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Enveloping Algebras and Geometric Representation Theory

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Workshop: Enveloping Algebras and Geometric Representation Theory

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Abstracts

Towards a Riemann-Hilbert correspondence for $\widehat{\mathcal{D}}$-modules

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(joint work with Simon Wadsley)

1. Background

Let R be a complete discrete valuation ring with uniformiser π, residue field $k := R/\pi R$ and field of fractions $K := R[\frac{1}{\pi}]$.

Definition. *Let $\mathfrak{g}$ be a finite dimensional Lie algebra over K.*

(a) A Lie lattice *in $\mathfrak{g}$ is a finitely generated R-submodule L of $\mathfrak{g}$ which satisfies $[L, L] \subset L$ and which spans $\mathfrak{g}$ as a K-vector space.*

(b) Let L be a Lie lattice in $\mathfrak{g}$. The affinoid enveloping algebra *of L is*

$$\widehat{U(L)_K} := \left(\varprojlim U(L)/(\pi^a)\right) \otimes_R K.$$

(c) The Arens-Michael envelope *of $U(\mathfrak{g})$ is*

$$\widehat{U(\mathfrak{g})} := \varprojlim \widehat{U(L)_K}$$

where the inverse limit is taken over all possible Lie lattices L in $\mathfrak{g}$.

For any Lie lattice L in $\mathfrak{g}$, its set of π-power multiples is cofinal in the set of all Lie lattices, so that

$$\widehat{U(\mathfrak{g})} \cong \varprojlim \widehat{U(\pi^n L)_K}.$$

Example. Suppose that $\mathfrak{g} = Kx$ is a one-dimensional Lie algebra, spanned by an element x. If $L = Rx$ then $U(L) = R[x]$ is just a polynomial ring in one variable over R, the π-adic completion $\widehat{U(L)} = \widehat{R[x]}$ can be identified with the following subset of $R[[x]]$:

$$\widehat{R[x]} = \left\{ \sum_{i=0}^{\infty} \lambda_i x^i \in R[[x]] : \lim_{i\to\infty} \lambda_i = 0 \right\}.$$

The affinoid enveloping algebra $\widehat{U(L)_K}$ consists of power series in $K[[x]]$ satisfying the same convergence condition:

$$\widehat{U(L)_K} = K\langle x\rangle := \left\{ \sum_{i=0}^{\infty} \lambda_i x^i \in K[[x]] : \lim_{i\to\infty} \lambda_i = 0 \right\}.$$

Similarly, $\widehat{U(\pi^n L)_K} = K\langle \pi^n x\rangle$ can be identified with the set of formal power series $\sum_{i=0}^{\infty} \lambda_i x^i \in K[[x]]$ satisfying the stronger convergence condition

$$\lim_{i\to\infty} \lambda_i/\pi^{ni} = 0 \quad \text{for all} \quad n \geq 0.$$

It follows that the Arens-Michael envelope $\widehat{K[x]}$ of $K[x]$ consists of formal power series $\sum_{i=0}^{\infty} \lambda_i x^i \in K[[x]]$ whose sequence of coefficients (λ_i) is *rapidly decreasing*:

$$\widehat{K[x]} = K\{x\} := \left\{ \sum_{i=0}^{\infty} \lambda_i x^i \in K[[x]] : \lim_{i\to\infty} \lambda_i/\pi^{ni} = 0 \quad \text{for all} \quad n \geq 0 \right\}$$

Motivation. Let G be a p-adic Lie group, and suppose that the ground field K is a finite extension of the field $\mathbb{Q}_p$ of p-adic numbers. In number theory [13], we study *admissible locally analytic K-representations of G*. This is an abelian category which is anti-equivalent to the category of *co-admissible $D(G,K)$-modules*. We do not recall the definition of the *locally analytic distribution algebra* $D(G,K)$ here, but simply note that it is a particular K-Fréchet-space completion of the abstract group ring $K[G]$. This completion is large enough to contain the enveloping algebra $U(\mathfrak{g})$ of the Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$, and the closure of $U(\mathfrak{g})$ in $D(G,K)$ turns out to be isomorphic to its Arens-Michael envelope $\widehat{U(\mathfrak{g})}$.

Unfortunately, Arens-Michael envelopes are non-Noetherian rings whenever $\mathfrak{g}$ is non-zero. To get around this, Schneider and Teitelbaum introduced the following

Definition.

(a) Suppose that $A_0 \leftarrow A_1 \leftarrow A_2 \leftarrow \cdots$ is a tower of Noetherian K-Banach algebras such that
- *A_{n+1} has dense image in A_n for all $n \geq 0$, and*
- *A_n is a flat right A_{n+1}-module for all $n \geq 0$.*

Then $A := \varprojlim A_n$ is said to be a Fréchet-Stein algebra*.*

(b) A left A-module M is said to be co-admissible *if $A_n \otimes_A M$ is a finitely generated A_n-module for all $n \geq 0$, and the natural map $M \to \varprojlim A_n \otimes_A M$ is a bijection.*

(c) We let $\mathcal{C}_A$ denote the full subcategory of left A-modules consisting of the co-admissible A-modules.

Schneider and Teitelbaum proved that $\mathcal{C}_A$ is always an abelian category whenever A is a Fréchet-Stein algebra. They also proved that the locally analytic distribution algebras $D(G,K)$ and the Arens-Michael envelopes $\widehat{U(\mathfrak{g})}$ are Fréchet-Stein.

Example. The algebras $A_n = K\langle \pi^n x\rangle := \widehat{R[\pi^n x]} \otimes_R K$ satisfy the conditions above, so their inverse limit $\widehat{K[x]} = \varprojlim A_n$ provides an example of a (commutative) Fréchet-Stein algebra.

2. $\widehat{\mathcal{D}}$-modules on rigid analytic spaces

Suppose now that $\mathfrak{g}$ is a split semisimple Lie algebra over K. Prompted by a desire to establish an analogue of the Beilinson-Bernstein localisation theorem for co-admissible modules over $\widehat{U(\mathfrak{g})}$, we introduced the sheaf $\widehat{\mathcal{D}}$ of infinite-order differential operators on rigid analytic spaces in [1]. For the necessary background on rigid analytic geometry, we refer the reader to the survey paper [14].

Definition. *Let X be an affinoid variety over K, and let $\mathcal{T}(X) := \mathrm{Der}_K\, \mathcal{O}(X)$.*

(a) A Lie lattice *on X is any finitely generated $\mathcal{O}(X)^\circ$-submodule L of $\mathcal{T}(X)$ such that $[L, L] \subset L$ and L spans $\mathcal{T}(X)$ as a K-vector space.*

(b) For any Lie lattice L on X we have the Noetherian Banach algebra

$$\widehat{U(L)_K} := \left(\varprojlim U(L)/(\pi^a)\right) \otimes_R K.$$

(c) $\widehat{\mathcal{D}}(X) := \varprojlim \widehat{U(L)_K}$, the inverse limit being taken over all possible Lie lattices L in $\mathcal{T}(X)$.

Any Lie lattice L on X can be viewed as a *Lie-Rinehart algebra* over $(R, \mathcal{O}(X)^\circ)$, and as such has an enveloping algebra $U(L)$. These concepts were introduced by George Rinehart in [11].

Example. If $X = \operatorname{Sp} K\langle x\rangle$ is the closed unit disc, then

$$\widehat{\mathcal{D}}(X) = K\langle x\rangle\{\partial\} := \left\{ \sum_{i=0}^{\infty} a_i \partial^i \in K\langle x\rangle[[\partial]] : \lim_{i\to\infty} a_i/\pi^{ni} = 0 \quad \text{for all} \quad n \geq 0 \right\}$$

is a particular K-Fréchet-space completion of the Weyl algebra $K[x; \partial]$.

Theorem 1 ([2])**.** *Let X be a smooth rigid analytic space.*

(1) $\widehat{\mathcal{D}}$ extends to a sheaf of K-Fréchet algebras on X.

(2) If X is affinoid and $\mathcal{T}(X)$ is a free $\mathcal{O}(X)$-module, then $\widehat{\mathcal{D}}(X)$ is a Fréchet-Stein algebra.

This basic result makes the following definition meaningful.

Definition. *Let X be a smooth rigid analytic space. A sheaf of $\widehat{\mathcal{D}}$-modules $\mathcal{M}$ on X is* co-admissible *if there is an admissible covering $\{X_i\}$ of X such that $\mathcal{T}(X_i)$ is a free $\mathcal{O}(X_i)$-module, and $\mathcal{M}(X_i) \in \mathcal{C}_{\widehat{\mathcal{D}}(X_i)}$ for all i. We denote the category of all co-admissible $\widehat{\mathcal{D}}$-modules on X by $\mathcal{C}_X$.*

Co-admissible $\widehat{\mathcal{D}}$-modules form a stack on smooth rigid analytic spaces. More precisely, we have the following analogue of Kiehl's Theorem in rigid analytic geometry.

Theorem 2 ([2])**.** *If X is a smooth affinoid variety such that $\mathcal{T}(X)$ is a free $\mathcal{O}(X)$-module, then the global sections functor induces an equivalence of categories*

$$\Gamma : \mathcal{C}_X \xrightarrow{\cong} \mathcal{C}_{\widehat{\mathcal{D}}(X)}.$$

We can now formulate our version of the Beilinson-Bernstein equivalence.

Theorem 3 ([4])**.** *Let $\mathbf{G}$ be a connected, simply connected, split semisimple algebraic group over K, let $\mathfrak{g}$ be its Lie algebra and let $\mathcal{B} := (\mathbf{G}/\mathbf{B})^{\mathrm{an}}$ be the rigid-analytic flag variety. Then $\mathcal{C}_{\mathcal{B}} \cong \mathcal{C}_{\widehat{\mathcal{D}(\mathcal{B})}}$ and $\widehat{\mathcal{D}(\mathcal{B})} \cong \widehat{U(\mathfrak{g})} \otimes_{Z(\mathfrak{g})} K$.*

3. Holonomicity and $\widehat{\mathcal{D}}$-module operations

Let us recall the classical Riemann-Hilbert correspondence.

Theorem 4 (Kashiwara-Mebkhout)**.** *Let X be a smooth complex algebraic variety. Then the de Rham functor is an equivalence of categories*

$$\mathrm{DR} : D^b_{\mathrm{rh}}(\mathcal{D}_X) \longrightarrow D^b_c(\mathbb{C}_{X^{\mathrm{an}}}).$$

It sends regular holonomic $\mathcal{D}_X$-modules to perverse sheaves on X.

We are still rather far away from a perfect analogue of this theorem in the world of $\widehat{\mathcal{D}}$-modules on rigid analytic spaces! Nevertheless, there are some mildly encouraging signs that *some* such analogue exists. Let us recall some necessary ingredients of the proof of Theorem 4.

1. (1) DR gives an equivalence between integrable connections and local systems,
2. (2) a classification theorem for holonomic $\mathcal{D}$-modules,
3. (3) preservation of holonomicity under f_+, f^+ and $\mathbb{D}$.

We will not say anything in the direction of (1), except point out that there is a very well-developed theory of p-adic differential equations, which in part seeks to find an appropriate generalisation of (1) in the rigid-analytic setting. See for example [6], [10] and [12, Theorem 7.2]. It follows from [3, Theorem B] that integrable connections on smooth rigid analytic spaces can be naturally identified with co-admissible $\widehat{\mathcal{D}}$-modules that are $\mathcal{O}$-coherent.

In the direction of (2), a currently unresolved problem is to develop a good theory of characteristic varieties for co-admissible $\widehat{\mathcal{D}}$-modules. Nevertheless, we can make the following

Definition. *Let X be a smooth affinoid variety such that $\mathcal{T}(X)$ is a free $\mathcal{O}(X)$-module, and let M be a co-admissible $D := \widehat{\mathcal{D}(X)}$-module.*

1. *(1) The* grade *of M is $j(M) = \min\{j \in \mathbb{N} : \mathrm{Ext}^j_D(M, D) \neq 0\}$.*
2. *(2) The* dimension *of M is $d(M) := 2\dim X - j(M)$.*
3. *(3) M is* weakly holonomic *if $d(M) = \dim X$.*

These are reasonable definitions because (a slight modification of) the theory in [13, §8] can be applied to co-admissible D-modules. This is permissible because of the following theorem, whose proof uses Hartl's result [7] on the existence of regular formal models for smooth rigid analytic spaces.

Theorem 5 ([5])**.** *Let X be a smooth affinoid variety such that $\mathcal{T}(X)$ is a free $\mathcal{O}(X)$-module. Then*

1. *(1) There is a Fréchet-Stein structure $\widehat{\mathcal{D}(X)} \cong \varprojlim A_n$ where each A_n is Auslander-Gorenstein with injective dimension bounded above by $2\dim X$.*
2. *(2) $d(M) \geq \dim X$ for every non-zero co-admissible $\widehat{\mathcal{D}(X)}$-module M.*

Weakly holonomic $\widehat{\mathcal{D}}$-modules need not have finite length, as the following example shows.

Example. Let $\theta_n(t) = \prod_{m=0}^{n}(1 - \pi^m t)$ and define

$$\theta(t) := \lim_{n\to\infty} \theta_n(t) = \prod_{m=0}^{\infty}(1 - \pi^m t) \in \widehat{K[t]}.$$

Let $X = \operatorname{Sp} K\langle x\rangle$ be the closed unit disc, let $D := \widehat{\mathcal{D}(X)}$ and define

$$M := D/D\theta(\partial).$$

Then $d(M) = 1$ so M is weakly holonomic. However for every $n \geq 0$, M surjects onto $D/D\theta_n(\partial)$ which is a direct sum of $n+1$ integrable connections of rank 1 on X. Hence M has infinite length.

In the direction of (3), there is an analogue of Kashiwara's Equivalence:

Theorem 6 ([3, Theorem A])**.** *Let $\iota : Y \hookrightarrow X$ be a closed embedding of smooth rigid analytic spaces. Then the $\widehat{\mathcal{D}}$-module push-forward functor*

$$\iota_+ : \mathcal{C}_Y \to \mathcal{C}_X$$

is fully faithful, and its essential image consists of the co-admissible $\widehat{\mathcal{D}}_X$-modules $\mathcal{M}$ supported on $\iota(Y)$.

It is straightforward to check that ι_+ preserves weakly holonomic $\widehat{\mathcal{D}}$-modules. However, the following examples show that weakly holonomic $\widehat{\mathcal{D}}$-modules are too large to be preserved $\widehat{\mathcal{D}}$-module pushforwards and pullbacks, in general.

Example.

(1) Consider the weakly holonomic D-module M on $X = \operatorname{Sp} K\langle x\rangle$ from the previous Example, and let $\iota : Y := \{0\} \hookrightarrow X = \operatorname{Sp} K\langle x\rangle$ be the inclusion of a point. It is natural to define the pull-back $\iota^+ M$ of M along ι to be

$$\iota^+ M := M/xM.$$

However this is *not* a finite dimensional K-vector space, because M admits surjections onto integrable connections of arbitrarily large rank. Thus $\iota^+ M$ is not weakly holonomic.

(2) Let $U = X \backslash \{0\}$ and let $N := \widehat{\mathcal{D}(U)}/\widehat{\mathcal{D}(U)}\theta(1/x)$. Then N is the global sections of a weakly holonomic $\widehat{\mathcal{D}}$-module on the smooth quasi-Stein variety U, but it can be shown that N is not even co-admissible as a $D = \widehat{\mathcal{D}(X)}$-module.

In this last example, the problem is caused by the fact that $\operatorname{Supp}(N)$ is not a proper subset of X, and already in the classical setting of $\mathcal{D}$-modules on complex analytic manifolds, holonomicity is not preserved under $\mathcal{D}$-module pushforwards along open embeddings. However, we do have the following positive result, whose proof relies on Temkin's rigid-analytic version [15] of Hironaka's theorem on the embedded resolution of singularities of complex analytic spaces.

Theorem 7 ([5])**.** *Let $j : U \hookrightarrow X$ be a Zariski open embedding of smooth rigid analytic spaces. Then $\mathbf{R}^i j_*(\mathcal{O}_U)$ is a co-admissible weakly holonomic $\widehat{\mathcal{D}}_X$-module for all $i \geq 0$.*

Corollary ([5])**.** *Let Z be a closed analytic subset of the smooth rigid analytic space X. Then the local cohomology sheaves with support in Z*

$$H^i_Z(\mathcal{O}_X)$$

are co-admissible $\widehat{\mathcal{D}}_X$-modules for all $i \geq 0$.

These results give new examples of interesting weakly holonomic $\widehat{\mathcal{D}}$-modules. Local cohomology sheaves in rigid analytic geometry were originally considered by Kisin in [9]; note that $H^i_Z(\mathcal{O}_X)$ is not in general a coherent $\mathcal{D}$-module.

We end with expressing the hope that there is some full subcategory $\mathcal{H}$ of weakly holonomic $\widehat{\mathcal{D}}$-modules containing all integrable connections, whose objects have finite length and have well-defined characteristic varieties, whose simple objects admit a classification similar to [8, Theorem 3.4.2], and which are stable under all appropriate $\widehat{\mathcal{D}}$-module pushforwards and pullbacks.

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