\delta_v})^{t=0} = \tilde H_0(\bar{\mathcal{M}}_{\delta_v})^{t=0}$, $F_v = -1_{-\delta_v}$ in $\tilde H_0(\bar{\mathcal{M}}_{-\delta_v})^{t=0}$ and $H_v = b_{0,v,1}$ in $\tilde H_0(\bar{\mathcal{M}}_0)^{t=0}$ for $v \in Q_0$. Then as in (2.7) we have

$$\begin{gathered} [H_v, H_w]^{t=0} = 0, \quad [E_v, F_w]^{t=0} = 0 \quad \text{if } v \neq w, \quad [E_v, F_v]^{t=0} = H_v, \\ [H_v, E_w]^{t=0} = a_{vw} E_w, \quad [H_v, F_w]^{t=0} = -a_{vw} F_w, \\ (\operatorname{ad} E_v)^{1-a_{vw}}(E_w) = 0 \quad \text{if } v \neq w, \quad (\operatorname{ad} F_v)^{1-a_{vw}}(F_w) = 0 \quad \text{if } v \neq w. \end{gathered} \tag{5.63}$$

(b) *Write $\mathfrak{g}'(A) = \mathfrak{g} = \mathfrak{h} \oplus \mathfrak{n}_+ \oplus \mathfrak{n}_-$ for the derived Kac–Moody algebra associated to Q, as in §2.1.2. Then there are unique, injective Lie algebra morphisms $\bar\Upsilon : \mathfrak{g} \to \tilde H_0(\bar{\mathcal{M}})^{t=0}$ and $\Upsilon : \mathfrak{n}_+ \to \tilde H_0(\mathcal{M})^{t=0}$ with $\bar\Upsilon(e_v) = \Upsilon(e_v) = E_v$, $\bar\Upsilon(f_v) = F_v$ and $\bar\Upsilon(h_v) = H_v$ for all $v \in Q_0$. These $\bar\Upsilon, \Upsilon$ are compatible with the gradings $\mathfrak{g} = \bigoplus_{\boldsymbol{d} \in \mathbb{Z}^{Q_0}} \mathfrak{g}_{\boldsymbol{d}}$ and $\tilde H_0(\bar{\mathcal{M}})^{t=0} = \bigoplus_{\boldsymbol{d} \in \mathbb{Z}^{Q_0}} \tilde H_0(\bar{\mathcal{M}}_{\boldsymbol{d}})^{t=0}$. Also*

$$\bar\Upsilon \circ \omega = \tilde H_0(\bar\Sigma)^{t=0} \circ \bar\Upsilon : \mathfrak{g} \longrightarrow \tilde H_0(\bar{\mathcal{M}})^{t=0}, \tag{5.64}$$

where $\omega:\mathfrak{g}\to\mathfrak{g}$ is the involution in Theorem 2.4(d), *and $\bar\Sigma:\bar{\mathcal M}\to\bar{\mathcal M}$ is induced by translation* $[1]:D^b\,\text{mod-}\mathbb{C}Q\to D^b\,\text{mod-}\mathbb{C}Q$ *as in Definition* 5.12.

(c) *Suppose that either Q is of finite type (i.e. its underlying graph is a disjoint union of Dynkin diagrams of type A, D or E), or Q is connected and of affine type (i.e. its underlying graph is an affine Dynkin diagram). Then $\bar\Upsilon$ and Υ in part* **(b)** *are isomorphisms.*

(d) *Let Q be nonempty. Then Proposition* 3.6 *implies that the inclusion $\bar{\mathcal M}'=\bar{\mathcal M}\setminus\{0\}\hookrightarrow\bar{\mathcal M}$ induces an isomorphism $\tilde H_*(\bar{\mathcal M}')^{t=0}\to\tilde H_*(\bar{\mathcal M})^{t=0}$, so* **(a)**–**(c)** *hold with $\tilde H_0(\bar{\mathcal M}')^{t=0}$ in place of $\tilde H_0(\bar{\mathcal M})^{t=0}$. Also Corollary* 3.27 *says that the Lie algebra morphism $\Pi^{\rm pl}_{t=0}:\tilde H_0(\bar{\mathcal M}')^{t=0}\to\tilde H_0(\bar{\mathcal M}^{\rm pl})$ from* (3.50) *and Theorem* 3.29(b) *is an isomorphism. Thus* **(a)**–**(c)** *also hold with $(\tilde H_0(\bar{\mathcal M}^{\rm pl}),[\,,\,]^{\rm pl})$ in place of $(\tilde H_0(\bar{\mathcal M})^{t=0},[\,,\,]^{t=0})$.*

Proof. For (a), as Q has no vertex loops we have $\chi(\delta_v,\delta_v)=a_{vv}=2$, so

$$\tilde H_0(\bar{\mathcal M}_{\delta_v})^{t=0}=H_0(\bar{\mathcal M}_{\delta_v})^{t=0}=H_0(\bar{\mathcal M}_{\delta_v})=\langle 1_{\delta_v}\rangle_R$$

by (3.36) and (3.43), and the same holds for $\tilde H_0(\bar{\mathcal M}_{-\delta_v})^{t=0}$. Also

$$\tilde H_0(\bar{\mathcal M}_0)^{t=0}=H_2(\bar{\mathcal M}_0)^{t=0}=H_2(\bar{\mathcal M}_0)/t\diamond H_0(\bar{\mathcal M}_0)=H_2(\bar{\mathcal M}_0)=\langle b_{0,v,1}:v\in Q_0\rangle_R$$

using (3.36), (3.43), (5.34), and $t\diamond 1_0=0$ by (5.48). This proves (5.62).

The first, third, fourth and fifth equations of (5.63) follow from the coefficients of z^{-1} in Propositions 5.17 and 5.18. The second, sixth and seventh equations of (5.63) hold essentially trivially by the same argument, as each takes values in $H_n(\bar{\mathcal M}_{\boldsymbol d})=0$ for $n<0$. For the second, if $v\neq w\in Q_0$ then

$$[E_v,F_w]^{t=0}\in\tilde H_0(\bar{\mathcal M}_{\delta_v-\delta_w})^{t=0}=H_{-2+2a_{vw}}(\bar{\mathcal M}_{\delta_v-\delta_w})^{t=0}=0,$$

using (3.43) and $a_{vv}=a_{ww}=2$, $a_{vw}\leqslant 0$. For the sixth, if $v\neq w\in Q_0$ then

$$(\mathrm{ad}\,E_v)^{1-a_{vw}}(E_w)\in\tilde H_0(\bar{\mathcal M}_{(1-a_{vw})\delta_v+\delta_w})^{t=0}=H_{-2}(\bar{\mathcal M}_{(1-a_{vw})\delta_v+\delta_w})^{t=0}=0,$$

using (3.43) and $2-\chi\bigl((1-a_{vw})\delta_v+\delta_w,(1-a_{vw})\delta_v+\delta_w\bigr)=-2$ as $a_{vv}=a_{ww}=2$. The seventh equation of (5.63) is the same. This proves part (a).

For (b), comparing (2.7) and (5.63) we see that by definition of $\mathfrak{g}'(A)=\mathfrak{g}$ in Definition 2.3, there is a unique Lie algebra morphism $\bar\Upsilon:\mathfrak{g}\to\tilde H_0(\bar{\mathcal M})^{t=0}$ with $\bar\Upsilon(e_v)=E_v$, $\bar\Upsilon(f_v)=F_v$ and $\bar\Upsilon(h_v)=H_v$ for all $v\in Q_0$. As $E_v\in H_0(\mathcal M_{\delta_v})=H_0(\bar{\mathcal M}_{\delta_v})$, this restricts to $\Upsilon:\mathfrak{n}_+\to\tilde H_0(\mathcal M)^{t=0}$. These $\bar\Upsilon,\Upsilon$ are compatible with the gradings $\mathfrak{g}=\bigoplus_{\boldsymbol d\in\mathbb{Z}^{Q_0}}\mathfrak{g}_{\boldsymbol d}$ and $\tilde H_0(\bar{\mathcal M})^{t=0}=\bigoplus_{\boldsymbol d\in\mathbb{Z}^{Q_0}}\tilde H_0(\bar{\mathcal M}_{\boldsymbol d})^{t=0}$, as they are compatible on generators e_v,f_v,h_v of $\mathfrak{g}$. Thus, $\mathrm{Ker}\,\bar\Upsilon$ is a graded ideal in $\mathfrak{g}$. But $\mathfrak{g}_0=\mathfrak{h}$ has basis $h_v,v\in Q_0$, and $\tilde H_0(\bar{\mathcal M}_0)^{t=0}$ has basis $H_v,v\in Q_0$ by (5.62), so $\Upsilon|_{\mathfrak h}:\mathfrak{h}\to\tilde H_0(\bar{\mathcal M}_0)^{t=0}$ is an isomorphism, and $\mathrm{Ker}\,\bar\Upsilon\cap\mathfrak{h}=0$. Hence $\mathrm{Ker}\,\bar\Upsilon=0$ by Theorem 2.4(e), and so $\bar\Upsilon,\Upsilon$ are injective.

Theorem 2.4(d) gives $\omega(e_v)=-f_v$, $\omega(f_v)=-e_v$, $\omega(h_v)=-h_v$ for $v\in Q_0$. Also $\tilde H_0(\bar\Sigma)^{t=0}(E_v)=-F_v$, $\tilde H_0(\bar\Sigma)^{t=0}(F_v)=-E_v$, $\tilde H_0(\bar\Sigma)^{t=0}(H_v)=-H_v$ for $v\in Q_0$ by (5.42) and $E_v=1_{\delta_v}$, $F_v=-1_{-\delta_v}$, $H_v=b_{0,v,1}$. As $\bar\Upsilon(e_v)=E_v$,

$\bar\Upsilon(f_v) = F_v$, $\bar\Upsilon(h_v) = H_v$, equation (5.64) holds on generators e_v, f_v, h_v of $\mathfrak{g}$, and so holds on all of $\mathfrak{g}$. This proves part (b).

For (c), as $\bar\Upsilon$ is injective and graded, we see that $\bar\Upsilon$ is an isomorphism if and only if $\dim \mathfrak{g}_{\boldsymbol{d}} = \dim \tilde H_0(\bar{\mathcal{M}}_{\boldsymbol{d}})^{t=0}$ for all $\boldsymbol{d} \in \mathbb{Z}^{Q_0}$, where $\dim \tilde H_0(\bar{\mathcal{M}}_{\boldsymbol{d}})^{t=0}$ is given in (5.49). If Q is of finite type and $\boldsymbol{d} \in \mathbb{Z}^{Q_0}$, then by the theory of semisimple Lie algebras, either:

(i) (Cartan subalgebra.) $\boldsymbol{d} = 0$ and $\dim \mathfrak{g}_0 = \dim \mathfrak{h} = |Q_0|$;

(ii) (Roots.) $\boldsymbol{d} \neq 0$, $\chi(\boldsymbol{d}, \boldsymbol{d}) = 2$ and $\dim \mathfrak{g}_{\boldsymbol{d}} = 1$; or

(iii) $\boldsymbol{d} \neq 0$, $\chi(\boldsymbol{d}, \boldsymbol{d}) > 2$ and $\dim \mathfrak{g}_{\boldsymbol{d}} = 0$.

By (5.49), the dimensions of $\tilde H_0(\bar{\mathcal{M}}_{\boldsymbol{d}})^{t=0}$ are the same, so $\bar\Upsilon$ is an isomorphism.

Similarly, if Q is connected and of affine type and $\boldsymbol{d} \in \mathbb{Z}^{Q_0}$, then by the theory of affine Lie algebras [84, §6], either:

(i) (Cartan subalgebra.) $\boldsymbol{d} = 0$ and $\dim \mathfrak{g}_0 = \dim \mathfrak{h} = |Q_0|$;

(ii) (Real roots.) $\boldsymbol{d} \neq 0$, $\chi(\boldsymbol{d}, \boldsymbol{d}) = 2$ and $\dim \mathfrak{g}_{\boldsymbol{d}} = 1$;

(iii) (Imaginary roots.) $\boldsymbol{d} \neq 0$, $\chi(\boldsymbol{d}, \boldsymbol{d}) = 0$ and $\dim \mathfrak{g}_{\boldsymbol{d}} = |Q_0| - 1$; or

(iv) $\boldsymbol{d} \neq 0$, $\chi(\boldsymbol{d}, \boldsymbol{d}) > 2$ and $\dim \mathfrak{g}_{\boldsymbol{d}} = 0$.

Again, by (5.49), $\bar\Upsilon$ is an isomorphism. The analogues for $\Upsilon : \mathfrak{n}_+ \to \tilde H_0(\mathcal{M})^{t=0}$ also hold, using (5.50). Part (d) is immediate. This completes the proof. □

Theorem 5.4 provides some evidence for the following conjecture:

Conjecture 5.22. *Suppose we can make the 'supported on indecomposables' version proposed in* §3.9.1 *work in the situation of Theorem* 5.21. *Then the Lie subalgebra of* $\tilde H_0(\mathcal{M})$ *'supported on indecomposables' is* $\Upsilon(\mathfrak{n}_+)$.

5.4 Examples with $\mathcal{A} = \text{mod-}\mathbb{C}Q$ and $\mathcal{T} = D^b \text{mod-}\mathbb{C}Q$

We discuss some examples of the situation considered in §5.3. Throughout we take $\mathbb{K} = \mathbb{C}$, and R to be a field of characteristic zero, and $H_*(-), H^*(-)$ to be the (co)homology theories of (higher) Artin $\mathbb{C}$-stacks over R from Example 2.35.

Example 5.23. Take Q to be the quiver $\overset{u}{\bullet}$ with one vertex u and no edges. Then $\text{mod-}\mathbb{C}Q = \mathrm{Vect}_{\mathbb{C}}$, the abelian category of finite-dimensional vector spaces, and $D^b \text{mod-}\mathbb{C}Q = D^b \mathrm{Vect}_{\mathbb{C}}$. As in §5.2 the moduli stacks $\mathcal{M}$ of objects in $\mathcal{A} = \mathrm{Vect}_{\mathbb{C}}$ and $\bar{\mathcal{M}}$ of objects in $\mathcal{T} = D^b \mathrm{Vect}_{\mathbb{C}}$ are

$$\mathcal{M} \cong \textstyle\coprod_{r\in\mathbb{N}} [*/\mathrm{GL}(r,\mathbb{C})], \qquad \bar{\mathcal{M}} \cong \mathbf{Perf}_{\mathbb{C}} = \textstyle\coprod_{r\in\mathbb{Z}} \mathbf{Perf}_{\mathbb{C}}^r.$$

As in §5.2.2 and §5.3.2 we have

$$\begin{aligned} H_*(\bar{\mathcal{M}}) &\cong \textstyle\bigoplus_{r\in\mathbb{Z}} H_*(\bar{\mathcal{M}}_r) = \bigoplus_{r\in\mathbb{Z}} R[b_{r,i} : i = 1, 2, \ldots], \\ H_*(\mathcal{M}) &\cong \textstyle\bigoplus_{r\in\mathbb{N}} H_*(\mathcal{M}_r) \cong \bigoplus_{r\in\mathbb{N}} I_r^\circ \subset H_*(\bar{\mathcal{M}}), \end{aligned} \tag{5.65}$$

where $I_r^\circ \subset R[b_{r,i},\ i \geqslant 1]$ is the annihilator of $I_r \subset H^*(\bar{\mathcal{M}}_r) \cong R[\beta_{r,i},\ i \geqslant 1]$. Here we omit u from notation $b_{r,u,i}, \beta_{r,u,i}$ in §5.3.2 as there is only one vertex u.

Thus, $H_*(\bar{\mathcal{M}})$ has basis the monomials $\mu_{r,\boldsymbol{n}} = \prod_{i\geqslant 1} b_{r,i}^{n_i}$ for $r \in \mathbb{Z}$ and $n_i \in \mathbb{N}$ with only finitely many n_i nonzero, where by (3.23) and (3.31) we have

$$\mu_{r,\boldsymbol{n}} \in H_{2\sum_i in_i}(\bar{\mathcal{M}}),\ \ \mu_{r,\boldsymbol{n}} \in \tilde{H}_{2r^2-2+2\sum_i in_i}(\bar{\mathcal{M}}),\ \ \mu_{r,\boldsymbol{n}} \in \check{H}_{r^2+\sum_i in_i}(\bar{\mathcal{M}}),$$

as the Euler form is $\chi(r,s) = 2rs$ for $r,s \in \mathbb{Z}$, since $a_{uu} = 2$. We have

$$\tilde{H}_0(\bar{\mathcal{M}}) = \langle b_{0,1}, 1_1, 1_{-1}\rangle_R \cong \tilde{H}_0(\bar{\mathcal{M}})^{t=0} \cong \tilde{H}_0(\bar{\mathcal{M}}^{\mathrm{pl}}),$$

and the Lie brackets $[\,,\,]^{t=0} \cong [\,,\,]^{\mathrm{pl}}$ on $\tilde{H}_0(\bar{\mathcal{M}})^{t=0} \cong \tilde{H}_0(\bar{\mathcal{M}}^{\mathrm{pl}})$ are given by

$$[b_{0,1}, 1_1]^{t=0} = 2\cdot 1_1, \quad [b_{0,1}, 1_{-1}]^{t=0} = -2\cdot 1_1, \quad [1_1, 1_{-1}]^{t=0} = b_{0,1}.$$

Thus the Lie algebras $\tilde{H}_0(\bar{\mathcal{M}})^{t=0}, \tilde{H}_0(\bar{\mathcal{M}}^{\mathrm{pl}})$ are isomorphic to $\mathfrak{sl}(2,R)$, with Cartan subalgebra $\mathfrak{h} = \langle b_{0,1}\rangle_R$ and root spaces $\mathfrak{g}_1 = \langle 1_1\rangle_R$ and $\mathfrak{g}_{-1} = \langle 1_{-1}\rangle_R$.

In §3.4 we discussed the maps $\Pi_{t=0}^{\mathrm{pl}} : H_k(\bar{\mathcal{M}}_r')^{t=0} \to H_k(\bar{\mathcal{M}}_r^{\mathrm{pl}})$ for $r \in \mathbb{Z} = K(\mathcal{T})$. Proposition 3.24(a),(b) proves these are isomorphisms for $r \neq 0$, and Proposition 3.26 shows $\Pi_{t=0}^{\mathrm{pl}} : H_k(\bar{\mathcal{M}}_0')^{t=0} \to H_k(\bar{\mathcal{M}}_0^{\mathrm{pl}})$ are isomorphisms for $k = 0,1,2$. The proof in §4.5 using the homology Leray–Serre spectral sequence involves a map $d_2 : E_{3,0}^2 = H_3(\bar{\mathcal{M}}_0^{\mathrm{pl}}) \to E_{0,2}^2 = H_0(\bar{\mathcal{M}}_0^{\mathrm{pl}})$. Since $t\diamond - : H_0(\bar{\mathcal{M}}_0') \to H_2(\bar{\mathcal{M}}_0')$ is zero by (5.48), equation (4.27) implies that $\operatorname{Im} d_2 = H_0(\mathcal{M}_0') \cong R$, and by further computation as in §4.5 we find that $H_3(\bar{\mathcal{M}}_0^{\mathrm{pl}}) \cong R$. But $H_3(\bar{\mathcal{M}}_0') = H_3(\bar{\mathcal{M}}_0) = 0$ by (3.13) and (5.65), so $H_3(\bar{\mathcal{M}}_0')^{t=0} = 0$. Hence $\Pi_{t=0}^{\mathrm{pl}} : H_3(\bar{\mathcal{M}}_0')^{t=0} \to H_3(\bar{\mathcal{M}}_0^{\mathrm{pl}})$ is *not* an isomorphism.

Example 5.24. Let Q be the quiver $\overset{v}{\bullet} \rightrightarrows \overset{w}{\bullet}$ with two vertices v, w and two edges from v to w. The underlying graph is the affine Dynkin diagram $\tilde{A}_1$. Thus Theorem 5.21(c),(d) say that the Lie algebras $\tilde{H}_0(\bar{\mathcal{M}})^{t=0}, \tilde{H}_0(\bar{\mathcal{M}}^{\mathrm{pl}})$ are isomorphic to the corresponding Kac–Moody algebra, which is the affine Lie algebra $\widehat{\mathfrak{sl}}(2,R)$, as in Kac [84, §6]. Explicitly, $\widehat{\mathfrak{sl}}(2,R) = \mathfrak{sl}(2,R)[t,t^{-1}] \oplus \langle c\rangle_R$, where $\mathfrak{sl}(2,R)[t,t^{-1}]$ is the loop algebra of $\mathfrak{sl}(2,R)$, and $\langle c\rangle_R$ is a central extension. Identify $\mathbb{Z}^{Q_0} \cong \mathbb{Z}^2$ by $\boldsymbol{d} \cong (\boldsymbol{d}(v), \boldsymbol{d}(w))$. Then for $\boldsymbol{d} \in \mathbb{Z}^{Q_0}$ we have

(i) $\chi(\boldsymbol{d},\boldsymbol{d}) = 0$ if and only if $\boldsymbol{d} = (n,n)$ for $n \in \mathbb{Z}$.

(ii) $\chi(\boldsymbol{d},\boldsymbol{d}) = 2$ if and only if $\boldsymbol{d} = (n,n+1)$ or $\boldsymbol{d} = (n,n-1)$ for $n \in \mathbb{Z}$.

(iii) $\chi(\boldsymbol{d},\boldsymbol{d}) \geqslant 4$ except in cases (i),(ii).

Hence from (5.34) we see that

$$\begin{aligned}\tilde{H}_{-2}(\bar{\mathcal{M}}) &= \textstyle\bigoplus_{n\in\mathbb{Z}} H_0(\bar{\mathcal{M}}_{(n,n)}) = \bigoplus_{n\in\mathbb{Z}} \langle 1_{(n,n)}\rangle_R,\\ \tilde{H}_0(\bar{\mathcal{M}}) &= \textstyle\bigoplus_{n\in\mathbb{Z}} \bigl(H_2(\bar{\mathcal{M}}_{(n,n)}) \oplus H_0(\bar{\mathcal{M}}_{(n,n+1)}) \oplus H_0(\bar{\mathcal{M}}_{(n,n-1)})\bigr)\\ &= \textstyle\bigoplus_{n\in\mathbb{Z}} \langle b_{(n,n),v,1}, b_{(n,n),w,1}, 1_{(n,n+1)}, 1_{(n,n-1)}\rangle_R.\end{aligned}$$

By (5.48), $t \diamond - : \tilde{H}_{-2}(\bar{\mathcal{M}}) \to \tilde{H}_0(\bar{\mathcal{M}})$ maps $1_{(n,n)} \mapsto n(b_{(n,n),v,1} + b_{(n,n),w,1})$. Therefore

$$\tilde{H}_0(\bar{\mathcal{M}})^{t=0} \cong \langle b_{(0,0),v,1} + b_{(0,0),w,1}\rangle_R \oplus \bigoplus_{n\in\mathbb{Z}} \langle b_{(n,n),v,1} - b_{(n,n),w,1}, 1_{(n,n+1)}, 1_{(n,n-1)}\rangle_R,$$

as we must quotient $\tilde{H}_0(\bar{\mathcal{M}})$ by the vectors $b_{(n,n),v,1} + b_{(n,n),w,1}$ for $n \neq 0$. The isomorphism $\mathfrak{sl}(2,R)[t,t^{-1}] \oplus \langle c\rangle_R \cong \tilde{H}_0(\bar{\mathcal{M}})^{t=0}$ identifies $\mathfrak{sl}(2,R)\otimes t^n$ with $\langle b_{(n,n),v,1} - b_{(n,n),w,1}, 1_{(n,n+1)}, 1_{(n,n-1)}\rangle_R$ and $\langle c\rangle_R$ with $\langle b_{(0,0),v,1} + b_{(0,0),w,1}\rangle_R$.

Example 5.25. Let Q be the quiver $\overset{u}{\bullet} \quad \overset{v}{\bullet} \rightrightarrows \overset{w}{\bullet}$ with three vertices u, v, w and two edges from v to w, the disjoint union of quivers in Examples 5.23 and 5.24. The corresponding Kac–Moody algebra is $\mathfrak{g} = \mathfrak{sl}(2,R) \oplus \widehat{\mathfrak{sl}}(2,R)$. As Q is not connected, Theorem 5.21(c) does not apply, so $\bar{\Upsilon} : \mathfrak{g} \to \tilde{H}_0(\bar{\mathcal{M}})^{t=0}$ and $\Upsilon : \mathfrak{n}_+ \to \tilde{H}_0(\mathcal{M})^{t=0}$ are injective, but need not be (and are not) surjective.

Identify $\mathbb{Z}^{Q_0} \cong \mathbb{Z}^3$ by $\boldsymbol{d} \cong (\boldsymbol{d}(u), \boldsymbol{d}(v), \boldsymbol{d}(w))$. Then $\boldsymbol{d} = (1,1,1)$ and $\boldsymbol{d} = (1,-1,-1)$ both have $\chi(\boldsymbol{d},\boldsymbol{d}) = 2$, so $H_0(\bar{\mathcal{M}}_{(1,1,1)}) = \langle 1_{(1,1,1)}\rangle_R$ and $H_0(\bar{\mathcal{M}}_{(1,-1,-1)}) = \langle 1_{(1,-1,-1)}\rangle_R$ are both subspaces of $\tilde{H}_0(\bar{\mathcal{M}})^{t=0}$. However, $(1,1,1)$ and $(1,-1,-1)$ are not roots of $\mathfrak{g}$, so $\langle 1_{(1,1,1)}\rangle_R$ and $\langle 1_{(1,-1,-1)}\rangle_R$ are not in the image of $\bar{\Upsilon}$. Hence $\bar{\Upsilon}$ is not surjective, and also Υ is not surjective as $\langle 1_{(1,1,1)}\rangle_R$ lies in $\tilde{H}_0(\mathcal{M})^{t=0} \setminus \Upsilon(\mathfrak{n}_+)$. Note too that Kac–Moody algebras have $\mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha\in\Delta_+\cup\Delta_-} \mathfrak{g}_\alpha$, where $\Delta_\pm \subset (\pm\mathbb{N})^{Q_0}$. But $(1,-1,-1)$ is a root of $\tilde{H}_0(\bar{\mathcal{M}})^{t=0}$ which lies in neither $\mathbb{N}^{Q_0}$ nor $(-\mathbb{N})^{Q_0}$.

To be continued.

A Binomial coefficients

Here we collect some definitions and facts about binomial coefficients $\binom{m}{n}$ for $m, n \in \mathbb{Z}$, since some readers may be unfamiliar with these if $m < 0$ or $n < 0$.

Definition A.1. Let $m, n \in \mathbb{Z}$. Define the *binomial coefficient* $\binom{m}{n}$ by

$$\binom{m}{n} = \begin{cases} 0, & n < 0, \\ 1, & n = 0, \\ \frac{m(m-1)(m-2)\cdots(m-n+1)}{n!}, & n > 0. \end{cases} \tag{A.1}$$

Binomial coefficients have the following well known properties:

$$\binom{m}{n} = 0 \qquad \text{if and only if } n < 0 \text{ or } 0 \leqslant m < n, \tag{A.2}$$

$$\binom{m}{n} = \frac{m!}{n!(m-n)!} \qquad \text{if } 0 \leqslant n \leqslant m, \tag{A.3}$$

$$\binom{m}{n} = (-1)^n \binom{n-m-1}{n} \qquad \text{for all } m, n \in \mathbb{Z}, \tag{A.4}$$

$$\binom{m}{n} = \binom{m}{m-n} \qquad \text{if } m \geqslant 0, \tag{A.5}$$

$$\binom{m+1}{n} = \binom{m}{n} + \binom{m}{n-1} \qquad \text{for all } m, n \in \mathbb{Z}. \tag{A.6}$$

The Binomial Theorem says that

$$(1+x)^m = \sum_{n \geqslant 0} \binom{m}{n} x^n, \tag{A.7}$$

which holds in polynomials in x if $m \geqslant 0$, and in power series convergent when $|x| < 1$ if $m < 0$.

We now prove some identities on binomial coefficients which will be used in the main text. First we show that for all $m \geqslant 0$ and $n, p \in \mathbb{Z}$ we have

$$\sum_{k \geqslant 0:\ k \leqslant p} (-1)^k \binom{k+m+n-p}{k} \binom{m}{p-k} = (-1)^p \binom{n}{p}. \tag{A.8}$$

We do this by induction on $m = 0, 1, \ldots$. Write $S_{m,n,p}$ for the left hand side of (A.8). For the first step $m = 0$, the only potentially nonzero term in the sum is $k = p$, if $p \geqslant 0$, giving $S_{0,n,p} = (-1)^p \binom{n}{p}$, as we want, where if $p < 0$ this still holds as both sides are zero. For the inductive step, suppose (A.8) holds for all $m \geqslant 0$ and $n, p \in \mathbb{Z}$ with $m \leqslant m'$. In the sum for $S_{m'+1,n,p}$, rewriting $\binom{m'+1}{p-k} = \binom{m'}{p-k} + \binom{m'}{p-k-1}$ by (A.6) and rearranging the sum gives

$$S_{m'+1,n,p} = S_{m',n+1,p} + S_{m',n,p-1} = (-1)^p \binom{n+1}{p} + (-1)^{p-1} \binom{n}{p-1} = (-1)^p \binom{n}{p},$$

using the inductive hypothesis in the second step and (A.6) in the third. Hence by induction (A.8) holds for all $m \geqslant 0$ and $n, p \in \mathbb{Z}$.

Next we will show that if $l, m, n \in \mathbb{Z}$ with $l + m + n = -1$ then

$$(-1)^l \binom{-l-1}{n} + (-1)^m \binom{-m-1}{l} + (-1)^n \binom{-n-1}{m} = 0. \tag{A.9}$$

As $l + m + n = -1$, one or two of l, m, n must be negative, and the others nonnegative. This gives six cases, which after cyclic permutations of l, m, n are equivalent to one of: (A) $l < 0$ and $m, n \geqslant 0$; and (B) $l, m < 0$ and $n \geqslant 0$.

In case (A) note that $\binom{-m-1}{l} = 0$ by (A.2) and

$$\begin{aligned}(-1)^l \binom{-l-1}{n} &= (-1)^l \binom{-l-1}{-l-1-n} = \binom{-l-1}{m} \\ &= (-1)^{l+m} \binom{m+l}{m} = (-1)^{1+n} \binom{-n-1}{m},\end{aligned}$$

using (A.5) in the first step, $l+m+n = -1$ in the second and fourth, and (A.4) in the third. Equation (A.9) follows in case (A). In case (B), all three terms in (A.9) are zero by (A.2). This proves (A.9).

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