The centre of completed group algebras of pro-p groups

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Abstract. We compute the centre of the completed group algebra of an arbitrary countably based pro-p group with coefficients in $\mathbb{F}_p$ or $\mathbb{Z}_p$. Some other results are obtained.

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1 Introduction

Let G be a pro-p group. In this paper we investigate some rings related to the completed group algebra of G over $\mathbb{F}_p$, which we denote by Ω_G:

$$\Omega_G = \mathbb{F}_p[[G]] := \varprojlim_{N \lhd_o G} \mathbb{F}_p[G/N].$$

When G is analytic in the sense of [3], Ω_G and its p-adic analogue Λ_G defined by

$$\Lambda_G = \mathbb{Z}_p[[G]] := \varprojlim_{N \lhd_o G} \mathbb{Z}_p[G/N]$$

are right and left Noetherian rings, which are in general noncommutative. If in addition G is torsion free, the results of Brumer, Neumann and others show that Ω_G and Λ_G have finite global dimension and have no zero-divisors; for an overview, see [1]. Moreover, under the name of Iwasawa algebras, these rings are frequently of interest to number theorists (see [2] for more details).
Our main result is

THEOREM A. *Let G be a countably based pro-p group. Then the centre of Ω_G is equal to the closure of the centre of $\mathbb{F}_p[G]$:*

$$Z(\Omega_G) = \overline{Z(\mathbb{F}_p[G])}.$$

Similarly, the centre of Λ_G is equal to the closure of the centre of $\mathbb{Z}_p[G]$:

$$Z(\Lambda_G) = \overline{Z(\mathbb{Z}_p[G])}.$$

When G is p-valued in the sense of Lazard [5, III.2.1.2], we obtain a cleaner result.

COROLLARY A. *Let G be a countably based p-valued pro-p group with centre Z. Then*

$$Z(\Omega_G) = \Omega_Z \quad \textit{and} \quad Z(\Lambda_G) = \Lambda_Z.$$

The class of p-valued pro-p groups is rather large; for example, every closed subgroup of a uniform pro-p group [3, 4,1] is p-valued. Also, any pro-p subgroup of $GL_n(\mathbb{Z}_p)$ is p-valued when $p > n+1$ [5, p. 101].

We remark that when G is an open pro-p subgroup of $\mathrm{GL}_2(\mathbb{Z}_p)$, a version of the above result was proved by Howson [4, 4.2] using similar techniques.

We also use the method used in the proof of Theorem A to compute endomorphism rings of certain induced modules for Ω_G, when G is an analytic pro-p group.

THEOREM B. *Let H be a closed subgroup of an analytic pro-p group G. Let $M = \mathbb{F}_p \otimes_{\Omega_H} \Omega_G$ and write $R = \mathrm{End}_{\Omega_G}(M)$. Then R is finite-dimensional over $\mathbb{F}_p$ if and only if $N_{\mathfrak{g}}(\mathfrak{h}) = \mathfrak{h}$, where $\mathfrak{h}$ and $\mathfrak{g}$ denote the $\mathbb{Q}_p$-Lie algebras of H and G, respectively.*

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2 Fixed Points

Let X be a group. For any (right) X-space S, let $S^X = \{s \in S : s.X = s\}$ denote the set of fixed points of X in S. Also, let $\mathcal{O}(S)$ denote the collection of all *finite* X-orbits on S, and for any orbit $\mathcal{C} \in \mathcal{O}(S)$, let $\hat{\mathcal{C}}$ denote the orbit sum

$$\hat{\mathcal{C}} = \sum_{s \in \mathcal{C}} s,$$

viewed as an element of the permutation module $\mathbb{F}_p[S]$. Thus, $\mathbb{F}_p[S]^X$ is spanned by all the $\hat{\mathcal{C}}$ as $\mathcal{C}$ ranges over $\mathcal{O}(S)$: $\mathbb{F}_p[S]^X = \mathbb{F}_p[\hat{\mathcal{O}(S)}]$.

Now let X be a pro-p group. Assume we are given an inverse system

$$\ldots \overset{\pi_{n+1}}{\twoheadrightarrow} A_n \overset{\pi_n}{\twoheadrightarrow} A_{n-1} \overset{\pi_{n-1}}{\twoheadrightarrow} \ldots \overset{\pi_2}{\twoheadrightarrow} A_1$$

of finite X-spaces. We can consider the natural inverse system of permutation modules associated with the A_i:

$$\ldots \overset{\pi_{n+1}}{\twoheadrightarrow} \mathbb{F}_p[A_n] \overset{\pi_n}{\twoheadrightarrow} \mathbb{F}_p[A_{n-1}] \overset{\pi_{n-1}}{\twoheadrightarrow} \ldots \overset{\pi_2}{\twoheadrightarrow} \mathbb{F}_p[A_1]$$

where we keep the same notation for the connecting maps π_n. Now, form the inverse limit

$$Y = \varprojlim A_n,$$

this is clearly an X-space. We can also form the inverse limit

$$\Omega_Y = \varprojlim \mathbb{F}_p[A_n],$$

which is easily seen to be an Ω_X-module.

Note that Ω_Y is a compact metric space, with metric given by $d(\alpha, \beta) = \|\alpha - \beta\|$, where $\|.\|$ is a norm on Ω_Y given by

$$\|(\alpha_n)_n\| = \sup\{p^{-n} : \alpha_n \neq 0\} \quad \text{and} \quad \|0\| = 0.$$

We are interested in the fixed points of Ω_Y, viewed as an X-space. It is straightforward to see that there is a natural embedding of $\mathbb{F}_p[Y]$ into Ω_Y and that $\mathbb{F}_p[Y]^X \subseteq \Omega_Y^X$.

PROPOSITION 2.1. *With the notations above,* $\Omega_Y^X = \overline{\mathbb{F}_p[Y]}^X = \overline{\mathbb{F}_p[Y]^X}$.

Proof. Because the action of X on Ω_Y is continuous, it is clear that Ω_Y^X is a closed subset of Ω_Y, so by the above remarks $\overline{\mathbb{F}_p[Y]^X} \subseteq \Omega_Y^X$. Let $\alpha = (\alpha_n)_n \in \Omega_Y^X$. Since the natural maps $\Omega_Y \twoheadrightarrow \mathbb{F}_p[A_n]$ are maps of X-spaces, we see that each α_n lies in $\mathbb{F}_p[A_n]^X$.

Let the integer r be least such that $\alpha_r \neq 0$. Consider $\alpha_r \in \mathbb{F}_p[A_r]^X$; thus $\alpha_r = \sum_{\mathcal{C}\in\mathcal{O}(A_r)} \lambda_{\mathcal{C}}\hat{\mathcal{C}}$ and not all the $\lambda_{\mathcal{C}}$ are zero.

Pick a $\mathcal{C} \in \mathcal{O}(A_r)$ with $\lambda_{\mathcal{C}} \neq 0$. Since π_{r+1} is a map of X-spaces, $\pi_{r+1}^{-1}(\mathcal{C}) = \mathcal{D}_1 \cup \mathcal{D}_2 \cup \ldots \cup \mathcal{D}_k$ is a union of X-orbits, with $\pi_{r+1}(\mathcal{D}_j) = \mathcal{C}$ for $j = 1,\ldots,k$ and $\pi_{r+1}(\mathcal{D}_j) \cap \mathcal{C} = \emptyset$ for $j > k$, if we let $\mathcal{D}_{k+1},\ldots,\mathcal{D}_m$ denote the remaining elements of $\mathcal{O}(A_{r+1})$.

We claim we can find a $\mathcal{D}_j$ with $1 \leq j \leq k$ such that $|\mathcal{D}_j| = |\mathcal{C}|$.

For, suppose not. Then $|\mathcal{D}_j| > |\mathcal{C}|$ for each $j = 1,\ldots,k$. As $\pi_{r+1} : \mathcal{D}_j \twoheadrightarrow \mathcal{C}$ is a surjective map of finite transitive X-spaces, and because X is a pro-p group, we deduce that each fibre $(\pi_{r+1}|\mathcal{D}_j)^{-1}(s)$ for $s \in \mathcal{C}$ has size a power of p greater than 1. But then, because we are working over $\mathbb{F}_p$, we must have $\pi_{r+1}(\hat{\mathcal{D}}_j) = 0$, for each $1 \leq j \leq k$.

Now, since $\alpha_{r+1} \in \mathbb{F}_p[A_{r+1}]^X$, we can write $\alpha_{r+1} = \sum_{j=1}^m \mu_j \hat{\mathcal{D}}_j$ for some $\mu_j \in \mathbb{F}_p$. So, $\alpha_r = \pi_{r+1}(\alpha_{r+1}) = \sum_{j=k+1}^m \mu_j \pi_{r+1}(\hat{\mathcal{D}}_j)$. But $\pi_{r+1}(\mathcal{D}_j) \cap \mathcal{C} = \emptyset$ for all $j > k$, contradicting the fact that $\mathcal{C} \subseteq \mathrm{supp}(\alpha_r)$.

Hence, we can find $\mathcal{C}_{r+1} \in \mathcal{O}(A_{r+1})$ with $|\mathcal{C}_{r+1}| = |\mathcal{C}_r|$ and $\pi_{r+1}(\mathcal{C}_{r+1}) = \mathcal{C}_r$, where we set $\mathcal{C}_r$ to be $\mathcal{C}$. It is clear that we can continue this process of "lifting" the X-orbits, without ever increasing the sizes. Thus, we get a sequence

$$\cdots \overset{\pi_{n+2}}{\twoheadrightarrow} \mathcal{C}_{n+1} \overset{\pi_{n+1}}{\twoheadrightarrow} \mathcal{C}_n \overset{\pi_n}{\twoheadrightarrow} \ldots \overset{\pi_{r+1}}{\twoheadrightarrow} \mathcal{C}_r$$

of X-orbits, each having the same size as $\mathcal{C}_r$.

Now, pick some $s_r \in \mathcal{C}_r$ and inductively choose lifts $s_n \in \mathcal{C}_n$ for each $n \geq r$. Let s be the element of Y determined by these lifts. It is then straightforward to see that the X-orbit of s in Y is finite and that the image of this orbit in A_r equals $\mathcal{C}$. Let $\mathcal{F}_{\mathcal{C}}$ denote this element of $\mathcal{O}(Y)$.

Finally, we can consider the element $\beta = \sum_{\mathcal{C}\in\mathcal{O}(A_r)} \lambda_{\mathcal{C}}\hat{\mathcal{F}}_{\mathcal{C}}$. Obviously β lies in $\mathbb{F}_p[Y]^X$, and the image of β in $\mathbb{F}_p[A_r]$ coincides with

α_r. Hence, $\alpha - \beta$ has norm strictly smaller than that of α and also lies in Ω_Y^X. Applying the argument above to $\alpha - \beta$ instead of α and iterating, we see that α can be approximated arbitrarily closely by elements of $\mathbb{F}_p[Y]^X$. $\square$

Next we turn to the analogous proposition over the p-adics. Let

$$\Lambda_Y = \varprojlim \mathbb{Z}_p[A_n].$$

This is naturally a Λ_X-module and there is a natural isomorphism $\Lambda_Y/p\Lambda_Y \cong \Omega_Y$ of Λ_X-modules. Moreover, since $\Lambda_Y \cong \varprojlim(\mathbb{Z}/p^n\mathbb{Z})[A_n]$ is a countably based pro-p group, it is a compact metric space.

PROPOSITION 2.2. *With the notations above,* $\Lambda_Y^X = \overline{\mathbb{Z}_p[Y]^X}$.

Proof. As in the proof of Proposition 2.1, the inclusion $\overline{\mathbb{Z}_p[Y]^X} \subseteq \Lambda_Y^X$ is clear. Let $\bar{}: \Lambda_Y \to \Omega_Y$ denote reduction mod p.
Let $\alpha \in \Lambda_Y^X$ so that $\overline{\alpha} \in \Omega_Y^X$. By Proposition 2.1, $\overline{\alpha} = \lim_{n\to\infty} u_n$ for some $u_n \in \mathbb{F}_p[Y]^X$. Since $\mathbb{F}_p[Y]^X = \mathbb{F}_p[\hat{\mathcal{O}(Y)}]$ and $\mathbb{Z}_p[Y]^X = \mathbb{Z}_p[\hat{\mathcal{O}(Y)}]$, we can choose $v_n \in \mathbb{Z}_p[Y]^X$ such that $\overline{v_n} = u_n$ for all n. Since Λ_Y is compact, by passing to a convergent subsequence we may assume that v_n converges to $\beta_0 \in \overline{\mathbb{Z}_p[Y]^X}$. Now $\overline{\alpha} = \lim_{n\to\infty} \overline{v_n} = \overline{\beta_0}$, so $\alpha - \beta_0 \in p\Lambda_Y \cap \Lambda_Y^X = p\Lambda_Y^X$, since Λ_Y is p-torsion free.
Hence we can write $\alpha = \beta_0 + p\alpha_1$ where $\alpha_1 \in \Lambda_Y^X$. Iterating the above argument, we obtain elements $\beta_1, \beta_2, \ldots \in \overline{\mathbb{Z}_p[Y]^X}$ and $\alpha_1, \alpha_2, \ldots \in \Lambda_Y^X$ such that $\alpha_n = \beta_n + p\alpha_{n+1}$ for all $n \geq 1$. So $\alpha = \sum_{n=0}^{\infty} \beta_n p^n \in \overline{\mathbb{Z}_p[Y]^X}$. $\square$

3 MAIN RESULTS

We immediately make use of the above Propositions.

Proof of Theorem A. Since G is countably based, we can write G as an inverse limit of the countable system $A_n = G/G_n$, for some suitable open normal subgroups $G_1 \supset G_2 \supset \ldots \supset G_n \supset \ldots$ of G. Each A_n is a finite G-space, where G acts by conjugation. Now apply Propositions 2.1 and 2.2. $\square$

Proof of Corollary A. $Z(\mathbb{F}_p[G])$ is spanned over $\mathbb{F}_p$ by all conjugacy class sums $\hat{\mathcal{C}}$, where $\mathcal{C}$ is a *finite* conjugacy class of G. Let $\mathcal{C}$ be such a conjugacy class and let $x \in \mathcal{C}$. Then the centralizer $C_G(x)$ of x in G is a closed subgroup of G of finite index.
Let $y \in G$; then $y^{p^n} \in C_G(x)$ for some n, so $(x^{-1}yx)^{p^n} = y^{p^n}$. Since G is p-valued, the map $g \mapsto g^p$ is injective on G by [5, Chapter III, Proposition 2.1.4], so $x^{-1}yx = y$ and $\mathcal{C} \subseteq Z$.
Hence $Z(\mathbb{F}_p[G]) = \mathbb{F}_p[Z]$, and similarly $Z(\mathbb{Z}_p[G]) = \mathbb{Z}_p[Z]$. The result follows from Theorem A. $\square$

We remark that Corollary A does not extend to arbitrary torsion free analytic pro-p groups. This can be easily checked for the group given in [5, Chapter III, Example 3.2.5].
We now turn to the proof of Theorem B. Let $G = \varprojlim G/G_n$ be a pro-p group, H a closed subgroup. Let $M = \mathbb{F}_p \otimes_{\Omega_H} \Omega_G$ be the induced module from the trivial module for Ω_H. G acts on the coset space $Y = H\backslash G$ by right translation and we can write $Y = \varprojlim HG_n\backslash G$ as an inverse limit of finite G-spaces. It is easy to see that Ω_Y is then naturally isomorphic to M.
Let R denote the endomorphism ring $\operatorname{End}_{\Omega_G} M$ of M. Each element $f \in R$ gives rise to a trivial Ω_H-submodule of M generated by $f(1 \otimes 1)$, when we view M as an Ω_H-module by restriction. This gives an isomorphism of $\mathbb{F}_p-$vector spaces

$$R = \operatorname{Hom}_{\Omega_G}(\mathbb{F}_p \otimes_{\Omega_H} \Omega_G, M) \cong \operatorname{Hom}_{\Omega_H}(\mathbb{F}_p, M),$$

expressing the fact "induction is left adjoint to restriction".
Now $\operatorname{Hom}_{\Omega_H}(\mathbb{F}_p, M)$ can be thought of as the sum of all trivial Ω_H-submodules of M, which is precisely the set $M^H = \Omega_Y^H$, where H acts on Y by right translation. In view of Proposition 2.1, we are interested in the finite $H-$orbits on Y; these are given by those double cosets of H in G which are finite unions of left cosets of H. Suppose HxH is such a double coset; then $\operatorname{Stab}_H(Hx) = \{h \in H : Hxh = Hx\} = H \cap H^x$ has finite index in H, so the set $\mathcal{N}_G(H) = \{x \in G : H \cap H^x \leq_o H\}$ is of interest; we observe that it contains the usual normalizer $N_G(H)$ of H in G. This set is sometimes called the *commensurator* of H in G.

Proof of Theorem B. Recall [3, 9.5] that G contains an open normal uniform subgroup K and that the $\mathbb{Q}_p$-Lie algebra of G can be defined by

$$\mathcal{L}(G) = K \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$$

where K is viewed as a $\mathbb{Z}_p$-module of finite rank [3, 4.17]. The conjugation action of G on K is $\mathbb{Z}_p$-linear and therefore extends to an action of G on $\mathcal{L}(G)$, which is easily checked to be independent of the choice of K. This is just the adjoint action of G on $\mathcal{L}(G)$. Next, we observe that when $x \in G$,

$$\begin{aligned} H \cap H^x \leq_o H &\Leftrightarrow \mathcal{L}(H \cap H^x) = \mathcal{L}(H) \cap \mathcal{L}(H)^x = \mathcal{L}(H) \\ &\Leftrightarrow \mathcal{L}(H)^x = \mathcal{L}(H), \end{aligned}$$

so $N := \mathcal{N}_G(H) = \mathrm{Stab}_G \mathfrak{h}$ *is* a (closed) subgroup of G.
By [3, Exercise 9.10], we see that the Lie algebra of N is equal to the normalizer $N_{\mathfrak{g}}(\mathfrak{h})$ of $\mathfrak{h}$ in $\mathfrak{g}$. We remark in passing that this implies that $N_G(H)$ has finite index in N when dealing with analytic pro-p groups; this is not true in general.
Now, by Proposition 2.1 and the above remarks, R is finite dimensional over $\mathbb{F}_p$ if and only if the number of finite H-orbits on $Y = H\backslash G$ is finite.
Clearly $\{Hx : HxH \text{ is a finite } H\text{-orbit}\} = H\backslash N$, so the number of finite H-orbits on Y is finite if and only if H has finite index in N. This happens if and only if $\mathfrak{h} = \mathcal{L}(H) = \mathcal{L}(N) = N_{\mathfrak{g}}(\mathfrak{h})$, as required. □

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