A CANONICAL DIMENSION ESTIMATE FOR NON-SPLIT SEMISIMPLE P-ADIC LIE GROUPS

KONSTANTIN ARDAKOV AND CHRISTIAN JOHANSSON

Abstract. We prove that the canonical dimension of an admissible Banach space or a locally analytic representation of an arbitrary semisimple p-adic Lie group is either zero or at least half the dimension of a non-zero coadjoint orbit. This extends the results of Ardakov-Wadsley and Schmidt in the split semisimple case.

1. Introduction

This paper can be regarded as a postscript to [AW1]. Its purpose is to record an argument that extends [AW1, Theorem A] to compact semisimple p-adic analytic groups whose Lie algebra is not necessarily split (p is a prime number). An analogue of [AW1, Theorem A] for distribution algebras was proved by Schmidt in [Sch2, Theorem 9.9]; our argument works to extend his result as well. For simplicity, we mostly focus on the setting of Iwasawa algebras in this introduction. Let us recall that if G is a compact p-adic analytic group and KG is its completed group ring (or Iwasawa algebra) with respect to a finite field extension $K/\mathbb{Q}_p$, then the category $\mathcal{M}$ of finitely generated KG-modules is an abelian category antiequivalent to the category of admissible K-Banach space representations of G ([ST1]). $\mathcal{M}$, and related categories of p-adic representations of (locally) compact p-adic analytic groups have received a lot of attention in the last decades, motivated by research in the Langlands programme and Iwasawa theory. Each object M in $\mathcal{M}$ has a non-negative integer $d(M) = d_{KG}(M)$ attached to it called its canonical dimension. This notion gives rise to a natural filtration of $\mathcal{M}$

$$\mathcal{M} = \mathcal{M}_d \supseteq \mathcal{M}_{d-1} \supseteq ... \supseteq \mathcal{M}_0$$

by Serre subcategories, where $d = \dim G$ and $M \in \mathcal{M}_i$ if and only $d(M) \leq i$. The canonical dimension $d(M)$ may be taken as a measure of the "size" of M; for example it is known that $d(M) = 0$ if and only M is finite dimensional as K-vector space, and $d(M) < d$ if and only if M is torsion. It may also be regarded as a noncommutative analogue of the dimension of the support of a module in the commutative setting. The following result, together with its analogue for distribution algebras (Theorem 2) is the main result of this note:

Theorem 1. *Let G be a compact p-adic analytic group whose Lie algebra $\mathcal{L}(G)$ is semisimple, and let $G_{\mathbb{C}}$ denote a complex semisimple algebraic group with the same root system as $\mathcal{L}(G) \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p$. Let M be a finitely generated KG-module which is infinite dimensional as a K-vector space, and assume that p is an odd very good prime for G. Then $d_{KG}(M) \geq r$, where r is half the smallest possible dimension of a nonzero coadjoint $G_{\mathbb{C}}$-orbit.*

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For the definition of a very good prime see [AW1, §6.8]. We recall that Theorem A of [AW1] proves the same conclusion under the additional hypothesis that $\mathcal{L}(G)$ is *split* semisimple. Since the Langlands programme often deals with groups that are not split, it seemed natural to wish to remove this assumption from the theorem. Examples of such nonsplit groups are $G = \mathbb{G}(K)$, where $\mathbb{G}/K$ is an anisotropic semisimple algebraic group.

The proof is a variation of that of the split case. Let us first give a short rough description of the split case and refer to the introduction of [AW1] for more details. A short argument reduces the problem to the case when G is a uniform pro-p group. This implies we may associate to G a certain $\mathbb{Z}_p$-Lie algebra $\mathfrak{h}$. The algebra KG comes with a morphism into an inverse system $(D_w = D_{p^{-1/w}}(G,K))_{w=p^n}$ of distribution algebras of G, whose inverse limit is the locally analytic distribution algebra $D(G,K)$ with coefficients in K studied in detail by Schneider and Teitelbaum. For large enough $w = p^n$, $D_w \otimes_{KG} M \neq 0$. The algebra D_w turns out to be a crossed product $\mathcal{U}_n * G/G^{p^n}$, where $\mathcal{U}_n = \widehat{U(p^n\mathfrak{h})}_K$ is an affinoid enveloping algebra, and it turns out that it is enough to prove the analogous result for the algebras $\mathcal{U}_n$. The proof of this analogue relies on some technical work: analogues of Beilinson-Bernstein localisation, Quillen's Lemma and Bernstein's Inequality. The assumption that $\mathcal{L}(G)$ is split comes up in the localisation theory, which works for affinoid enveloping algebras of $\mathcal{O}_L$-Lie algebras that are of the form $\pi^m\mathfrak{g}$, with $L/\mathbb{Q}_p$ a finite extension, π a uniformizer in L and $\mathfrak{g}$ a split Lie algebra over $\mathcal{O}_L$.

In this paper we follow the same reduction steps to get nonzero $\mathcal{U}_n$-modules $D_{p^n} \otimes_{KG} M$ for sufficiently large n. Since the techniques of [AW1] are not strong enough to deduce a canonical dimension estimate for the affinoid enveloping algebra $\mathcal{U}_n$ in this case, we base change to a finite extension $L/\mathbb{Q}_p$ such that the Lie algebra $\mathcal{L}(G) \otimes_{\mathbb{Q}_p} L$ is split and hence has a split lattice $\mathfrak{g}$ in addition to the lattices $p^n\mathfrak{h} \otimes_{\mathbb{Z}_p} \mathcal{O}_L$. We then sandwich a p-power multiple of $\mathfrak{g}$ in between two sufficiently large p-power multiples of $\mathfrak{h} \otimes_{\mathbb{Z}_p} \mathcal{O}_L$ in order to base change one of the $D_{p^n} \otimes_{KG} M$ to an affinoid enveloping algebra for which the theory of [AW1] applies.

As remarked earlier, Schmidt gave an analogue of [AW1, Theorem A] for coadmissible modules of the distribution algebra $D(G,K)$, using the methods of [AW1]. Here we may now allow G to be a locally L-analytic group for some intermediate extension $K/L/\mathbb{Q}_p$. Our argument works equally well in this setting to remove the splitness hypothesis of [Sch2, Theorem 9.9], giving us the following theorem:

Theorem 2. *Let G be a locally L-analytic group whose Lie algebra $\mathcal{L}(G)$ is semisimple, and let $G_{\mathbb{C}}$ denote a complex semisimple algebraic group with the same root system as $\mathcal{L}(G) \otimes_L \overline{\mathbb{Q}}_p$. Let M be a coadmissible $D(G,K)$-module such that $d_{D(G,K)}(M) \geq 1$ and assume that p is an odd very good prime for G. Then $d_{D(G,K)}(M) \geq r$, where r is half the smallest possible dimension of a nonzero coadjoint $G_{\mathbb{C}}$-orbit.*

We remark that Theorem 1 follows from Theorem 2 by the faithful flatness of the distribution algebra over the Iwasawa algebra ([ST2, Theorem 5.2]). Nevertheless we have opted to give independent arguments, keeping the proof of Theorem 1 independent of [Sch2, Corollary 5.13] (in the case $L = \mathbb{Q}_p$), which was conjectured in [AW1] but neither proved nor needed. We also remark that, for a coadmissible $D(G,K)$-module M, $d_{D(G,K)}(M) = 0$ if and only if $D_w \otimes_{D(G,K)} M$ is a finite-dimensional K-vector space for all w.

Let us now describe the contents of this paper. Section 2 proves the necessary flatness results needed make the sandwich argument work and recalls some generalities on crossed products. The flatness result is a consequence of a general result of Berthelot and Emerton. Section 3 introduces the main players, discusses some complements to results in [AW1], and then proves Theorem 1, fleshing out the strategy described above. Finally, section 4 proves Theorem 2.

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2. Flatness and Crossed Products

Fix a prime number p and let K be a finite extension of $\mathbb{Q}_p$. In this section we give a condition for the natural map between affinoid enveloping algebras induced from an inclusion of finite free $\mathcal{O}_K$-Lie algebras of the same rank to be left and right flat. All completions arising in this paper are p-adic completions. We use a technique due to Berthelot, abstracted by Emerton in [Em, Proposition 5.3.10].

Lemma 3. *Let $\mathfrak{h} \subseteq \mathfrak{g}$ be $\mathcal{O}_K$-Lie algebras, both finite free of rank d. Assume that $[\mathfrak{g}, \mathfrak{h}] \subseteq \mathfrak{h}$. Then* $\widehat{U(\mathfrak{g})}_K$ is a flat left and right $\widehat{U(\mathfrak{h})}_K$-module.

Proof. The proof is very close to that of [SS, Proposition 3.4], which contains the case $\mathfrak{h} = p\mathfrak{g}$ as a special case. First we prove right flatness. Put $A = U(\mathfrak{h})$ and $B = U(\mathfrak{g})$ and let $F_\bullet A$ resp. $F_\bullet B$ denote the usual Poincaré-Birkhoff-Witt filtrations on A and B respectively. A and B are well known to be p-torsion-free and p-adically separated left Noetherian $\mathbb{Z}_p$-algebras. Define a new increasing filtration on B by the $\mathbb{Z}_p$-submodules

$$F'_i B = A \cdot F_i B = \sum_{j \leq i} A\mathfrak{g}^j.$$

In order to apply [Em, Proposition 5.3.10], we need to verify conditions (i)-(iii) in the statement of [Em, Lemma 5.3.9]. Condition (ii), that $F'_0 B = A$, is clear. Condition (i) asks that $F'_i B \cdot F'_j B \subseteq F'_{i+j} B$. To see this, note that because $\mathfrak{g}$ normalises $\mathfrak{h}$, we have $[\mathfrak{g}, A] \subseteq A$, so $\mathfrak{g}A \subseteq A\mathfrak{g} + A$. Hence $\mathfrak{g}^i A \subset F'_i B$ by an easy induction, so $A\mathfrak{g}^i A\mathfrak{g}^j \subset F'_{i+j} B$ as required. Condition (iii) asks that $Gr^{F'}_\bullet B$ is finitely generated over A by central elements. Since $F_\bullet B \subseteq F'_\bullet B$, the identity map induces a natural map $f : Gr^F_\bullet B \to Gr^{F'}_\bullet B$ and since $Gr^{F'}_0 B = A$ we see that $Gr^{F'}_\bullet B$ is generated (as a ring) by $f(Gr^F_\bullet B)$ and A. Now $Gr^F_\bullet B = \mathrm{Sym}(\mathfrak{g})$, thus we may find $X_1, ..., X_d \in \mathfrak{g}$ whose symbols $gr^{F'} X_k \in Gr^{F'}_1 B$ generate $Gr^{F'}_\bullet B$ over A. Thus we have shown finite generation and to show that the $gr^{F'} X_k$ are central it suffices to prove that for any $X \in \mathfrak{g}$, $gr^{F'} X$ commutes with A in $Gr^{F'}_\bullet B$. Since A is generated by $\mathfrak{h}$ it suffices verify this for $Y \in \mathfrak{h} \subseteq Gr^{F'}_0 B$. Computing, we see that $(gr^{F'} X)(gr^{F'} Y) - (gr^{F'} Y)(gr^{F'} X)$ is equal to the image of the bracket $[X, Y]$ inside $Gr^{F'}_1 B$. By assumption $[X, Y] \in \mathfrak{h}$, so it becomes 0 in $Gr^{F'}_1 B$. Having verified the assumptions of [Em, Proposition 5.3.10] we may conclude that $\widehat{B}_K$ is a flat right $\widehat{A}_K$-module.

To prove left flatness we remark that it is well known that $U(\mathfrak{h})^{\mathrm{op}} \to U(\mathfrak{g})^{\mathrm{op}}$ is canonically isomorphic to $U(\mathfrak{h}^{\mathrm{op}}) \to U(\mathfrak{g}^{\mathrm{op}})$, where we recall that the opposite Lie algebra $\mathfrak{h}^{\mathrm{op}}$ of $\mathfrak{h}$ is the $\mathcal{O}_K$-module $\mathfrak{h}$ together with a new bracket $[-,-]'$ defined by $[X, Y]' = [Y, X]$. Thus, having proved right flatness of $U(\mathfrak{h}^{\mathrm{op}}) \to U(\mathfrak{g}^{\mathrm{op}})$ above we may deduce left flatness of $U(\mathfrak{h}) \to U(\mathfrak{g})$. $\square$

Next let us recall the notion of a crossed product from e.g. [Pa, §1]. Let S be a ring and let H be a group. A *crossed product* $S*H$ is a ring containing S as a subring and a subset $\overline{H}=\{\bar{h}\mid h\in H\}$ of units bijective with H, that satisfies the following conditions:

- $S*H$ is a free right S-module with basis $\overline{H}$.
- We have $\bar{h}S=S\bar{h}$ and $\overline{gh}S=\bar{g}\bar{h}S$ for all $g,h\in H$.

Given a crossed product $S*H$ we obtain functions $\sigma : H\to \mathrm{Aut}(S)$ and $\tau : H\times H\to S^{\times}$, called the action and the twisting, defined by

$$\sigma(h)(s):=(\bar{h})^{-1}s\bar{h},$$

$$\tau(g,h)=(\overline{gh})^{-1}\bar{g}\bar{h}.$$

The associativity of $S*H$ is equivalent to certain relations between σ and τ ([Pa, Lemma 1.1]) and conversely, given S, H, σ and τ satisfying these relations we may construct $S*H$ as the free right S-module on a set $\overline{H}=\{\bar{h}:h\in H\}$ which is in bijection with H, and multiplication being defined by extending the rule

$$(\bar{g}r)(\bar{h}s)=\overline{gh}\tau(g,h)r^{\sigma(h)}s$$

additively ($g,h\in H,\ r,s\in S$). Here we use the notation s^f to denote the (right) action of $f\in\mathrm{Aut}(S)$ on $s\in S$. Let us record how crossed products behave with respect to taking opposites.

Lemma 4. *Let $S*H$ be a crossed product of ring S by a group H, with action $\sigma : H\to\mathrm{Aut}(S)$ and twisting $\tau : H\times H\to S^{\times}$.*

1) $(S*H)^{\mathrm{op}}=S^{\mathrm{op}}*H^{\mathrm{op}}$, with action σ^{op} and twisting τ^{op} on the right hand side given by

$$\sigma^{\mathrm{op}}(h)=\sigma(h)^{-1},$$

$$\tau^{\mathrm{op}}(g,h)=\tau(h,g)^{\sigma(g)^{-1}\sigma(h)^{-1}}.$$

2) Let T be another ring with a homomorphism $\phi : S\to T$ and let $\Gamma\subseteq Aut(S)$ be a subgroup of the automorphisms of S that contains the inner automorphisms defined by $\tau(g,h)$ for $g,h\in H$ and the $\sigma(h)$ for $h\in H$. We assume that there is a compatible homomorphism $\psi : \Gamma\to\mathrm{Aut}(T)$, where by compatible, we mean that for $f\in\Gamma$, $\phi\circ f=\psi(f)\circ\phi$, and additionally that ψ sends the inner automorphism defined by $\tau(g,h)\in S^{\times}$ to the inner automorphism defined by $\phi(\tau(g,h))$ for all $g,h\in H$. Then we may form a crossed product $T*H$ with action $\psi\circ\sigma$ and twisting $\phi\circ\tau$. Moreover ϕ extends naturally to a homomorphism $\Phi : S*H\to T*H$.

3) The constructions in 1) and 2) are compatible: Given the situation in 2), we also have $\phi : S^{\mathrm{op}}\to T^{\mathrm{op}}$ and $S^{\mathrm{op}}*H^{\mathrm{op}}$; we may form $T^{\mathrm{op}}*H^{\mathrm{op}}$ and we obtain a natural map $S^{\mathrm{op}}*H^{\mathrm{op}}\to T^{\mathrm{op}}*H^{\mathrm{op}}$. Under the identification in 1) this agrees with $(T*H)^{\mathrm{op}}$ and Φ.

Proof. 1) $(S*H)^{\mathrm{op}}$ contains S^{op} and as a subring as well as the set $\overline{H}$ which is bijective with H^{op}. To verify that $(S*H)^{\mathrm{op}}$ is a crossed product $S^{\mathrm{op}}*H^{\mathrm{op}}$ it remains to verify that, working in $S*H$, $S\bar{h}$ is a free left S-module and that $S\overline{gh}=(S\bar{g})\bar{h}$ for all $g,h\in H$. The first assertion follows from $s\bar{h}=\bar{h}s^{\sigma(h)}$ and that $\bar{h}S$ is a free right S-module and $\sigma(h)$ is an automorphism of S. The second follows similarly using the formula

$$\bar{g}\bar{h}=\tau(g,h)^{\sigma(gh)^{-1}}\overline{gh}.$$

We may then compute the formulae for σ^{op} and τ^{op}, using $\cdot$ to distinguish the multiplication in $(S*H)^{\mathrm{op}}$ or H^{op} from that in $S*H$ or H:

$$s^{\sigma^{\mathrm{op}}(h)} = (\bar{h})^{-1} \cdot s \cdot \bar{h} = \bar{h}s(\bar{h})^{-1} = s^{\sigma(h)^{-1}};$$

$$\tau^{\mathrm{op}}(g,h) = (\overline{g \cdot h})^{-1} \cdot \bar{g} \cdot \bar{h} = \bar{h}\bar{g}(\overline{hg})^{-1} = \tau(h,g)^{\sigma(g)^{-1}\sigma(h)^{-1}}.$$

This finishes the proof of (1). We now prove (2). First, it is straightforward to verify that $\psi \circ \sigma$ and $\phi \circ \tau$ satisfy [Pa, Lemma 1.1] using the compatibility condition. Thus we may form $T * H$ as above, and ϕ induces an additive group homomorphism $\Phi : S*H \to T*H$ given by $\Phi(\bar{h}s) = \bar{h}\phi(s)$ which is easily checked to be a ring homomorphism. Finally, to check 3) we note that

$$(\psi \circ \sigma)^{\mathrm{op}}(h) = \psi(\sigma(h))^{-1} = \psi(\sigma(h)^{-1}) = \psi(\sigma^{\mathrm{op}}(h))$$

and

$$(\phi \circ \tau)^{\mathrm{op}}(g,h) = \phi(\tau(h,g))^{\psi(\sigma(g))^{-1}\psi(\sigma(h))^{-1}} = \phi(\tau(h,g))^{\psi(\sigma(g)^{-1}\sigma(h)^{-1})} = $$
$$= \phi\left(\tau(h,g)^{\sigma(g)^{-1}\sigma(h)^{-1}}\right) = \phi(\tau^{\mathrm{op}}(g,h)).$$

This shows that $(T*H)^{\mathrm{op}} = T^{\mathrm{op}} * H^{\mathrm{op}}$, and checking that the maps agree is another straightforward computation. □

Remark 5. This Lemma implies that the natural left module analogue of [AB, Lemma 5.4] holds (this may of course also be proved directly). References to [AB, Lemma 5.4] below will often implicitly be to its left module analogue.

3. Canonical Dimensions for Iwasawa algebras

Let us first fix some notation and terminology. As in the previous section we let K be a finite field extension of $\mathbb{Q}_p$. In this section we let G denote a compact p-adic analytic group. For the definition, see [DDMS, §8] ; note that for us a p-adic analytic group is the same as a $\mathbb{Q}_p$-analytic group. Put $d = \dim G$ (this is the dimension of G as a $\mathbb{Q}_p$-analytic group, so e.g. $\dim SL_2(K) = 3[K:\mathbb{Q}_p]$). As in [DDMS, §9.5] we let $\mathcal{L}(G)$ denote the Lie algebra of G; this is a $\mathbb{Q}_p$-Lie algebra of dimension d. We let KG denote the Iwasawa algebra of G with coefficients in K, defined as $KG = \left(\varprojlim \mathcal{O}_K[G/N]\right) \otimes_{\mathcal{O}_K} K$, where the inverse limit runs through all open normal subgroups $N \subseteq G$. KG is Auslander-Gorenstein with self-injective dimension d (see [AW1, Definition 2.5] for this notion). This follows for example from [AB, Lemma 5.4] , [AW1, Lemma 10.13] and the existence of a uniform normal open subgroup of G (see below for this notion). For any Auslander-Gorenstein ring A, we define the grade $j_A(M)$ and canonical dimension $d_A(M)$ of any nonzero finitely generated A-module M by

$$j_A(M) = \min\left\{ j \mid \mathrm{Ext}^j_A(M,A) \neq 0 \right\},$$
$$d_A(M) = \mathrm{inj\,dim}_A A - j_A(M).$$

See [AW1, §2.5] for more definitions and details. We will also use the notion of a *uniform pro-p group*, for which we refer to [DDMS, §4 and §8.3]. A uniform pro-p group H has an associated $\mathbb{Z}_p$-Lie algebra L_H — see [DDMS, §4.5] — which is free of rank $\dim H$. We also define $\mathcal{L}(H) = L_H \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. The remainder of this section will be devoted to the proof of Theorem 1. First, we record a simple reduction:

Lemma 6. *It suffices to prove Theorem 1 for uniform pro-p groups and for $K = \mathbb{Q}_p$.*

Proof. Pick a uniform normal open subgroup H of G. Let M be a finitely generated KG-module. Then $\mathcal{L}(G) = \mathcal{L}(H)$, so $r_G = r_H$, and [AB, Lemma 5.4] implies that $d_{KG}(M) = d_{KH}(M)$. Hence we may reduce to uniform pro-p groups. Then [AW1, Lemma 2.6] implies that it suffices to prove it for $K = \mathbb{Q}_p$. □

Before we proceed we need to recall the rings $\mathcal{U}_n * H_n$, introduced in [AW1, §10.6] . From now on let us assume that G is uniform and $K = \mathbb{Q}_p$. We may then set $\mathfrak{h} = p^{-1}L_G$; since L_G satisfies $[L_G, L_G] \subseteq pL_G$, $\mathfrak{h}$ is a $\mathbb{Z}_p$-sub Lie algebra of $\mathcal{L}(G)$, finite free of rank d. Recall that, for all $n \in \mathbb{Z}_{\geq 0}$, the groups G^{p^n} are normal uniform pro-p subgroups of G with $L_{G^{p^n}} = p^n L_G$ and we set $H_n = G/G^{p^n}$. In [AW1, §10.6], $\mathcal{U}_n$ is defined to be the microlocalisation of $\mathbb{Z}_pG^{p^n}$ with respect to the microlocal Ore set $S_n = \bigcup_{a\geq 0} p^a + \mathfrak{m}_n^{a+1}$, where $\mathfrak{m}_n$ is the unique (left and right) maximal ideal of $\mathbb{Z}_pG^{p^n}$. If we need to emphasize the group G we will write $\mathcal{U}_n(G)$. By [AW1, Theorem 10.4], $\mathcal{U}_n$ is isomorphic to $\widehat{U(p^n\mathfrak{h})}_{\mathbb{Q}_p}$. $\mathbb{Z}_pG$ is a crossed product $\mathbb{Z}_pG^{p^n} * H_n$ and since S_n is invariant under automorphisms we obtain a canonical homomorphism $\mathrm{Aut}(\mathbb{Z}_pG^{p^n}) \to \mathrm{Aut}(\mathcal{U}_n)$. Note that if $f_1, f_2 \in \mathrm{Aut}(\mathcal{U}_n)$ come from $\mathrm{Aut}(\mathbb{Z}_pG^{p^n})$ and agree on $\mathbb{Z}_pG^{p^n}$ then they are equal (this is immediate from the construction). Thus we may form $\mathcal{U}_n * H_n$ by Lemma 4(2) and we get an induced homomorphism $\mathbb{Z}_pG \to \mathcal{U}_n * H_n$. Since p is invertible in $\mathcal{U}_n * H_n$, we have a homomorphism $\mathbb{Q}_pG \to \mathcal{U}_n * H_n$. We will need some left/right complements to various results in [AW1]:

Proposition 7. *1) The natural map $\mathbb{Q}_pG \to \mathcal{U}_n * H_n$ is left and right flat.*

2) If M is a finitely generated left $\mathbb{Q}_pG$-module, then $(\mathcal{U}_n * H_n) \otimes_{\mathbb{Q}_pG} M = 0$ if and only if M is S_n-torsion.

3) If M is a finitely generated right $\mathbb{Q}_pG$-module, then $M \otimes_{\mathbb{Q}_pG} (\mathcal{U}_n * H_n) = 0$ if and only if M is S_n-torsion.

4) If M is a finitely generated p-torsion free left or right $\mathbb{Z}_pG$-module, then there exists an $n_0 \in \mathbb{Z}_{\geq 0}$ such that M is S_n-torsion-free for all $n \geq n_0$.

Proof. 1) Right flatness is [AW1, Proposition 10.6(d)] and from this we also get right flatness of $\mathbb{Q}_pG^{\mathrm{op}} \to \mathcal{U}_n(G^{\mathrm{op}}) * H_n^{\mathrm{op}}$ (i.e. performing the same constructions for the opposite group). However we have identifications $\mathbb{Q}_pG^{\mathrm{op}} = (\mathbb{Q}_pG)^{\mathrm{op}}$ and $\mathcal{U}_n(G^{\mathrm{op}}) * H_n^{\mathrm{op}} = (\mathcal{U}_n(G) * H_n)^{\mathrm{op}}$ identifying $\mathbb{Q}_pG^{\mathrm{op}} \to \mathcal{U}_n(G^{\mathrm{op}}) * H_n^{\mathrm{op}}$ with $(\mathbb{Q}_pG)^{\mathrm{op}} \to (\mathcal{U}_n(G) * H_n)^{\mathrm{op}}$ using Lemma 4, so we get the desired left flatness.

2) is [AW1, Proposition 10.6(e)] and 3) follows from 2) applied to G^{op} with the identifications in the proof of 1). Similarly the left part of 4) is [AW1, Corollary 10.11] and the right part follows as above. □

Before we get to the proof of the main theorem we will abstract a short calculation from the proof:

Lemma 8. *Let A, B, S and T be rings and assume that we have a commutative diagram*

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ S \arrow[r] & T \end{tikzcd}$$

where $A \to S$ makes S into a crossed product $A * G$ for some finite group G. Let M be a finitely generated left S-module and assume that $\mathrm{Ext}^i_S(M,S) \otimes_S T \neq 0$ for some i. Then

$$\mathrm{Ext}^i_A(M,A) \otimes_A B \neq 0.$$

Proof. Assume that $\mathrm{Ext}^i_A(M,A) \otimes_A B = 0$. Then $\mathrm{Ext}^i_A(M,A) \otimes_A T = 0$. However, by Remark 5 $\mathrm{Ext}^i_A(M,A) = \mathrm{Ext}^i_S(M,S)$ as right A-modules. Thus $\mathrm{Ext}^i_S(M,S) \otimes_A T = 0$. But $\mathrm{Ext}^i_S(M,S) \otimes_A T$ surjects onto $\mathrm{Ext}^i_S(M,S) \otimes_S T$, a contradiction. □

We now come to the proof of Theorem 1. Recall from [AW1, Theorem 3.3, Proposition 9.1(b)] and the paragraph below it that $\mathcal{U}_n$ is Auslander-Gorenstein with self-injective dimension d (in fact it is Auslander regular). By [AB, Lemma 5.4] $\mathcal{U}_n * H_n$ is also Auslander-Gorenstein of self-injective dimension d ($\mathcal{U}_n * H_n$ is also Auslander regular). We will use this freely in the proof (in particular the fact that the self-injective dimensions agree).

Proposition 9. *Theorem 1 is true when G is uniform and $K = \mathbb{Q}_p$.*

Proof. Let M be a finitely generated $\mathbb{Q}_pG$-module which is not finite dimensional as a $\mathbb{Q}_p$-vector space. Let $j = j_{\mathbb{Q}_pG}(M)$ and set $N := \mathrm{Ext}^j_{\mathbb{Q}_pG}(M, \mathbb{Q}_pG)$, this is a finitely generated right $\mathbb{Q}_pG$-module. By Proposition 7 we can find $t \geq 0$ such that N is S_n-torsion-free for all $n \geq t$. Now $N = \mathrm{Ext}^j_{\mathbb{Q}_pG^{p^n}}(M, \mathbb{Q}_pG^{p^n})$ and $d_{\mathbb{Q}_pG}(M) = d_{\mathbb{Q}_pG^{p^n}}(M)$ for any $n \geq 0$ by [AB, Lemma 5.4], so by replacing G by G^{p^t} if necessary we may assume $t = 0$. Thus we can assume that N is S_0-torsion-free. Let F be a finite extension of $\mathbb{Q}_p$ such that $\mathcal{L}(G) \otimes_{\mathbb{Q}_p} F$ is a split F-Lie algebra. Then we may find a split semisimple simply connected algebraic group $\mathbb{G}/\mathcal{O}_F$ whose $\mathcal{O}_F$-Lie algebra $\mathfrak{g}$ satisfies $\mathfrak{g} \otimes_{\mathcal{O}_F} F \cong \mathcal{L}(G) \otimes_{\mathbb{Q}_p} F$. We fix this isomorphism throughout and consider $\mathfrak{g}$ as a subset of $\mathfrak{g} \otimes_{\mathcal{O}_F} F$. Recall that $\mathfrak{h} = p^{-1}L_G$; we can find integers $n, m \geq 0$ such that

$$p^n(\mathfrak{h} \otimes_{\mathbb{Z}_p} \mathcal{O}_F) \subseteq p^m\mathfrak{g} \subseteq \mathfrak{h} \otimes_{\mathbb{Z}_p} \mathcal{O}_F.$$

Now because N is S_0-torsion-free, $N \otimes_{\mathbb{Q}_pG} \mathcal{U}_0$ is non-zero, and therefore $N \otimes_{\mathbb{Q}_pG} (\mathcal{U}_n * H_n)$ is also non-zero. By applying Proposition 7 together with [AW1, Proposition 2.6] we deduce that

$$d_{\mathbb{Q}_pG}(M) = d_{\mathcal{U}_0}(M_0) = d_{\mathcal{U}_n * H_n}(M_n)$$

where $M_n := (\mathcal{U}_n * H_n) \otimes_{\mathbb{Q}_pG} M$. We may now again apply [AB, Lemma 5.4] to deduce that

$$d_{\mathcal{U}_n * H_n}(M_n) = d_{\mathcal{U}_n}(M_n).$$

Furthermore [AW1, Lemmas 2.6 and 3.9] give us that

$$d_{\mathcal{U}_n}(M_n) = d_{F \otimes_{\mathbb{Q}_p} \mathcal{U}_n}(F \otimes_{\mathbb{Q}_p} M) \quad \text{and} \quad F \otimes_{\mathbb{Q}_p} \mathcal{U}_n = U(p^n \widehat{\mathfrak{h} \otimes_{\mathbb{Z}_p}} \mathcal{O}_F)_F.$$

Since $F \otimes_{\mathbb{Q}_p} \mathcal{U}_n \to F \otimes_{\mathbb{Q}_p} \mathcal{U}_0$ factors through $\widehat{U(p^m\mathfrak{g})}_F$, there is a commutative diagram

$$\begin{tikzcd}
F \otimes_{\mathbb{Q}_p} \mathcal{U}_n \arrow[r] \arrow[d] & \widehat{U(p^m\mathfrak{g})}_F \arrow[d] \\
(F \otimes_{\mathbb{Q}_p} \mathcal{U}_n) * H_n \arrow[r] & F \otimes_{\mathbb{Q}_p} \mathcal{U}_0.
\end{tikzcd}$$

Let $S = (F \otimes_{\mathbb{Q}_p} \mathcal{U}_n) * H_n$ and $T = F \otimes_{\mathbb{Q}_p} \mathcal{U}_0$, and note that $F \otimes_{\mathbb{Q}_p} M_n = S \otimes_{\mathbb{Q}_pG} M$, so that

$$\mathrm{Ext}^j_S(F \otimes_{\mathbb{Q}_p} M_n, S) \otimes_S T \cong \mathrm{Ext}^j_S(S \otimes_{\mathbb{Q}_pG} M, S) \otimes_S T \cong \mathrm{Ext}^j_{\mathbb{Q}_pG}(M, \mathbb{Q}_pG) \otimes_{\mathbb{Q}_pG} T = N \otimes_{\mathbb{Q}_pG} T \neq 0$$

because $N \otimes_{\mathbb{Q}_p G} \mathcal{U}_0 \neq 0$ by construction. Hence Lemma 8 implies that

$$\mathrm{Ext}^j_{F\otimes_{\mathbb{Q}_p}\mathcal{U}_n}(F \otimes_{\mathbb{Q}_p} M_n, F \otimes_{\mathbb{Q}_p} \mathcal{U}_n) \otimes_{F\otimes_{\mathbb{Q}_p}\mathcal{U}_n} \widehat{U(p^m\mathfrak{g})}_F \neq 0.$$

Since $p^n\mathfrak{h} \otimes_{\mathbb{Z}_p} \mathcal{O}_F \subseteq p^m\mathfrak{g}$ and $[p^m\mathfrak{g}, p^n\mathfrak{h} \otimes_{\mathbb{Z}_p} \mathcal{O}_F] \subseteq p^n\mathfrak{h} \otimes_{\mathbb{Z}_p} \mathcal{O}_F$, Lemma 3 implies that the top arrow in the above commutative diagram $F \otimes_{\mathbb{Q}_p} \mathcal{U}_n \to \widehat{U(p^m\mathfrak{g})}_F$ is left and right flat. Thus [AW1, Proposition 2.6] applied to the top arrow in the commutative diagram above gives

$$d_{\mathbb{Q}_p G}(M) = d_{F\otimes_{\mathbb{Q}_p}\mathcal{U}_n}(F \otimes_{\mathbb{Q}_p} M_n) = d_{\widehat{U(p^m\mathfrak{g})}_F}(V)$$

where $V := \widehat{U(p^m\mathfrak{g})}_F \otimes_{F\otimes_{\mathbb{Q}_p}\mathcal{U}_n} (F \otimes_{\mathbb{Q}_p} M_n)$. Here we have used the fact that $\widehat{U(p^m\mathfrak{g})}_F$ is Auslander regular of self-injective dimension d. Now [AW1, Lemma 10.13] implies that $d_{\mathbb{Q}_p G}(M) \geq 1$ since M is not finite dimensional as a $\mathbb{Q}_p$-vector space (we note here that of [AW1, Lemma 10.13] does not require G to satisfy the assumptions of of [AW1, §10.12]). Finally, [AW1, Theorem 9.10] implies that $d_{\widehat{U(p^m\mathfrak{g})}_F}(V) \geq r$, so $d_{\mathbb{Q}_p G}(M) \geq r$ as desired. □

4. Canonical dimensions for distribution algebras

In this section we let $L \subseteq K$ be finite extensions of $\mathbb{Q}_p$ and let G be a locally L-analytic group. We let $D(G, K)$ denote the algebra of K-valued distributions on G, studied in depth in [ST2], which we refer to for more details and some terminology.

Following the notation of [ST2] we let G_0 denote the underlying $\mathbb{Q}_p$-analytic group obtained from G by forgetting the L-structure. Recall (see e.g. [Sch2, §5.6]) that G is said to be L-uniform if G_0 is uniform and $L_{G_0} \subseteq \mathcal{L}(G)$ is an $\mathcal{O}_L$-lattice. If this holds we will write L_G for L_{G_0} with its $\mathcal{O}_L$-module structure. When G is L-uniform, $D(G, K)$ carries a Fréchet-Stein structure given by the inverse system $(D_r(G, K))_{r\in[p^{-1},1)\cap p^{\mathbb{Q}}}$ (see e.g. [Sch2, §5.17]). We put $r_n = p^{-1/p^n}$ and will from now on only consider the inverse system $(D_{r_n}(G, K))_{n\in\mathbb{Z}_{\geq 0}}$, which is final in the previous inverse system. The rings $D_{r_n}(G, K)$ are Auslander regular of self-injective and global dimension $d = \dim_L G$ when $n \geq 1$ by [Sch2, Proposition 9.3]. Thus, the category of coadmissible $D(G, K)$-modules ([ST2, §6]) has a well defined dimension theory ([ST2, §8]). Note that if G is L-uniform then so is G^{p^n} for all $n \geq 0$, and we have $D_{r_0}(G^{p^n}, K) * H_n = D_{r^n}(G, K)$ by [Sch2, Corollary 5.13] with $H_n = G/G^{p^n}$ as in the previous section.

When G is a general locally L-analytic group, G has at least one (and hence many) open L-uniform subgroup(s); indeed any compact open locally L-analytic subgroup of G has a basis of neighbourhoods of the identity consisting of open normal L-uniform subgroups by [Sch1, Corollary 4.4]. Thus the category of coadmissible $D(G, K)$-modules has a well defined dimension theory using the formalism of [ST2, §8], defined by restriction to an arbitrary open L-uniform subgroup. Therefore it suffices, by definition, to prove Theorem 2 for L-uniform groups.

We recall the link between affinoid enveloping algebras and distribution algebras. The first part of the following version of the Lazard isomorphism is essentially proved in [Sch2, §6.6] but stated only in a special case; it was then proven (in somewhat more generality) in [AW2, Lemma 5.2]. We give a brief sketch of the proof for the convenience of the reader.

Proposition 10. *Assume that G is L-uniform. Then there is an isomorphism $\Psi_G : \widehat{U(\mathfrak{g})}_K \to D_{r_0}(G, K)$, where $\mathfrak{g} = p^{-1}L_G$. It is compatible with morphisms $\alpha : G \to H$ in the sense that the* *diagram*

$$\begin{array}{ccc} \widehat{U(\mathfrak{g})}_K & \xrightarrow{U(\alpha)} & \widehat{U(\mathfrak{h})}_K \\ \downarrow{\scriptstyle \Psi_G} & & \downarrow{\scriptstyle \Psi_H} \\ D_{r_0}(G,K) & \xrightarrow{D(\alpha)} & D_{r_0}(H,K) \end{array}$$

commutes where $U(\alpha)$ and $D(\alpha)$ are the natural maps induced by α.

Proof. We let $\mathfrak{g}_0$ denote $\mathfrak{g}$ thought of as a $\mathbb{Z}_p$-Lie algebra by forgetting the $\mathcal{O}_L$-structure. Recall that there is a natural morphism $\mathcal{L}(G_0) \to D_{r_0}(G_0, K)$ factoring through $D(G_0, K)$ defined by

$$Xf = \frac{d}{dt}\left(f(\exp(-tX))\right)|_{t=0}$$

for $f \in \mathcal{C}^{\mathrm{la}}(G_0, K)$ and $X \in \mathcal{L}(G_0)$ (see e.g. [Sch2, §5.2]); it gives an inclusion $\mathfrak{g} \to D_{r_0}(G,K)$. By e.g. [Sch2, Proposition 6.3] there is an isomorphism

$$\Psi_{G_0} : \widehat{U(\mathfrak{g}_0)}_K = U(\mathfrak{g}_0 \widehat{\otimes_{\mathbb{Z}_p}} \mathcal{O}_L)_K \to D_{r_0}(G_0, K)$$

which is compatible with the embeddings of $\mathfrak{g}_0$ on both sides. Put $\mathfrak{k} = \ker(\mathfrak{g}_0 \otimes_{\mathbb{Z}_p} \mathcal{O}_L \to \mathfrak{g})$ where the map is given by $a \otimes X \mapsto aX$. Then $D_{r_0}(G,K)$ is the quotient of $D_{r_0}(G_0,K)$ by the ideal generated by $\mathfrak{k}$ by [Sch1, Lemma 5.1]. Similarly, $\widehat{U(\mathfrak{g})}_K$ is the quotient of $U(\mathfrak{g}_0 \widehat{\otimes_{\mathbb{Z}_p}} \mathcal{O}_L)_K$ by the ideal generated by $\mathfrak{k}$ by a straightforward argument: using the PBW filtration one sees that the sequence

$$0 \to \mathfrak{k}.U(\mathfrak{g}_0 \otimes_{\mathbb{Z}_p} \mathcal{O}_L) \to U(\mathfrak{g}_0 \otimes_{\mathbb{Z}_p} \mathcal{O}_L) \to U(\mathfrak{g}) \to 0$$

is an exact sequence of finitely generated $U(\mathfrak{g}_0 \otimes_{\mathbb{Z}_p} \mathcal{O}_L)$-modules. Now use the Artin-Rees Lemma and tensor with K to conclude. Thus Ψ_{G_0} induces the desired isomorphism Ψ_G by quotienting out by the ideal generated by $\mathfrak{k}$ on the source and target. Finally, the compatibility with morphisms follows from the functoriality of the morphism $\mathcal{L}(G_0) \to D_{r_0}(G_0, K)$, which is straightforward to check from the defining formula. □

With these preparations we may now prove Theorem 2 (which, we recall, one only needs to show for L-uniform groups).

Proposition 11. *Theorem 2 holds for L-uniform groups G.*

Proof. The proof is very similar to that of Proposition 9. First of all note that without loss of generality $L = K$ by [AW1, Lemma 2.6]. For the purposes of this proof we put $D := D(G,L)$ and $D_n := D_{r_n}(G,L)$. Let M be a coadmissible left D-module and put $N = \mathrm{Ext}^j_D(M,D)$ with $j = j_D(M)$. By [ST2, Lemma 8.4] we may find an integer $t \geq 0$ such that $N_n := N \otimes_D D_n = \mathrm{Ext}_{D_n}(M_n, D_n) \neq 0$ for all $n \geq t$, where $M_n := D_n \otimes_D M$. Because of the bimodule isomorphisms $D_n \cong D_{r_0}(G^{p^n}, L) \otimes_{D(G^{p^n},L)} D$ and $D_n \cong D \otimes_{D(G^{p^n},L)} D_{r_0}(G^{p^n}, L)$, we see that $M_n \cong D_{r_0}(G^{p^n}, L) \otimes_{D(G^{p^n},L)} M$ and $N_n \cong N \otimes_{D(G^{p^n},L)} D_{r_0}(G^{p^n}, L)$ for any $n \geq 0$. Therefore we may, as in the proof of Proposition 9, replace G by G^{p^t} and without loss of generality assume that $t = 0$.

Pick a finite extension F/L such that $\mathcal{L}(G) \otimes_L F$ is split and let $\mathfrak{g} \subseteq \mathcal{L}(G) \otimes_L F$ be a split $\mathcal{O}_F$-sub-Lie algebra. Write $\mathfrak{h} := p^{-1} L_G$ and pick positive integers m and n such that

$$p^n \mathfrak{h} \otimes_{\mathcal{O}_L} \mathcal{O}_F \subseteq p^m \mathfrak{g} \subseteq \mathfrak{h} \otimes_{\mathcal{O}_L} \mathcal{O}_F.$$

We have that $d_D(M) = d_{D_n}(M_n)$ and from here on we argue exactly as in the proof of Proposition 9, using that $D_n = D_{r_0}(G^{p^n}, L) * G/G^{p^n} = \widehat{U(p^n\mathfrak{h})}_L * G/G^{p^n}$ using Proposition 10. Note that the compatibility statement in Proposition 10 ensures that the diagram that Lemma 8 has to be applied to, namely

$$\begin{array}{ccc} D_{r_0}(G^{p^n}, F) & \longrightarrow & \widehat{U(p^m\mathfrak{g})}_F \\ \downarrow & & \downarrow \\ D_{r_n}(G, F) & \longrightarrow & F \otimes_L \widehat{U(\mathfrak{h})}_L \end{array}$$

is commutative. □

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