A VERSION OF THE FUNDAMENTAL THEOREM FOR YOUNG MEASURES

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1. Introduction.

Let $\Omega \subset \mathbb{R}^n$ be measurable and let $z^{(j)}:\Omega \longrightarrow \mathbb{R}^m$ be a given sequence of functions. The fundamental theorem concerning Young measures asserts that under appropriate hypotheses there exists a subsequence $z^{(\mu)}$ of $z^{(j)}$ and a family of probability measures (ν_x) on $\mathbb{R}^m$ (the *Young measure*) such that the weak limit of $f(z^{(\mu)})$ is given by the expectation

$$<\nu_x, f> = \int_{\mathbb{R}^m} f(\lambda)\ d\nu_x(\lambda) \qquad \text{a.e. } x \in \Omega, \tag{1}$$

for any continuous function $f:\mathbb{R}^m \longrightarrow \mathbb{R}$. The purpose of this note is to give a version of the theorem that is convenient for some applications to nonlinear partial differential equations and variational problems of mechanics, together with a reasonably self-contained proof. Although sharper than some statements in the literature as regards the hypotheses on the $z^{(j)}$ and f, the form of the theorem given here is not essentially new. In particular, E.J.Balder has shown me how it may be regarded as a consequence of a very general lower semicontinuity theorem he has proved in [3 Theorem 2.1]. Nevertheless, I hope that the statement given here may be of some use.

The Young measure (ν_x) can intuitively be thought of as giving the limiting probability distribution as $\mu \longrightarrow \infty$ of the values of $z^{(\mu)}$ near x. To be more precise, suppose that Ω is open and that $x \in \Omega$. Denote by $B(x,\delta)$ the open ball with centre x and radius $\delta > 0$. Keeping x, μ and δ fixed, let $\nu^{(\mu)}_{x,\delta}$ be the probability distribution of the values of $z^{(\mu)}(y)$ as y is chosen uniformly at random from $B(x,\delta)$. Then it is shown below that

$$\nu_x = \lim_{\delta\to 0}\lim_{\mu\to\infty} \nu^{(\mu)}_{x,\delta}, \tag{2}$$

almost everywhere, where the convergence is weak* in the sense of measures.

The Young measure was introduced by L.C.Young (see [39,40]) as a means of treating problems of the calculus of variations for which there does not exist a minimizer in a classical sense. Many applications and developments to the calculus of variations and optimal control theory have been made by MacShane [28], Gamkrelidze [24] and others. More abstract ideas in the same spirit are the integral currents of Federer & Fleming [23] and varifolds (see, for example, Allard [1]). The Young measure was developed as a tool for analysing nonlinear partial differential equations by Tartar [36], who suggested how it could help to prove existence theorems for nonlinear hyperbolic systems, a programme he carried out for a single hyperbolic equation. These ideas were then significantly developed by DiPerna [18] to prove the existence of solutions for a system of two hyperbolic equations in one space dimension. Further results along these lines have been given by, for example, Dafermos [15], DiPerna [18], Rascle [31], Roytburd & Slemrod [32], Schonbek [33] and Serre [34]. Applications of the Young measure to variational problems of continuum mechanics have been made by Ball & Knowles [9], Ball & James [7,8], Chipot & Kinderlehrer [14], and Kinderlehrer [27]. In the last four papers cited, the application is to the description of the microstructure of crystals. (The order in which the limits are taken in (2) corresponds to the way in which the microstructure is experimentally observed; namely, a small region of diameter δ of the crystal is examined microscopically, but this region is typically larger than the length scale μ^{-1} of the microstructure.)

The method used here to prove the fundamental theorem delivers (ν_x) directly via duality rather than by disintegration of a measure on a product space; the principal idea can be found in Castaing & Valadier [13], Warga [38,39], Balder [3,4], and recent descriptions appear in Capuzzo Dolcetta & Ishii [12], and Slemrod & Roytburd [35]. An alternative direct construction is suggested in Tartar [37 pp268-9]. For the more usual method see Berliocchi & Lasry [11], Tartar [36] and Balakrishnan [2]. A Young measure corresponding to a bounded sequence in L^p capable of detecting some concentrations as well as oscillations has been introduced by DiPerna & Majda [19,20] as a tool for studying vortex dynamics.

2. The fundamental theorem for Young measures.

Our aim is to prove the following result:

Theorem

Let $\Omega \subset \mathbb{R}^n$ be Lebesgue measurable, let $K \subset \mathbb{R}^m$ be closed, and let $z^{(j)}: \Omega \longrightarrow \mathbb{R}^m$, $j = 1,2,\ldots$, be a sequence of Lebesgue measurable functions satisfying $z^{(j)}(\cdot) \longrightarrow K$ in measure as $j \longrightarrow \infty$, i.e. given any open neighbourhood U of K in $\mathbb{R}^m$

$$\lim_{j\to\infty} \text{meas } \{x \in \Omega : z^{(j)}(x) \notin U\} = 0.$$

Then there exists a subsequence $z^{(\mu)}$ of $z^{(j)}$ and a family (ν_x), $x \in \Omega$, of positive measures on $\mathbb{R}^m$, depending measurably on x, such that (*i*) $\|\nu_x\|_M := \int_{\mathbb{R}^m} d\nu_x \le 1$ for a.e. $x \in \Omega$,

(*ii*) $\text{supp}\ \nu_x \subset K$ for a.e. $x \in \Omega$, and

(*iii*) $f(z^{(\mu)}) \xrightarrow{*} \langle \nu_x, f\rangle = \int_{\mathbb{R}^m} f(\lambda)\, d\nu_x(\lambda)$

in $L^\infty(\Omega)$ for each continuous function $f : \mathbb{R}^m \longrightarrow \mathbb{R}$ satisfying $\lim_{|\lambda|\to\infty} f(\lambda) = 0$.

Suppose further that $\{z^{(\mu)}\}$ satisfies the boundedness condition

$$\lim_{k\to\infty} \sup_\mu \text{meas}\ \{x \in \Omega \cap B_R : |z^{(\mu)}(x)| \ge k\} = 0, \tag{3}$$

for every $R > 0$, where $B_R = B(0,R)$. Then $\|\nu_x\|_M = 1$ for a.e. $x \in \Omega$ (i.e. ν_x is a probability measure), and given any measurable subset A of Ω

$$f(z^{(\mu)}) \longrightarrow \langle \nu_x, f\rangle \quad \text{in } L^1(A) \tag{4}$$

for any continuous function $f : \mathbb{R}^m \longrightarrow \mathbb{R}$ such that $\{f(z^{(\mu)})\}$ is sequentially weakly relatively compact in $L^1(A)$.

Remarks

1. The condition (3) is very weak, and is equivalent to the following: given any $R > 0$ there exists a continuous nondecreasing function $g_R : [0,\infty) \longrightarrow \mathbb{R}$, with $\lim_{t\to\infty} g_R(t) = \infty$, such that

$$\sup_\mu \int_{\Omega\cap B_R} g_R(|z^{(\mu)}(x)|)\, dx < \infty. \tag{5}$$

In fact suppose that (5) holds. Then, since g_R is nondecreasing,

$$\sup_\mu \text{meas}\ \{x \in \Omega \cap B_R : |z^{(\mu)}(x)| \ge t\}\cdot g_R(t) \le \sup_\mu \int_{\Omega\cap B_R} g_R(|z^{(\mu)}(x)|)\, dx.$$

Since $\lim_{t\to\infty} g_R(t) = \infty$, we obtain (3).

Conversely, if (3) holds, we may choose $0 < t_j < t_{j+1}$, $j=1,2,\dots$, so that

$$\sup_\mu \text{meas}\ \{x \in \Omega\cap B_R : |z^{(\mu)}(x)| \ge t_j\} \le j^{-3},$$

and let

$$\bar{g}_R(t) = \begin{cases} 0 & \text{if } t \in [0, t_1), \\ j & \text{if } t \in [t_j, t_{j+1}). \end{cases}$$

Then

$$\sup_\mu \int_{\Omega\cap B_R} \bar{g}_R(|z^{(\mu)}(x)|)\, dx = \sup_\mu \sum_{j=1}^{\infty} j\ \text{meas}\ \{x \in \Omega \cap B_R : t_{j+1} > |z^{(\mu)}(x)| > t_j\}$$

$$\le \sum_{j=1}^{\infty} j^{-2} < \infty .$$

Choosing a suitable continuous $g_R \le \bar{g}_R$ we thus obtain (5).

Conditions similar to (5) are used by Berliocchi & Lasry [11 Proposition 5], and by Balder [3 Section 2] (who calls it 'tightness'; see also Balder [5]).

An application of the theorem to the case when the $z^{(j)}$ are bounded in $L^1(\Omega;\mathbb{R}^m)$ (i.e. with $g_R(t) = t$) appears in Ball & Murat [10].

2. If the functions $z^{(j)}$ are uniformly bounded in $L^\infty(\Omega;\mathbb{R}^m)$ then the functions $f(z^{(j)})$ are uniformly bounded in $L^\infty(\Omega)$ for any continuous $f:\mathbb{R}^m \longrightarrow \mathbb{R}$. Hence by the theorem there is a family of probability measures (ν_x) and a subsequence $z^{(\mu)}$ such that

$$f(z^{(\mu)}) \overset{*}{\longrightarrow} \langle \nu_x, f \rangle \quad \text{in } L^\infty(\Omega)$$

for all such f. In this way we recover the form of the theorem given by Tartar [36]. If Ω is bounded and if the $z^{(j)}$ are uniformly bounded in $L^p(\Omega;\mathbb{R}^m)$ for some p, $1 < p < \infty$, then we obtain from the theorem the existence of a family of probability measures (ν_x) and a subsequence $z^{(\mu)}$ such that

$$f(z^{(\mu)}) \longrightarrow \langle \nu_x, f \rangle \quad \text{in } L^r(\Omega) \tag{6}$$

for any continuous $f : \mathbb{R}^m \longrightarrow \mathbb{R}$ satisfying

$$|f(\lambda)| \le const.\ (1 + |\lambda|^q), \qquad \lambda \in \mathbb{R}^m, \tag{7}$$

where $q > 0$ and $1 < r < p/q$ (see Schonbek [33]).

3. If A is bounded, the condition that $\{f(z^{(\mu)})\}$ be sequentially weakly relatively compact in $L^1(A)$ is satisfied if and only if

$$\sup_\mu \int_A \psi(|f(z^{(\mu)})|)\, dx < \infty$$

for some continuous function $\psi : [0,\infty) \longrightarrow \mathbb{R}$ with $\lim_{\lambda\to\infty} \psi(\lambda)/\lambda = \infty$ (de la Vallée Poussin's criterion; cf. MacShane [29], Dellacherie & Meyer [16]).

4. As explained in the introduction, the Young measure (ν_x) can be thought of as the limiting probability distribution of the values of $z^{(\mu)}$ near the point x. In fact if Ω is open and $x \in \Omega$ then for $\delta > 0$ sufficiently small the open ball $B(x,\delta)$ with centre x and radius δ is contained in Ω, and the formula

$$\langle \nu^{(\mu)}_{x,\delta}, f \rangle = ⨍_{B(x,\delta)} f(z^{(\mu)}(y))\, dy \tag{8}$$

defines a continuous linear form on continuous functions $f : \mathbb{R}^m \longrightarrow \mathbb{R}$ of compact support. (Here and below $⨍_E (\cdot)\, dx$ denotes $(\text{meas } E)^{-1}\int_E (\cdot)\, dx$.) The Radon measure $\nu^{(\mu)}_{x,\delta}$ is a probability measure giving the distribution of the values of $z^{(\mu)}$ in $B(x,\delta)$, and can be written

$$\nu^{(\mu)}_{x,\delta} = ⨍_{B(x,\delta)} \delta_{z^{(\mu)}(y)}\, dy, \tag{9}$$

where δ_a denotes the Dirac mass at $a \in \mathbb{R}^m$. As $\mu \longrightarrow \infty$, $\nu^{(\mu)}_{x,\delta} \overset{*}{\longrightarrow} \nu_{x,\delta}$ in the sense of measures, where by the theorem

$$<\nu_{x,\delta}, f> \;=\; ⨍_{B(x,\delta)} <\nu_y, f> \, dy \,, \tag{10}$$

that is,

$$\nu_{x,\delta} \;=\; ⨍_{B(x,\delta)} \nu_y \, dy \,. \tag{11}$$

By Lebesgue's differentiation theorem, for any fixed f

$$<\nu_{x,\delta}, f> \;\longrightarrow\; <\nu_x, f> \qquad \text{as } \delta \longrightarrow 0 \quad \text{for a.e. } x \in \Omega.$$

Choosing a countable dense set of functions f we deduce easily that $\nu_{x,\delta} \overset{*}{\longrightarrow} \nu_x$ in the sense of measures as $\delta \longrightarrow 0$ for a.e. $x \in \Omega$. The number $1 - \|\nu_x\|_M$ represents the limiting proportion of the points in $B(x,\delta)$ at which $z^{(\mu)}$ becomes unbounded.

Proof of the theorem

We denote by $C_0(\mathbb{R}^m)$ the Banach space of continuous functions $f : \mathbb{R}^m \longrightarrow \mathbb{R}$ satisfying $\lim_{|\lambda|\to\infty} f(\lambda) = 0$, with the norm $\|f\|_{C^0} = \sup_{\lambda\in\mathbb{R}^m} |f(\lambda)|$. A well known form of the Riesz representation theorem (Hewitt & Stromberg [25 p364]) asserts that there is a isometric isomorphism between the dual space $C_0(\mathbb{R}^m)^*$ of $C_0(\mathbb{R}^m)$ and the Banach space $M(\mathbb{R}^m)$ of bounded Radon measures on $\mathbb{R}^m$ obtained by associating with each $\nu \in M(\mathbb{R}^m)$ the linear form $f \longmapsto \int_{\mathbb{R}^m} f(\lambda)\, d\nu$ on $C_0(\mathbb{R}^m)$. The norm on $M(\mathbb{R}^m)$ is given by $\|\nu\|_M = \int_{\mathbb{R}^m} d|\nu|$. We associate with $z^{(j)}$ the mapping $\nu^{(j)}: \Omega \longrightarrow M(\mathbb{R}^m)$ defined by

$$\nu^{(j)}(x) \;=\; \delta_{z^{(j)}(x)} \,. \tag{12}$$

For each j, $\nu^{(j)}$ belongs to the space $L^\infty_w(\Omega; M(\mathbb{R}^m))$ of equivalence classes of weak* measurable mappings $\mu : \Omega \longrightarrow M(\mathbb{R}^m)$ that are essentially bounded, i.e.

$$\|\mu\|_{\infty,M} := \operatorname*{ess\,sup}_{x \in \Omega} \|\mu(x)\|_M \;<\; \infty \,. \tag{13}$$

(We say that μ is weak* measurable if $<\mu(x), f>$ is measurable with respect to x for every $f \in C_0(\mathbb{R}^m)$.) Under the norm $\|\cdot\|_{\infty,M}$, $L^\infty_w(\Omega; M(\mathbb{R}^m))$ is a Banach space. Since $C_0(\mathbb{R}^m)$ is separable, there is an isometric isomorphism between the dual space of $L^1(\Omega; C_0(\mathbb{R}^m))$ and $L^\infty_w(\Omega; M(\mathbb{R}^m))$ obtained by associating with each $\mu \in L^\infty_w(\Omega; M(\mathbb{R}^m))$ the linear form

$$\psi \;\longmapsto\; \int_\Omega <\mu(x), \psi(x,\cdot)> \, dx \tag{14}$$

on $L^1(\Omega; C_0(\mathbb{R}^m))$ (see Edwards [22 p588], A. & C. Ionescu Tulcea [26 p93], Meyer [30 p244]). But

$$\|\nu^{(j)}\|_{\infty,M} = 1 \quad \text{for all } j.$$

Since $C_0(\mathbb{R}^m)$ is separable, so is $L^1(\Omega; C_0(\mathbb{R}^m))$, and hence (cf. Dunford &

Schwartz [21 pp 424-426]) there exists a subsequence $\nu^{(\mu)}$ of $\nu^{(j)}$ and an element $\nu = (\nu_x)$ of $L^\infty_w(\Omega;M(\mathbb{R}^m))$ such that $\nu^{(\mu)} \overset{*}{\longrightarrow} \nu$ in $L^\infty_w(\Omega;M(\mathbb{R}^m))$. By (14) this implies that

$$\int_\Omega \psi(x,z^{(\mu)}(x))\,dx \longrightarrow \int_\Omega \langle\nu_x,\psi(x,\cdot)\rangle\,dx \tag{15}$$

as $\mu \longrightarrow \infty$ for every $\psi \in L^1(\Omega;C_0(\mathbb{R}^m))$. In particular, taking $\psi(x,\lambda) = \phi(x)f(\lambda)$, where $\phi \in L^1(\Omega)$ and $f \in C_0(\mathbb{R}^m)$, we have that

$$f(z^{(\mu)}) \overset{*}{\longrightarrow} \langle\nu_x,f\rangle \quad \text{in } L^\infty(\Omega) \tag{16}$$

for every $f \in C_0(\mathbb{R}^m)$, which is (*iii*). By weak* lower semicontinuity of the norm, $\|\nu\|_{\infty,M} \le 1$, which is (*i*). To prove (*ii*), assume that $K \ne \mathbb{R}^m$ and let $f \in C_0^K(\mathbb{R}^m) := \{g \in C_0(\mathbb{R}^m)\ ;\ g|_K = 0\}$. Then, taking $U = \{z \in \mathbb{R}^m\colon\ |f(z)| < \varepsilon\}$, it follows from the hypothesis that $z^{(j)}(\cdot) \longrightarrow K$ in measure that $f(z^{(\mu)}(\cdot)) \longrightarrow 0$ in measure. Since f is bounded, we deduce that

$$\begin{aligned}\int_\Omega \phi(x)\,\langle\nu_x,f\rangle\,dx &= \lim_{\mu\to\infty}\int_\Omega \phi(x)\,f(z^{(\mu)}(x))\,dx\\ &= 0\end{aligned}$$

for every $\phi \in L^1(\Omega)$. Since $C_0(\mathbb{R}^m)$ is separable so is $C_0^K(\mathbb{R}^m)$, and it follows that for a.e. $x \in \Omega$ we have $\langle\nu_x,f\rangle = 0$ for all $f \in C_0^K(\mathbb{R}^m)$, i.e. supp $\nu_x \subset K$. Since $\nu^{(j)}(x) \ge 0$ a.e. we deduce similarly that $\nu_x \ge 0$ a.e..

Now suppose that (3) holds. Define $\vartheta^k \in C_0(\mathbb{R}^m)$ by

$$\vartheta^k(\lambda) = \begin{cases} 1 & \text{for } |\lambda| \le k\,,\\ 1+k-|\lambda| & \text{for } k \le |\lambda| \le k+1\,,\\ 0 & \text{for } |\lambda| \ge k+1\,. \end{cases} \tag{17}$$

Then if $E \subset \Omega$ is bounded and measurable

$$\begin{aligned}\lim_{\mu\to\infty} ⨍_E \vartheta^{(k)}(z^{(\mu)}(x))\,dx &= ⨍_E \langle\nu_x,\vartheta^{(k)}\rangle\,dx\\ &\le ⨍_E \|\nu_x\|_M\,dx\,.\end{aligned} \tag{18}$$

But

$$0 \le ⨍_E \left[1-\vartheta^{(k)}(z^{(\mu)}(x))\right]dx \le \frac{\text{meas}\{x \in E\colon\ |z^{(\mu)}(x)| \ge k\}}{\text{meas } E},$$

so that letting $k \longrightarrow \infty$ we get from (3) and (18) that

$$1 \le ⨍_E \|\nu_x\|_M\,dx\,. \tag{19}$$

Since $\|\nu_x\|_M \le 1$ a.e. and E is arbitrary, (19) implies that $\|\nu_x\|_M = 1$ a.e..

Suppose further that $f : \mathbb{R}^m \longrightarrow \mathbb{R}$ is continuous and that $f(z^{(\mu)})$ is sequentially weakly relatively compact in $L^1(\Omega)$. Let $f^+ = \max(f,0)$, $f^- = \max(-f,0)$, so that $f = f^+ - f^-$. By the Dunford-Pettis theorem (Dunford & Schwartz [21 p292]) $f^+(z^{(\mu)})$ and $f^-(z^{(\mu)})$ are both sequentially weakly relatively compact in $L^1(\Omega)$. Hence to prove (4) we can and will suppose that

$f \geq 0$. It also clearly suffices to prove (4) for the case when A is bounded and $f(z^{(\mu)}) \longrightarrow \chi$, say, in $L^1(A)$. Define $f^k \in C_0(\mathbb{R}^m)$ by $f^k = \vartheta^k f$, where ϑ^k is given by (17). Let $\phi \in L^\infty(A)$. We claim that

$$\int_A \phi\, f^k(z^{(\mu)})\, dx \longrightarrow \int_A \phi\, f(z^{(\mu)})\, dx \tag{20}$$

as $k \longrightarrow \infty$, uniformly in μ. In fact

$$\left| \int_A \phi\, (f^k(z^{(\mu)}) - f(z^{(\mu)}))\, dx \right|$$

$$\leq const. \int_{\{x\in A:\, |z^{(\mu)}(x)|\geq k\}} f(z^{(\mu)})\, dx, \tag{21}$$

and, given $\varepsilon > 0$, by the Dunford-Pettis theorem there exists an $M > 0$ such that

$$\sup_\mu \int_{\{x\in A:\, f(z^{(\mu)}(x))\geq M\}} f(z^{(\mu)})\, dx \leq \varepsilon .$$

Hence by (3)

$$\int_{\{x\in A:\, |z^{(\mu)}(x)|\geq k\}} f(z^{(\mu)})\, dx \leq \varepsilon + M \operatorname{meas}\{x\in A : |z^{(\mu)}(x)| \geq k\}$$

$$\leq 2\varepsilon ,$$

for all μ if k is sufficiently large, which together with (21) gives (20). Since also by *(iii)*

$$\lim_{\mu\to\infty} \int_A \phi\, f^k(z^{(\mu)})\, dx = \int_A \phi\, \langle \nu_x, f^k\rangle\, dx \tag{22}$$

it follows that

$$\lim_{k\to\infty} \int_A \phi\, \langle \nu_x, f^k\rangle\, dx = \int_A \phi\, \chi\, dx . \tag{23}$$

Choosing $\phi \geq 0$ and noting that f^k is increasing, we deduce from the monotone convergence theorem that $\langle \nu_x, f\rangle = \chi$ a.e. in A as required.

□

Remarks on the proof

1. The proof makes contact in several places with that of Balder [3 Theorem 2.1].

2. The reader is warned that the mappings $\nu^{(j)}$ do not in general belong to the space $L^\infty(\Omega; M(\mathbb{R}^m))$ of essentially bounded <u>strongly</u> measurable mappings from Ω to $M(\mathbb{R}^m)$ even when the $z^{(j)}$ are smooth. This space cannot be identified with the dual of $L^1(\Omega; C_0(\mathbb{R}^m))$ via (14).

3. The same argument as in the proof shows that under the hypothesis (3), for any measurable $A \subset \Omega$,

$$f(\cdot, z^{(\mu)}) \longrightarrow \langle \nu_x, f(x,\cdot)\rangle \text{ in } L^1(A) \tag{24}$$

for every Carathéodory function $f : A \times \mathbb{R}^m \longrightarrow \mathbb{R}$ such that $\{f(\cdot, z^{(\mu)})\}$ is sequentially weakly relatively compact in $L^1(A)$.

Acknowledgements

This research was carried out during a visit to the Institute for Mathematics and its Applications at the University of Minnesota in 1987. I am grateful to M.Luskin for having made this visit possible and pleasant, to J.L.Ericksen and R.D.James for having prompted the investigation, and to D.Kinderlehrer for discussions. I also wish to thank E.J.Balder for his comments on an earlier version. The paper was completed while I visited the Laboratoire d'Analyse Numérique, Université Pierre et Marie Curie, Paris with the support of C.N.R.S..

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