Krull dimension of Iwasawa algebras

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1 Introduction

Let G be a compact p-adic Lie group. In the recent years, there has been an increased amount of interest in completed group algebras (Iwasawa algebras)

$$\Lambda_G = \mathbb{Z}_p[[G]] := \varprojlim_{N \triangleleft_o G} \mathbb{Z}_p[G/N],$$

for example, because of their connections with number theory and arithmetic geometry; see the paper by Coates, Schneider and Sujatha ([4]) for more details.

When G is a uniform pro-p group, Λ_G is a concrete example of a complete local Noetherian ring (noncommutative, in general) with good homological properties: it is known that Λ_G has finite global dimension and is an Auslander regular ring. Thus, Λ_G falls into the class of rings studied by Brown, Hajarnavis and MacEacharn in [1]. There they consider various properties of Noetherian rings R of finite global dimension, including the Krull(-Gabriel-Rentschler) dimension $\mathcal{K}(R)$ - a module-theoretic dimension which measures how far R is from being Artinian. They also posed the following question:

Question ([1], Section 5). *Let R be a local right Noetherian ring, whose Jacobson radical satisfies the Artin-Rees property. Is the Krull dimension of R always equal to the global dimension of R?*

In this paper, we address the problem of computing $\mathcal{K}(\Lambda_G)$. We establish lower and upper bounds for $\mathcal{K}(\Lambda_G)$ in terms of the Lie algebra $\mathfrak{g} = \mathcal{L}(G)$ of G:

Theorem A. *Let $\lambda(\mathfrak{g})$ be the maximum length m of chains $0 = \mathfrak{g}_0 < \mathfrak{g}_1 < \ldots < \mathfrak{g}_m = \mathfrak{g}$ of sub-Lie-algebras of $\mathfrak{g}$. Then*

$$\lambda(\mathfrak{g}) + 1 \leq \mathcal{K}(\Lambda_G) \leq \dim \mathfrak{g} + 1.$$

For some groups, the two bounds coincide:

Corollary A. *Let $\mathfrak{r}$ be the solvable radical of $\mathfrak{g}$ and suppose that the semisimple part $\mathfrak{g}/\mathfrak{r}$ of $\mathfrak{g}$ is isomorphic to a direct sum of copies of of $\mathfrak{sl}_2(\mathbb{Q}_p)$. Then $\mathcal{K}(\Lambda_G) = \dim \mathfrak{g} + 1$.*

We also establish a better upper bound for $\mathcal{K}(\Lambda_G)$ when $\mathcal{L}(G)$ is simple and split over $\mathbb{Q}_p$:

Theorem B. *Let $p \geq 5$ and suppose $\mathfrak{g} \neq \mathfrak{sl}_2(\mathbb{Q}_p)$ is split simple over $\mathbb{Q}_p$ with a Cartan subalgebra $\mathfrak{t}$ and a Borel subalgebra $\mathfrak{b}$. Then*

$$\dim \mathfrak{b} + \dim \mathfrak{t} + 1 \leq \mathcal{K}(\Lambda_G) \leq \dim \mathfrak{g} < \mathrm{gld}(\Lambda_G).$$

The author believes that $\dim \mathfrak{b} + \dim \mathfrak{t} + 1$ is the true value of $\mathcal{K}(\Lambda_G)$, with G as above. Applying Theorem B to a particular case allows us to obtain a negative answer to the question posed above:

Corollary B. *Let $p \geq 5$ and let $G = \ker(SL_3(\mathbb{Z}_p) \to SL_3(\mathbb{F}_p))$. Then Λ_G is a local right Noetherian ring whose Jacobson radical satisfies the Artin Rees Property, but*

$$\mathcal{K}(\Lambda_G) = 8 < \mathrm{gld}(\Lambda_G) = 9.$$

In addition, we reprove a general result of R. Walker connecting $\mathcal{K}(R)$ with $\mathcal{K}(R/I)$ for a certain ring R and a suitable ideal I:

Theorem C (Walker, [10]). *Suppose R is right Noetherian and x is a right regular normal element belonging to the Jacobson radical of R. If $\mathcal{K}(R) < \infty$ then*

$$\mathcal{K}(R) = \mathcal{K}(R/xR) + 1.$$

The reader might like to compare this result with the corresponding one on global dimensions; see Theorem 7.3.7 of [7].

We will denote the completed group algebra of G over $\mathbb{F}_p$ by Ω_G:

$$\Omega_G := \varprojlim_{N \triangleleft_o G} \mathbb{F}_p[G/N]$$

Theorem C applies directly to Iwasawa algebras, since it is easy to see that $\Omega_G \cong \Lambda_G / p\Lambda_G$:

Corollary C. $\mathcal{K}(\Lambda_G) = \mathcal{K}(\Omega_G) + 1.$

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Notation. *All rings are assumed to be associative and to possess a unit, but are not necessarily commutative. $J(R)$ always denotes the Jacobson radical of the ring R. All modules are right modules, unless stated otherwise; Mod-R denotes the category of all right modules over R. The symbol p will always mean a fixed prime.*

2 Preliminaries

2.1 Filtrations

We will conform with the definitions and notations used in the book [6] throughout this paper. In this section, we briefly recall the most relevant concepts.

A *filtration* on a ring R is a set of additive subgroups $FR = \{F_nR : n \in \mathbb{Z}\}$, satisfying $1 \in F_0R$, $F_nR \subseteq F_{n+1}R$, $F_nR.F_mR \subseteq F_{n+m}R$ for all $n, m \in \mathbb{Z}$, and $\cup_{n\in\mathbb{Z}} F_nR = R$. If R has a filtration, R is said to be a *filtered ring*. In what follows, we assume R is a filtered ring.

Let M be an R-module. A *filtration* on M is a set of additive subgroups of M, $FM = \{F_nM : n \in \mathbb{Z}\}$, satisfying $F_nM \subseteq F_{n+1}M$, $F_nM.F_mR \subseteq F_{n+m}M$ for all $n, m \in \mathbb{Z}$ and $\cup_{n\in\mathbb{Z}} F_nM = M$. If M has a filtration, M is said to be a *filtered R-module*. The filtration on M is said to be *separated* if $\cap_{n\in\mathbb{Z}} F_nM = 0$.

Let I be a two-sided ideal of R. A notable example of a filtration on R is the *I-adic filtration* given by $F_nR := I^{-n}$ if $n \leq 0$ and $F_nR = R$ otherwise.

The *associated graded ring* of R is defined to be $\operatorname{gr} R = \oplus_{n\in\mathbb{Z}} F_nR/F_{n-1}R$. If $x \in R$, the *symbol* of x in $\operatorname{gr} R$ is $\sigma(x) := x + F_{n-1}R \in F_nR/F_{n-1}R$, where n is such that $x \in F_nR \backslash F_{n-1}R$. If $x \in \cap_{n\in\mathbb{Z}} F_nR$, define $\sigma(x) = 0$.

The *Rees ring* of R is defined to be $\widetilde{R} = \bigoplus_{n\in\mathbb{Z}} F_nR$, which we view to be a subring of the Laurent polynomial ring $R[t, t^{-1}]$.

The *associated graded module* and *Rees module* of a filtered R-module M are defined similarly. We say that the filtration FM on M is *good* if and only if $\widetilde{M}$ is a finitely generated $\widetilde{R}$-module. Note that a finitely generated R-module M always possesses a good filtration, for example the *deduced filtration* given by $F_nM = M.F_nR$ for $n \in \mathbb{Z}$.

2.2 Iwasawa algebras

By a well-known result of Lazard (see, for example, Theorem 8.36 of [5]), any compact p-adic Lie group G has an open normal uniform pro-p subgroup H. Since H has finite index in G, any open normal subgroup of H contains an open normal subgroup of G. Hence

$$\Lambda_H = \varprojlim_{N\in\mathcal{C}} \mathbb{Z}_p[H/N] \quad \text{and} \quad \Lambda_G = \varprojlim_{N\in\mathcal{C}} \mathbb{Z}_p[G/N],$$

where $\mathcal{C} = \{N \triangleleft_o G : N \subseteq H\}$. It follows that Λ_G is a free right and left Λ_H-module of finite rank (an appropriate transversal for H in G will serve as a basis), so $\mathcal{K}(\Lambda_G) = \mathcal{K}(\Lambda_H)$ by Corollary 6.5.3 of [7].

Thus restricting ourselves to the class of uniform pro-p groups does not lose any generality and we will assume that G denotes a uniform pro-p group throughout this paper. For more information about these groups, see the excellent book [5].

Following [5], we will write L_G for the $\mathbb{Z}_p$-Lie algebra of G ([5], 4.29) and $\mathcal{L}(G) = \mathfrak{g}$ for the $\mathbb{Q}_p$-Lie algebra of G ([5], 9.5).

The following properties of Λ_G and Ω_G are more or less well known:

Lemma 2.1. *Let $R = \Lambda_G$ or Ω_G and let $d = \dim G$. Then:*

(i) R is a local right Noetherian ring with maximal ideal $J = \ker(R \twoheadrightarrow \mathbb{F}_p)$.

(ii) R is complete with respect to the J-adic filtration.

(iii) $\operatorname{gr}_J \Omega_G \cong \mathbb{F}_p[X_1, \ldots, X_d]$.

(iv) $\operatorname{gld}(\Lambda_G) = \operatorname{gld}(\Omega_G) + 1 = \dim G + 1$.

(v) J satisfies the right (and left) Artin Rees Property.

Proof. Proofs of (i),(ii) and (iii) can be found in Chapter 7 of [5]. Part (iv) is established in [2]. By Theorem 2.2 of Chapter II of [6], the J-adic filtration has the Artin Rees property, which is easily seen to imply that the ideal J has the Artin Rees Property in the sense of 4.2.3 of [7]. □

Henceforth, J_G will always denote the maximal ideal of Ω_G. We will require the following characterization of Artinian modules of Ω_G:

Proposition 2.2. *Let G be a uniform pro-p group with lower p-series $\{G_n, n \geq 1\}$. Let $M = \Omega_G/I$ be a cyclic Ω_G-module. The following are equivalent:*

(i) M is Artinian.

(ii) $J_G^n \subseteq I$ for some $n \in \mathbb{N}$.

(iii) $J_{G_m} \subseteq I$ for some $m \geq 1$.

(iv) M is finite dimensional over $\mathbb{F}_p$.

Proof. Note that by Theorem 3.6 of [5], G_n is uniform for each $n \geq 1$.

(i) $\Rightarrow$ (ii). As Ω_G is Noetherian, M has finite length. Also Ω_G/J_G is the unique simple Ω_G-module, as Ω_G is local. Hence $MJ_G^n = 0$.

(ii) $\Rightarrow$ (iii). Suppose $J_G^n \subseteq I$. Choose m such that $p^{m-1} \geq n$. Then $g^{p^{m-1}} - 1 = (g-1)^{p^{m-1}} \in J_G^n \subseteq I$ for all $g \in G$. As $G_m = G^{p^{m-1}}$, we see that $G_m - 1 \subseteq I$ so $J_{G_m} \subseteq I$ as required.

(iii) $\Rightarrow$ (iv). If $J_{G_m} \subseteq I$, $J_{G_m}\Omega_G \subseteq I$ as I is a right ideal of Ω_G. Hence $\mathbb{F}_p[G/G_m] \cong \Omega_G/J_{G_m}\Omega_G \twoheadrightarrow \Omega_G/I = M$. Since $|G : G_m|$ is finite, the result follows.

(iv) $\Rightarrow$ (i). This is clear. □

2.3 Krull dimension

The definitions and basic facts about the Krull(-Gabriel-Rentschler) dimension can be found in Chapter 6 of [7]. Recall that an R-module M is said to be n-critical if $\mathcal{K}(M) = n$ and $\mathcal{K}(M/N) < n$ for all nonzero submodules N of M; thus a 0-critical module is nothing other than a simple module.

The following (well known) Lemma is the basis for many arguments involving the Krull dimension. Since we shall not require the general case of ordinal-valued Krull dimensions, we restrict ourselves to the case when the dimension is finite. We write $\mathrm{Lat}(R)$ for the lattice of all right ideals of a ring R.

Lemma 2.3. *Let R and S be rings, with R Noetherian of finite Krull dimension. Let $f : \mathrm{Lat}(R) \to \mathrm{Lat}(S)$ be an increasing function and let $k, n \in \mathbb{N}$, with $\mathcal{K}_R(R) \geq n$. Let X and Y be right ideals of R with $Y \subseteq X$ and suppose $\mathcal{K}_R(X/Y)+k \leq \mathcal{K}_S(f(X)/f(Y))$ whenever X/Y is n-critical. Then $\mathcal{K}_R(X/Y)+k \leq \mathcal{K}_S(f(X)/f(Y))$ whenever $\mathcal{K}_R(X/Y) \geq n$.*

In particular, $\mathcal{K}_R(R) + k \leq \mathcal{K}_S(S)$.

Proof. This follows from [7], 6.1.17. □

3 Main Results

We now proceed to prove the main theorems stated in the introduction. We prove Theorem C in Section 3.1; the argument is a straightforward induction based on Nakayama's Lemma and is different to the one used by Walker in [10].

Theorem A is proved in Section 3.2, where we also consider the length function $\lambda(\mathfrak{g})$ of a finite dimensional Lie algebra $\mathfrak{g}$. It is also shown that Corollary A follows from Theorem A.

The remainder of the paper is devoted to proving Theorem B.

3.1 Reduction to Ω_G

Let R be a ring. Suppose x is a normal element of R and M is an R-module. It's clear that Mx is an R-submodule of M; recall that M is said to be *x-torsion free* if $mx = 0 \Rightarrow m = 0$ for all $m \in M$.

The following result summarizes various elementary properties of modules.

Lemma 3.1. *Let x be a normal element of a ring R and let $B \subseteq A$ be right R-modules with Krull dimension. Then:*

(a) If A/B and B are x-torsion free then A is also x-torsion free.

(b) If A/B is x-torsion free then $Ax \cap B = Bx$ and $\mathcal{K}(B/Bx) \leq \mathcal{K}(A/Ax)$.

(c) If A is x-torsion free then $\mathcal{K}(A/Ax) = \mathcal{K}(Ax^{n-1}/Ax^n) = \mathcal{K}(A/Ax^n)$ for all $n \geq 1$.

The main step comes next.

Lemma 3.2. *Let R be a right Noetherian ring, x a normal element of $J(R)$. Suppose M is a finitely generated x-torsion free R-module with finite Krull dimension. Then $\mathcal{K}(M/Mx) \geq \mathcal{K}(M) - 1$.*

Proof. Proceed by induction on $\mathcal{K}(M) = \beta$. Note that $\beta \geq 1$ since M is x-torsion free. Since $x \in J(R)$, the base case $\beta = 1$ follows from Nakayama's Lemma. We can find a chain $M = M_1 > M_2 > \ldots > M_k > \ldots$ such that M_i/M_{i+1} is $(\beta - 1)$-critical for all $i \geq 1$.

Case 1: $\exists i \geq 1$ such that M_i/M_{i+1} is not x-torsion free.

Pick a least such i. Let N/M_{i+1} be the x-torsion part of M_i/M_{i+1}; thus M_i/N is x-torsion free.

As each M_j/M_{j+1} is x-torsion free for all $j < i$, M/N is also x-torsion free by Lemma 3.1(a). Hence, by Lemma 3.1(b), $\mathcal{K}(M/Mx) \geq \mathcal{K}(N/Nx)$.

Since M is x-torsion free and $0 < N \subseteq M$, N is also x-torsion free. Hence, by Lemma 3.1 (c), $\mathcal{K}(N/Nx) = \mathcal{K}(N/Nx^n)$ for all $n \geq 1$.

As M is Noetherian and N/M_{i+1} is x-torsion, there exists $n \geq 1$ such that $(N/M_{i+1})x^n = 0$. Hence $Nx^n \subseteq M_{i+1}$, so $N/Nx^n \twoheadrightarrow N/M_{i+1}$ and $\mathcal{K}(N/Nx^n) \geq \mathcal{K}(N/M_{i+1})$.

Since N/M_{i+1} is a nonzero submodule of the $(\beta - 1)$-critical M_i/M_{i+1}, we deduce that $\mathcal{K}(N/M_{i+1}) = \beta - 1 = \mathcal{K}(M) - 1$. The result follows.

Case 2: M_i/M_{i+1} is x-torsion free $\forall i \geq 1$.

Consider the chain

$$M = Mx + M_1 \geq Mx + M_2 \geq \ldots \geq Mx. \qquad (\dagger)$$

Now, M_i/M_{i+1} is x-torsion free and has Krull dimension $\beta - 1$, so by induction, $\mathcal{K}((M_i/M_{i+1})/(M_i/M_{i+1}).x) \geq \beta - 2$. But

$$\frac{M_i/M_{i+1}}{(M_i/M_{i+1}).x} = \frac{M_i/M_{i+1}}{(M_i x + M_{i+1})/M_{i+1}} \cong \frac{M_i}{M_i x + M_{i+1}}, \quad \text{and}$$

$$\frac{M_i + Mx}{M_{i+1} + Mx} \cong \frac{M_i}{(M_{i+1} + Mx) \cap M_i} = \frac{M_i}{M_{i+1} + (M_i \cap Mx)}.$$

Since M/M_i is x-torsion free by Lemma 3.1 (a), $M_i \cap Mx = M_i x$ by Lemma 3.1(b), so every factor of (†) has Krull dimension $\geq \beta - 2$. Hence $\mathcal{K}(M/Mx) \geq \beta - 1 = \mathcal{K}(M) - 1$. □

Proof of Theorem C. Since x is right regular, R_R is x-torsion free. By Lemma 3.1 (c), the chain $R > xR > \ldots > x^k R > \ldots$ has infinitely many factors with Krull dimension equal to $\mathcal{K}(R/xR)$, so $\mathcal{K}(R) > \mathcal{K}(R/xR)$. The result follows from Lemma 3.2. □

We remark that as x is normal, xR is an ideal of R and so the Krull dimensions of R/xR over R and over the ring R/xR coincide.

3.2 A lower bound for the Krull dimension

Proposition 3.3. *Let G be a uniform pro-p group and let H be a closed uniform subgroup such that $|G : H| = \infty$. Then:*

(i) The induced module $M = \mathbb{F}_p \otimes_{\Omega_H} \Omega_G$ is not Artinian over Ω_G.

(ii) $\mathcal{K}(\Omega_H) < \mathcal{K}(\Omega_G)$.

Proof. (i) Since $\mathbb{F}_p \cong \Omega_H / J_H$ and since $- \otimes_{\Omega_H} \Omega_G$ is flat by Lemma 4.5 of [2], we see that $M \cong \Omega_G / J_H \Omega_G$ as right Ω_G-modules.

Suppose M is Artinian. Then $J_{G_m} \subseteq J_H \Omega_G$ for some $m \geq 1$, by Proposition 2.2. It is easy to check that $(1 + J_H \Omega_G) \cap G = H$ for any closed subgroup H of any profinite group G. Hence

$$G_m = (1 + J_{G_m} \Omega_G) \cap G \subseteq (1 + J_H \Omega_G) \cap G = H$$

which forces $|G : H|$ to be finite, a contradiction.

(ii) Consider the increasing function $f : \mathrm{Lat}(\Omega_H) \to \mathrm{Lat}(\Omega_G)$, given by $I \mapsto I \otimes_{\Omega_H} \Omega_G$. Suppose X and Y are right ideals of R such that $Y \subseteq X$ and such that X/Y is simple. Since Ω_H is local, $X/Y \cong \mathbb{F}_p$ so $f(X)/f(Y) \cong \mathbb{F}_p \otimes_{\Omega_H} \Omega_G \cong M$ as Ω_G is a flat Ω_H-module. As M is not Artinian by part (i), $\mathcal{K}(f(X)/f(Y)) \geq 1$, so by Lemma 2.3 $\mathcal{K}(\Omega_H) + 1 \leq \mathcal{K}(\Omega_G)$, as required. □

Note that the analogous proposition for universal enveloping algebras is false: for example, the Verma module of highest weight zero for $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$ is Artinian, and indeed, $\mathcal{K}(\mathcal{U}(\mathfrak{g})) = \mathcal{K}(\mathcal{U}(\mathfrak{b})) = 2$, where $\mathfrak{b}$ is a Borel subalgebra of $\mathfrak{g}$.

We can now give a proof of the first result stated in the Introduction:

Proof of Theorem A. By Theorem C, it is sufficient to show $\lambda(\mathfrak{g}) \leq \mathcal{K}(\Omega_G) \leq d$, where $d = \dim \mathfrak{g}$. First, we show that $\lambda(\mathfrak{g}) \leq \mathcal{K}(\Omega_G)$.

Proceed by induction on $\lambda(\mathfrak{g})$. Let $0 = \mathfrak{g}_0 < \mathfrak{g}_1 < \ldots < \mathfrak{g}_k = \mathfrak{g}$ be a chain of maximal length $k = \lambda(\mathfrak{g})$ in $\mathfrak{g}$.

We can find a closed uniform subgroup H of G with Lie algebra $\mathfrak{g}_{k-1}$. Since $\mathfrak{g}_{k-1} < \mathfrak{g}$, $|G : H| = \infty$.

By the inductive hypothesis, $k-1 = \lambda(\mathfrak{g}_{k-1}) \leq \mathcal{K}(\Omega_H)$. By Proposition 3.3, $\mathcal{K}(\Omega_H) < \mathcal{K}(\Omega_G)$, so $k = \lambda(\mathfrak{g}) \leq \mathcal{K}(\Omega_G)$.

By Lemma 2.1, we see that Ω_G is a complete filtered ring with $\operatorname{gr} \Omega_G \cong \mathbb{F}_p[X_1, \ldots, X_d]$. It follows from Proposition 7.1.2 of Chapter I of [6] and Corollary 6.4.8 of [7] that $\mathcal{K}(\Omega_G) \leq \mathcal{K}(\operatorname{gr} \Omega_G) = d$, as required. □

Theorem A stimulates interest in the length $\lambda(\mathfrak{g})$ of a finite dimensional Lie algebra $\mathfrak{g}$. The following facts about this invariant are known:

Proposition 3.4. *Let $\mathfrak{g}$ be a finite dimensional Lie algebra over a field k.*

(i) If $\mathfrak{h}$ is an ideal of $\mathfrak{g}$, $\lambda(\mathfrak{g}) = \lambda(\mathfrak{h}) + \lambda(\mathfrak{g}/\mathfrak{h})$.

(ii) If $\mathfrak{g}$ is solvable, $\lambda(\mathfrak{g}) = \dim_k(\mathfrak{g})$,

(iii) If $\mathfrak{g}$ is split semisimple, $\lambda(\mathfrak{g}) \geq \dim \mathfrak{b} + \dim \mathfrak{t}$, where $\mathfrak{t}$ and $\mathfrak{b}$ are some Cartan and Borel subalgebras of $\mathfrak{g}$, respectively.

(iv) $\lambda(\mathfrak{sl}_2(k)) = 3$.

Proof. (i)Putting together two chains of maximal length in $\mathfrak{h}$ and $\mathfrak{g}/\mathfrak{h}$ shows that $\lambda(\mathfrak{g}) \geq \lambda(\mathfrak{h}) + \lambda(\mathfrak{g}/\mathfrak{h})$. The reverse inequality follows by considering the chains $0 = \mathfrak{g}_0 \cap \mathfrak{h} \subseteq \ldots \subseteq \mathfrak{g}_i \cap \mathfrak{h} \subseteq \ldots \subseteq \mathfrak{h}$ and $\mathfrak{h} \subseteq \mathfrak{g}_1 + \mathfrak{h} \subseteq \ldots \subseteq \mathfrak{g}_i + \mathfrak{h} \subseteq \ldots \subseteq \mathfrak{g}$ whenever $0 = \mathfrak{g}_0 < \ldots < \mathfrak{g}_i < \ldots < \mathfrak{g}_n = \mathfrak{g}$ is a chain of subalgebras of maximal length in $\mathfrak{g}$.

(ii) This follows directly from (i).

(iii) Let $l = \dim \mathfrak{t}$. Given a Borel subalgebra $\mathfrak{b}$, there are exactly 2^l parabolic subalgebras containing it, corresponding 1-1 with the subsets of the set of simple roots of $\mathfrak{g}$. This correspondence preserves inclusions, so we can find a chain of subalgebras of length l starting with $\mathfrak{b}$. Combining this together with a maximal chain of length $\dim \mathfrak{b}$ in $\mathfrak{b}$ gives the result.

(iv) This follows from (iii), since for $\mathfrak{g} = \mathfrak{sl}_2(k)$, $\dim \mathfrak{t} = 1, \dim \mathfrak{b} = 2$ and $\dim \mathfrak{g} = 3$. □

Proof of Corollary A. This now follows directly from Theorem A and Proposition 3.4. □

3.3 An upper bound

The method of proof of Theorem B is similar in spirit to that used by S. P. Smith in his proof of the following theorem, providing an analogous better upper bound for $\mathcal{K}(\mathcal{U}(\mathfrak{g}))$ when $\mathfrak{g}$ is semisimple:

Theorem 3.5 (Smith). *Let $\mathfrak{g}$ be a complex semisimple Lie algebra. Let $2r+1$ be the dimension of the largest Heisenberg Lie algebra contained in $\mathfrak{g}$. Then $\mathcal{K}(\mathcal{U}(\mathfrak{g})) \leq \dim \mathfrak{g} - r - 1$.*

Proof. See Corollary 4.3 of [8], bearing in mind the comments contained in section 3.1 of that paper. □

Definition 3.6. *Let k be a field. The* Heisenberg k-Lie algebra *of dimension $2r+1$ is defined by the presentation*

$$\mathfrak{h}_{2r+1} = k < w, u_1, \ldots, u_r, v_1, \ldots, v_r : \quad [u_i, v_j] = \delta_{ij} w, [w, u_i] = [w, v_i] = 0, \\ [u_i, u_j] = [v_i, v_j] = 0 > .$$

Here δ_{ij} is the Kronecker delta.

First we establish a useful fact about uniform pro-p groups H with $\mathbb{Q}_p$-Lie algebra isomorphic to a Heisenberg Lie algebra.

Lemma 3.7. *Let H be a uniform pro-p group such that $\mathcal{L}(H)$ is isomorphic to $\mathfrak{h}_{2r+1}$. Let the centre $Z(H)$ of H be topologically generated by z. Then there exist $x, y \in H$ and $k \in \mathbb{N}$ such that $[x, y] = z^{p^k}$.*

Proof. By Theorem 9.10 of [5], we may assume that the group law on H is given by the Campbell-Hausdorff formula on L_H. Let $(,)$ denote the Lie bracket on $\mathcal{L}(H) = \mathfrak{h}_{2r+1}$.

Since $(L_H, (L_H, L_H)) \subseteq (\mathfrak{h}_{2r+1}, (\mathfrak{h}_{2r+1}, \mathfrak{h}_{2r+1})) = 0$, the group law on L_H given by the Campbell-Hausdorff series reduces to

$$\alpha * \beta = \alpha + \beta + \frac{1}{2}(\alpha, \beta)$$

for $\alpha, \beta \in L_H$. It's then easily checked that the group commutator satisfies

$$[\alpha, \beta] = \alpha^{-1} * \beta^{-1} * \alpha * \beta = (\alpha, \beta). \qquad (\dagger)$$

Now as $\mathbb{Q}_p L_H = \mathfrak{h}_{2r+1}$ there exists $n \in \mathbb{N}$ such that $p^n u_1, p^n v_1 \in L_H$, whence $(p^n u_1, p^n v_1) \in L_H \cap \mathbb{Q}_p w = \mathbb{Z}_p z$. Hence $(p^n u_1, p^n v_1) = p^k \lambda z$ for some unit $\lambda \in \mathbb{Z}_p$ and some $k \in \mathbb{N}$, an equation inside L_H. We may now take $x = p^n \lambda^{-1} u_1$, $y = p^n v_1$ and apply $(\dagger)$. □

Next we develop some dimension theory for finitely generated Ω_G-modules, where G is an arbitrary uniform pro-p group. Recall that the J_G-adic filtration on Ω_G gives rise to a polynomial associated graded ring.

Definition 3.8. *Let M be a finitely generated Ω_G-module, equipped with some good filtration FM. The* characteristic ideal *of M is defined to be*

$$J(M) := \sqrt{\operatorname{Ann}\operatorname{gr} M}.$$

The graded dimension *of M is defined to be*

$$d(M) := \mathcal{K}(\operatorname{gr}\Omega_G / J(M)).$$

Lemma 4.1.9 of Chapter III of [6] shows that $J(M)$ and hence $d(M)$ does not depend on the choice of a good filtration for M. It is easy to prove that $d(M) = \mathcal{K}(\operatorname{gr} M)$ for any good filtration FM on M.

Let $\mathfrak{h}$ be a $\mathbb{Q}_p$-Lie subalgebra of $\mathfrak{g}$, the $\mathbb{Q}_p$-Lie algebra of G. Let $H = \mathfrak{h} \cap L_G$; since L_G/H injects into $\mathfrak{g}/\mathfrak{h}$ which is torsion-free, we see that H is actually a closed uniform subgroup of G, by Theorem 7.15 of [5].

We will call H the *isolated* uniform subgroup of G with $\mathbb{Q}_p$-Lie algebra $\mathfrak{h}$.

The following proposition is the main step in our proof of the upper bound for $\mathcal{K}(\Omega_G)$. Recall that J_G denotes the maximal ideal of Ω_G.

Proposition 3.9. *Let G be a uniform pro-p group with $\mathbb{Q}_p$-Lie algebra $\mathfrak{g}$ such that $\mathfrak{h}_3 \subseteq \mathfrak{g}$. Let H be the isolated uniform subgroup of G with Lie algebra $\mathfrak{h}_3$. Let $Z = Z(H) = \overline{< z >}$, say. Let M be a finitely generated Ω_G-module such that $d(M) \leq 1$. Then $\sigma(z-1) \in J(M)$.*

Proof. Let A be a uniform subgroup of G with torsion-free L_G/L_A. Using Theorem 7.23(ii) of [5] it is easy to check that the subspace filtration on Ω_A induced from the J_G-adic filtration on Ω_G coincides with the J_A-adic filtration.

It follows that the Rees ring $\widetilde{\Omega}_A$ of Ω_A embeds into $\widetilde{\Omega}_G$ and that $\widetilde{\Omega}_A \cap t\widetilde{\Omega}_G = t\widetilde{\Omega}_A$, so this embedding induces a natural embedding of graded rings

$$\operatorname{gr}\Omega_A = \widetilde{\Omega}_A / t\widetilde{\Omega}_A \hookrightarrow \widetilde{\Omega}_G / t\widetilde{\Omega}_G = \operatorname{gr}\Omega_G.$$

It's easy to see that L_H/L_Z is torsion-free. Since L_G/L_H is torsion-free by assumption on H, L_G/L_Z is also torsion-free so the above discussion applies to both Z and H.

Now, equip M with a good filtration FM and consider the Rees module $\widetilde{M}$. This is an $\widetilde{\Omega}_G$-module, so we can view it as an $\widetilde{\Omega}_H$-module by restriction.

Let $S = \widetilde{\Omega}_Z - t\widetilde{\Omega}_Z$. This is a central multiplicatively closed subset of the domain $\widetilde{\Omega}_H$, so we may form the localisations $\widetilde{\Omega}_Z S^{-1} \hookrightarrow \widetilde{\Omega}_H S^{-1}$ and the localised $\widetilde{\Omega}_H.S^{-1}$-module $\widetilde{M}S^{-1}$.

Let $R = \varprojlim \widetilde{\Omega}_Z S^{-1}/t^n.\widetilde{\Omega}_Z S^{-1}$ and let $N = \varprojlim \widetilde{M}S^{-1}/t^n.\widetilde{M}S^{-1}$.

It's clear that N is an R-module. Also, as t is central in $\widetilde{\Omega}_H S^{-1}$, N has the structure of a $\widetilde{\Omega}_H S^{-1}$-module. In particular, as H embeds into $\widetilde{\Omega}_H S^{-1}$, N is an H-module.

Now, consider the t-adic filtration on R. It's easy to see that

$$R/tR = \widetilde{\Omega}_Z S^{-1}/t\widetilde{\Omega}_Z S^{-1} \cong \operatorname{gr}\Omega_Z.\bar{S}^{-1},$$

where $\bar{S} = \operatorname{gr}\Omega_Z - \{0\}$. Thus $R/tR \cong k$, the field of fractions of $\operatorname{gr}\Omega_Z$.

As t acts injectively on $\widetilde{\Omega}_Z S^{-1}$, $t^nR/t^{n+1}R \cong k$ for all $n \geq 0$. Hence the graded ring of R with respect to the t-adic filtration is

$$\operatorname{gr}_t R = \bigoplus_{n=0}^{\infty} \frac{t^nR}{t^{n+1}R} \cong k[s],$$

where $s = t + t^2R \in tR/t^2R$.

We can also consider the t-adic filtration on N. Again, we see that $N/tN \cong t^nN/t^{n+1}N \cong \operatorname{gr} M.\bar{S}^{-1}$. Hence

$$\operatorname{gr}_t N = \bigoplus_{n=0}^{\infty} t^nN/t^{n+1}N \cong (\operatorname{gr} M.\bar{S}^{-1}) \otimes_k k[s].$$

Now, because $d(M) \leq 1$, $\operatorname{gr} M.\bar{S}^{-1}$ is finite dimensional over k. It follows that $\operatorname{gr}_t N$ is a finitely generated $\operatorname{gr}_t R$-module.

Because N is complete with respect to the t-adic filtration, this filtration on N is separated. Also R is complete, so by Theorem 5.7 of Chapter I of [6], N is finitely generated over R.

Now $\widetilde{\Omega}_Z S^{-1}$ is a local ring with maximal ideal $t\widetilde{\Omega}_Z S^{-1}$. Hence R is a commutative local ring with maximal ideal tR; since $\cap_{n=0}^{\infty} t^nR = 0$, the only ideals of R are $\{t^nR : n \geq 0\}$.

Hence R is a commutative PID and N is a finitely generated t-torsionfree R-module. This forces N to be free over R, say $N \cong R^n$, for some $n \geq 0$.

Now, Z embeds into R and the action of R commutes with the action of H on N. Hence we get a group homomorphism

$$\rho : H \to GL_n(R)$$

such that $\rho(z) = zI$, where I is the $n \times n$ identity matrix.

But H is a uniform pro-p group with $\mathbb{Q}_p$-Lie algebra $\mathfrak{h}_3$, so by Lemma 3.7 we can find elements $x, y \in H$ such that $[x, y] = z^{p^k}$ for some $k \geq 1$.

Hence $[\rho(x), \rho(y)] = \rho(z)^{p^k} = z^{p^k}.I$. Taking determinants yields $z^{np^k} = 1$. Since $Z = \overline{< z >} \cong \mathbb{Z}_p$, this is only possible if $n = 0$.

Therefore $N = 0$ and so $N/tN = \operatorname{gr} M.\bar{S}^{-1} = 0$. Hence $Q \cap \bar{S} \neq \emptyset$, where $Q = \operatorname{Ann}_{\operatorname{gr}\Omega_G} \operatorname{gr} M$. Because Q is graded and because $\operatorname{gr}\Omega_Z \cong \mathbb{F}_p[\sigma(z-1)]$, we see that $\sigma(z-1)^m \in Q$ for some $m \geq 0$. Hence $\sigma(z-1) \in J(M) = \sqrt{Q}$. $\square$

The above result should be compared to the Bernstein inequality for finitely generated modules M for the Weyl algebra $A_1(\mathbb{C})$, which gives a restriction on the possible values of the dimension of M. When $\mathfrak{g}$ is itself a Heisenberg Lie algebra, a stronger result has been proved by Wadsley ([9], Theorem B):

Theorem 3.10. *Let G be a uniform pro-p group with $\mathbb{Q}_p$-Lie algebra $\mathfrak{h}_{2r+1}$ and let M be a finitely generated Ω_G-module. If $d(M) \leq r$, then $\operatorname{Ann}_{\Omega_G}(M) \cap \Omega_Z \neq 0$, where $Z = Z(G)$.*

We are tempted to conjecture that the following generalization of Proposition 3.9 holds:

Conjecture. *Let G be a uniform pro-p group with $\mathbb{Q}_p$-Lie algebra $\mathfrak{g}$ such that $\mathfrak{h}_{2r+1} \subseteq \mathfrak{g}$. Let H be the isolated uniform subgroup of G with Lie algebra $\mathfrak{h}_{2r+1}$ and let $Z = Z(H) = \overline{< z >}$, say. Let M be a finitely generated Ω_G-module such that $d(M) \leq r$. Then $\sigma(z-1) \in J(M)$.*

This is a more general analogue of Lemma 3.2 of [8] corresponding to the Bernstein inequality for $A_r(\mathbb{C})$. If this conjecture is correct, we would be able to sharpen the upper bound on $\mathcal{K}(\Omega_G)$ from $\dim \mathfrak{g} - 1$ to $\dim \mathfrak{g} - r$, when G is as in Theorem B.

Let G be a uniform pro-p group, and consider the set G/G_2, where $G_2 = P_2(G) = G^p$. We know that G/G_2 is a vector space over $\mathbb{F}_p$ of dimension $d = \dim(G)$. The automorphism group $\operatorname{Aut}(G)$ of G acts naturally on G/G_2; this action commutes with the $\mathbb{F}_p$-linear structure on G/G_2. Because $[G, G] \subseteq G_2$ the action of $\operatorname{Inn}(G)$ is trivial, so we see that G/G_2 is naturally an $\mathbb{F}_p[\operatorname{Out}(G)]$-module.

Similarly, we obtain an action of $\operatorname{Aut}(G)$ on J/J^2 where $J = J_G \triangleleft \Omega_G$; it's easy to see that $\operatorname{Inn}(G)$ again acts trivially, so J/J^2 is also an $\mathbb{F}_p[\operatorname{Out}(G)]$-module.

Lemma 3.11. *The map $\varphi : G/G_2 \to J/J^2$ given by $\varphi(gG_2) = \sigma(g-1) = g - 1 + J^2$ is an isomorphism of $\mathbb{F}_p[\operatorname{Out}(G)]$-modules.*

Proof. It is easy to check that φ is an $\mathbb{F}_p$-linear map preserving the $\mathrm{Out}(G)$-structure.

Now $\{g_1G_2, \ldots, g_dG_2\}$ is a basis for G/G_2, if $\{g_1, \ldots, g_d\}$ is a topological generating set for G. By Theorem 7.24 of [5], $\{X_1, \ldots, X_d\}$ is a basis for J/J^2, where $X_i = \sigma(g_i - 1) = \varphi(g_iG_2)$. The result follows. $\square$

Theorem 3.12. *Let G, H, z be as in Proposition 3.9. Suppose zG_2 generates the $\mathbb{F}_p[\mathrm{Out}(G)]$-module G/G_2. Then*

(i) Ω_G has no finitely generated modules M with $d(M) = 1$

(ii) $\mathcal{K}(\Omega_G) \leq \dim \mathfrak{g} - 1$.

Proof. Let M be a finitely generated Ω_G-module with $d(M) \leq 1$. By Lemma 3.11, $G/G_2 \cong J/J^2$ as $\mathbb{F}_p[\mathrm{Out}(G)]$-modules. Because zG_2 generates G/G_2, $\varphi(zG_2) = \sigma(z - 1) \in J/J^2$ generates J/J^2. In other words, $\mathbb{F}_p.\{\sigma(z - 1)^\alpha : \alpha \in \mathrm{Out}(G)\} = J/J^2$.

Let $\theta \in \mathrm{Aut}(G)$. By Proposition 3.9 applied to H^θ, $\sigma(z^\theta - 1) = \sigma(z - 1)^{\bar{\theta}} \in J(M)$, where $^-$: $\mathrm{Aut}(G) \to \mathrm{Out}(G)$ is the natural surjection.

Hence $J/J^2 = \mathbb{F}_p.\{\sigma(z - 1)^\alpha : \alpha \in \mathrm{Out}(G)\} \subseteq J(M)$. This forces

$$(X_1, \ldots, X_d) \subseteq J(M) \subseteq \mathbb{F}_p[X_1, \ldots, X_d] = \mathrm{gr}\, \Omega_G,$$

whence $d(M) = 0$ and part (i) follows.

Consider the increasing map $\mathrm{gr} : \mathrm{Lat}(\Omega_G) \to \mathrm{Lat}(\mathrm{gr}\, \Omega_G)$, where we endow each right ideal of Ω_G with the subspace filtration from the J_G-adic filtration on G. If $X, Y \lhd_r \Omega_G$ are such that $M = X/Y$ is 1-critical, then $\mathcal{K}(\mathrm{gr}\, M) = \mathcal{K}(\mathrm{gr}\, X/\, \mathrm{gr}\, Y) \geq 1$, giving M the subquotient filtration from Ω_G.

Now, by Proposition 1.2.3 of Chapter II of [6], this subquotient filtration is good, since Ω_G is a complete filtered ring with Noetherian $\mathrm{gr}\, \Omega_G$. Hence $\mathcal{K}(\mathrm{gr}\, M) = d(M) \geq 1$ by the remarks following Definition 3.8. By part (i), $\mathcal{K}(\mathrm{gr}\, X/\, \mathrm{gr}\, Y) \geq 2$ so part (ii) follows from Lemma 2.3. $\square$

We will use this result to deduce Theorem B.

3.4 Chevalley groups over $\mathbb{Z}_p$

We recall some facts from the theory of Chevalley groups:

Let $X \in \{A_l, B_l, C_l, D_l, E_6, E_7, E_8, F_4, G_2\}$ be an indecomposable root system and let R be a commutative ring. Let $\mathcal{B} = \{h_r : r \in \Pi\} \cup \{e_r : r \in X\}$ be the *Chevalley basis* for the R-Lie algebra X_R.

Let $X(R) = \quad < x_r(t) : r \in X, t \in R > \quad \subseteq \mathrm{Aut}(X_R)$ be the adjoint *Chevalley group* over R. Here $x_r(t) \in \mathrm{Aut}(X_R)$ is given by

$$\begin{aligned} &x_r(t).e_r = e_r \\ &x_r(t).e_{-r} = e_{-r} + th_r - t^2 e_r \\ &x_r(t).h_s = h_s - A_{sr} t e_r \\ &x_r(t).e_s = \textstyle\sum_{i=0}^{b} M_{r,s,i} t^i e_{ir+s} \end{aligned}$$

where $s \in X$ is a root linearly independent from r, $a \in \mathbb{N}$ is the largest integer such that $s - ar \in X$, $b \in \mathbb{N}$ is the largest integer such that $s + br \in X$, $A_{sr} = \frac{2(s,r)}{(r,r)}$ and $M_{r,s,i} = \pm\binom{a+i}{i}$.

Let R^* denote the group of units of R. When $t \in R^*$ and $r \in X$, define

$$\begin{aligned} &n_r(t) = x_r(t) x_{-r}(-t^{-1}) x_r(t) \quad \text{and} \\ &h_r(t) = n_r(t) n_r(-1). \end{aligned}$$

The actions of $h_r(t)$ and $n_r = n_r(1)$ on X_R are as follows:

$$\begin{aligned} &h_r(t).h_s = h_s, && s \in \Pi \\ &h_r(t).e_s = t^{A_{rs}} e_s, && s \in X \\ &n_r.h_s = h_{w_r(s)} \\ &n_r.e_s = \eta_{r,s} e_{w_r(s)} \end{aligned}$$

Here w_r is the Weyl reflection on X corresponding to the root r and $\eta_{r,s} = \pm 1$.

The Steinberg relations hold in $X(R)$:

$$\begin{aligned} &h_r(t_1) h_r(t_2) = h_r(t_1 t_2), && t_1, t_1 \in R^*, r \in X \\ &x_r(t) x_s(u) x_r(t)^{-1} = x_s(u).\textstyle\prod_{i,j>0} x_{ir+js}(C_{ijrs} t^i u^j), && t, u \in R, r, s \in X \\ &h_s(u) x_r(t) h_s(u)^{-1} = x_r(u^{A_{sr}} t), && t \in R, u \in R^*, r, s \in X. \end{aligned}$$

Here C_{ijrs} are certain integers such that $C_{i1rs} = M_{r,s,i}$.

For more details on the above, see [3].

Now, consider the $\mathbb{Z}_p$-Lie algebra $X_{\mathbb{Z}_p}$. Since $[pX_{\mathbb{Z}_p}, pX_{\mathbb{Z}_p}] = p^2[X_{\mathbb{Z}_p}, X_{\mathbb{Z}_p}] \subseteq p.pX_{\mathbb{Z}_p}$, we see that $pX_{\mathbb{Z}_p}$ is a powerful $\mathbb{Z}_p$-Lie algebra. Let $Y = (pX_{\mathbb{Z}_p}, *)$ be the uniform pro-p group constructed from $pX_{\mathbb{Z}_p}$ using the Campbell-Hausdorff formula.

We have a group homomorphism $\mathrm{Ad} : Y \to GL(pX_{\mathbb{Z}_p})$ given by $\mathrm{Ad}(g)(u) = gug^{-1}$. It is shown in Exercise 9.10 of [5] that

$$\mathrm{Ad} = \exp \circ \mathrm{ad}$$

where $\exp : \mathfrak{gl}(pX_{\mathbb{Z}_p}) \to GL(pX_{\mathbb{Z}_p})$ is the exponential map.

It's clear that $\ker \mathrm{Ad} = Z(Y)$. Since the Lie algebra $X_{\mathbb{Q}_p}$ of Y is simple, it's easy to see that $\mathcal{L}(Z(Y)) = Z(\mathcal{L}(Y)) = 0$; hence $\ker \mathrm{Ad} = 1$ and Ad is an injection.

Lemma 3.13. *Let $N = \mathrm{Ad}(Y)$ and $G = X(\mathbb{Z}_p)$. Then $N \triangleleft G$.*

Proof. First we show that $N \subseteq G$. It's clear that the $\mathbb{Z}_p$-linear action of N on $pX_{\mathbb{Z}_p}$ extends naturally to a $\mathbb{Z}_p$-linear action of N on $X_{\mathbb{Z}_p}$. Now, direct computation shows that

$$\begin{aligned} \mathrm{Ad}(te_r) &= x_r(t), && t \in p\mathbb{Z}_p, r \in X \text{ and} \\ \mathrm{Ad}(th_r) &= h_r(\exp(t)), && t \in p\mathbb{Z}_p, r \in \Pi. \end{aligned}$$

Hence $\mathrm{Ad}(pu\mathbb{Z}_p) \subseteq G$ for all $u \in \mathcal{B}$. The set $p\mathcal{B}$ is a $\mathbb{Z}_p$-basis for $pX_{\mathbb{Z}_p}$ and hence a topological generating set for Y by Theorem 9.8 of [5]. By Proposition 3.7 of [5], Y is equal to the product of the procyclic subgroups $pu\mathbb{Z}_p$ as u ranges over $\mathcal{B}$. Hence $N \subseteq G$.

Now, let $r, s \in X$, $t \in \mathbb{Z}_p$ and $u \in p\mathbb{Z}_p$. By the Steinberg relations, we have

$$x_r(t)x_s(u)x_r(t)^{-1} = x_s(u).\prod_{i,j>0} x_{ir+js}(C_{ijrs}t^iu^j) \in N$$

and

$$x_r(t)h_s(\exp(u))x_r(t)^{-1} = h_s(\exp(u))x_r(\exp(-A_{sr}u)t)x_r(-t) \in N$$

since $C_{ijrs}t^iu^j \in p\mathbb{Z}_p$ and $\exp(-A_{sr}u) - 1 \in p\mathbb{Z}_p$, whenever $u \in p\mathbb{Z}_p$.

Hence $N \triangleleft G$, as required. □

Theorem 3.14. *Let G, N be as in Lemma 3.13. There exists a commutative diagram of group homomorphisms:*

$$\begin{array}{ccccc} G & \xrightarrow{\alpha} & X(\mathbb{F}_p) & \xrightarrow{\iota} & \mathrm{Aut}(X_{\mathbb{F}_p}) \\ {\scriptstyle\beta}\downarrow & & & & \downarrow{\scriptstyle\varphi^*} \\ \mathrm{Aut}(N) & \xrightarrow[\pi]{} & \mathrm{Out}(N) & \xrightarrow[\gamma]{} & \mathrm{Aut}(N/N_2) \end{array}$$

Proof. We begin by defining all the relevant maps. Any automorphism f of $X_{\mathbb{Z}_p}$ must fix $pX_{\mathbb{Z}_p}$ and hence induces an automorphism $\alpha(f)$ of $X_{\mathbb{F}_p} \cong X_{\mathbb{Z}_p}/pX_{\mathbb{Z}_p}$. It's clear from the definition of the Chevalley groups that $\alpha(x_r(t)) = x_r(\bar{t})$ where $^-\colon \mathbb{Z}_p \to \mathbb{F}_p$ is reduction mod p and that α is a surjection.

Since Ad is an isomorphism of Y onto N, N is a uniform pro-p group, and we have an $\mathbb{F}_p$-linear bijection $\varphi : X_{\mathbb{F}_p} \to N/N_2$ given by $\varphi(\bar{x}) = \mathrm{Ad}(px)N_2$, where $^-\colon X_{\mathbb{Z}_p} \to X_{\mathbb{F}_p}$ is the natural map. This induces an isomorphism $\varphi^* : \mathrm{Aut}(X_{\mathbb{F}_p}) \to \mathrm{Aut}(N/N_2)$ given by $\varphi^*(f) = \varphi f \varphi^{-1}$.

We have observed in the remarks preceding Lemma 3.11 that $\mathrm{Out}(N)$ acts naturally on N/N_2; we denote this action by γ. By Lemma 3.13 N is normal in G, and we denote the conjugation action of G on N by β.

Finally, ι is the natural injection of $X(\mathbb{F}_p)$ into $\mathrm{Aut}(X_{\mathbb{F}_p})$ and π is the natural projection of $\mathrm{Aut}(N)$ onto $\mathrm{Out}(N)$.

It remains to check that $\varphi^*\iota\alpha = \gamma\pi\beta$. It is sufficient to show $\varphi^*\iota\alpha(x_r(t)) = \gamma\pi\beta(x_r(t))$ for any $r \in X$ and $t \in \mathbb{Z}_p$. We check these maps agree on the basis $\{\mathrm{Ad}(pu).N_2 : u \in \mathcal{B}\}$ of N/N_2. On the one hand, we have

$$\begin{aligned}
\varphi^*\iota\alpha(x_r(t))(\mathrm{Ad}(pe_s)N_2) &= \varphi^*(x_r(\bar{t}))(\mathrm{Ad}(pe_s)N_2) = \varphi(x_r(\bar{t})(\overline{e_s})) = \\
&= \varphi(\sum_{i=0}^{b} M_{r,s,i}\bar{t}^i\overline{e_{ir+s}}) = \\
&= \prod_{i=0}^{b} \mathrm{Ad}(pM_{r,s,i}t^i e_{ir+s})N_2 = \\
&= \prod_{i=0}^{b} x_{ir+s}(pM_{r,s,i}t^i)N_2, (\dagger)
\end{aligned}$$

using the definition of the action of $x_r(\bar{t})$ on $X_{\mathbb{F}_p}$. On the other hand,

$$\begin{aligned}
\gamma\pi\beta(x_r(t))(\mathrm{Ad}(pe_s)N_2) &= x_r(t)x_s(p)x_r(-t)N_2 = \\
&= x_s(p)\prod_{i,j>0} x_{ir+js}(C_{ijrs}t^ip^j)N_2,
\end{aligned}$$

using the Steinberg relations.

Since $x_\alpha(p^2) \in N_2$ for any $\alpha \in X$, we see that the all the terms in the above product with $j > 1$ vanish, and the remaining expression is equal to the result of $(\dagger)$, since $C_{i1rs} = M_{r,s,i}$.

A similar computation shows that $\varphi^*\iota\alpha(x_r(t))$ also agrees with $\gamma\pi\beta(x_r(t))$ on $\mathrm{Ad}(ph_s)N_2$ for any $s \in \Pi$, and the result follows. □

The above theorem shows that the action of $\mathrm{Out}(N)$ on N/N_2 which was of interest in the preceding section is linked to the natural action of $X(\mathbb{F}_p)$ on $X_{\mathbb{F}_p}$. Since α is surjective, we see that if $\bar{e_r}$ generates $X_{\mathbb{F}_p}$ as an $\mathbb{F}_p[X(\mathbb{F}_p)]$-module, then $\mathrm{Ad}(pe_r)N_2$ generates N/N_2 as an $\mathbb{F}_p[\mathrm{Out}(N)]$-module. We drop the bars in the following proposition.

Proposition 3.15. *Suppose $p \geq 5$ and let $R = \mathbb{F}_p[X(\mathbb{F}_p)]$. Then $X_{\mathbb{F}_p} = R.e_r$ for any $r \in X$.*

Proof. This is probably well known and is purely a matter of computation. Let W denote the Weyl group of X.

Note that $(x_{-r}(1) + \eta_{r,r} n_r - 1).e_r = h_{-r} \in R.e_r$, whence $h_r = -h_{-r} \in R.e_r$ also.

By Proposition 2.1.8 of [3], we can choose $w \in W$ such that $w(r) \in \Pi$. Hence $n_w.h_r = h_{w(r)} \in R.e_r$.

Let α, β be adjacent fundamental roots. Then $n_\alpha.h_\beta = h_{w_\alpha(\beta)} = h_\beta - A_{\beta\alpha} h_\alpha$ where $A_{\beta\alpha} = -1, -2$ or -3. The condition on p implies that if $h_\beta \in R.e_r$ then $h_\alpha \in R.e_r$ also.

Since X is indecomposable, $h_\alpha \subseteq R.e_r$ for any $\alpha \in \Pi$. Since the fundamental coroots span the Cartan subalgebra, $h_s \in R.e_r$ for any $s \in X$.

Finally. $x_s(1).h_s = h_s - 2e_s$, whence $e_s \in R.e_r$ for any $s \in X$, since $p \neq 2$. Since $\{e_s, h_r : s \in X, r \in \Pi\}$ is a basis for $X_{\mathbb{F}_p}$, the result follows. □

The condition on p in the above proposition can be relaxed somewhat - it might even be the case that it can be dropped altogether. Since this is a small detail of no interest to us, we restrict ourselves to the case $p \geq 5$.

We can finally provide a proof of our main result.

Proof of Theorem B. In view of Theorem C and Lemma 2.1, it is sufficient to prove that

$$\dim \mathfrak{b} + \dim \mathfrak{t} \leq \mathcal{K}(\Omega_G) \leq \dim \mathfrak{g} - 1.$$

Note that the lower bound on $\mathcal{K}(\Omega_G)$ follows from Proposition 3.4 and Theorem A.

Let X be the root system of $\mathfrak{g}$; thus $\mathfrak{g} = X_{\mathbb{Q}_p}$. Since X is not of type A_1 by assumption on $\mathfrak{g}$, we can find two roots $r, s \in X$ such that $r + s \in X$ but $r + 2s, 2r + s \notin X$; it's then easy to see that the root spaces of r and s generate a subalgebra of $\mathfrak{g}$ isomorphic to $\mathfrak{h}_3$ with centre $\mathbb{Q}_p e_{r+s}$.

Let N be the uniform pro-p group appearing in the statement of Theorem 3.14. By construction, $\mathfrak{g}$ is the Lie algebra of N. By Proposition 3.15 and the remarks preceding it, we see that $\mathrm{Ad}(pe_{r+s})N_2 \in N/N_2$ generates the $\mathbb{F}_p[\mathrm{Out}(N)]$-module N/N_2. Hence $\mathcal{K}(\Omega_N) \leq \dim \mathfrak{g} - 1$ by Theorem 3.12.

Since the Lie algebra of G is $\mathfrak{g} = \mathbb{Q}_p L_G = \mathbb{Q}_p L_N$, we see that $N \cap G$ is an open subgroup of both N and G, whence $\mathcal{K}(\Omega_G) = \mathcal{K}(\Omega_N) \leq \dim \mathfrak{g} - 1$, as required. □

Proof of Corollary B. It is readily seen that G is a uniform pro-p group with $\mathbb{Q}_p$-Lie algebra $\mathfrak{sl}_3(\mathbb{Q}_p)$ which is split simple over $\mathbb{Q}_p$. We have observed in Lemma 2.1 that Λ_G is a local right Noetherian ring whose Jacobson radical satisfies the right Artin Rees Property, and that $\mathrm{gld}(\Lambda_G) = \dim \mathfrak{g} + 1 = 9$.

If $\mathfrak{b}$ and $\mathfrak{t}$ denote the Borel and Cartan subalgebras of $\mathfrak{g}$, then $\dim \mathfrak{b} = 5$ and $\dim \mathfrak{t} = 2$. The result follows from Theorems B and C. □

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