LOCALISATION AT AUGMENTATION IDEALS IN IWASAWA ALGEBRAS

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Abstract. Let G be a compact $p-$adic analytic group and let Λ_G be its completed group algebra with coefficient ring the $p-$adic integers $\mathbb{Z}_p$. We show that the augmentation ideal in Λ_G of a closed normal subgroup H of G is localisable if and only if H is finite-by-nilpotent, answering a question of Sujatha. The localisations are shown to be Auslander-regular rings with Krull and global dimensions equal to $\dim H$. It is also shown that the minimal prime ideals and the prime radical of the $\mathbb{F}_p-$version Ω_G of Λ_G are controlled by Ω_{Δ^+}, where Δ^+ is the largest finite normal subgroup of G. Finally, we prove a conjecture of Ardakov and Brown[1].

1. Introduction

1.1. **Iwasawa algebras.** Let G be a compact $p-$adic analytic group and let Λ_G and Ω_G denote the completed group algebras of G with coefficients in $\mathbb{Z}_p$ and $\mathbb{F}_p$, respectively. Otherwise known as Iwasawa algebras, these rings were first defined by Lazard [16] and have been the focus of increasing attention in recent years, primarily because of their connections to number theory and arithmetic geometry. We refer the reader to [11] and [12] for more information about these connections.

1.2. Iwasawa algebras also form a natural class of Noetherian algebras, analogous to the classes of group algebras kH of polycyclic-by-finite groups H and enveloping algebras $\mathcal{U}(\mathfrak{g})$ of finite dimensional Lie algebras $\mathfrak{g}$. These rings have been extensively investigated during the '60s, '70s and '80s using the well-developed theory of noncommutative Noetherian rings [18], and a great deal is known about them. In contrast, much less is known about the structure of Λ_G and Ω_G (but see [2] for a summary of what *is* known).

1.3. **Localisation at semiprime ideals.** When R is a commutative ring and P is a prime ideal, one can always localise at P. This process is fundamental to commutative algebra and algebraic geometry and consists of "inverting" all elements of $R\backslash P$. When R is not necessarily commutative this is no longer possible in general, and the question of when prime (and more generally, semiprime) ideals are localisable plays a major role in the theory of noncommutative Noetherian rings.

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The book [15] is a definitive reference on this subject and [18, Chapter 4] provides a more condensed account.

1.4. Together with Ken Brown [1], the author studied localisation at ideals in Ω_G arising from a closed normal subgroup H of G. It was shown [1, Theorem D] that a certain semiprime ideal P_H of Ω_G is always localisable. Moreover, necessary and sufficient group-theoretic conditions on G and H were found to ensure that the augmentation ideal

$$w_{H,G} = \ker(\Omega_G \twoheadrightarrow \Omega_{G/H})$$

of Ω_G is localisable [1, Theorem E], provided it is semiprime.

1.5. Having done this, it was natural to ask whether augmentation ideals in Λ_G are localisable. Here there are two types of augmentation ideals, namely the $\mathbb{Z}_p-$augmentation ideal

$$I_{H,G} = \ker(\Lambda_G \twoheadrightarrow \Lambda_{G/H})$$

and the $\mathbb{F}_p-$version

$$v_{H,G} = \ker(\Lambda_G \twoheadrightarrow \Omega_{G/H}).$$

A straightforward lifting argument shows that when $v_{H,G}$ is semiprime, it is localisable in Λ_G if and only if $w_{H,G}$ is localisable in Ω_G [1, Theorem H].

1.6. Sujatha [23] asked the following question:

Question. *When is $I_{H,G}$ localisable?*

Note that because $\Lambda_{G/H}$ is semiprime [20], $I_{H,G}$ is always a semiprime ideal of Λ_G. Our first result, originally a conjecture of Brown [4], provides a complete answer to this question:

Theorem A. *$I_{H,G}$ is localisable if and only if H is finite-by-nilpotent.*

Recall that a group H is said to be *finite-by-nilpotent* if it has a finite normal subgroup F such that H/F is nilpotent. The proof is given in (3.7) and (3.15).

Note that a compact $p-$adic analytic group H is finite-by-nilpotent if and only if $H/\Delta^+(H)$ is nilpotent, where as in [1, 1.3] $\Delta^+(H)$ denotes the unique maximal finite normal subgroup of H.

1.7. Let G be a compact $p-$adic analytic group and suppose that H is a closed normal subgroup such that G/H is isomorphic to $\mathbb{Z}_p$. Then even though $w_{H,G}$ might not be localisable in Ω_G, we can ensure it becomes localisable by passing to an open subgroup: for example, choose any open pro-p subgroup G_1 of G and set $H_1 = H \cap G_1$; then w_{H_1,G_1} *is* localisable in Ω_{G_1} by [1, Theorem E].

On the other hand, the condition on H imposed by Theorem A is much more restrictive. For example, if G is open in $\mathrm{GL}_2(\mathbb{Z}_p)$ then neither $I_{H,G}$ nor "the" $\mathbb{Z}_p-$augmentation ideal $I_{G,G}$ is localisable in Λ_G.

1.8. There are a number of analogous results dealing with localisation in group algebras kG of polycyclic-by-finite groups G over a field k of characteristic zero. See for example [15, Theorem A.4.8], [5], [6] and [21, §11.2]. We have been unable to find the exact analogue of Theorem A in the literature.

1.9. **Properties of the localisations.** We next study ring-theoretic properties of the localisation $\Lambda_{G,H}$ of Λ_G at $I_{H,G}$ provided it exists. Our second result is an analogue of [1, Theorems I and J]:

Theorem B. *Suppose that $H/\Delta^+(H)$ is nilpotent. Then*

(a) $\Lambda_{G,H}$ *is Auslander regular,*
(b) $\mathrm{gld}(\Lambda_{G,H}) = \dim H$*,*
(c) $\mathcal{K}(\Lambda_{G,H}) = \dim H$*.*

The proof is given in (4.7). The method of proof is unusual in that the global dimension $\mathrm{gld}(\Lambda_{G,H})$ and the Krull dimension $\mathcal{K}(\Lambda_{G,H})$ are computed simultaneously. The crucial fact used here is a result of Roos [8, Corollary 1.3] which ensures that $\mathcal{K}(T) \leqslant \mathrm{gld}(T)$ for any Auslander-regular ring T.

1.10. Having obtained Theorem B, the author realised that the same method can be used to answer a question left open in [1]. This asks when the localisation $\Omega_{G,H}$ of Ω_G at the semiprime ideal P_H mentioned in (1.4) has finite global dimension.

Theorem C. *Let H be a closed normal subgroup of the compact $p-$adic analytic group G. Then $\Omega_{G,H}$ has finite global dimension if and only if the inverse image L of $\Delta^+(G/H)$ in G has no elements of order p.*

We proved in [1, Theorem J(iv)] that the group-theoretic condition appearing above is necessary and conjectured that it is also sufficient. It follows immediately from the proof of [1, Theorem J(iii)] that the global dimension of $\Omega_{G,H}$ equals $\dim H$ whenever it is finite. The proof of Theorem C is given in (6.4).

1.11. **Minimal primes of Ω_G.** The extra ingredient in the proof of Theorem C is a control theorem, Proposition 6.3. This states that P_H is controlled by Ω_L:

$$P_H = (P_H \cap \Omega_L).\Omega_G$$

and follows easily from our last result:

Theorem D. *Let $\Delta^+ = \Delta^+(G)$ denote the largest finite normal subgroup of the compact $p-$adic analytic group G. The minimal prime ideals of Ω_G are controlled by Ω_{Δ^+}. There is a bijective correspondence between minimal prime ideals of Ω_G and $G-$prime ideals of Ω_{Δ^+}, given by*

$$\begin{array}{ccc} P & \mapsto & P \cap \Omega_{\Delta^+} \\ Q.\Omega_G & \leftarrow & Q. \end{array}$$

The prime radical $N(\Omega_G)$ of Ω_G is also controlled by Ω_{Δ^+}:

$$N(\Omega_G) = (N(\Omega_G) \cap \Omega_{\Delta^+}).\Omega_G = J(\Omega_{\Delta^+}).\Omega_G.$$

Here $J(\Omega_{\Delta^+})$ denotes the Jacobson radical of Ω_{Δ^+}.

The proof is given in (5.6) and (5.7). Theorem D can be thought of as a generalization of [1, Theorem A] and [3, Theorem 9.2].

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1.13. **Conventions.** All rings are assumed to be associative and to have a unit element. All modules are assumed to be right modules, unless explicitly stated otherwise. When we speak of a ring-theoretic property like "Noetherian" or "localisable", we implicitly mean that both the right and left handed properties hold. $J(R)$ denotes the Jacobson radical of the ring R. For technical and aesthetic reasons, we prove most of the results stated in the Introduction for completed group algebras with slightly more general coefficients than $\mathbb{F}_p$ and $\mathbb{Z}_p$; see (2.2) and (2.3).

2. Preliminaries

2.1. **Background on localisation.** Let R be a Noetherian ring and let S be a multiplicatively closed subset of R containing 1. Recall [18, 2.1.3] that a *right localisation* of R at S is a ring R_S together with a ring homomorphism $\varphi : R \to R_S$ such that

- $\varphi(s)$ is a unit in R_S for all $s \in S$,
- every element of R_S can be written in the form $\varphi(r)\varphi(s)^{-1}$ and
- $\ker \varphi = \operatorname{ass}(S)$, where $\operatorname{ass}(S) := \{x \in R : xs = 0 \text{ for some } s \in S\}$ is the *assassinator* of S.

If R_S exists, it satisfies a universal property [18, Lemma 2.1.4] and is therefore unique up to isomorphism. A *left localisation* is defined similarly and is isomorphic to the right localisation whenever both exist by [18, Corollary 2.1.4].

Recall that S is said to be a *right Ore set* if for all $r \in R$ and $s \in S$ there exist $r' \in R$ and $s' \in S$ such that $rs' = r's$; left Ore sets are defined similarly. By Ore's Theorem [18, Theorem 2.1.12, Lemma 2.1.13], R_S exists if and only if S is a right Ore set. Note that in this case, R embeds into R_S if and only if S consists of *regular elements*: $r \in R$ is *regular* if it is not a zero-divisor. Of course, if R is commutative then every multiplicatively closed set S is automatically an Ore set, and localisation at S is always possible.

If I is an ideal of R, $\mathcal{C}_R(I)$ denotes the multiplicatively closed subset of R consisting of all elements of R which regular modulo I; recall that I is said to be *(right) localisable* if and only if $\mathcal{C}_R(I)$ is a (right) Ore set. Note that if R happens to be commutative and I is a prime ideal of R then $\mathcal{C}_R(I)$ is just $R \backslash I$.

2.2. **Notation.** Let K be a finite field extension of $\mathbb{Q}_p$. Let $\mathcal{O}$ be the ring of integers of K; this is a finite extension of $\mathbb{Z}_p$ and a complete local discrete valuation ring. We fix a uniformizer π of $\mathcal{O}$ and write $k = \mathcal{O}/\pi\mathcal{O}$ for the residue field of $\mathcal{O}$; this is a finite field of characteristic p.

Throughout this paper, G denotes a compact $p-$adic analytic group and H denotes a closed normal subgroup of G.

2.3. **Completed group algebras.** Let $\mathcal{O}[[G]]$ be the completed group algebra of G with coefficients in $\mathcal{O}$:

$$\mathcal{O}[[G]] = \varprojlim \mathcal{O}[G/N]$$

where N runs over all the open normal subgroups of G. Similarly,

$$k[[G]] = \varprojlim k[G/N]$$

is the completed group algebra of G with coefficients in k. Note that these are just the usual group algebras when G is finite. Since we will not be considering the usual group algebra kG or $\mathcal{O}G$ if G is infinite, we will usually denote $k[[G]]$ by kG and $\mathcal{O}[[G]]$ by $\mathcal{O}G$.

We will also write KG for the localisation of $\mathcal{O}[[G]]$ at $\{1, \pi, \pi^2, \cdots\}$:

$$KG := K \otimes_{\mathcal{O}} \mathcal{O}[[G]].$$

Note that if G is finite, this again coincides with the usual group algebra, however in general KG is not complete.

2.4. **The graded ring.** By a well-known result of Lazard [13, Corollary 8.34], any compact $p-$adic analytic group G contains an open normal *uniform* pro-p subgroup N of finite index; we will use this fact in what follows without further mention. Uniform pro-p groups are defined at [13, 4.1].

Lemma. *Let G be a uniform pro-p group and let J be the augmentation ideal of kG. Then $\operatorname{gr}_J kG \cong k[X_1, \ldots, X_d]$, where $d = \dim G$.*

Proof. This follows from [13, Theorem 7.24]. □

When $\mathcal{O} = \mathbb{Z}_p$, the following result is due to Venjakob [24, Theorem 3.26].

Proposition. *$\mathcal{O}G$ and kG are Auslander-Gorenstein. In particular, both rings are Noetherian.*

Proof. Suppose first that G is uniform. Then kG is Auslander-Gorenstein by the Lemma and [10, Theorem 2.4]. For $\mathcal{O}G$ use [10, Theorem 2.2] and the fact that $kG \cong \mathcal{O}G/\pi\mathcal{O}G$.

In general, choose an open normal uniform subgroup N of G; then $\mathcal{O}G$ is a crossed product of $\mathcal{O}N$ with the finite group G/N, so $\mathcal{O}G$ is Auslander-Gorenstein by the remarks preceeding [1, Lemma 5.4]. A similar argument deals with kG. □

2.5. **Primeness and semiprimeness for $\mathcal{O}G$.** We will need the following minor generalization of [1, Theorem F].

Proposition. *(i) $\mathcal{O}G$ is semiprime,*
(ii) $\mathcal{O}G$ is prime if and only if $\Delta^+(G) = 1$,
(iii) $\mathcal{O}G$ is a domain if and only if G is torsionfree.

Proof. For part (i), use the argument of Neumann [20]. That the group-theoretic conditions in parts (ii) and (iii) are necessary is obvious, so we just indicate how to prove that they are sufficient.

For part (ii), note that kG is prime by [3, Theorem 9.2]. Now the lifting argument used in the proof of [1, Theorem F](ii) works. For part (iii), imitate the proof of [1, Theorem C]. $\square$

3. Localisable ideals in $\mathcal{O}G$

3.1. As in the Introduction, define the ideal $I_{H,G}$ of $\mathcal{O}G$ by

$$I_{H,G} = \ker(\mathcal{O}G \twoheadrightarrow \mathcal{O}[[G/H]]).$$

Let $I_H = I_{H,H}$. We have $I_H = (H-1)\mathcal{O}H$ and $I_{H,G} = (H-1)\mathcal{O}G = I_H \cdot \mathcal{O}G = \mathcal{O}G \cdot I_H$. Because $\mathcal{O}[[G/H]]$ is semiprime by Proposition 2.5(i), $I_{H,G}$ is always a semiprime ideal of $\mathcal{O}G$.

3.2. We first prove the ($\Rightarrow$) part of Theorem A. The proof is given in (3.7) and requires some preliminary results.

Define a sequence of closed normal subgroups of G by setting $H_1 = H$ and $H_{n+1} = \overline{(H_n, H)}$ for $n \geqslant 1$, where $(x, y) = x^{-1}y^{-1}xy$ is the group commutator. Clearly H_n contains $\gamma_n(H)$, the $n-$th term of the lower central series of H. Now, the descending chain of closed subgroups

$$H = H_1 \geqslant H_2 \geqslant H_3 \geqslant \cdots$$

has a subgroup H_m of least dimension. Hence $|H_n/H_{n+1}| < \infty$ whenever $n \geqslant m$.

3.3. **Lemma.** Every right (and left) ideal of $\mathcal{O}G$ is closed with respect to the natural topology of $\mathcal{O}G$.

Proof. By Proposition 2.4, $\mathcal{O}G$ is Noetherian. Hence every one-sided ideal of $\mathcal{O}G$ is the continous image of $(\mathcal{O}G)^n$ for some $n \geqslant 1$. Since $\mathcal{O}G$ is compact and Hausdorff, the result follows. $\square$

3.4. **Lemma.** Let $P = I_{H_m,G}$, $Q = I_{H_{m+1},G}$ and $I = I_{H,G}$. Then $Q \subseteq PI$.

Proof. It is enough to show that $g - 1 \in PI$ for all $g \in H_{m+1} = \overline{(H_m, H)}$.

Since PI is closed by Lemma 3.3, it is enough to show that $(H_m, H) \equiv 1 \mod PI$. We will show that in fact $(x, y) \equiv 1 \mod PI$ for any $x \in H_m$ and $y \in H$.

This is true because if $x \in H_m$ and $y \in H$, then $x-1 \in P$ and $y-1 \in I$, so $[x-1, y-1] \in PI$ and therefore $xy \equiv yx \mod PI$. □

3.5. **Lemma.** If $S = \mathcal{C}_{\mathcal{O}G}(I)$ is a right Ore set in $\mathcal{O}G$, then P/Q is $S-$torsion.

Proof. P is generated as a right ideal by elements of the form $g-1$, where $g \in H_m$. Since S is a right Ore set, in view of [18, Lemma 2.1.8] it is sufficient to prove that for all $g \in H_m$ there exists $s \in S$ such that $(g-1)s \in Q$.

But if $g \in H_m$ then $g^n \in H_{m+1}$ for some $n \geqslant 1$ since H_m/H_{m+1} is finite. So

$$g^n - 1 = (g-1)(1 + g + \cdots + g^{n-1}) \in Q.$$

Now $s := 1 + g + \cdots + g^{n-1} \equiv n \mod I$ because $g \in H_m \subseteq H$. Since $\mathcal{O}G/I \cong \mathcal{O}[[G/H]]$ is torsionfree as a $\mathbb{Z}_p-$module, s is regular modulo I as required. □

3.6. **Lemma.** Let C be a right Ore set in $\mathcal{O}G$ such that $0 \notin C$. Then $(1+\text{ass}(C)) \cap G$ is a finite normal subgroup of G.

Proof. Since C is a right Ore set, $\text{ass}(C)$ is a two-sided ideal of $\mathcal{O}G$ by [18, Lemma 2.1.9]. Hence $F = (1 + \text{ass}(C)) \cap G$ is a normal subgroup of G.

Now, let U be an open normal uniform subgroup of G. If $g \in U \backslash 1$ then $g-1$ is a regular element in $\mathcal{O}U$ because $\mathcal{O}U$ is a domain by Proposition 2.5(iii). Since $\mathcal{O}G = \mathcal{O}U * (G/U)$ is a free right and left $\mathcal{O}U-$module, we see that $g-1$ is also a regular element of $\mathcal{O}G$ whenever $g \in U \backslash 1$. Since $0 \notin C$, we have

$$U \cap F = \{g \in U : (g-1)s = 0 \quad \text{for some} \quad s \in C\} = 1,$$

so $F \hookrightarrow G/U$ and F is finite. □

3.7. **Proof of Theorem A($\Rightarrow$).** Suppose $I = I_{H,G}$ is right localisable, so that $S = \mathcal{C}_{\mathcal{O}G}(I)$ is a right Ore set. If M is a right $\mathcal{O}G-$module, we will denote the localisation of M at S by M_S. By Lemma 3.4 we see that

$$Q_S \subseteq (PI)_S \subseteq P_S.I_S \subseteq P_S.$$

Also $Q_S = P_S$ since P/Q is $S-$torsion by Lemma 3.5, so $P_S.I_S = P_S$. But I_S is the Jacobson radical of $\mathcal{O}G_S$ [15, Theorem 3.2.3(a)], so $P_S = 0$ by Nakayama's Lemma. Therefore $P \subseteq \text{ass}(S)$.

Hence $H_m \leqslant (1 + P) \cap G \leqslant (1 + \text{ass}(S)) \cap G$ which is finite by Lemma 3.6, so $H_m \subseteq \Delta^+(H)$. It follows that the $m-$th term of the lower central series of $H/\Delta^+(H)$ is trivial, as required. □

3.8. We now turn to the proof of the converse assertion in Theorem A. The first step is a reduction to the case when $\Delta^+(H) = 1$.

Let $F = \Delta^+(H)$. Since H is normal in G and F is characteristic in H, F is a finite normal subgroup of G. Let $\overline{\cdot} : G \to G/F$ denote the natural surjection.

Lemma. *$I_{H,G} \lhd \mathcal{O}G$ is right localisable if and only if $I_{\overline{H},\overline{G}} \lhd \mathcal{O}\overline{G}$ is.*

Proof. Since π is always regular modulo $I_{H,G}$ and is a central regular element of $\mathcal{O}G$, $I_{H,G}$ is right localisable in $\mathcal{O}G$ if and only if $KI_{H,G}$ is right localisable in KG. Similarly, $I_{\overline{H},\overline{G}}$ is right localisable if and only if $KI_{\overline{H},\overline{G}}$ is right localisable.

It is well known that KI_F, the augmentation ideal of the ordinary group algebra KF, is generated as a right ideal by the central idempotent

$$f = 1 - \frac{1}{|F|}\sum_{g\in F} g.$$

Because F is normal in G, f is central in KG. Moreover, $KI_{F,G} = KI_F\mathcal{O}G = fKG$ is contained in $KI_{H,G}$. By [1, Proposition 3.6], $KI_{H,G}$ is right localisable in KG if and only if $KI_{H,G}/fKG$ is right localisable in KG/fKG. But the latter ring is isomorphic to $K\overline{G}$, and $KI_{H,G}/fKG \cong KI_{\overline{H},\overline{G}}$ under this isomorphism. $\square$

3.9. **Until the end of this section,** we assume that H is nilpotent and that $\Delta^+(H) = 1$. To avoid trivialities, assume further that $H \neq 1$.

Let $Z = Z(H)$ be the centre of H. Since $\Delta^+(Z)$ is a finite characteristic subgroup of Z, it is a finite normal subgroup of H and therefore is trivial. Hence Z is a torsionfree abelian pro-p group of finite rank $d \geqslant 1$, say: $Z \cong \mathbb{Z}_p^d$.

3.10. Choose a topological generating set $\{a_1, \ldots, a_d\}$ for Z and let $b_i = a_i - 1 \in \mathcal{O}Z$. Because $\mathcal{O}$ is a finitely generated $\mathbb{Z}_p-$module, one can check that

$$\mathcal{O}Z \cong \mathcal{O}\otimes_{\mathbb{Z}_p}\mathbb{Z}_pZ \cong \mathcal{O}[[b_1, \ldots, b_d]].$$

as $\mathcal{O}-$modules. Hence every element of $\mathcal{O}Z$ can be written in the form

$$\sum_{\alpha\in\mathbb{N}^d}\lambda_\alpha \mathbf{b}^\alpha$$

where $\lambda_\alpha \in \mathcal{O}$ for all $\alpha \in \mathbb{N}^d$. Clearly

$$I_Z = \{\sum_{\alpha\in\mathbb{N}^d}\lambda_\alpha\mathbf{b}^\alpha \in \mathcal{O}Z : \lambda_0 = 0\} = \sum_{i=1}^d b_i\mathcal{O}Z.$$

Since the b_i's commute, we have

$$I_Z^n = \sum_{\substack{\alpha\in\mathbb{N}^d\\ \langle\alpha\rangle = n}} \mathbf{b}^\alpha\mathcal{O}Z$$

for any $n \geqslant 0$. It is straightforward to see that I_Z^n/I_Z^{n+1} is $\pi-$torsionfree. Therefore I_Z^n/I_Z^{n+1} is a free $\mathcal{O}-$module of finite rank:

$$I_Z^n/I_Z^{n+1} = \bigoplus_{\substack{\alpha\in\mathbb{N}^d\\ \langle\alpha\rangle = n}} \mathcal{O}(\mathbf{b}^\alpha + I_Z^{n+1}).$$

3.11. Now consider the graded ring $\mathrm{gr}_I\,\mathcal{O}G$ of $\mathcal{O}G$ with respect to the $I-$adic filtration, where $I = I_{Z,G}$:

$$\mathrm{gr}_I\,\mathcal{O}G = \bigoplus_{n=0}^{\infty} I^n/I^{n+1}.$$

Since $I = I_Z \cdot \mathcal{O}G$ and Z is normal in G, $I^n = I_Z^n \cdot \mathcal{O}G$. By the flatness of $\mathcal{O}G$ viewed as a left $\mathcal{O}Z-$module [7, Lemma 4.5], we have

$$I^n/I^{n+1} = \frac{I_Z^n \cdot \mathcal{O}G}{I_Z^{n+1} \cdot \mathcal{O}G} \cong \left(\frac{I_Z^n}{I_Z^{n+1}}\right) \otimes_{\mathcal{O}Z} \mathcal{O}G$$

as right $\mathcal{O}G-$modules, and similarly on the left. In fact, I^n/I^{n+1} is a $\mathcal{O}[[G/Z]] - \mathcal{O}[[G/Z]]-$bimodule and (3.10) shows that I^n/I^{n+1} is free of finite rank both as a left and as a right $\mathcal{O}[[G/Z]]-$module.

3.12. **Lemma.** If A denotes $\mathrm{gr}_I\,\mathcal{O}G$, then $A/\pi A$ is right Noetherian.

Proof. Write $A_n = I^n/I^{n+1}$ so that $A = \bigoplus_{n=0}^{\infty} A_n$. Then

$$A/\pi A = \bigoplus_{n=0}^{\infty} \left(\frac{A_n}{\pi A_n}\right).$$

Next, $A_n/\pi A_n \cong (w_Z^n/w_Z^{n+1}) \otimes_{kZ} kG$ as right $kG-$modules, where w_Z is the image of I_Z in kZ. Also, w_Z is the augmentation ideal of kZ, so w_Z/w_Z^2 is a finite dimensional $k-$vector space and we can find an open normal subgroup N of G containing Z which acts trivially on w_Z/w_Z^2 by conjugation. Hence N acts trivially on every w_Z^n/w_Z^{n+1}. Therefore, using Lemma 2.4, we obtain a ring isomorphism

$$\begin{aligned} A/\pi A &\cong \left(\bigoplus_{n=0}^{\infty} \left(\frac{w_Z^n}{w_Z^{n+1}}\right) \otimes_{kZ} kN\right) \otimes_{kN} kG \\ &\cong k[[N/Z]][X_1, \ldots, X_d] * \left(\frac{G}{N}\right). \end{aligned}$$

The result follows from Hilbert's basis theorem, Proposition 2.4 and the fact that G/N is finite. □

3.13. **Proposition.** $A = \mathrm{gr}_I\,\mathcal{O}G$ is right Noetherian.

Proof. In view of [19, Theorem II.3.5], to prove that A is right Noetherian it is sufficient to show that every graded right ideal $J = \bigoplus_{n=0}^{\infty} J_n$ of A is finitely generated.

Suppose first that J is such that A/J is $\pi-$torsion free. Now, the image of J in $A/\pi A$ is a graded right ideal and $A/\pi A$ is right Noetherian, so we can find $z_1, \ldots, z_t \in J$ such that $z_i \in A_{d_i}$ for some integers $d_1, \ldots, d_t$ and

$$J + \pi A = \sum_{i=1}^{t} z_i A + \pi A.$$

Equating graded components gives

$$J_n + \pi A_n = \sum_{i=1}^{t} z_i A_{n-d_i} + \pi A_n$$

for all $n \in \mathbb{Z}$, setting $A_n = 0$ for $n < 0$. Since A_n and A_n/J_n are $\pi-$torsionfree, $J_n \cap \pi^a A_n = \pi^a J_n$ for any $a \geqslant 0$. Hence

$$(\dagger) \qquad (J_n \cap \pi^a A_n) + \pi^{a+1} A_n = \left(\sum_{i=1}^{t} z_i \pi^a A_{n-d_i} \right) + \pi^{a+1} A_n$$

for all a and n. Fix n and let $K_n = \sum_{i=1}^{t} z_i A_{n-d_i}$. Clearly $K_n \subseteq J_n$; we claim that in fact $K_n = J_n$. To see this, let $x \in J_n$. By (†) with $a = 0$, we can find $r_{i0} \in A_{n-d_i}$ for $i = 1, \ldots, t$ and $x_1 \in A_n$ such that

$$x = \sum_{i=1}^{t} z_i r_{i0} + \pi x_1.$$

Since $\sum_{i=1}^{t} z_i r_{i0} \in K_n \subseteq J_n$, we see that $\pi x_1 \in J_n \cap \pi A_n$. Applying (†) with $a = 1$ gives elements $r_{i1} \in A_{n-d_i}$ and $x_2 \in A_n$ such that

$$\pi x_1 = \sum_{i=1}^{t} z_i \pi r_{i1} + \pi^2 x_2.$$

Continuing like this, we obtain elements $r_{i0}, r_{i1}, \ldots \in A_{n-d_i}$ for each $i = 1, \ldots, t$ such that

$$x - \sum_{i=1}^{t} z_i \left(\sum_{a=0}^{k} \pi^a r_{ia} \right) \in \pi^{k+1} A_n$$

for all $k \in \mathbb{N}$. Since A_{n-d_i} is a free right $\mathcal{O}[[G/Z]]-$module of finite rank by (3.11), it is complete with respect to the $\pi-$adic filtration, so for each $i = 1, \ldots, t$ the partial sums $\sum_{a=0}^{k} \pi^a r_{ia}$ converge to an element $r_i = \sum_{a=0}^{\infty} \pi^a r_{ia} \in A_{n-d_i}$. It follows that

$$x = \sum_{i=1}^{t} z_i r_i \in K_n$$

so $J_n = K_n$ as claimed. Taking the direct sum over all $n \in \mathbb{Z}$ gives $J = \sum_{i=1}^{t} z_i A$, so J is finitely generated.

Now if J is an arbitrary graded right ideal of A, let $L = \{x \in A : \pi^m x \in J$ for some $m \geqslant 0\}$. This is again a graded right ideal of A and A/L is $\pi-$torsionfree so L is finitely generated by the above. Hence we can find some $m \geqslant 0$ such that $\pi^m L \subseteq J \subseteq L$. Now $J/\pi^m L$ is a right ideal of the ring $A/\pi^m A$ which is right Noetherian because $A/\pi A$ is, so $J/\pi^m L$ is a finitely generated right $A-$module. It follows that J is finitely generated, as required. □

3.14. **Lemma.** Let R be a right Noetherian ring and let P be a right localisable semiprime ideal of R. Let ${}_RV_R$ be a bimodule which is free of finite rank on both sides and such that $VP = PV$. Then for any $v \in V$ and any $s \in S = \mathcal{C}_R(P)$, there exists $v' \in V$ and $s' \in S$ such that

$$vs' = sv'.$$

Proof. Let $\overline{\cdot} : R \to R/P$ denote the natural surjection. We have a map of right $R-$modules $\varphi_s : V_R \to V_R$ given by left multiplication by s. This induces a map of right $\overline{R}-$ modules

$$\overline{\varphi_s} : V/VP \to V/VP.$$

Since $V/VP = V/PV$ is a free left $\overline{R}-$module and $s \in S$ is right regular modulo P, we see that $\ker(\overline{\varphi_s}) = 0$. Now $(V/VP)_S$ is a free right $(R/P)_S-$module of finite rank, and $(R/P)_S$ is semisimple Artinian by Goldie's Theorem [18, Theorem 2.3.6]. Hence $(V/VP)_S$ is a right Artinian R_S-module, so the localised map

$$(\overline{\varphi_s})_S : (V/VP)_S \to (V/VP)_S$$

is surjective. Hence $sV_S + V_SP_S = V_S$ and therefore $(V_S/sV_S).P_S = V_S/sV_S$. Now P_S is the Jacobson radical of R_S and V_S/sV_S is a finitely generated right R_S-module, so $V_S = sV_S$ by Nakayama's Lemma.

Now if $v \in V$, we can find $w \in V$ and $t \in S$ such that $v1^{-1} = swt^{-1}$ inside V_S. Hence $vt - sw$ lies in the right $S-$torsion submodule of V and therefore we can find some $u \in S$ such that $(vt - sw)u = 0$. Now set $s' = tu \in S$ and $v' = wu \in V$. □

3.15. **Proof of Theorem A($\Leftarrow$).** By Lemma 3.8, we can assume that $\Delta^+(H) = 1$, so H is nilpotent. Proceed by induction on the nilpotency class of H. When this is zero, $H = 1$ and $I_{H,G} = 0$ is right localisable by Goldie's Theorem [18, Theorem 2.3.6] because $\mathcal{O}G$ is semiprime by Proposition 2.5(i).

Assume therefore that the nilpotency class of H is nonzero, so that the centre Z of H is nontrivial. Let $\overline{\cdot} : G \to G/Z$ denote the natural surjection and $I = I_{Z,G}$. Also, let $R = \mathcal{O}\overline{G} \cong \mathcal{O}G/I$ and $P = I_{H,G}/I = I_{\overline{H},\overline{G}}$. Since $\overline{H}$ is nilpotent of class strictly smaller than that of H, P is a semiprime right localisable ideal in R by induction.

Thus $S = \mathcal{C}_R(P)$ is a right Ore set in R and we have to show that $T = \mathcal{C}_{\mathcal{O}G}(I_{H,G})$ is a right Ore set in $\mathcal{O}G$.

We can view S as a subset of

$$\operatorname{gr}_I \mathcal{O}G = R \oplus \frac{I}{I^2} \oplus \frac{I^2}{I^3} \oplus \cdots$$

consisting of homogeneous elements of degree 0. It is straightforward to see that T is the "saturated lift of S" in the language of [14]:

$$T = \{t \in \mathcal{O}G : \sigma_I(t) \in S\}.$$

Moreover, that T is a right Ore set in $\mathcal{O}G$ follows from [14, Corollary 2.2], provided

(a) the Rees ring $\widetilde{\mathcal{O}G}$ with respect to the $I-$adic filtration is right Noetherian, and
(b) S is a right Ore set in $\operatorname{gr}_I \mathcal{O}G$.

Now, $\mathcal{O}G$ is complete with respect to the $I-$adic filtration and $\operatorname{gr}_I \mathcal{O}G$ is right Noetherian by Proposition 3.13, so (a) follows from [17, Chapter II, Proposition 2.2.1].

To show (b) holds, let $r \in \operatorname{gr}_I \mathcal{O}G$ and let $s \in S$. Then $r \in V := \bigoplus_{n=0}^{m} I^n/I^{n+1}$ for some $m \in \mathbb{N}$. Note that V is an $R-R-$bimodule, free of finite rank on both sides by the remarks made in (3.11).

Since Z commutes with H, $I^n I_{H,G} = I_{H,G} I^n$ for any $n \geqslant 0$, so

$$\left(\frac{I^n}{I^{n+1}}\right).P = P.\left(\frac{I^n}{I^{n+1}}\right)$$

for all $n \geqslant 0$. Hence $VP = PV$, so by Lemma 3.14, we can find $r' \in V \subseteq \operatorname{gr}_I \mathcal{O}G$ and $s' \in S$ such that $rs' = sr'$ as required.

Of course, everything above has a left-handed version, so we also have proved that $I_{H,G}$ is left localisable if $H/\Delta^+(H)$ is nilpotent. The result follows. □

4. Properties of the localisations

4.1. This section is devoted to the proof of Theorem B, which is given in (4.7). As in §3, we begin with a reduction to the case when $\Delta^+(H) = 1$. First, a well-known group theoretic result.

Lemma. *Suppose H is nilpotent and that $\Delta^+(H) = 1$. Then H is torsionfree.*

Proof. The subset F of H consisting of all elements of finite order is a subgroup by the proof of [13, 0.4(vii)]. Choose an open normal uniform subgroup U of H; then U is torsionfree by [13, Theorem 4.5], so $U \cap F = 1$ and F is finite. Clearly F is normal, so $F = \Delta^+(H)$ and the result follows. □

4.2. **Until the end of this section,** we will assume that $I_{H,G}$ is localisable, or equivalently that $H/\Delta^+(H)$ is nilpotent by Theorem A.

Choose an open normal torsionfree subgroup L/H of G/H. The next result is a direct analogue of [1, Lemma 5.1]. We will denote the localisation of $\mathcal{O}G$ at $I_{H,G}$ by $\mathcal{O}G_H$.

Lemma. *Suppose that $\Delta^+(H) = 1$. Then*

(i) $\mathcal{O}G_H$ is a crossed product of $\mathcal{O}L_H$ with the finite group G/L:

$$\mathcal{O}G_H \cong \mathcal{O}L_H * (G/L),$$

(ii) $T := \mathcal{C}_{\mathcal{O}G}(I_{H,G})$ consists of regular elements of $\mathcal{O}G$.

Proof. The proof of part (i) is very similar to the proof of [1, Lemma 5.1] so we will omit the details. The main things to note are:

- H is torsionfree by Lemma 4.1, so L is also torsionfree,

- $\mathcal{O}L$ is a domain by Proposition 2.5(iii) because L is torsionfree,
- $S := \mathcal{C}_{\mathcal{O}L}(I_{H,L})$ is an Ore set in $\mathcal{O}L$ consisting of regular elements,
- S is also an Ore set in $\mathcal{O}G$ consisting of regular elements of $\mathcal{O}G$, and
- $\mathcal{O}G_T \cong \mathcal{O}G_S$.

The last isomorphism shows that $\mathrm{ass}(T) = \mathrm{ass}(S)$, but S consists of regular elements so $\mathrm{ass}(T) = 0$ and part (ii) follows. □

4.3. We will need the following result in the proof of Theorem B.

Proposition. *Let* $F = \Delta^+(H)$. *Then* $\mathrm{ass}(\mathcal{C}_{\mathcal{O}G}(I_{H,G})) = I_{F,G}$.

Proof. Let $\overline{\cdot} : G \to G/F$ denote the natural surjection. We have seen in the proof of Lemma 3.8 that $KI_{H,G}$ is localisable in KG and that it contains the ideal $KI_{F,G}$ which is generated by the central idempotent f. Note that because $\pi \in \mathcal{C}_{\mathcal{O}G}(I_{H,G})$, $\mathcal{O}G_H$ is isomorphic to the localisation of KG at $KI_{H,G}$. Let u denote the image of f in $\mathcal{O}G_H$.

Now, the Jacobson radical of $\mathcal{O}G_H$ is equal to $I_{H,G}.\mathcal{O}G_H$ so it contains u. But then $1-u$ is invertible and u is an idempotent, so $u = 0$. Hence

$$I_{F,G} \subseteq \mathcal{O}G \cap fKG \subseteq \mathrm{ass}(\mathcal{C}_{\mathcal{O}G}(I_{H,G})).$$

In order to prove the reverse inclusion, we may assume that $F = 1$. But now $\mathcal{C}_{\mathcal{O}G}(I_{H,G})$ consists of regular elements of $\mathcal{O}G$ by Lemma 4.2(ii) and the result follows. □

Corollary. $\mathcal{O}G_H \cong \mathcal{O}\overline{G}_{\overline{H}}$.

Until the end of this section, we will assume that $\Delta^+(H) = 1$.

4.4. **Proposition.** $\dim H \leqslant \mathcal{K}(\mathcal{O}H_H)$.

Proof. We will construct a chain of prime ideals in $R := \mathcal{O}H_H$ of length $d = \dim H$.

We can assume that $d \geqslant 1$ to avoid trivialities. Now $\dim Z \geqslant 1$ where Z is the centre of H as in (3.9). Let A be a closed subgroup of Z with $\dim A = 1$; then A is normal in H. Let H_1 be the inverse image in H of $\Delta^+(H/A)$, this is again a normal subgroup of H with $\dim H_1 = 1$. Note that $I_{H_1,H}R$ is a nonzero ideal in R by Proposition 4.3 because H_1 is infinite.

Since $\Delta^+(H/H_1) = 1$ and $\dim H/H_1 = d-1$, we can iterate the above construction and choose a sequence of closed normal subgroups

$$1 = H_0 < H_1 < \cdots < H_d = H$$

such that $\Delta^+(H/H_i) = 1$ for all $i = 0, \ldots, d-1$. This gives rise to a sequence of ideals in R

$$0 \subset I_{H_1,H}R \subset I_{H_2,H}R \subset \cdots \subset I_{H_d,H}R$$

and each inclusion is strict by the above remarks. It is easy to see that

$$R/I_{H_i,H}R \cong \mathcal{O}[[H/H_i]]_{H/H_i},$$

a localisation of $\mathcal{O}[[H/H_i]]$. Since $\Delta^+(H/H_i) = 1$ by construction, $\mathcal{O}[[H/H_i]]$ is a prime ring by by Proposition 2.5(ii), so each $I_{H_i,H}R$ is a prime ideal in R.

The result now follows from [18, Lemma 6.4.5]. □

4.5. **Lemma.** $\mathcal{O}H_H \hookrightarrow \mathcal{O}G_H$ and $\mathcal{O}G_H$ is a faithfully flat $\mathcal{O}H_H-$module.

Proof. Let $R = \mathcal{O}H_H$ and $T = \mathcal{O}G_H$. Since $\Delta^+(H) = 1$, $\mathcal{O}H \hookrightarrow R$ and $\mathcal{O}G \hookrightarrow T$ by Proposition 4.3.

Because $\mathcal{C}_{\mathcal{O}H}(I_H) \subseteq \mathcal{C}_{\mathcal{O}G}(I_{H,G})$, the natural injection $\mathcal{O}H \to \mathcal{O}G \to T$ factors through $\mathcal{O}H \to R$ by the universal property of localisation [18, Lemma 2.1.4]. Hence $R \hookrightarrow T$ as required for the first part.

Now, $\mathcal{O}G$ is a flat $\mathcal{O}H-$module. Because localisation is exact, T is a flat $\mathcal{O}H-$module. For any $R-$module X, we have $X \otimes_{\mathcal{O}H} R \cong X$ by [18, Proposition 7.4.2(i)], so

$$X \otimes_R T \cong X \otimes_{\mathcal{O}H} R \otimes_R T \cong X \otimes_{\mathcal{O}H} T.$$

Hence T is a flat $R-$module.

Finally, $\mathcal{O}H/I_H \cong \mathcal{O}$ is a domain, so the localisation R/I_HR is a division ring (in fact, it is isomorphic to the field K). Since $J := I_HR$ is the Jacobson radical of R, J is the unique maximal right and left ideal of R. But now $JT = I_H \cdot \mathcal{O}G \cdot T = I_{H,G}T$ is the Jacobson radical of T so $JT \neq T$. Hence T is a faithfully flat $R-$module by [18, Proposition 7.2.3]. □

Corollary. $\mathcal{K}(\mathcal{O}H_H) \leqslant \mathcal{K}(\mathcal{O}G_H)$.

Proof. Apply [18, Lemma 6.5.3(ii)]. □

4.6. **Lemma.** $\mathcal{O}G_H$ is Auslander-Gorenstein.

Proof. Since $\Delta^+(H) = 1$, $\mathcal{C}_{\mathcal{O}G}(I_{H,G})$ consists of regular elements of $\mathcal{O}G$ by Lemma 4.2(ii). Now $\mathcal{O}G$ is Auslander-Gorenstein by Proposition 2.4, so we can apply [9, Proposition 2.1]. □

4.7. **Proof of Theorem B.** By the right exactness of tensor product, we see that $(\mathcal{O}H/I_H) \otimes_{\mathcal{O}H} T \cong T/I_HT$. By the proof of Lemma 4.5, $T = \mathcal{O}G_H$ is a flat $\mathcal{O}H-$module, so

$$\mathrm{pd}_T(T/I_HT) \leqslant \mathrm{pd}_{\mathcal{O}H}(\mathcal{O}H/I_H) = \mathrm{pd}_{\mathcal{O}H}(\mathcal{O}).$$

By [7, Corollary 4.4] and [22, Corollaire 1], we have $\mathrm{pd}_{\mathcal{O}H}(\mathcal{O}) = \dim H$. Now $J(T) = I_HT$ and T is semilocal, so [18, Theorem 7.3.14(ii)] gives

$$\mathrm{gld}(T) \leqslant \mathrm{gld}(T/I_HT) + \mathrm{pd}_T(T/I_HT) \leqslant \dim H,$$

bearing in mind [18, 7.1.5]. Thus T has finite global dimension. Because T is Auslander-Gorenstein by Lemma 4.6, part (a) follows.

We prove parts (b) and (c) simultaneously. By Proposition 4.4 and Corollary 4.5, we have $\dim H \leqslant \mathcal{K}(\mathcal{O}H_H) \leqslant \mathcal{K}(T)$. But $\mathcal{K}(T) \leqslant \mathrm{gld}(T)$ by [8, Corollary 1.3] because T is Auslander-Gorenstein, and we have seen above that $\mathrm{gld}(T) \leqslant \dim H$. The result follows. □

5. Minimal primes in kG

5.1. We now turn to the study of the minimal prime ideals of kG. First, a very general result.

Lemma. *Let A be a closed subgroup of G. Then kG is a faithfully flat $kA-$module. For any right ideal I of kA, we have $I = IkG \cap kA$.*

Proof. Note that kG is a flat $kA-$module by [7, Lemma 4.5]. For any right $kA-$module M we have an isomorphism of $k-$vector spaces

$$M \otimes_{kA} kG \cong M \otimes_k k \otimes_{kA} kG \cong M \otimes_k \mathrm{Ind}_A^G k.$$

Since k is a field and $\mathrm{Ind}_A^G k$ is always nonzero, $\mathrm{Ind}_A^G k$ is a faithfully flat $k-$module. Hence if $M \otimes_{kA} kG = 0$ then $M = 0$ as required. The second statement follows from [18, Lemma 7.2.5] applied to the $kA-$module kA/I. □

5.2. Let $\Delta^+ = \Delta^+(G)$ and $J = J(k\Delta^+)$. Because Δ^+ is finite, this is a nilpotent ideal. Moreover, J is clearly invariant under conjugation by elements of G, so the right ideal JkG of kG is actually two-sided:

$$J \cdot kG = kG \cdot J.$$

This makes it easy to see that JkG is nilpotent and hence contained in the prime radical of kG. In what follows, we let $\overline{\cdot} : kG \to kG/JkG$ denote the canonical map. By Lemma 5.1, we see that $\overline{k\Delta^+} \cong k\Delta^+/J$ is a semisimple Artinian subring of $\overline{kG}$.

5.3. The group G acts on the centrally primitive idempotents of $\overline{k\Delta^+}$. Whenever $\mathcal{C}$ is a $G-$orbit on these idempotents, $\widehat{\mathcal{C}} = \sum_{e \in \mathcal{C}} e$ is a central idempotent in $\overline{kG}$. Let $f_1, \ldots, f_r$ be the central idempotents of kG obtained in this way. It is easy to see that they are pairwise orthogonal and that $1 = f_1 + f_2 + \cdots + f_r$.

Theorem. *$f_i.\overline{kG}$ is a prime ring for all $i = 1, \ldots, r$.*

Proof. When $J = 0$, this is precisely the content of [3, Theorem 9.2]. It is possible to make appropriate modifications to the proof in [3, §10] to cover the general case. Because this is straightforward, long and mainly consists of replacing kG by $\overline{kG}$ and $k\Delta^+$ by $\overline{k\Delta^+}$ in various places, we will omit the details. □

5.4. As in [3, §9], Theorem 5.3 has many consequences. Before we deduce them, we need an elementary Lemma.

Lemma. *Let R be a ring and let $1 \neq e \in R$ be a central idempotent. Suppose that eR is a prime ideal in R. Then eR is a minimal prime.*

Proof. Let Q be a prime ideal of R contained in eR and let $f = 1 - e$. Now if $f \in Q$, $f = f^2 \in fQ \subseteq feR = 0$ so $f = 0$ and $e = 1$, a contradiction. Because $0 = eR.fR \subseteq Q$ and Q is prime, $e \in Q$ so $Q = eR$ as required. □

5.5. Keeping the notation of (5.3), let $e_i = 1 - f_i \in \overline{k\Delta^+}$, $i = 1, \ldots, r$. These are again nonzero central idempotents of $\overline{kG}$.

Theorem. *The minimal primes of $\overline{kG}$ are precisely $\{e_1.\overline{kG}, \ldots, e_r.\overline{kG}\}$.*

Proof. Since $\overline{kG}/e_i.\overline{kG} \cong f_i.\overline{kG}$ is a prime ring by Theorem 5.3, each $e_i.\overline{kG}$ is prime. By Lemma 5.4 each $e_i.\overline{kG}$ is minimal.

Now if Q is a minimal prime of $\overline{kG}$, Q contains $(e_1.\overline{kG}) \cdots (e_r.\overline{kG}) = 0$ so $e_i.\overline{kG} \subseteq Q$ for some i. By the minimality of Q, $Q = e_i.\overline{kG}$ and the result follows. □

Corollary. *$\overline{kG}$ is semiprime.*

Proof. If $x \in \bigcap_{i=1}^r e_i\overline{kG}$, then $f_i x = 0$ for all $i = 1, \ldots, r$. Hence $x = (f_1 + \cdots + f_r)x = 0$, so 0 is a semiprime ideal of $\overline{kG}$. □

5.6. **Prime radical of kG.** It is now straightforward to lift information to kG:

Theorem. *The prime radical $N(kG)$ of kG is controlled by $k\Delta^+$:*

$$N(kG) = JkG = (N(kG) \cap k\Delta^+).kG.$$

Proof. We have seen in (5.2) that $JkG \subseteq N(kG)$. But $\overline{kG} = kG/JkG$ is semiprime by Corollary 5.5, so $N(kG) = JkG$. The last equality follows from Lemma 5.1. □

5.7. **Minimal primes of kG.** Recall [18, 10.5.3] that if G is a group acting by ring automorphisms on a ring R, then an ideal P of R is said to be $G-$*prime* if whenever A and B are $G-$invariant ideals of R such that $AB \subseteq P$, then either $A \subseteq P$ or $B \subseteq P$. By [18, Corollary 10.5.7], the minimal primes above P form a single $G-$orbit and P is the intersection of the minimal primes above it; thus any $G-$prime ideal is semiprime, but not necessarily prime.

Theorem. *The minimal prime ideals of kG are controlled by $k\Delta^+$. There is a bijective correspondence between minimal prime ideals of kG and $G-$prime ideals of $k\Delta^+$, given by*

$$\begin{array}{ccc} P & \mapsto & P \cap k\Delta^+ \\ Q.kG & \leftarrow\!\shortmid & Q. \end{array}$$

Proof. Choose $a_i \in k\Delta^+$ such that $\overline{a_i} = e_i \in \overline{k\Delta^+}$, and let $P_i = a_i kG + JkG$ be the inverse image of $e_i\overline{kG}$ in kG. By Theorems 5.5 and 5.6, $\overline{P_i}$ is a minimal prime of $\overline{kG}$ and JkG is the prime radical of kG, so P_i is a minimal prime of kG. Moreover, all the minimal primes arise in this way, so $\{P_1, \ldots, P_r\}$ is the complete list of minimal primes of kG.

Now $P_i = (a_i k\Delta^+ + J).kG$ so $P_i \cap k\Delta^+ = a_i k\Delta^+ + J$ and

$$P_i = (P_i \cap k\Delta^+).kG$$

by Lemma 5.1. Thus the P_i's are all controlled by $k\Delta^+$.

Finally $\overline{P_i \cap k\Delta^+} = e_i.\overline{k\Delta^+}$ is a $G-$prime ideal of $\overline{k\Delta^+}$ by construction, so $P_i \cap k\Delta^+$ is a $G-$prime ideal of $k\Delta^+$. If Q is a $G-$prime ideal of $k\Delta^+$, then $k\Delta^+/Q$ is semisimple Artinian because Q is semiprime. Now the centrally primitive idempotents of $k\Delta^+/Q$ must lie in a single $G-$orbit, so $Q = a_i k\Delta^+ + J$ for some i. Hence all $G-$prime ideals of $k\Delta^+$ are of the form $P_i \cap k\Delta^+$ for some i, and the result follows. □

6. An application

6.1. In [1], Ken Brown and the author studied a certain semiprime ideal P_H of Ω_G coming from a closed normal subgroup H of a compact $p-$adic analytic group G. We briefly recall the definition. Recall (1.4) that $w_{H,G}$ denotes the kernel of the map $\Omega_G \twoheadrightarrow \Omega_{G/H}$.

6.2. Choose an open pro-p subgroup N of H which is normal in G. Then P_H is defined to be the prime radical of $w_{N,G}$:

$$P_H := \sqrt{w_{N,G}}.$$

It is shown in [1, Lemma 3.2] that P_H is independent of the choice of N and is thus well-defined. Now let L be the inverse image of $\Delta^+(G/H)$ in G. Then any open subgroup of H is open in L, so $P_H = P_L$.

6.3. We can use this observation to obtain a more explicit description of P_H.

Proposition. *The ideal P_H is controlled by Ω_L:*

$$P_H = J(\Omega_L)\Omega_G = (P_H \cap \Omega_L)\Omega_G.$$

Proof. Choose N as in (6.2). Then $P_H/w_{N,G}$ is the prime radical of the ring $\Omega_G/w_{N,G} \cong \Omega_{G/N}$. It is easy to see that $\Delta^+(G/N) = L/N$, so by Theorem 5.6

$$P_H/w_{N,G} = J(\Omega_{L/N})\Omega_{G/N}.$$

Now $w_{N,L} = w_{N,N}\Omega_L$ is contained in $J(\Omega_L)$ by the remarks in [1, 1.2], because N is an open pro-p subgroup of L. Hence $J(\Omega_{L/N}) = J(\Omega_L)/w_{N,L}$. Because Ω_G is a flat Ω_L-module, we obtain

$$J(\Omega_{L/N})\Omega_{G/N} \cong \frac{J(\Omega_L)}{w_{N,L}} \otimes_{\Omega_L} \Omega_G \cong \frac{J(\Omega_L)\Omega_G}{w_{N,L}\Omega_G} = \frac{J(\Omega_L)\Omega_G}{w_{N,G}},$$

so $P_H = J(\Omega_L)\Omega_G$. The second equality follows from Lemma 5.1. □

6.4. **Proof of Theorem C.** It was proved in [1, Theorem J(iv)] that the finiteness of the global dimension of $\Omega_{G,H}$ forces L to have no elements of order p.

Conversely, suppose that L has no elements of order p and let $R = \Omega_L$. By the well-known result of Brumer [7], R has finite global dimension. Now, $T := \Omega_{G,H}$ is a semilocal Noetherian ring with Jacobson radical $J(T) = P_H T = J(R)T$ by Proposition 6.3. Because Ω_G is a flat $R-$module and because localisation is exact, T is a flat $R-$module. Hence

$$\mathrm{pd}_T(T/J(T)) \leqslant \mathrm{pd}_R(R/J(R)) \leqslant \mathrm{gld}\, R < \infty.$$

Now apply [18, Theorem 7.3.14(ii)].

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