THE BERNSTEIN CENTRE IN NATURAL CHARACTERISTIC

KONSTANTIN ARDAKOV, PETER SCHNEIDER

ABSTRACT. Let G be a locally profinite group and let k be a field of positive characteristic p. Let $Z(G)$ denote the centre of G and let $\mathfrak{Z}(G)$ denote the Bernstein centre of G, that is, the k-algebra of natural endomorphisms of the identity functor on the category of smooth k-linear representations of G. We show that if G contains an open pro-p subgroup but no proper open centralisers, then there is a natural isomorphism of k-algebras $\mathfrak{Z}(Z(G)) \xrightarrow{\cong} \mathfrak{Z}(G)$. We also describe $\mathfrak{Z}(Z(G))$ explicitly as a particular completion of the abstract group ring $k[Z(G)]$. Both conditions on G are satisfied whenever G is the group of points of any connected smooth algebraic group defined over a local field of residue characteristic p. In particular, when the algebraic group is semisimple, we show that $\mathfrak{Z}(G) = k[Z(G)]$.

1. Introduction

Let G be a locally profinite group and let k be any field. Recall that a k-linear representation V of G is said to be *smooth* if every vector in V is fixed by an open subgroup of G. The smooth k-linear representations of G form an abelian category $\mathrm{Mod}(G)$ and the *Bernstein centre* $\mathfrak{Z}(G)$ of $\mathrm{Mod}(G)$ is by definition the ring of natural endomorphisms of the identity functor on $\mathrm{Mod}(G)$: it is naturally a commutative k-algebra which acts on every V in $\mathrm{Mod}(G)$ by k-linear endomorphisms commuting with the action of G on V.

When G is a reductive group over a local non-archimedean field of mixed characteristic $(0, p)$ and k is the complex numbers, the Bernstein centre $\mathfrak{Z}(G)$ was studied in detail by Bernstein in [Ber]. He found that $\mathfrak{Z}(G)$ decomposes as a direct product of smaller algebras that are now called the *Bernstein components*, and he found a parametrisation of these components in terms of G-conjugacy classes of cuspidal pairs on the Levi subgroups of G. Consequently, in this classical case, the Bernstein centre is rather large, and it plays a fundamental role in the classical local Langlands correspondence [Hen].

Recently, there has been a lot of interest in a possible mod-p local Langlands correspondence, where instead one works with smooth representations defined over a field k of characteristic p; see, for example, [Vig04], [BP], [Sch] and [BH+]. It is therefore natural to enquire about the structure of $\mathfrak{Z}(G)$. There has been an expectation that in this case $\mathfrak{Z}(G)$ is small, which we will confirm in this paper by a precise explicit calculation.

We begin the description of our results by introducing the following completion of the group algebra $k[Z(G)]$ of the centre $Z(G)$ of G:

$$k\widehat{[Z(G)]} := \varprojlim k[Z(G)/Z']$$

where Z' runs over all compact open subgroups of $Z(G)$. By considering the natural action of $k[Z(G)]$ on the subspace V^U of U-fixed points of a given object V in $\mathrm{Mod}(G)$ as U ranges over all compact open subgroups of G, it is not difficult to see that the algebra $k\widehat{[Z(G)]}$ acts by $k[G]$-linear endomorphisms on every V in $\mathrm{Mod}(G)$ in a natural way. This defines a k-algebra

Date: February 28, 2023.

homomorphism $\Phi_G : k\widehat{[Z(G)]} \to \mathfrak{Z}(G)$ which is easily seen to be injective. Using this map we can now state our main result.

Theorem 1.1. *Let G be a locally profinite group that contains an open pro-p subgroup and let k be a field of characteristic p. Suppose that G contains no proper open centralisers. Then the natural map $\Phi_G : k\widehat{[Z(G)]} \xrightarrow{\cong} \mathfrak{Z}(G)$ is an isomorphism.*

We show in Thm. 6.13 below that the assumptions on G are satisfied whenever $G = \mathbf{G}(\mathfrak{F})$ for some connected smooth algebraic group $\mathbf{G}$ defined over a local nonarchimedean field $\mathfrak{F}$ of residue characteristic p. By applying Theorem 1.1 to G and to $Z(G)$ in turn, we obtain the following

Corollary 1.2. *Under the assumptions of Theorem 1.1, the natural map*

$$\Phi_G \circ \Phi_Z^{-1} : \mathfrak{Z}(Z(G)) \xrightarrow{\cong} \mathfrak{Z}(G)$$

is an isomorphism of k-algebras.

Here is a brief summary of the contents of this paper. In §3, we use a result of Positselski [Pos] to show that $\mathfrak{Z}(G)$ is naturally isomorphic to the k-algebra of bi-equivariant k-linear endomorphisms $\mathrm{End}_{\mathrm{Mod}(G\times G)}(C_c^\infty(G,k))$ of the space $C_c^\infty(G,k)$ of compactly supported locally constant k-valued functions on G. In fact, for the purposes of Theorem 1.1, we only need the injectivity of the natural map from $\mathfrak{Z}(G)$ to this endomorphism ring. In §4, we use the fact that $C_c^\infty(G,k)$ is isomorphic in $\mathrm{Mod}(G)$ to the compact induction from U to G of the corresponding space of locally constant k-valued functions on some fixed open pro-p subgroup U of G, together with a version of the Mackey decomposition, to reduce the calculation of the bi-equivariant endomorphism algebra to the calculation of the fixed points in some profinite permutation modules $k[[U]]$ under a twisted conjugation action of various open subgroups U_w of U. In §5 we crucially use the hypothesis that k has characteristic p to show that in this situation, all fixed points in such profinite permutation modules arise from the *finite* orbits. In §6 we use the hypothesis on open centralisers in G to show that there are no such finite U_w-orbits in U — unless w happens to be central in G — and deduce Theorem 1.1 which reappears as Theorem 6.10. We also explain how to verify the hypothesis on open centralisers in the case of connected algebraic groups. Finally, in §7 we give a self-contained proof of Positselski's Theorem, for the convenience of the reader unfamiliar with the categorical machinery in [Pos].

We thank F. Pop and M.-F. Vignéras for pointing out that Prop. 6.12 is known to the experts and for indicating an argument. After this paper was finished, A. Dotto has informed us that he had proved in [Dot] (using different methods) that for certain p-adic reductive groups, the Bernstein centre is a local ring when one fixes a central character.

The second author acknowledges support from Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under Germany's Excellence Strategy EXC 2044 –390685587, Mathematics Münster: Dynamics–Geometry–Structure.

2. Notation

Throughout G is a locally profinite group, and k is any field. If G contains an open pro-p subgroup then we call G locally pro-p. By $\mathrm{Op}_c(G)$ we denote the set of open compact subgroups of G. Furthermore, $\mathrm{Mod}(G)$, resp. Vect, denotes the abelian category of all smooth G-representations in k-vector spaces, resp. of all k-vector spaces. As usual, $Z(G)$ is the centre of G. Similarly, $Z(R)$, for a ring R, is the centre of R. For any topological space X we let $C_c^\infty(X,k)$ denote the k-vector space of k-valued locally constant functions with compact support on X. If X is compact, resp. discrete, we simply write $C^\infty(X,k)$, resp. $C_c(X,k)$. For a compact open subset $C \subseteq X$ let $\mathrm{char}_C \in C_c^\infty(X,k)$ denote the characteristic function of C. For any group Γ acting on a set Y and any subset P of Y, we write $\Gamma_P = \{g \in \Gamma : gP = P\}$ to denote the stabiliser of P in Γ.

3. The Bernstein centre of Mod(G)

The Bernstein centre $\mathfrak{Z}(G)$ of the abelian category $\mathrm{Mod}(G)$ by definition is the ring of natural endomorphisms of the identity functor on $\mathrm{Mod}(G)$. It obviously is a commutative k-algebra, which acts functorially on any object in $\mathrm{Mod}(G)$. In order to describe it in more down to earth terms we observe that the group $G \times G$ acts smoothly on $C_c^\infty(G,k)$ by ${}^{(g_1,g_2)}F(-) := F(g_1^{-1} - g_2)$. In the following we write G_ℓ, resp. G_r, and correspondingly g_ℓ and g_r, if we refer to the action of the left, resp. right, factor G, and correspondingly $g \in G$.

We first make the larger k-algebra $\mathfrak{A}(G)$ of all natural endomorphisms of the forgetful functor $\mathrm{Mod}(G) \to \mathrm{Vect}$ more explicit. An element of $\mathfrak{A}(G)$, of course, is a family of k-linear endomorphisms T_V for any V in $\mathrm{Mod}(G)$ which commute, in an obvious sense, with any morphism in $\mathrm{Mod}(G)$. Any $g \in G$ via its action on the V can be seen as an element $(g_V)_V \in \mathfrak{A}(G)$. Viewing $C_c^\infty(G,k)$ as an object in $\mathrm{Mod}(G)$ via the G_ℓ-action we consider the k-algebra homomorphism

$$\begin{aligned} \mathfrak{A}(G) &\longrightarrow \mathrm{End}_k(C_c^\infty(G,k)) \\ (T_V)_V &\longmapsto T_{C_c^\infty(G,k)} \ . \end{aligned}$$

Observing that the G_r-action on $C_c^\infty(G,k)$ is by G_ℓ-equivariant maps we see that any $T_{C_c^\infty(G,k)}$ has to commute with this G_r-action. The above map therefore is, in fact, a map

$$\Theta : \mathfrak{A}(G) \longrightarrow \mathrm{End}_{\mathrm{Mod}(G_r)}(C_c^\infty(G,k)) \ .$$

We then have the following result due to Positselski ([Pos] Prop. 3.6(a)). See section 7 for an alternative proof.

Proposition 3.1. *The map Θ is an isomorphism.*

Corollary 3.2. $\Theta(\mathfrak{Z}(G)) = \mathrm{End}_{\mathrm{Mod}(G\times G)}(C_c^\infty(G,k))$.

Proof. Note that $\mathfrak{Z}(G) = \{(T_V)_V \in \mathfrak{A}(G) : (T_V \circ g_V)_V = (g_V \circ T_V)_V$ for any $g \in G\}$. The map Θ sends $(T_V)_V \in \mathfrak{A}(G)$ to T_C where $C := C_c^\infty(G,k)$. Hence $\Theta(\mathfrak{Z}(G))$ is contained in $\{T \in \mathrm{End}_{\mathrm{Mod}(G_r)}(C) : T \circ g_C = g_C \circ T\} = \mathrm{End}_{\mathrm{Mod}(G\times G)}(C)$, and we have to show that equality holds. Suppose that $T \in \mathrm{End}_{\mathrm{Mod}(G\times G)}(C)$; using Proposition 3.1 we can find $(T_V)_V \in \mathfrak{A}(G)$ such that $T_C = T$. Fix $g \in G$ and define the k-linear map $S_V := g_V T_V g_V^{-1} : V \to V$ for each $V \in \mathrm{Mod}(G)$. Then for every morphism $\varphi : V \to W$ in $\mathrm{Mod}(G)$, we have $\varphi g_V = g_W \varphi$, so $\varphi S_V = \varphi g_V T_V g_V^{-1} = g_W \varphi T_V g_V^{-1} = g_W T_W \varphi g_V^{-1} = g_W T_W g_W^{-1} \varphi = S_W \varphi$. Hence $(S_V)_V \in \mathfrak{A}(G)$ as well, and $\Theta((S_V)_V) = S_C = T$ because T commutes with g_C by assumption. Since Θ is injective, we conclude that $(S_V)_V = (T_V)_V$ which means that $(T_V)_V \in \mathfrak{Z}(G)$. Hence $T = \Theta((T_V)_V) \in \Theta(\mathfrak{Z}(G))$ for all $T \in \mathrm{End}_{\mathrm{Mod}(G\times G)}(C)$ as required. □

We will now write down some obvious elements in $\mathfrak{Z}(G)$.

Definition 3.3. *Let Z be a locally profinite abelian group, and let $\widehat{k[Z]}$ denote the following completion of $k[Z]$:*

$$\widehat{k[Z]} := \varprojlim_{Z' \in \mathrm{Op}_c(Z)} k[Z/Z'].$$

To get some feeling for this completion, consider the following special case.

Remark 3.4. Let Z be a locally profinite abelian group. Suppose there is a compact open subgroup Z_0 of Z and a discrete subgroup A of Z such that $Z = A \times Z_0$. Then $\widehat{k[Z]}$ is isomorphic to the usual completed group ring of the profinite group Z_0 but with coefficients taken in the discrete k-algebra $k[A]$:

$$\widehat{k[Z]} \cong k[A][[Z_0]] := \varprojlim_{Z_1 \in \mathrm{Op}_c(Z_0)} k[A][Z_0/Z_1]$$

If A is finite this simplifies to $\widehat{k[Z]} = k[A] \otimes_k k[[Z_0]]$.

Example 3.5. Suppose that $Z = \mathfrak{F}^\times$ for some finite extension $\mathfrak{F}$ of $\mathbb{Q}_p$. Fix a uniformiser $\pi \in o_{\mathfrak{F}}$, let A be the subgroup of Z generated by π and let $Z_0 := o_{\mathfrak{F}}^\times$. Then Z, A and Z_0 satisfy the conditions of Remark 3.4. Therefore in this case, $\widehat{k[\mathfrak{F}^\times]}$ is the completed group ring of the compact abelian p-adic Lie group $o_{\mathfrak{F}}^\times$ with coefficients in the Laurent polynomial ring $k[A] \cong k[\pi, \pi^{-1}]$ in the variable π.

Lemma 3.6. *$\widehat{k[Z(G)]}$ acts naturally on every $V \in \mathrm{Mod}(G)$ by $k[G]$-linear endomorphisms.*

Proof. For every compact open subgroup H of G, restrict the $Z := Z(G)$-action on V to $V^H = \{v \in V : h \cdot v = v \text{ for all } h \in H\}$. Since the action of Z on V commutes with the action of H, V^H is a Z-submodule of V. The normal subgroup $Z \cap H$ of Z acts trivially on V^H and therefore the Z-action on V^H factors descends to a well-defined action of $Z/(Z \cap H)$ on V^H. By the definition of the completion $\widehat{k[Z]}$, there is a canonical map from this completion to $k[Z/(Z \cap H)]$, because $Z \cap H$ is a compact open subgroup of Z. In this way, every V^H is naturally a $\widehat{k[Z]}$-module, via this homomorphism. Let J be another compact open subgroup of G, contained in H. Then V^H is contained in V^J, and both are $\widehat{k[Z]}$-modules as explained above. We observe that the inclusion map $V^H \to V^J$ is in fact $\widehat{k[Z]}$-linear.

We have now defined a $\widehat{k[Z]}$-action on V^H for every compact open subgroup H of G and we have checked that these actions are compatible with the inclusions $V^H \to V^J$ whenever $J \subseteq H$ is a smaller compact open subgroup. Therefore there is a well-defined action of $\widehat{k[Z]}$ on $V = \bigcup_H V^H$.

It remains to check that the $\widehat{k[Z]}$-action we have constructed commutes with the given G-action on V. Fix $x \in \widehat{k[Z]}$, $g \in G$ and $v \in V$. Choose a compact open subgroup H of G such that $v \in V^H$; then v and $g \cdot v$ both lie in V^J where $J := H \cap gHg^{-1} \in \mathrm{Op}_c(G)$. The action of x on V^J is equal to the action of its image $\overline{x}$ in $k[ZJ/J]$. Now $zJ \cdot (g \cdot v) = g \cdot (zJ \cdot v)$ for every $zJ \in ZJ/J$ so $x \cdot (g \cdot v) = \overline{x} \cdot (g \cdot v) = g \cdot (\overline{x} \cdot v) = g \cdot (x \cdot v)$ as required. ◻

Let $x \mapsto \Phi_V(x) \in \mathrm{End}_{\mathrm{Mod}(G)}(V)$ denote the action of $x \in \widehat{k[Z(G)]}$ on $V \in \mathrm{Mod}(G)$ that was constructed in Lemma 3.6.

Lemma 3.7. *The map* $x \mapsto \Phi(x) := (\Phi_V(x))_V$ *is a* k*-algebra homomorphism*

$$\Phi : k\widehat{[Z(G)]} \to \mathfrak{Z}(G).$$

Proof. The action of $k[Z(G)]$ commutes with every morphism $\varphi : V \to W$ in $\mathrm{Mod}(G)$. This implies that $\Phi_W(x)\circ\varphi = \varphi\circ\Phi_V(x)$ for every $x \in k\widehat{[Z(G)]}$. So $(\Phi_V(x))_V$ does define an element $\Phi(x)$ in $\mathfrak{Z}(G)$. The verification that Φ is a k-algebra homomorphism is straightforward. □

Proposition 3.8. *Let* $C := C_c^\infty(G,k)$*, regarded as an object in* $\mathrm{Mod}(G_\ell)$*. Then*

(1) $\Phi_C : k\widehat{[Z(G)]} \to \mathrm{End}_{\mathrm{Mod}(G_\ell)}(C)$ *is injective, and*
(2) $\mathrm{im}(\Phi_C) \subseteq \mathrm{End}_{\mathrm{Mod}(G\times G)}(C)$.

Proof. (1) Write $Z := Z(G)$. It is enough to show that $k[Z/(Z\cap U)]$ acts faithfully on $C^U = C_c^\infty(G,k)^{U_\ell} = C_c(U\backslash G,k)$ for every compact open subgroup U of G. For any $x \in G$, consider the characteristic function $\mathrm{char}_{Ux} \in C_c(U\backslash G,k)$ of the coset $Ux \in U\backslash G$; if $z \in Z$ then ${}^{z_\ell}\mathrm{char}_U = \mathrm{char}_{zU} = \mathrm{char}_{Uz}$ shows that char_U generates a $k[Z/(Z\cap U)]$-submodule of C^U which is free of rank one. Hence $k[Z/(Z\cap U)]$ acts faithfully on C^U.

(2) This follows from the construction of $\Phi_C : \widehat{k[Z]} \to \mathrm{End}_{\mathrm{Mod}(G_\ell)}(C)$: it suffices to see that $\Phi_C(k[Z])$ commutes with the action of G_r, but $\Phi_C(z)(f) = {}^{z_\ell}f$ for any $f \in C$ and any $z \in Z$, and z_ℓ commutes with G_r. □

We now can easily compute the Bernstein centre for commutative groups.

Proposition 3.9. *If* $G = Z$ *is commutative then the maps*

$$\widehat{k[Z]} \xrightarrow[\cong]{\Phi} \mathfrak{Z}(Z) \xrightarrow[\cong]{\Theta} \mathrm{End}_{\mathrm{Mod}(Z)}(C_c^\infty(Z,k))$$

are isomorphisms.

Proof. We already know from Cor. 3.2 that the right hand map is an isomorphism. But for this proof we only use its injectivity which is very easy to see (cf. section 7). It therefore suffices to establish that the composed map $\Theta\circ\Phi = \Phi_{C_c^\infty(Z,k)}$ is bijective. We observe that

$$\begin{aligned}\mathrm{End}_{\mathrm{Mod}(Z)}(C_c^\infty(Z,k)) &= \mathrm{Hom}_{\mathrm{Mod}(Z)}\Big(\bigcup_{Z'\in\mathrm{Op}_c(Z)} C_c(Z/Z',k), C_c^\infty(Z,k)\Big) \\ &= \varprojlim_{Z'\in\mathrm{Op}_c(Z)} \mathrm{Hom}_{\mathrm{Mod}(Z)}(C_c(Z/Z',k), C_c^\infty(Z,k)) \\ &= \varprojlim_{Z'\in\mathrm{Op}_c(Z)} \mathrm{End}_{\mathrm{Mod}(Z)}(C_c(Z/Z',k))\ .\end{aligned}$$

This shows that $\Phi_{C_c^\infty(Z,k)} = \varprojlim_{Z'\in\mathrm{Op}_c(Z)} \Phi_{C_c(Z/Z',k)}$ and reduces us to the case of a discrete commutative group Z. But in this case $C_c(Z,k)$ is a free module of rank one over $k[Z]$. □

Corollary 3.10. *The composed map*

$$\mathfrak{Z}(Z(G)) \xrightarrow[\cong]{\Phi^{-1}} k\widehat{[Z(G)]} \xrightarrow{\Phi} \mathfrak{Z}(G)$$

between Bernstein centres is an injective homomorphism of k*-algebras.*

4. The universal endomorphism ring

One of our goals is the computation of the endomorphism ring $\mathrm{End}_{\mathrm{Mod}(G\times G)}(C_c^\infty(G,k))$.

We fix a compact open subgroup $U \subseteq G$. We also choose a set $\mathcal{W} \subseteq G$ of representatives of the double cosets of U in G. For the background for the subsequent points **A)** to **D)** see [Vig96] Chap. I.5; note that we use the following definition of compact induction:

$$\mathrm{ind}_U^G(V) := \{f \in C_c^\infty(G,V) : f(gu) = u^{-1}f(g) \quad \text{for all} \quad u \in U\},$$

which translates to the formula (1) found on p.38 in [Vig96] via the map $f \mapsto [g \mapsto f(g^{-1})]$.

A) Transitivity of induction: We have the G_ℓ-equivariant isomorphism

$$\begin{aligned} \mathrm{ind}_{U_\ell}^{G_\ell}(C^\infty(U,k)) &\xrightarrow{\cong} C_c^\infty(G,k) \\ \phi &\longmapsto F_\phi(g) := \phi(g)(1) \ . \end{aligned} \tag{1}$$

Note that $\phi(g)(u) = F_\phi(gu)$ for $u \in U$. One checks that the G_r-action on $C_c^\infty(G,k)$ corresponds under this isomorphism to the G_r-action on $\mathrm{ind}_{U_\ell}^{G_\ell}(C^\infty(U,k))$ given by

$$(h,\phi) \longmapsto h_r(\phi)(g)(u) := \phi(guh)(1) \ . \tag{2}$$

In particular, for $h = v \in U$ we have

$$v_r(\phi)(g)(u) = \phi(guv)(1) = {}^{(uv)_\ell^{-1}}(\phi(g))(1) = \phi(g)(uv) = {}^{v_r}(\phi(g))(u)$$

and hence

$$v_r(\phi)(g) = {}^{v_r}(\phi(g)) \ . \tag{3}$$

B) Frobenius reciprocity: We have the U_ℓ-equivariant embedding

$$\begin{aligned} C^\infty(U,k) &\longrightarrow \mathrm{ind}_{U_\ell}^{G_\ell}(C^\infty(U,k)) \\ f &\longmapsto \iota(f)(g) := \begin{cases} {}^{g_\ell^{-1}}f & \text{if } g \in U, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

It gives rise to the isomorphism

$$\begin{aligned} \mathrm{End}_{\mathrm{Mod}(G_\ell)}(\mathrm{ind}_{U_\ell}^{G_\ell}(C^\infty(U,k))) &\xrightarrow{\cong} \mathrm{Hom}_{\mathrm{Mod}(U_\ell)}(C^\infty(U,k), \mathrm{ind}_{U_\ell}^{G_\ell}(C^\infty(U,k))) \\ \alpha &\longmapsto \alpha^\flat := \alpha \circ \iota \ . \end{aligned} \tag{4}$$

Later on we will be interested only in G_r-equivariant α's. The main point being, as it will turn out, the U_r-equivariance we note here only the following. For $v \in U$ we compute, using (3) in the last equality,

$$\iota({}^{v_r}f)(g) = \begin{cases} {}^{g_\ell^{-1}}({}^{v_r}f) \\ 0 \end{cases} = \begin{cases} {}^{v_r}({}^{g_\ell^{-1}}f) \\ 0 \end{cases} = {}^{v_r}(\iota(f)(g)) = v_r(\iota(f))(g) \ ,$$

which means that the map ι is U_r-equivariant. Hence with α also $\alpha^\flat$ is U_r-equivariant.

C) Mackey decomposition: The disjoint decomposition $G = \dot{\bigcup}_{w\in\mathcal{W}} UwU$ gives rise to the U_ℓ-invariant decomposition

$$\mathrm{ind}_{U_\ell}^{G_\ell}(C^\infty(U,k)) = \oplus_{w\in\mathcal{W}}\, \mathrm{ind}_{U_\ell}^{(UwU)_\ell}(C^\infty(U,k)) \ .$$

We put $U_w := U \cap wUw^{-1}$. For any $w \in \mathcal{W}$ the map

$$\begin{aligned}\mathrm{ind}_{U_\ell}^{(UwU)_\ell}(C^\infty(U,k)) &\xrightarrow{\cong} \mathrm{ind}_{(U_w)_\ell}^{U_\ell}(w_* \,\mathrm{res}_{(U_{w^{-1}})_\ell}^{U_\ell}(C^\infty(U,k))) \\ \phi &\longmapsto \tilde{\phi}(u) := \phi(uw)\end{aligned}$$

is a U_ℓ-equivariant isomorphism.[1] Here the functor w_* sends a $U_{w^{-1}}$-representation Y to the U_w-representation $w_*Y := Y$ but on which U_w acts through the homomorphism $u \mapsto w^{-1}uw$. We see that

$$\begin{aligned}\mathrm{ind}_{U_\ell}^{G_\ell}(C^\infty(U,k)) &\xrightarrow{\cong} \oplus_{w\in\mathcal{W}}\, \mathrm{ind}_{(U_w)_\ell}^{U_\ell}(w_* C^\infty(U,k)) \\ \phi &\longmapsto (\phi_w(u) := \phi(uw))_w\end{aligned} \tag{5}$$

is a U_ℓ-equivariant isomorphism. Using (3) one checks that this map also is U_r-equivariant where U_r acts on the target in a way which is induced in each summand by the U_r-action on $C^\infty(U,k)$.

D) Second Frobenius reciprocity: Let Y be any object in $\mathrm{Mod}(U_w)$. Since U_w is open and hence of finite index in U the U_w-equivariant map

$$\begin{aligned}\mathrm{ind}_{U_w}^{U}(Y) &\longrightarrow Y \\ \phi &\longmapsto \phi(1)\end{aligned} \tag{6}$$

induces the isomorphism

$$\begin{aligned}\mathrm{Hom}_{\mathrm{Mod}(U_\ell)}(C^\infty(U,k), \mathrm{ind}_{U_w}^{U}(Y)) &\xrightarrow{\cong} \mathrm{Hom}_{\mathrm{Mod}((U_w)_\ell)}(C^\infty(U,k), Y) \\ \beta &\longmapsto [f \mapsto \beta(f)(1)]\ .\end{aligned}$$

For $Y = w_*C^\infty(U,k)$ this becomes the isomorphism

$$\begin{aligned}\mathrm{Hom}_{\mathrm{Mod}(U_\ell)}(C^\infty(U,k), \mathrm{ind}_{(U_w)_\ell}^{U_\ell}(w_*C^\infty(U,k))) &\xrightarrow{\cong} \mathrm{Hom}_{\mathrm{Mod}((U_w)_\ell)}(C^\infty(U,k), w_*C^\infty(U,k)) \\ \beta &\longmapsto [f \mapsto \beta(f)(1)]\ .\end{aligned} \tag{7}$$

In this case the map (6) is visibly U_r-equivariant. Hence (7) respects U_r-equivariance.

[1] This map does depend on the choice of the set $\mathcal{W}$.

The combination of (4), (5), and (7) leads to the following chain of maps:

$$\begin{array}{c}
\mathrm{End}_{\mathrm{Mod}(G_\ell)}(\mathrm{ind}_{U_\ell}^{G_\ell}(C^\infty(U,k))) \\
\Big\downarrow \cong \\
\mathrm{Hom}_{\mathrm{Mod}(U_\ell)}(C^\infty(U,k), \mathrm{ind}_{U_\ell}^{G_\ell}(C^\infty(U,k))) \\
\Big\downarrow \cong \\
\mathrm{Hom}_{\mathrm{Mod}(U_\ell)}(C^\infty(U,k), \oplus_{w\in\mathcal{W}} \mathrm{ind}_{(U_w)_\ell}^{U_\ell}(w_*C^\infty(U,k))) \\
\Big\downarrow \subseteq \\
\mathrm{Hom}_{\mathrm{Mod}(U_\ell)}(C^\infty(U,k), \prod_{w\in\mathcal{W}} \mathrm{ind}_{(U_w)_\ell}^{U_\ell}(w_*C^\infty(U,k))) \\
\Big\downarrow = \\
\prod_{w\in\mathcal{W}} \mathrm{Hom}_{\mathrm{Mod}(U_\ell)}(C^\infty(U,k), \mathrm{ind}_{(U_w)_\ell}^{U_\ell}(w_*C^\infty(U,k))) \\
\Big\downarrow \cong \\
\prod_{w\in\mathcal{W}} \mathrm{Hom}_{\mathrm{Mod}((U_w)_\ell)}(C^\infty(U,k), w_*C^\infty(U,k))
\end{array} \tag{8}$$

We denote the composite map by $\alpha \longmapsto \Omega(\alpha) = (\Omega_w(\alpha))_{w\in\mathcal{W}}$. Its explicit description is

$$\Omega_w(\alpha)(f) = (\alpha \circ \iota(f))(w) \ . \tag{9}$$

We have seen already that Ω respects U_r-equivariance. It is clear that the image of Ω is

$$\begin{aligned} &\mathrm{im}(\Omega) = \\ &\qquad \{(\omega_w)_w : \text{for any } f \in C^\infty(U,k) \text{ we have } \omega_w(f) = 0 \text{ for all but finitely many } w \in \mathcal{W}\}. \end{aligned} \tag{10}$$

To reformulate the target of Ω under the presence of the U_r-equivariance condition we first observe that

$$\begin{aligned} &\mathrm{Hom}_{\mathrm{Mod}(U_w\times U)}(C^\infty(U,k), w_*C^\infty(U,k)) = \\ &\qquad \{\beta \in \mathrm{End}_{\mathrm{Mod}(U_r)}(C^\infty(U,k)) : \beta({}^{u_\ell}(-)) = {}^{(w^{-1}uw)_\ell}(\beta(-)) \text{ for any } u \in U_w\}. \end{aligned} \tag{11}$$

The U_ℓ-action on $C^\infty(U,k)$ extends uniquely to an action of the completed group ring $k[[U_\ell]]$. The U_r-action is equivariant for this $k[[U_\ell]]$-action. This gives rise to a ring homomorphism

$$k[[U_\ell]] \xrightarrow{\cong} \mathrm{End}_{\mathrm{Mod}(U_r)}(C^\infty(U,k))$$

which is easily seen to be an isomorphism. Using this identification the equality (11) becomes the equality

$$\begin{aligned} &\mathrm{Hom}_{\mathrm{Mod}(U_w\times U)}(C^\infty(U,k), w_*C^\infty(U,k)) = \\ &\qquad \{\lambda \in k[[U]] : \lambda u = (w^{-1}uw)\lambda \text{ for any } u \in U_w\} =: k[[U]]_w \ . \end{aligned} \tag{12}$$

We conclude that Ω restricts to an injective map

$$\Omega_U : \mathrm{End}_{\mathrm{Mod}(G\times U)}(\mathrm{ind}_{U_\ell}^{G_\ell}(C^\infty(U,k))) \longrightarrow \prod_{w\in\mathcal{W}} k[[U]]_w \tag{13}$$

with the additional property, by (10), that any element $(\lambda_w)_w$ in the image of Ω_U satisfies:

(14) For any $f \in C^\infty(U,k)$ we have $\lambda_w(f) = 0$ for all but finitely many $w \in \mathcal{W}$.

In section 6 we will study the vector spaces $k[[U]]_w$ based upon a formalism developed in the next section.

5. Fixed points in profinite permutation modules

In this section, we extend [Ard] Prop. 2.1 to a more general setting.

Let Γ be a profinite group and let Y be a profinite set equipped with a continuous action $\mathrm{act} : \Gamma \times Y \to Y$ of Γ. We begin with some topological preliminaries.

Lemma 5.1. *Let P be a closed and open subset of Y. Then the stabiliser Γ_P is open in Γ.*

Proof. Note that $\mathrm{Op}_c(\Gamma)$ is a directed set under reverse inclusion. First, we show that

$$\bigcap_{H \in \mathrm{Op}_c(\Gamma)} HP = P.$$

Let x lie in the intersection; then for each $H \in \mathrm{Op}_c(\Gamma)$ we can find $y_H \in P$ and $h_H \in H$ such that $x = h_H y_H$. Regarding $H \mapsto h_H$ as a net in Γ, we observe that $h_H \to 1$ because $\bigcap_{H \in \mathrm{Op}_c(\Gamma)} H = \{1\}$. Now P is compact because P is closed in Y and Y is compact. So, by [Kel] Thm. 5.2, we can find a subnet $\varphi : B \to \mathrm{Op}_c(\Gamma)$ such that $y_{\varphi(b)} \to y$ for some $y \in P$. Note that $h_{\varphi(b)} \to 1$ still, so $(h_{\varphi(b)}, y_{\varphi(b)}) \to (1, y)$ in $\Gamma \times Y$. Since $\mathrm{act} : \Gamma \times Y \to Y$ is continuous, we conclude that $x = h_{\varphi(b)} \cdot y_{\varphi(b)} \to 1 \cdot y = y$. Therefore $x = y \in P$ because Y is Hausdorff.

Now each HP is the continuous image in Y of the compact space $H \times P$ under act, and is therefore compact. Since Y is Hausdorff, each HP is closed in Y. By the above,

$$Y \backslash P = \bigcup_{H \in \mathrm{Op}_c(\Gamma)} Y \backslash HP$$

is an open covering of $Y \backslash P$. Since P is also open, its complement $Y \backslash P$ is closed and hence compact. This implies that $P = H_1 P \cap \cdots \cap H_n P$ for some $H_1, \cdots, H_n \in \mathrm{Op}_c(\Gamma)$. But then the open subgroup $H_1 \cap \cdots \cap H_n$ of Γ stabilises P. $\square$

Now we can record a Γ-equivariant version of the well-known topological characterisation of profinite sets. Let $\mathfrak{P}$ be the set of open partitions $\mathcal{P}$ of Y so that every $\mathcal{P} \in \mathfrak{P}$ is a finite discrete quotient space of Y. These form an inverse system if we order $\mathfrak{P}$ by refinement, so that $\mathcal{P}_2 \geq \mathcal{P}_1$ if and only if every $P_2 \in \mathcal{P}_2$ is contained in some $P_1 \in \mathcal{P}_1$. By [Wil] Prop. 1.1.7, the natural map $Y \xrightarrow{\cong} \varprojlim_{\mathcal{P} \in \mathfrak{P}} \mathcal{P}$ is an isomorphism of profinite sets.

We say that $\mathcal{Q} \in \mathfrak{P}$ is Γ-*stable* if $gQ \in \mathcal{Q}$ for all $Q \in \mathcal{Q}$ and all $g \in \Gamma$. Like $\mathfrak{P}$, the set $\mathfrak{Q} \subset \mathfrak{P}$ of Γ-stable open partitions of Y forms a directed set under refinement.

Lemma 5.2. *Every $\mathcal{P} \in \mathfrak{P}$ admits a Γ-stable refinement $\mathcal{Q} \in \mathfrak{Q}$.*

Proof. Because Y is compact, $\mathcal{P}$ is a finite set, so using Lemma 5.1 we can find an open subgroup H of Γ which stabilises each member of $\mathcal{P}$. Fix a complete set $\{g_1, \cdots, g_k\}$ of left coset representatives of H in Γ, and define

$$\mathcal{Q} := \{g_1 P_1 \cap \cdots \cap g_k P_k : P_1, \cdots, P_k \in \mathcal{P}\} \backslash \{\emptyset\}.$$

Then every member of $\mathcal{Q}$ is a non-empty closed and open subset of Γ, and since $Y = \bigcup_{P\in\mathcal{P}} g_iP$ for each $i = 1, \cdots, k$, we see that

$$Y = \bigcap_{i=1}^{k} \bigcup_{P\in\mathcal{P}} g_iP = \bigcup_{P_1,\cdots,P_k\in\mathcal{P}} g_1P_1 \cap \cdots \cap g_kP_k = \bigcup_{Q\in\mathcal{Q}} Q.$$

So, $\mathcal{Q}$ forms a covering of Y by non-empty closed and open subsets. Let $Q = g_1P_1 \cap \cdots \cap g_kP_k$ and $Q' = g_1P'_1 \cap \cdots \cap g_kP'_k$ be two members of $\mathcal{Q}$ such that $Q \cap Q' \neq \emptyset$. Then

$$\emptyset \neq Q \cap Q' = (g_1P_1 \cap \cdots \cap g_kP_k) \cap \left(g_1P'_1 \cap \cdots \cap g_kP'_k\right) = g_1(P_1 \cap P'_1) \cap \cdots \cap g_k(P_k \cap P'_k)$$

shows that $P_i \cap P'_i \neq \emptyset$ for each $i = 1, \cdots, k$. Since $\mathcal{P}$ is a partition, $P_i = P'_i$ for each $i = 1, \cdots, k$ and hence $Q = Q'$. So, $\mathcal{Q}$ is a partition of Y.

Finally, to see that $\mathcal{Q}$ is Γ-stable, consider the permutation action of Γ on Γ/H. Fix $x \in \Gamma$ and write $xg_iH = g_{x\cdot i}H$ for some permutation $i \mapsto x \cdot i$ of $\{1, \cdots, k\}$. Since H stabilises each $P \in \mathcal{P}$, we have $xg_iP = g_{x\cdot i}P$ for all $i = 1, \cdots, k$ and all $P \in \mathcal{P}$. Then for any $Q = gP_1 \cap \cdots \cap g_kP_k \in \mathcal{Q}$, we see that

$$xQ = x(g_1P_1 \cap \cdots \cap g_kP_k) = g_{x\cdot 1}P_1 \cap \cdots \cap g_{x\cdot k}P_k = g_1P_{x^{-1}.1} \cap \cdots \cap g_kP_{x^{-1}.k} \in \mathcal{Q}.$$

Thus $\mathcal{Q}$ is the required Γ-stable refinement of $\mathcal{P}$. □

We regard each $\mathcal{Q} \in \mathfrak{Q}$ as a finite discrete space; it admits a natural Γ-action. Lemma 5.1 implies that this action is continuous. If $\mathcal{Q} \in \mathfrak{Q}$ is a refinement of $\mathcal{R} \in \mathfrak{Q}$, then the natural map $\pi_{\mathcal{QR}} : \mathcal{Q} \to \mathcal{R}$ which sends $Q \in \mathcal{Q}$ to the unique element $R \in \mathcal{R}$ containing Q is Γ-equivariant, so the profinite set $\varprojlim_{\mathcal{Q}\in\mathfrak{Q}} \mathcal{Q}$ carries a natural continuous Γ-action. From Lemma 5.2 we easily deduce the following

Corollary 5.3. *The natural* Γ*-equivariant map* $Y \xrightarrow{\cong} \varprojlim_{\mathcal{Q}\in\mathfrak{Q}} \mathcal{Q}$ *is an isomorphism.*

Let R be any possibly noncommutative ring.

Definition 5.4. *The* permutation module $R[[Y]]$ *of* Y *is the projective limit*

$$R[[Y]] := \varprojlim_{\mathcal{Q}\in\mathfrak{Q}} R[\mathcal{Q}]$$

of the usual permutation $R[\Gamma]$*-modules* $R[\mathcal{Q}]$ *as* $\mathcal{Q}$ *ranges over the set* $\mathfrak{Q}$ *of* Γ*-stable open partitions of* Y*. It is equipped with the projective limit topology of the discrete topologies.*

By functoriality, there is a natural Γ-action on $R[[Y]]$ and we would like to describe the R-submodule of Γ-invariants $R[[Y]]^\Gamma$ in $R[[Y]]$.

Definition 5.5.

1. *Let* $\Gamma\backslash Y$ *denote the set of* Γ*-orbits in* Y*.*
2. *Let* $\Delta_\Gamma(Y) := \{y \in Y : |\Gamma \cdot y| < \infty\}$ *be the set of points whose* Γ*-orbit is finite.*
3. *For each* $\mathcal{O} \in \Delta_\Gamma(Y)$*, let* $\upsilon_Y(\mathcal{O}) := \sum_{y\in\mathcal{O}} y \in R[Y] \subset R[[Y]]$ *denote the* orbit sum *of* $\mathcal{O}$*.*

Any such finite orbit sum is Γ-invariant, so we obtain a natural R-linear map

$$\upsilon_Y : R[\Gamma\backslash\Delta_\Gamma(Y)] \to R[[Y]]^\Gamma$$

which sends the basis vector $\mathcal{O} \in R[\Gamma\backslash\Delta_\Gamma(Y)]$ to its orbit sum $\upsilon_Y(\mathcal{O})$. Evidently this map is injective.

Remark 5.6. Since the action of Γ on Y is continuous, the stabiliser Γ_y of any point $y \in Y$ is closed. Since a closed subgroup of finite index in Γ is open, we see that $y \in Y$ lies in $\Delta_\Gamma(Y)$ if and only if y is fixed by some open subgroup H of Γ, which we may as well take to be normal:

$$\Delta_\Gamma(Y) = \bigcup_{H \in \mathrm{Op}_c(\Gamma)} Y^H = \bigcup_{N \in \mathrm{Op}_c(\Gamma) \text{ normal}} Y^N.$$

We begin with the well-known, but instructive case where Y is finite.

Lemma 5.7. *Suppose that Y is finite. Then υ_Y is a bijection.*

For any two $\mathcal{Q} \geq \mathcal{R}$ in $\mathfrak{Q}$ we introduce the R-linear map

$$\sigma_{\mathcal{Q}\mathcal{R}} : R[\Gamma\backslash\mathcal{Q}] \longrightarrow R[\Gamma\backslash\mathcal{R}]$$
$$\mathcal{C} \longmapsto \frac{|\mathcal{C}|}{|\pi_{\mathcal{Q}\mathcal{R}}(\mathcal{C})|}\pi_{\mathcal{Q}\mathcal{R}}(\mathcal{C}) \ .$$

Note that the fraction actually is an integer. It is easy the check that the diagram

$$\begin{array}{ccc} R[\Gamma\backslash\mathcal{Q}] & \xrightarrow{\sigma_{\mathcal{Q}\mathcal{R}}} & R[\Gamma\backslash\mathcal{R}] \\ \upsilon_{\mathcal{Q}}\big\downarrow\cong & & \cong\big\downarrow\upsilon_{\mathcal{R}} \\ R[\mathcal{Q}]^\Gamma & \xrightarrow[R[\pi_{\mathcal{Q}\mathcal{R}}]]{} & R[\mathcal{R}]^\Gamma \end{array} \tag{15}$$

is commutative. By passing to the projective limit we obtain an R-linear topological isomorphism

$$\varprojlim_{\mathcal{Q}\in\mathfrak{Q}} R[\Gamma\backslash\mathcal{Q}] \xrightarrow{\cong} R[[Y]]^\Gamma \tag{16}$$

w.r.t. the projective limit topologies. Note that on the target this topology coincides with the subspace topology from $R[[Y]]$.

We now specialise to the case where R is a ring of characteristic p and Γ is a pro-p group. In this case, we have the following description.

Lemma 5.8. *Let $x = (x_\mathcal{Q})_{\mathcal{Q}\in\mathfrak{Q}} \in \prod_{\mathcal{Q}\in\mathfrak{Q}} R[\Gamma\backslash\mathcal{Q}]$ and write $x_\mathcal{Q} = \sum_{\mathcal{C}\in\Gamma\backslash\mathcal{Q}} x(\mathcal{C})\,\mathcal{C}$ for all $\mathcal{Q} \in \mathfrak{Q}$. Then $x \in \varprojlim_{\mathcal{Q}\in\mathfrak{Q}} R[\Gamma\backslash\mathcal{Q}]$ if and only if for all $\mathcal{P} \geq \mathcal{Q}$ in $\mathfrak{Q}$ and all $\mathcal{C} \in \Gamma\backslash\mathcal{Q}$ we have*

$$x(\mathcal{C}) = \sum_{\mathcal{B}} x(\mathcal{B})$$

where the sum runs over the finite set $\{\mathcal{B} \in \Gamma\backslash\mathcal{P} : \pi_{\mathcal{P}\mathcal{Q}}(\mathcal{B}) = \mathcal{C}$ and $|\mathcal{B}| = |\mathcal{C}|\}$.

Proof. Let $\mathcal{P} \geq \mathcal{Q}$ and write $x_\mathcal{P} = \sum_{\mathcal{B}\in\Gamma\backslash\mathcal{P}} x(\mathcal{B})\mathcal{B}$. Then by (15), we have

$$\sigma_{\mathcal{P}\mathcal{Q}}(x_\mathcal{P}) = \sum_{\mathcal{C}\in\Gamma\backslash\mathcal{Q}} \left(\sum_{\pi_{\mathcal{P}\mathcal{Q}}(\mathcal{B})=\mathcal{C}} x(\mathcal{B})\frac{|\mathcal{B}|}{|\mathcal{C}|} \right) \mathcal{C}.$$

Let $\mathcal{B} \in \Gamma\backslash\mathcal{P}$ and let $\mathcal{C} = \pi_{\mathcal{P}\mathcal{Q}}(\mathcal{B})$. Then $|\mathcal{B}|/|\mathcal{C}| = [\Gamma_{\pi_{\mathcal{P}\mathcal{Q}}(P)} : \Gamma_P]$ for any $P \in \mathcal{B}$. Since Γ is a pro-p group, the index $[\Gamma_{\pi_{\mathcal{P}\mathcal{Q}}(P)} : \Gamma_P]$ is a power of p. Since R is a ring of characteristic p,

$|\mathcal{B}|/|\mathcal{C}|$ is zero in R if $|\mathcal{B}| > |\mathcal{C}|$. Hence

$$\sigma_{\mathcal{P}\mathcal{Q}}(x_{\mathcal{P}}) = \sum_{\mathcal{C}\in\Gamma\backslash\mathcal{Q}} \left(\sum_{\substack{\pi_{\mathcal{P}\mathcal{Q}}(\mathcal{B})=\mathcal{C} \\ |\mathcal{B}|=|\mathcal{C}|}} x(\mathcal{B}) \right) \mathcal{C}.$$

The result follows, because $x \in \varprojlim_{\mathcal{Q}\in\mathfrak{Q}} R[\Gamma\backslash\mathcal{Q}]$ if and only if $\sigma_{\mathcal{P}\mathcal{Q}}(x_{\mathcal{P}}) = x_{\mathcal{Q}}$ for all $\mathcal{P} \geq \mathcal{Q}$. $\square$

Here is our generalisation of [Ard] Prop. 2.1.

Proposition 5.9. *Suppose that R is a ring of characteristic p and that Γ is a pro-p group. Then the map $\upsilon_Y : R[\Gamma\backslash\Delta_\Gamma(Y)] \to R[[Y]]^\Gamma$ has dense image.*

Proof. In view of (16), it is enough to show that the natural R-linear map

$$\eta : R[\Gamma\backslash\Delta_\Gamma(Y)] \to \varprojlim_{\mathcal{Q}\in\mathfrak{Q}} R[\Gamma\backslash\mathcal{Q}]$$

has dense image. Let $\sigma_{\mathcal{R}} : \varprojlim_{\mathcal{Q}\in\mathfrak{Q}} R[\Gamma\backslash\mathcal{Q}] \to R[\Gamma\backslash\mathcal{R}]$ be the canonical projection maps; by definition of the projective limit topology, it will be enough to show that

$$\sigma_{\mathcal{R}}(\operatorname{im}\eta) = \operatorname{im}\sigma_{\mathcal{R}} \quad \text{for all } \mathcal{R} \in \mathfrak{Q}. \tag{17}$$

Fix $\mathcal{R} \in \mathfrak{Q}$ and $\mathcal{D} \in \Gamma\backslash\mathcal{R}$. For each $\mathcal{Q} \geq \mathcal{R}$ in $\mathfrak{Q}$, define

$$\mathcal{S}(\mathcal{Q}) := \{\mathcal{C} \in \Gamma\backslash\mathcal{Q} : \pi_{\mathcal{Q}\mathcal{R}}(\mathcal{C}) = \mathcal{D} \text{ and } |\mathcal{C}| = |\mathcal{D}|\}.$$

Suppose $\mathcal{P} \geq \mathcal{Q} \geq \mathcal{R}$ and pick $\mathcal{B} \in \mathcal{S}(\mathcal{P})$. Let $\mathcal{C} := \pi_{\mathcal{P}\mathcal{Q}}(\mathcal{B}) \in \Gamma\backslash\mathcal{Q}$; then $\pi_{\mathcal{Q}\mathcal{R}}(\mathcal{C}) = \pi_{\mathcal{Q}\mathcal{R}}\pi_{\mathcal{P}\mathcal{Q}}(\mathcal{B}) = \pi_{\mathcal{P}\mathcal{R}}(\mathcal{B}) = \mathcal{D}$ and hence $|\mathcal{B}| \geq |\pi_{\mathcal{P}\mathcal{Q}}(\mathcal{B})| = |\mathcal{C}| \geq |\pi_{\mathcal{Q}\mathcal{R}}(\mathcal{C})| = |\mathcal{D}| = |\mathcal{B}|$ which implies that $|\mathcal{C}| = |\mathcal{D}|$. Hence $\mathcal{C} \in \mathcal{S}(\mathcal{Q})$. In view of (15), we see that the transition map $\sigma_{\mathcal{P}\mathcal{Q}} : R[\Gamma\backslash\mathcal{P}] \to R[\Gamma\backslash\mathcal{Q}]$ restricts to a well-defined map $\sigma_{\mathcal{P}\mathcal{Q}} : \mathcal{S}(\mathcal{P}) \to \mathcal{S}(\mathcal{Q})$, so the finite sets $(\mathcal{S}(\mathcal{Q}))_{\mathcal{Q}\geq\mathcal{R}}$ form a projective system.

Now, suppose that there exists $x \in \varprojlim_{\mathcal{Q}\in\mathfrak{Q}} R[\Gamma\backslash\mathcal{Q}]$ such that $x(\mathcal{D}) \neq 0$ (notation as in Lemma 5.8). For every $\mathcal{Q} \geq \mathcal{R}$, we deduce from Lemma 5.8 that $x(\mathcal{C}) \neq 0$ for at least one $\mathcal{C} \in \Gamma\backslash\mathcal{Q}$ with $\pi_{\mathcal{Q}\mathcal{R}}(\mathcal{C}) = \mathcal{D}$ and $|\mathcal{C}| = |\mathcal{D}|$. Hence $\mathcal{S}(\mathcal{Q})$ is non-empty for every $\mathcal{Q} \geq \mathcal{R}$. Since the indexing set $\{\mathcal{Q} \in \mathfrak{Q} : \mathcal{Q} \geq \mathcal{R}\}$ of this projective system is directed, we deduce from [Bou] Chap. I, §9.6, Prop. 8(b) that the projective limit $\varprojlim_{\mathcal{Q}\geq\mathcal{R}} \mathcal{S}(\mathcal{Q})$ is non-empty. Choose a family $(\mathcal{C}(\mathcal{Q}))_{\mathcal{Q}\geq\mathcal{R}}$ in this limit. Then since each member of this family is a finite set of size $|\mathcal{D}|$, it follows from the definition of $\mathcal{S}(\mathcal{Q})$ that the transition maps $\pi_{\mathcal{P}\mathcal{Q}} : \mathcal{P} \to \mathcal{Q}$ restrict to bijections $\pi_{\mathcal{P}\mathcal{Q}} : \mathcal{C}(\mathcal{P}) \to \mathcal{C}(\mathcal{Q})$ for every $\mathcal{P} \geq \mathcal{Q} \geq \mathcal{R}$. Therefore the projective limit $\varprojlim_{\mathcal{Q}\geq\mathcal{R}} \mathcal{C}(\mathcal{Q})$ is non-empty. So we may choose an element $(P(\mathcal{Q}))_{\mathcal{Q}\geq\mathcal{R}}$ in this projective limit. Although we think of $P(\mathcal{Q})$ as a point in the finite set $\mathcal{Q}$ we may view $P(\mathcal{Q})$ correspondingly as a non-empty open and closed subset of Y. By definition of $\pi_{\mathcal{Q}\mathcal{R}}$, we see that for each $\mathcal{P} \geq \mathcal{Q} \geq \mathcal{R}$, $P(\mathcal{P})$ is contained in $P(\mathcal{Q})$. Since $\mathfrak{Q}$ is directed and forms a cofinal family of finite open partitions of Y by Lemma 5.2 and since Y is a profinite set, we see that $\bigcap_{\mathcal{Q}\geq\mathcal{R}} P(\mathcal{Q})$ is a singleton point, $\{y_{\mathcal{D}}\}$ say. Now, $\Gamma_{P(\mathcal{P})} \subseteq \Gamma_{P(\mathcal{Q})}$ for all $\mathcal{P} \geq \mathcal{Q} \geq \mathcal{R}$ since $\pi_{\mathcal{P}\mathcal{Q}} : \mathcal{P} \to \mathcal{Q}$ is Γ-equivariant and since its restriction $\pi_{\mathcal{P}\mathcal{Q}} : \mathcal{C}(\mathcal{P}) \to \mathcal{C}(\mathcal{Q})$ is a bijection mapping $P(\mathcal{Q})$ to $P(\mathcal{P})$. On the other hand, $[\Gamma : \Gamma_{P(\mathcal{Q})}] = |\Gamma \cdot P(\mathcal{Q})| = |\mathcal{C}(\mathcal{Q})| = |\mathcal{D}|$ shows that in fact $\Gamma_{P(\mathcal{Q})} = \Gamma_{P(\mathcal{R})}$ for all $\mathcal{Q} \geq \mathcal{R}$. Since $\bigcap_{\mathcal{Q}\geq\mathcal{R}} \Gamma_{P(\mathcal{Q})}$ evidently stabilises $\bigcap_{\mathcal{Q}\geq\mathcal{R}} P(\mathcal{Q})$, we conclude that the finite index subgroup $\Gamma_{P(\mathcal{R})}$ of Γ fixes $y_{\mathcal{D}}$ and hence $y_{\mathcal{D}} \in \Delta_\Gamma(Y)$. Hence $\sigma_{\mathcal{R}}(\eta(\Gamma \cdot y_{\mathcal{D}})) = \Gamma \cdot P(\mathcal{R}) = \mathcal{D}$.

Finally, since the inclusion $\sigma_{\mathcal{R}}(\operatorname{im}\eta) \subseteq \operatorname{im}\sigma_{\mathcal{R}}$ in (17) is clear, take $x = (x_{\mathcal{Q}})_{\mathcal{Q}\in\mathfrak{Q}} \in \varprojlim_{\mathcal{Q}\in\mathfrak{Q}} R[\Gamma\backslash\mathcal{Q}]$ and consider its image $\sigma_{\mathcal{R}}(x) = x_{\mathcal{R}} = \sum_{x(\mathcal{D})\neq 0} x(\mathcal{D})\mathcal{D} \in R[\Gamma\backslash\mathcal{R}]$. Then

$$x_{\mathcal{R}} = \sigma_{\mathcal{R}}\eta\left(\sum_{x(\mathcal{D})\neq 0} x(\mathcal{D})\,\Gamma\cdot y_{\mathcal{D}}\right) \in \sigma_{\mathcal{R}}(\operatorname{im}\eta).$$

This shows that $\operatorname{im}\sigma_{\mathcal{R}} \subseteq \sigma_{\mathcal{R}}(\operatorname{im}\eta)$ and completes the proof. □

Theorem 5.10. *Let Γ be a pro-p group which acts continuously on the profinite set Y and let R be a ring of characteristic p. Suppose that $\Delta_\Gamma(Y)$ is closed in Y. Then $\Delta_\Gamma(Y)$ is also a profinite set, and the natural maps*

$$i : R[[\Delta_\Gamma(Y)]]^\Gamma \to R[[Y]]^\Gamma \quad \textit{and} \quad j : R[[\Gamma\backslash\Delta_\Gamma(Y)]] \to R[[\Delta_\Gamma(Y)]]^\Gamma$$

are both isomorphisms.

Proof. The first assertion is clear since any closed subspace of a profinite set is again profinite.

Write $\Delta := \Delta_\Gamma(Y)$. We will first show that the map $i : R[[\Delta]]^\Gamma \to R[[Y]]^\Gamma$ has closed image. Write $Y = \varprojlim Y_\alpha$ as a projective limit of finite sets. Since Δ is a profinite set, its image in Y is closed, so we can write $\Delta = \varprojlim \Delta_\alpha$ where Δ_α is the image of Δ in Y_α. Let $\pi_\alpha : R[[Y]] \to R[Y_\alpha]$ be the natural projection; then the image of $R[[\Delta]]$ in $R[[Y]]$ is equal to $\bigcap_\alpha \pi_\alpha^{-1}(R[\Delta_\alpha])$ and is therefore closed because each π_α is continuous. Hence the image of $i : R[[\Delta]]^\Gamma \to R[[Y]]^\Gamma$ is also closed. Now, the commutative triangle

$$\begin{array}{ccc} R[\Gamma\backslash\Delta] & \xrightarrow{\upsilon_\Delta} & R[[\Delta]]^\Gamma \\ & {\scriptstyle\upsilon_Y}\searrow \quad \swarrow{\scriptstyle i} & \\ & R[[Y]]^\Gamma & \end{array}$$

shows that the image of i contains the image of υ_Y. Since this last image is dense by Prop. 5.9 and since the image of i is closed, we conclude that i is surjective. Its injectivity follows from the injectivity of the natural map $R[[\Delta]] \to R[[Y]]$.

Since Δ is a profinite set, by Corollary 5.3 we can assume that $\Delta = \varprojlim \Delta_\alpha$ is the projective limit of finite discrete spaces Δ_α equipped with a continuous Γ-action. The map j appears in the following commutative diagram

$$\begin{array}{ccc} R[[\Gamma\backslash\Delta]] & \xrightarrow{\cong} & \varprojlim R[\Gamma\backslash\Delta_\alpha] \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle \varprojlim \upsilon_{\Delta_\alpha}} \\ R[[\Delta]]^\Gamma & \xrightarrow[\cong]{} & \varprojlim R[\Delta_\alpha]^\Gamma. \end{array}$$

The rows in this diagram are isomorphisms by definition of the profinite permutation modules $R[[\Gamma\backslash\Delta]]$ and $R[[\Delta]]$. We can now use Lemma 5.7 to see that j is an isomorphism. □

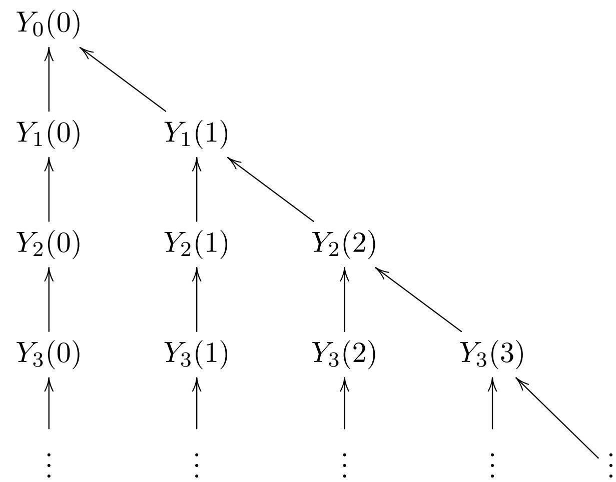
In general, $\Delta_\Gamma(Y)$ need not be closed in Y as the following example shows.

Example 5.11. Let $\Gamma = \mathbb{Z}_p$. For each $n \geq 0$, let Y_n be the disjoint union of $n+1$ Γ-orbits $Y_n(0), Y_n(1), \cdots, Y_n(n)$, where $|Y_n(m)| = p^m$ for all $n \geq m \geq 0$. Choose base points

$y_n(m) \in Y_n(m)$ for each $n \geq m \geq 0$. For each $n \geq 0$, define $\pi_{n+1} : Y_{n+1} \to Y_n$ to be the unique Γ-equivariant map such that

$$\pi_{n+1}(y_{n+1}(m)) = \begin{cases} y_n(m) & \text{if} \quad n \geq m \geq 0 \\ y_n(n) & \text{if} \quad m = n+1. \end{cases}$$

The restriction of π_{n+1} to $Y_{n+1}(m)$ is an isomorphism onto $Y_n(m)$ whenever $n \geq m \geq 0$. The projective limit $Y = \varprojlim Y_n$ is naturally a profinite set equipped with a continuous Γ-action.



For each $m \geq 0$, let $y_\infty(m) := (y_n(m))_{n=0}^\infty \in Y$. Then the Γ-orbit of $y_\infty(m)$ has size exactly p^m for all $m \geq 0$, so that $y_\infty(m) \in \Delta_\Gamma(Y)$ for all $m \geq 0$.

Suppose now that $z \in Y$ has infinite Γ-orbit. Write $z = (z_n)_{n=0}^\infty$ where $z_n \in Y_n$ and suppose for a contradiction that for some $n \geq 0$, $z_n \notin Y_n(n+1)$. Then $z_n \in Y_n(m)$ for some $m \leq n$. Then for all $k \geq n$, z_k must lie in the preimage of $Y_n(m)$ in Y_k, which is $Y_k(m)$ by construction. But then $p^m\mathbb{Z}_p$ fixes z_k for all $k \geq n$ and hence $p^m\mathbb{Z}_p$ fixes z as well. This contradicts the hypothesis that the Γ-orbit of z was infinite. Therefore z_n must lie in $Y_n(n+1)$ for each $n \geq 0$, and we conclude that $Y \backslash \Delta_\Gamma(Y)$ consists of a single infinite Γ-orbit. Since this orbit is necessarily closed, it follows that $\Delta_\Gamma(Y) = \bigcup_{m=0}^{\infty} \Gamma \cdot y_\infty(m)$ is open.

Finally, for any $n \geq 0$, the image of both z and $y_\infty(n+1)$ in Y_n is contained in $Y_n(n+1)$. Hence the image of z in Y_n is contained in the image of $\Delta_\Gamma(Y)$. Hence z lies in the closure of $\Delta_\Gamma(Y)$, but $z \notin \Delta_\Gamma(Y)$. So, $\Delta_\Gamma(Y)$ is not closed.

6. Twisted actions on completed group rings

From here on we assume that the field k has **characteristic** p**.** We keep our fixed compact open subgroup U of G but assume that it is pro-p, and we let $w \in G$ be any fixed element. Recall that $U_w = U \cap wUw^{-1}$ is an open subgroup of U. We introduce the twisted conjugation action of U_w on U, given by

$$\begin{aligned} U_w \times U &\longrightarrow U \\ (u, x) &\longmapsto (w^{-1}uw)xu^{-1} \ . \end{aligned}$$

Let $\Delta_{U_w}(U)$ be the set of finite U_w-orbits in U under this action and for any such orbit $\mathcal{O}$ recall that $\upsilon_U(\mathcal{O}) \in k[U]$ denotes the sum of all its elements. This action extends to a twisted conjugation action of U_w on $k[[U]]$, and $k[[U]]_w = k[[U]]^{U_w,*}$ simply is the subspace of fixed points under this action. Since U is pro-p, Prop. 5.9 gives the following result.

Proposition 6.1. *The k-vector subspace of $k[U]$ spanned by the orbit sums $v_U(\mathcal{O})$ for $\mathcal{O} \in \Delta_{U_w}(U)$ is dense in $k[[U]]^{U_w,*} = k[[U]]_w$.*

Next we make the following simple observation.

Lemma 6.2. *The U_w-orbit of an element $x \in U$ under the twisted action is finite if and only if the centraliser $C_G(wx)$ of the element $wx \in G$ is open in G.*

Proof. Obviously the orbit of x is finite if and only if the stabiliser $S_x := \{u \in U_w : xu = (w^{-1}uw)x\}$ is open in U_w. But for a general $g \in G$ (with w, x fixed) we have $xg = (w^{-1}gw)x$ if and only if $(wx)g = g(wx)$ if and only if $g \in C_G(wx)$. So, $S_x = C_G(wx) \cap U_w$. Thus, the orbit of x is finite if and only if S_x is open in U_w if and only if $C_G(wx)$ is open in G. □

Lemma 6.3. *Suppose that G contains no proper open centralisers. Then $Z(U) = Z(G) \cap U$.*

Proof. Obviously $Z(G) \cap U \subseteq Z(U)$. Let therefore $x \in Z(U)$. Then $U \subseteq C_G(x)$ so that $C_G(x)$ is open in G. Since G contains no proper open centralisers, we conclude that $x \in Z(G)$. □

We assume from now on that G contains no proper open centralisers.

Proposition 6.4. *For any $w \in \mathcal{W}$, we have $\Delta_{U_w}(U) = U \cap w^{-1}Z(G)$.*

Proof. The U_w-orbit of $x \in U$ is finite if and only if the centraliser $C_G(wx)$ is open in G by Lemma 6.2. But this is equivalent to $wx \in Z(G)$ because G contains no proper open centralisers. □

Corollary 6.5. *Let $w \in \mathcal{W}$. Then*

$$\Delta_{U_w}(U) = \begin{cases} U \cap Z(G) & \text{if } w \in Z(G), \\ \emptyset & \text{if } w \notin Z(G)U. \end{cases}$$

Proof. This follows immediately from Prop. 6.4. □

For convenience we choose a set of representatives $\mathcal{Z} \subseteq Z(G)$ for the double cosets of U contained in $Z(G)U$. We further assume that $\mathcal{Z} \subseteq \mathcal{W}$.

Proposition 6.6. *For any $w \in \mathcal{W}$, we have*

$$k[[U]]_w = \begin{cases} k[[Z(U)]] & \text{if } w \in \mathcal{Z}, \\ 0 & \text{if } w \notin \mathcal{Z}. \end{cases}$$

Proof. Prop. 6.4 implies that $\Delta_{U_w}(U)$ is closed in U, so we may apply Thm. 5.10 to obtain

$$k[[U]]_w = k[[U]]^{U_w,*} = k[[\Delta_{U_w}(U)]]^{U_w,*}.$$

Since $\Delta_{U_w}(U) = \emptyset$ when $w \notin \mathcal{Z}$ by Cor. 6.5, we may assume that $w \in \mathcal{Z}$. But then since $w \in Z(G)$, we have $U_w = U$ and the twisted action is the usual conjugation action of U on itself. The result follows, because $\Delta_{U_w}(U) = Z(U)$ by Cor. 6.5 and Lemma 6.3. □

Write $Z := Z(G)$. The following is straightforward.

Lemma 6.7. *The restriction map $\mathrm{Res}^G_Z : C^\infty_c(G,k) \to C^\infty_c(Z,k)$ is surjective, and $\widehat{k[Z]}$-equivariant.*

Let $C := \mathrm{ind}_{U_\ell}^{G_\ell}(C^\infty(U,k))$ and let $C_Z := \mathrm{ind}_{Z(U)_\ell}^{Z_\ell}(C^\infty(Z(U),k))$. From (13) together with Prop. 6.6 we have the injective map

$$\Omega_U : \mathrm{End}_{\mathrm{Mod}(G\times U)}(C) \longrightarrow \prod_{z\in\mathcal{Z}} \mathrm{End}_{\mathrm{Mod}(Z(U))}(C^\infty(Z(U),k)) = \prod_{z\in\mathcal{Z}} k[[Z(U)]] \ .$$

On the other hand, since $\mathcal{Z}$ is a transversal for $Z(U)$ in Z, replacing G by Z gives us the map $\Omega_{Z(U)}$ out of $\mathrm{End}_{\mathrm{Mod}(Z)}(C_Z)$ with the same target. These two maps appear in the following diagram:

$$\begin{array}{ccccc} \widehat{k[Z]} & \xrightarrow{\quad\Phi_C\quad} & \mathrm{End}_{\mathrm{Mod}(G\times G)}(C) & \xrightarrow{\subseteq} & \mathrm{End}_{\mathrm{Mod}(G\times U)}(C) \\ \downarrow{\scriptstyle \Phi_{C_Z}} & & & & \downarrow{\scriptstyle \Omega_U} \\ \mathrm{End}_{\mathrm{Mod}(Z)}(C_Z) & & \xrightarrow[\quad\Omega_{Z(U)}\quad]{} & & \prod_{z\in\mathcal{Z}} k[[Z(U)]] \end{array} \tag{18}$$

Lemma 6.8. *The diagram* (18) *is commutative.*

Proof. First, consider the following diagram:

$$\begin{array}{ccccc} C^\infty(U,k) & \xrightarrow{\quad\iota^G\quad} & \mathrm{ind}_{U_\ell}^{G_\ell}(C^\infty(U,k)) & \xrightarrow[\cong]{F_G} & C_c^\infty(G,k) \\ \downarrow{\scriptstyle \mathrm{Res}^U_{Z(U)}} & & \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \mathrm{Res}^G_Z} \\ C^\infty(Z(U),k) & \xrightarrow[\iota^Z]{} & \mathrm{ind}_{Z(U)}^{Z}(C^\infty(Z(U),k)) & \xrightarrow[F_Z]{\cong} & C_c^\infty(Z,k) \end{array} \tag{19}$$

where F_G and F_Z are the maps of Transitivity of Induction from (1), and ι^G and ι^Z are the Frobenius reciprocity embeddings appearing in (4). From the definitions of F and ι it is straightforward to verify that the composed horizontal maps are simply the extension by zero maps. As $Z(U) = U \cap Z$ by Lemma 6.3 the outer rectangle is commutative. Since F_G and F_Z are isomorphisms in $\mathrm{Mod}(G_\ell)$ and $\mathrm{Mod}(Z_\ell)$, respectively, and since $\mathrm{Res}^G_Z : C_c^\infty(G,k) \to C_c^\infty(Z,k)$ is $\widehat{k[Z]}$-linear by Lemma 6.7, there is a unique $\widehat{k[Z]}$-linear map ψ making the entire diagram commutative. Note that the inverse of F_G is given as follows: $F_G^{-1}(f)(g)(u) = f(gu)$ for all $f \in C_c^\infty(G,k), g \in G$ and $u \in U$. Now, given $q \in \mathrm{ind}_{U_\ell}^{G_\ell} C^\infty(U,k)$, $z \in Z$ and $y \in Z(U)$, we calculate

$$\begin{aligned} \psi(q)(z)(y) &= (F_Z^{-1} \circ \mathrm{Res}^G_Z \circ F_G)(q)(z)(y) = (\mathrm{Res}^G_Z F_G(q))(zy) \\ &= F_G(q)(zy) = q(zy)(1) = ({}^{y_\ell^{-1}}q(z))(1) = q(z)(y). \end{aligned}$$

Therefore the map ψ is given on $q \in \mathrm{ind}_{U_\ell}^{G_\ell}(C^\infty(U,k))$ by the explicit formula

$$\psi(q) = \mathrm{Res}^U_{Z(U)} \circ q_{|Z}. \tag{20}$$

Fix $x \in \widehat{k[Z]}$. Let $\alpha := \Phi_C(x)$ denote the action of x on C, and let $\beta := \Phi_{C_Z}(x)$ denote the action of x on C_Z. Using formula (9), we see that to prove the Lemma, it will be enough to show that $\Omega_z(\alpha) = \Omega_z(\beta)$ for every $z \in \mathcal{Z}$. To this end, fix $h \in C^\infty(Z(U),k)$ and choose $f \in C^\infty(U,k)$ such that $h = \mathrm{Res}^U_{Z(U)} f$ using Lemma 6.7 applied to U in place of G. Then we can calculate as follows:

$$\begin{aligned}
\Omega_z(\beta)(h) &= \beta(\iota^Z(\mathrm{Res}^U_{Z(U)} f))(z) && \text{by (9)}\\
&= \beta(\psi(\iota^G f))(z) && \text{by (19)}\\
&= \psi(\alpha(\iota^G f))(z) && \\
&= \mathrm{Res}^U_{Z(U)}(\alpha(\iota^G f)(z)) && \text{by (20)}\\
&= \mathrm{Res}^U_{Z(U)}\, \Omega_z(\alpha)(f) && \text{by (9)}\\
&= \Omega_z(\alpha)(\mathrm{Res}^U_{Z(U)} f) && \\
&= \Omega_z(\alpha)(h)
\end{aligned}$$

where on the third line we used the $\widehat{k[Z]}$-linearity of ψ, and on the sixth line we used the $k[[Z(U)]]$-linearity of $\mathrm{Res}^U_{Z(U)} : C^\infty(U,k) \to C^\infty(Z(U),k)$. The result follows, since $k[[Z(U)]]$ acts faithfully on $C^\infty(Z(U),k)$. □

Proposition 6.9. *Let G be a locally pro-p group which contains no proper open centralisers and let k be a field of characteristic p. For $C := \mathrm{ind}^{G_\ell}_{U_\ell}(C^\infty(U,k))$ the natural map*

$$\Phi_C : \widehat{k[Z(G)]} \xrightarrow{\cong} \mathrm{End}_{\mathrm{Mod}(G\times G)}(C)$$

is an isomorphism.

Proof. Suppose given $(\lambda_z)_{z\in\mathcal{Z}} \in \prod_{z\in\mathcal{Z}} k[[Z(U)]]$ such that for all $h \in C^\infty(U,k)$, $\lambda_z(h) = 0$ for all but finitely many $z \in \mathcal{Z}$. Let $f \in C^\infty(Z(U),k)$ be given and choose $h \in C^\infty(U,k)$ extending f using Lemma 6.7. Then $\lambda_z(f) = \lambda_z(\mathrm{Res}^U_{Z(U)}(h)) = \mathrm{Res}^U_{Z(U)}(\lambda_z(g)) = 0$ for all but finitely many $z \in \mathcal{Z}$. In view of condition (14), this shows that in the diagram (18), the image of Ω_U is contained in the image of $\Omega_{Z(U)}$. Write $C_Z := \mathrm{ind}^{Z_\ell}_{Z(U)_\ell}(C^\infty_c(Z,k))$. Since Ω_U is injective by (13), this gives us an injective map θ which appears as the diagonal arrow in the diagram (18):

$$\begin{array}{ccccc}
\widehat{k[Z]} & \xrightarrow{\Phi_C} & \mathrm{End}_{\mathrm{Mod}(G\times G)}(C) & \xrightarrow{\subseteq} & \mathrm{End}_{\mathrm{Mod}(G\times U)}(C)\\
\downarrow{\scriptstyle \Phi_{C_Z}} & & {\scriptstyle \theta}\swarrow & & \downarrow{\scriptstyle \Omega_U}\\
\mathrm{End}_{\mathrm{Mod}(Z)}(C_Z) & & \xrightarrow[\Omega_{Z(U)}]{} & & \prod_{z\in\mathcal{Z}} k[[Z(U)]]
\end{array}$$

and which makes the bottom triangle commutative. Since the rectangle commutes by Lemma 6.8 and since $\Omega_{Z(U)}$ is injective, the top triangle commutes. Now, C_Z is isomorphic to $C^\infty_c(Z,k)$ by (1), so Prop. 3.9 implies that Φ_{C_Z} is an isomorphism. Since θ is injective, we can now conclude that Φ_C is surjective. But since C is isomorphic to $C^\infty_c(G,k)$ in $\mathrm{Mod}(G)$ by (1), Φ_C is also injective by Prop. 3.8(1), and this completes the proof. □

Theorem 6.10. *Let G be a locally pro-p group which contains no proper open centralisers and let k be a field of characteristic p. Then the natural map*

$$\Phi : \widehat{k[Z(G)]} \xrightarrow{\cong} \mathfrak{Z}(G)$$

from Lemma 3.7 is an isomorphism.

Proof. By Prop. 3.8(2), we have the commutative diagram

$$\begin{array}{ccc} k[\widehat{Z(G)}] & \xrightarrow{\Phi} & \mathfrak{Z}(G) \\ & \searrow^{\Phi_C} & \downarrow \Theta \\ & & \mathrm{End}_{\mathrm{Mod}(G\times G)}(C). \end{array}$$

Now Φ_C is an isomorphism by Proposition 6.9, and Θ is injective by the injectivity part of Prop. 3.1. Hence Φ is an isomorphism. □

Corollary 6.11. *Let G be a locally pro-p group which contains no proper open centralisers and let k be a field of characteristic p. Then the k-algebra homomorphism $\mathfrak{Z}(Z(G)) \xrightarrow{\cong} \mathfrak{Z}(G)$ from Cor. 3.10 is an isomorphism.*

Finally, we specialise to the case where $G = \mathbf{G}(\mathfrak{F})$ for some connected smooth algebraic group $\mathbf{G}$ over a local nonarchimedean field $\mathfrak{F}$ of residue characteristic p. Certainly then G is locally pro-p (cf. [Ser] Thm. in §II.IV.8 and Cor. 2 in §II.IV.9). We quickly remind the reader that the smoothness condition is automatic if $\mathfrak{F}$ has characteristic zero ([DG] II §6.1.1). We need the following fact which is known to the experts. We provide a proof since we could not find one in the literature.

Proposition 6.12. *Let X be a smooth scheme over the local field $\mathfrak{F}$, and let $Y \subseteq X$ be a Zariski open dense subset. Then $Y(\mathfrak{F})$ is dense in $X(\mathfrak{F})$ w.r.t. the valuation topology.*

Proof. First we recall the elementary fact that the density of Y in X implies the density of $Y \cap X'$ in any open subset X' of X (check that any nonempty open subset of X' must meet Y non-trivially). Hence we may assume that X is affine.

By passing to one of the finitely many connected components of X we may further assume that X is connected. If $X(\mathfrak{F}) = \emptyset$ then there is nothing to prove. We therefore assume that $X(\mathfrak{F}) \neq \emptyset$. Then X is geometrically connected by [Sta] Lemma 33.7.14. The smoothness finally implies that X is geometrically irreducible. Therefore we may use the implicit function theorem in the form of [GPR] Thm. 9.2. It says the following:

Let $x \in X(\mathfrak{F})$ be a point and let $\mathbf{t} = (t_1, \ldots, t_d)$ be a system of local parameters at x. Choose a sufficiently small open neighbourhood $x \in U \subseteq X$ on which the t_i are defined. This gives us a morphism $U \to \mathbf{A}^d$ to the affine space over $\mathfrak{F}$ which sends x to 0. Then there exists an open neighbourhood $x \in B \subseteq U(\mathfrak{F})$ w.r.t. the valuation topology which is mapped by $\mathbf{t}$ homeomorphically onto a polydisk $B_r(0) \subseteq \mathbf{A}^d(\mathfrak{F})$ around 0 of sufficiently small radius r.

We apply this now to a point $x \in X(\mathfrak{F}) \setminus Y(\mathfrak{F})$. The open neighbourhood $x \in U \subseteq X$ can be chosen to be affine. By the irreducibility of X the Zariski closed subset $U \setminus Y$ of U cannot be equal to U. Hence the restriction map $\mathcal{O}(U) \to \mathcal{O}(U \setminus Y)$ is not injective. We pick a function $f \neq 0$ in the kernel. By possibly making the polydisk B smaller we may view $f = F(t_1, \ldots, t_d)$ as a nonzero formal power series convergent on B. The zero set of this power series is nowhere dense in B by the comment after Cor. 5 on p. 198 in [BGR]. Hence we can find a sequence $(x_n)_{n\in\mathbb{N}}$ of points in $B \subseteq U(\mathfrak{F})$ such that

$$\lim_{n\to\infty} x_n = x \text{ w.r.t. the valuation topology and } f(x_n) \neq 0 \text{ for any } n \in \mathbb{N}.$$

The latter says that each $x_n \in Y(\mathfrak{F})$, because f vanishes on $U \setminus Y$. The former then implies that x lies in the closure of $Y(\mathfrak{F})$ w.r.t. the valuation topology. □

Theorem 6.13. *Let* $\mathbf{G}$ *be a connected smooth algebraic group over a local nonarchimedean field* $\mathfrak{F}$ *of residue characteristic* p*, and let* k *be a field of characteristic* p*. Then* $G = \mathbf{G}(\mathfrak{F})$ *contains no proper open centralisers, and we have* $k[\widehat{Z(G)}] \cong \mathfrak{Z}(Z(G)) \cong \mathfrak{Z}(G)$*.*

Proof. Suppose that the centraliser $C_G(y)$ of some $y \in G$ is proper; then $y \notin Z(G)$. Now, $C_G(y) = C_{\mathbf{G}}(y)(\mathfrak{F})$, where the centraliser $C_{\mathbf{G}}(y)$ in $\mathbf{G}$ of the closed point y is a closed algebraic $\mathfrak{F}$-subgroup of $\mathbf{G}$ by [DG] II §1 Cor. 3.7. Since $y \notin Z(G)$, $\mathbf{G} \setminus C_{\mathbf{G}}(y)$ is a nonempty Zariski open subset of $\mathbf{G}$. Since $\mathbf{G}$ is irreducible by [DG] II §5 Thm. 1.1 it also is Zariski dense in $\mathbf{G}$. So we may apply Prop. 6.12 with $X = \mathbf{G}$ and $Y = \mathbf{G} \setminus C_{\mathbf{G}}(y)$ to obtain that $G \setminus C_G(y) = (\mathbf{G} \setminus C_{\mathbf{G}}(y))(\mathfrak{F})$ is dense in $\mathbf{G}(\mathfrak{F}) = G$. Hence $C_G(y)$ cannot be open in G.

The second part of the assertion now follows from Thm. 6.10 and Cor. 6.11. □

Corollary 6.14. *Suppose that the connected smooth algebraic group* $\mathbf{G}$ *is semisimple and that* k *has characteristic* p*; then the Bernstein centre of* $\mathrm{Mod}(G)$ *is* $\mathfrak{Z}(G) = k[Z(G)]$*.*

Proof. Under the present assumption $Z(G)$ is finite. □

7. Appendix: Positselski's Theorem

We give a proof of Prop. 3.1 which is more direct than the one in [Pos].

For the **injectivity** of Θ let $(T_V)_V \in \mathfrak{A}(G)$ such that $T_{C_c^\infty(G,k)} = 0$. Since $C_c(G/U,k)$, for any compact open subgroup $U \subseteq G$, is a G_ℓ-subrepresentation of $C_c^\infty(G,k)$ it follows that $T_{C_c^\infty(G/U,k)} = 0$. Now consider any object V in $\mathrm{Mod}(G)$. Any vector $v \in V$ is fixed by some compact open subgroup $U_v \subseteq G$. By Frobenius reciprocity we obtain a G-equivariant map $\rho_v : C_c(G/U_v,k) \to V$ sending the characteristic function of U_v to v. We deduce that $T_{\mathrm{im}(\rho_v)} = 0$. By varying the vector v finally obtain that $T_V = 0$.

The argument for the **surjectivity** of Θ is slightly more involved.

Lemma 7.1. *Inside the algebra* $\mathrm{End}_k(C_c^\infty(G,k))$ *the subalgebras* $\mathrm{End}_{\mathrm{Mod}(G_\ell)}(C_c^\infty(G,k))$ *and* $\mathrm{End}_{\mathrm{Mod}(G_r)}(C_c^\infty(G,k))$ *are mutually centralisers of each other.*

Proof. Since $k[G_r] \subseteq \mathrm{End}_{\mathrm{Mod}(G_\ell)}(C_c^\infty(G,k))$ and $k[G_\ell] \subseteq \mathrm{End}_{\mathrm{Mod}(G_r)}(C_c^\infty(G,k))$ it is immediate that the centraliser of one algebra is contained in the other algebra. For the opposite inclusions we have to show that $\beta \circ \alpha = \alpha \circ \beta$ for any $\alpha \in \mathrm{End}_{\mathrm{Mod}(G_\ell)}(C_c^\infty(G,k))$ and $\beta \in \mathrm{End}_{\mathrm{Mod}(G_r)}(C_c^\infty(G,k))$. Since

$$\bigcup_U C_c(U\backslash G,k) = C_c^\infty(G,k) = \bigcup_U C_c(G/U,k) ,$$

where U runs over all compact open subgroups of G, both homomorphisms α and β are completely determined by their values on the characteristic functions char_U. Moreover, $\alpha(\mathrm{char}_U) \in C_c(U\backslash G,k)$ and $\beta(\mathrm{char}_U) \in C_c(G/U,k)$. We have to check that $\beta\circ\alpha(\mathrm{char}_{xU}) = \alpha\circ\beta(\mathrm{char}_{xU})$ for any U and any $x \in G$. Let $\alpha(\mathrm{char}_U) = \sum_i c_i \,\mathrm{char}_{Ug_i}$ and $\beta(\mathrm{char}_{xUx^{-1}}) = \sum_j d_j \,\mathrm{char}_{h_j xUx^{-1}}$ for finitely many constants $c_i, d_j \in k$ and elements $g_i, h_j \in G$ (depending on x and U). We compute

$$\begin{aligned}\beta\circ\alpha(\mathrm{char}_{xU}) &= \beta(\sum_i c_i\,\mathrm{char}_{xUg_i}) = \sum_i c_i\beta(\mathrm{char}_{xUg_i}) = \sum_i c_i\beta(\mathrm{char}_{xUx^{-1}xg_i})\\ &= \sum_i c_i\beta({}^{(xg_i)_r^{-1}}\,\mathrm{char}_{xUx^{-1}}) = \sum_i c_i{}^{(xg_i)_r^{-1}}\beta(\mathrm{char}_{xUx^{-1}}) =\\ &= \sum_{i,j} c_i d_j{}^{(xg_i)_r^{-1}}\,\mathrm{char}_{h_jxUx^{-1}} = \sum_{i,j} c_i d_j\,\mathrm{char}_{h_jxUg_i}\end{aligned}$$

using that α is G_ℓ-equivariant, resp. β is G_r-equivariant, in the first, resp. fifth, equality. Correspondingly

$$\begin{aligned}\alpha\circ\beta(\mathrm{char}_{xU}) &= \alpha(\beta({}^{x_r^{-1}}\,\mathrm{char}_{xUx^{-1}})) = \alpha({}^{x_r^{-1}}\beta(\mathrm{char}_{xUx^{-1}})) = \alpha({}^{x_r^{-1}}\sum_j d_j\,\mathrm{char}_{h_jxUx^{-1}})\\ &= \sum_j d_j\alpha(\mathrm{char}_{h_jxU}) = \sum_{i,j} c_i d_j\,\mathrm{char}_{h_jxUg_i}\ .\end{aligned}$$

□

Lemma 7.2. *Suppose that U_i, for $i = 1, 2$, are compact open subgroups of G; any map $\rho \in \mathrm{Hom}_{\mathrm{Mod}(G)}(C_c(G/U_1, k), C_c(G/U_2, k))$ is the restriction of a map $\tilde\rho \in \mathrm{End}_{\mathrm{Mod}(G_\ell)}(C_c^\infty(G, k))$.*

Proof. By Frobenius reciprocity we have

$$\mathrm{Hom}_{\mathrm{Mod}(G)}(C_c(G/U_1, k), C_c(G/U_2, k)) = C_c(U_1\backslash G/U_2, k)$$

with ρ corresponding to $\rho(\mathrm{char}_{U_1})$ and

$$\begin{aligned}\mathrm{End}_{\mathrm{Mod}(G_\ell)}(C_c^\infty(G, k)) &= \mathrm{Hom}_{\mathrm{Mod}(G_\ell)}(\bigcup_{U\subseteq U_1} C_c(G/U, k), C_c^\infty(G, k))\\ &= \varprojlim_{U\subseteq U_1} \mathrm{Hom}_{\mathrm{Mod}(G_\ell)}(C_c(G/U, k), C_c^\infty(G, k))\\ &= \varprojlim_{U\subseteq U_1} C_c(U\backslash G, k)\end{aligned}$$

with $\tilde\rho$ corresponding to $(\tilde\rho(\mathrm{char}_U))_U$. The transition map $C_c(U'\backslash G, k) \to C_c(U\backslash G, k)$, for $U' \subseteq U$, in the projective system is given by $\mathrm{char}_{U'g} \mapsto \mathrm{char}_{Ug}$. Suppose that $\rho(\mathrm{char}_{U_1}) = \sum_i c_i\,\mathrm{char}_{U_1g_i}$. Then $(\sum_i c_i\,\mathrm{char}_{Ug_i})_U$ defines a $\tilde\rho$ as in the assertion which restricts to ρ. □

We now fix a $T \in \mathrm{End}_{\mathrm{Mod}(G_r)}(C_c^\infty(G, k))$, and we have to construct a $(T_V)_V \in \mathfrak{A}(G)$ such that $T_{C_c^\infty(G,k)} = T$. Of course, we turn the latter condition into the definition $T_{C_c^\infty(G,k)} := T$ and extend it in several steps to all of $\mathrm{Mod}(G)$.

Step 1: Let $\mathfrak{C}_0$ denote the full subcategory of $\mathrm{Mod}(G)$ with the single object $C_c^\infty(G, k)$. Then Lemma 7.1 tells us that $T_{C_c^\infty(G,k)} = T$ is a natural transformation of the forgetful functor $\mathfrak{C}_0 \to \mathrm{Vect}$.

Step 2: Let $\mathfrak{C}_1$ denote the full subcategory of $\mathrm{Mod}(G)$ with objects representations of the form $C_c(G/U, k)$ for $U \subseteq G$ some compact open subgroup. We have $C_c(G/U, k) = C_c^\infty(G, k)^{U_r} \subseteq C_c^\infty(G, k)$. The map T being G_r-equivariant it respects the subrepresentation $C_c(G/U, k)$, and we may define $T_{C_c(G/U,k)} := T|C_c(G/U, k)$. It follows from Lemma 7.2 and Step 1 that this defines a natural transformation of the forgetful functor $\mathfrak{C}_1 \to \mathrm{Vect}$.

Step 3: Let $\mathfrak{C}$ denote the full subcategory of $\mathrm{Mod}(G)$ with objects arbitrary direct sums of objects in $\mathfrak{C}_1$. If $V = \oplus_{i\in I}V_i$ with V_i in $\mathfrak{C}_1$ then we simply define $T_V := \oplus_{i\in I}T_{V_i}$. Let

$V' = \oplus_{j \in J} V'_j$ be another object in $\mathfrak{C}$ with V'_j in $\mathfrak{C}_1$ and let $\gamma : V \to V'$ be a homomorphism in $\mathrm{Mod}(G)$. There are homomorphisms $\gamma_{i,j} : V_i \to V'_j$ in $\mathrm{Mod}(G)$ such that $\gamma((v_i)_i) = (\sum_i \gamma_{i,j}(v_i))_j$. Using Step 2 in the fourth equality we compute

$$\begin{aligned} T_{V'} \circ \gamma((v_i)_i) &= T_{V'}((\sum_i \gamma_{i,j}(v_i))_j) = (T_{V'_j}(\sum_i \gamma_{i,j}(v_i)))_j = (\sum_i T_{V'_j}(\gamma_{i,j}(v_i)))_j \\ &= (\sum_i \gamma_{i,j}(T_{V_i}(v_i)))_j = \gamma((T_{V_i}(v_i))_i) = \gamma \circ T_V((v_i)_i) \ . \end{aligned}$$

This shows that our definition extends T further to a natural transformation of the forgetful functor $\mathfrak{C} \to \mathrm{Vect}$.

Step 4: In the proof of injectivity we have seen already that for any V in $\mathrm{Mod}(G)$ there is a surjective homomorphism $C \twoheadrightarrow V$ in $\mathrm{Mod}(G)$ with C in $\mathfrak{C}$.

Step 5: Consider a surjective homomorphism $\beta : C \twoheadrightarrow V$ as well as a surjective homomorphism $\gamma : C' \twoheadrightarrow \ker(\beta)$ as in Step 4. By Step 3 we then have the commutative diagram

$$\begin{array}{ccccc} C' & \twoheadrightarrow & \ker(\beta) & \xrightarrow{\subseteq} & C \\ {\scriptstyle T_{C'}}\downarrow & & & & \downarrow{\scriptstyle T_C} \\ C' & \twoheadrightarrow & \ker(\beta) & \xrightarrow{\subseteq} & C. \end{array}$$

This shows that T_C respects $\ker(\beta)$ and induces therefore a map T_V on V. We first check that T_V is independent of the choices. For this let $\beta_i : C_i \twoheadrightarrow V$, for $i = 1, 2$, be two surjective homomorphisms as in Step 4. At first we consider the case that there is a homomorphism $\gamma : C_1 \to C_2$ in $\mathrm{Mod}(G)$ such that $\beta_2 \circ \gamma = \beta_1$. Let τ_i be the linear endomorphism of V induced by T_{C_i}. Given a $v \in V$ we choose a preimage $x \in C_1$ such that $\beta_1(x) = v$. Using Step 3 in the third equality we compute

$$\tau_1(v) = \beta_1(T_{C_1}(x)) = \beta_2 \circ \gamma \circ T_{C_1}(x) = \beta_2 \circ T_{C_2} \circ \gamma(x) = \tau_2(\beta_2(\gamma(x))) = \tau_2(\beta_1(x)) = \tau_2(v).$$

In general without the presence of a γ we consider the commutative diagram

$$\begin{array}{ccccc} C_1 & \longrightarrow & C_1 \oplus C_2 & \longleftarrow & C_2 \\ & {\scriptstyle \beta_1}\searrow & \downarrow{\scriptstyle \beta_1 + \beta_2} & \swarrow{\scriptstyle \beta} & \\ & & V. & & \end{array}$$

with the horizontal maps being the canonical ones and apply the previous case to both halves.

Step 6: Finally we have to show that the thus defined T_V constitute a natural transformation of the forgetful functor $\mathrm{Mod}(G) \to \mathrm{Vect}$, i.e., we have to check that, for any homomorphism $\gamma : V_1 \to V_2$ in $\mathrm{Mod}(G)$ the diagram

$$\begin{array}{ccc} V_1 & \xrightarrow{\gamma} & V_2 \\ {\scriptstyle T_{V_1}}\downarrow & & \downarrow{\scriptstyle T_{V_2}} \\ V_1 & \xrightarrow{\gamma} & V_2 \end{array}$$

is commutative. We choose two surjective homomorphisms $\beta_i : C_i \twoheadrightarrow V_i$, for $i = 1, 2$, as in Step 4 and obtain the commutative diagram

$$\begin{array}{ccc} C_1 & \xrightarrow{\subseteq} & C_1 \oplus C_2 \\ {\scriptstyle \beta_1}\big\downarrow & & \big\downarrow{\scriptstyle \gamma\circ\beta_1+\beta_2} \\ V_1 & \xrightarrow{\gamma} & V_2. \end{array}$$

It then follows from Step 3 that $\gamma \circ T_{V_1} = T_{V_2} \circ \gamma$.

References

[Ard] Ardakov K.: *The centre of completed group algebras of pro-p-groups*. Doc. Math. 9, 599 - 606 (2004)

[Ber] Bernstein J: *Le "centre" de Bernstein*. In *Representations of reductive groups over a local field*, edited by Pierre Deligne. pp. 1–32, Hermann 1984

[BGR] Bosch S., Güntzer U., Remmert R.: *Non-Archimedean Analysis*. Springer Grundlehren der math. Wissenschaften, vol. 261 (1984)

[Bou] Bourbaki N.: *General Topology*. Chapters 1-4, Springer 1995

[BH+] Breuil C., Herzig F., Hu Y., Morra S, Schraen B.: *Conjectures and results on modular representations of $GL_n(K)$ for a p-adic field K*, preprint. (2021) https://arxiv.org/abs/2102.06188

[BP] Breuil, C., Paškūnas, V.: *Towards a modulo p Langlands correspondence for GL_2*. Mem. Amer. Math. Soc. 216(1016) (2012)

[Dot] Dotto, A.: *mod-p Bernstein centres of p-adic groups*, preprint. (2021) https://arxiv.org/pdf/2105.09367.pdf

[DG] Demazure M., Gabriel P.: *Groupes Algébriques*. North-Holland: Amsterdam 1970

[GPR] Green B., Pop F., Roquette P.: *On Rumely's Local-Global Principle*. Jber. Dt. Math.-Verein. 97, 43 - 74 (1995)

[Hen] Henniart, G.: *Représentations des groupes réductifs p-adiques*. Séminaire Bourbaki, 736, 193 - 219, 1990-1991

[Kel] Kelley, J.L.: *General topology*. Graduate Texts in Mathematics 27, Springer 1975

[Pos] Positselski L.: *Smooth duality and co-contra correspondence*. J. Lie Theory 30(1), 85 - 144 (2020)

[Sch] Schneider P.: *Smooth representations and Hecke modules in characteristic p*. Pacific J. Math. 279, 447 - 464 (2015)

[Ser] Serre J.-P.: *Lie Algebras and Lie groups*. Springer Lect. Notes Math. 1500 (1992)

[Sta] Stacksproject: stacks.math.columbia.edu/tag/04KV

[Vig96] Vignéras M.-F.: *Représentations ℓ-modulaires d'un groupe réductif p-adique avec $\ell \neq p$*. Progress in Math. 137, Birkhäuser: Boston 1996

[Vig04] Vignéras M.-F.: *Representations modulo p of the p-adic group $GL(2, F)$*. Compos. Math., 140(2), 333 – 358 (2004)

[Wil] Wilson J. S.: *Profinite Groups*. LMS Monographs. New Series, 19. The Clarendon Press, OUP, New York, 1998

Mathematical Institute, Woodstock Road, University of Oxford, Oxford OX2 6GG, UK
Email address: ardakov@maths.ox.ac.uk
URL: http://people.maths.ox.ac.uk/ardakov/

Universität Münster, Mathematisches Institut, Einsteinstr. 62, 48149 Münster, Germany
Email address: pschnei@wwu.de
URL: https://www.uni-muenster.de/Arithm/schneider/