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Saddle Point Analysis for an Ordinary Differential Equation in a Banach Space, and an Application to Dynamic Buckling of a Beam

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CONTENTS

1. Introduction

This paper was motivated by studies of the nonlinear beam equation

$$(1.1)\qquad \frac{\partial^2 u}{\partial t^2} + \alpha \frac{\partial^4 u}{\partial x^4} - \left(\beta + k\int_0^{\ell} \left[\frac{\partial u(\xi,t)}{\partial \xi}\right]^2 d\xi\right)\frac{\partial^2 u}{\partial x^2} + \delta \frac{\partial u}{\partial t} = 0 \; .$$

In (1.1) α, k and δ are positive constants. We regard the equation as an approximate model for the transverse motion of a linearly elastic extensible beam with ends fixed in space. There are, of course, more satisfactory models available and the reader is referred to the article by Antman [1]; nevertheless the relative simplicity and tractability of (1.1) make it a useful prototype for study. Bennett and Eisley [5] and Ray and Bert [28] have shown that under carefully controlled experimental conditions models based on (1.1) give satisfactory quantitative accuracy, at least when the tensile load H induced in the beam when it is constrained to lie straight is greater than the (negative) Euler critical load H_E.

In a previous paper [3], it was shown that for both clamped and hinged end conditions (1.1) generates a dynamical system, on a Banach space Σ, which is continuous in both the strong and weak topologies on Σ. (Actually, in [3], extra terms with positive coefficients were included in (1.1), but the results carry over in an obvious way.) An invariance principle (in the weak topology) was then used to show that any orbit converges strongly to some equilibrium position. Thus Σ may be partitioned into a finite number of regions of attraction, each region corresponding to one equilibrium position. In this paper we investigate the structure of this partition, and hence prove some stability results for (1.1). A natural approach, adopted, for instance, by Hsu, Kuo and Lee [20], is to linearize (1.1) about each equilibrium position in turn. To carry through this procedure, however, some method is required of relating the behavior of the nonlinear and linearized equations in a neighbourhood of the relevant equilibrium position. Saddle point analysis is the tool we use for this purpose.

Saddle point analysis originated in the context of an ordinary differential equation (ODE) in $\mathbb{R}^n$, but has been extended to neutral functional differential equations by Cruz and Hale [10]. In section 2 we present an extension to a class (which includes (1.1)) of ODE in a Banach space. As hypotheses we assume both that the equation may be written in 'variation of constants' form and that an 'exponential decomposition' holds for the linearized equation. With these hypotheses certain proofs for ODE in $\mathbb{R}^n$ carry over to ODE in a Banach space almost word for word. For instance, the reader may readily obtain a proof of our corollary to Theorem 2.2 by replacing the symbol $|\cdot|$ with $\|\cdot\|$ in the proof given by Hale [16, Chapter 3]. We need a more general treatment than this, however, so as to be able to treat bifurcation situations. To prove the existence of the various manifolds we combine techniques due to Hale [17] and Kelley [21]. The proofs have been included because there seems to be no strictly equivalent body of results in the literature even for ODE in $\mathbb{R}^n$. Kelley, for instance, treats differentiable perturbations rather than our Lipschitz ones, and works with the differential equation rather than the variation of constants formula. We treat only autonomous equations, although some techniques for nonautonomous ODE in $\mathbb{R}^n$ (e.g. Hale [16, Chapter 4]) carry over to nonautonomous ODE in Banach space.

Whether a solution to an ODE in a Banach space satisfies the corresponding variation of constants formula, and vice versa, seem to be delicate questions. There are a number of relevant theorems with varying hypotheses and different definitions of 'solution'; the reader is referred to Carroll [7] for a comprehensive survey. The hypothesis that a variation of constants formula holds avoids these difficulties. Our hypothesis of the existence of an exponential decomposition also conceals difficulties. It would be useful to have conditions on the spectrum of the infinitesimal generator of the semigroup appearing in the variation of constants formula for such a decomposition to exist, but in general we know of no nontrivial conditions of this type.

In applying the saddle point analysis to (1.1) the main difficulties are in satisfying the two hypotheses we have discussed. The exponential decomposition is established by explicitly solving the linearized equations, and the dimensions of the unstable manifolds are calculated using a theorem on a linear eigenvalue problem in Hilbert space which is proved in section 3. The explicit solution of the linearized equations is then used to prove the equivalence of a 'weak' form of (1.1) and the corresponding variation of constants formula.

When $H \geq H_E$ the only equilibrium position is the trivial one. When $H < H_E$, however, there are $2n+1$ equilibrium positions, two of which, $\pm v_1$, minimize the potential energy of the beam. In the latter case the saddle point analysis enables us to show that the regions of attraction of $\pm v_1$ have union dense in Σ. Thus any orbit in the region of attraction of another equilibrium state is Lyapunov unstable. By contrast, orbits in the regions of attraction of $\pm v_1$ (or zero if $H \geq H_E$) are asymptotically stable. Weaker forms of the last result were proved by Dickey [12] for hinged end conditions, and in [3] for both hinged and clamped end conditions. Dickey also obtained exponential rate-of-decay estimates for convergence to $\pm v_1$ - byproducts of our saddle point analysis are more and improved estimates of this type.

We make a particular study of the case $n = 1$. Then there are unique orbits 'connecting' 0 and $\pm v_1$. It is shown that if the beam has initial position close to zero in Σ then in general the motion stays close to one of these two orbits. For $n > 1$ the orbits connecting equilibrium states are helpful in visualizing the complicated structure of the regions of attraction. We include a conjecture about the nature of these orbits (see Figure 2) - to settle this conjecture new techniques would seem to be necessary.

The combined use of an invariance principle and saddle point analysis may find other applications to problems in continuum mechanics whose characteristic feature is a multiplicity of equilibrium states, steady states, or periodic solutions.

Sections 2 and 3 of this paper are self-contained and may be read independently of each other and the rest of the paper. In the remaining sections the reader may find helpful some familiarity with the methods used in [2] and [3].

2. The Saddle Point Property for an Ordinary Differential Equation in a Banach Space

Suppose A is a constant $n \times n$ matrix and $f : \mathbb{R}^n \to \mathbb{R}^n$ is continuous with $f(0) = 0$. is the purpose of classical autonomous saddle point analysis (Hale [16, Chapter 3], Hartman [18], Urabe [33]) to compare, in a neighbourhood of $x = 0$, the properties of the nonlinear ordinary differential equation

$$\dot{x} = Ax + f(x) \tag{2.1}$$

with those of the linear equation

$$\dot{x} = Ax. \tag{2.2}$$

In this section we extend this analysis to the case of certain ODE in a Banach space, for which the operator A may be unbounded.

Let X be a Banach space with norm $|\ |$. Let $\{T(t)\}_{t \geq 0}$ be a strongly continuous semigroup of linear operators on X. That is

(i) for each $t \geq 0$, $T(t) : X \to X$ is a bounded linear operator,

(ii) $T(s)T(t) = T(s+t)$ for all $s, t \geq 0$,

(iii) $T(0) = I$, where I denotes the identity operator,

and (iv) for each $\varphi \in X$, $T(t)\varphi$ is continuous in $t \geq 0$.

Thus $\{T(t)\}\ t \geq 0$ possesses a densely defined infinitesimal generator A (Hille and Phillips [19], Dunford and Schwartz [14, Vol. 1]) though we shall not need this fact. Also, there exist constants $M > 0$, $\mu > 0$ such that

(2.3) $$|T(t)\varphi| \leq Me^{\mu t}|\varphi| \quad \text{for all } \varphi \in X, \quad t \geq 0.$$

From time to time we shall need the following hypothesis:

Hypothesis (BU) (backwards uniqueness): for each $t \geq 0$ $T(t)$ is injective.

We assume that $\{T(t)\}\ t \geq 0$ induces an exponential decomposition of x. That is

(a) $X = \pi_- X \oplus \pi_0 X \oplus \pi_+ X$, where π_-, π_0, π_+ are continuous linear projection operators on X.

Condition (a) implies the relations

$$\pi_-\pi_0 = \pi_0\pi_- = \pi_-\pi_+ = \pi_+\pi_- = \pi_0\pi_+ = \pi_+\pi_0 = 0,$$

$$\pi_-\pi_- = \pi_-, \quad \pi_0\pi_0 = \pi_0, \quad \pi_+\pi_+ = \pi_+, \quad \pi_- + \pi_0 + \pi_+ = I .$$

If $\varphi \in X$ we write $\varphi_- = \pi_-\varphi$, $\varphi_0 = \pi_0\varphi$, $\varphi_+ = \pi_+\varphi$. Throughout we shall use the equivalent norm $\| \ \|$ on X, where $\|\varphi\| = |\varphi_-| + |\varphi_0| + |\varphi_+|$.

(b) For each $t \geq 0$ $T(t)$ commutes with the operators π_-, π_0, π_+ so that each of the subspaces $\pi_- X$, $\pi_0 X$, $\pi_+ X$ are invariant under $T(t)$. Futhermore $\{T(t)\}$ may be extended to form a continuous group of linear operators on $\pi_0 X \oplus \pi_+ X$. That is

(i) for each $t \in \mathbb{R}$, $T(t) : \pi_0 X \oplus \pi_+ X \to \pi_0 X \oplus \pi_+ X$ is a bounded linear operator,

(ii) $T(s)T(t)\varphi = T(s+t)$ for all $s, t \in \mathbb{R}$, $\varphi \in \pi_0 X \oplus \pi_+ X$,

and (iii) for each $\varphi \in \pi_0 X \oplus \pi_+ X$, $T(t)\varphi$ is continuous in $t \in \mathbb{R}$.

(Necessary and sufficient conditions in terms of A are known for a strongly continuous semigroup to form a group (Dunford and Schwartz [14, Vol. 1]). In applications (b) may follow from finite-dimensionality of $\pi_0 X \oplus \pi_+ X$.)

(c) There exist constants $a_- > 0$, $a_+ > 0$, $\min(a_-, a_+) > a_0 \geq 0$, $K \geq 1$ such that

(2.4i) $\quad \|T(t)\varphi_-\| \leq Ke^{-a_- t}\|\varphi_-\|$ for all $\varphi \in X$, $t \geq 0$,

(2.4ii) $\quad \|T(t)\varphi_0\| \leq Ke^{a_0 |t|}\|\varphi_0\|$ for all $\varphi \in X$, $t \in \mathbb{R}$,

(2.4iii) $\quad \|T(t)\varphi_+\| \leq Ke^{a_+ t}\|\varphi_+\|$ for all $\varphi \in X$, $t \leq 0$.

Without loss of generality we assume $a_0 > 0$.

For (2.2) $X = \mathbb{R}^n$, $T(t) = e^{At}$ and the subspaces $\pi_-\mathbb{R}^n$, $\pi_0\mathbb{R}^n$, $\pi_+\mathbb{R}^n$ correspond to the eigenvalues of A with negative, zero and positive real parts respectively. When $\pi_0 \equiv 0$ we say that an exponential dichotomy holds for $\{T(t)\}$. Exponential dichotomies for ODE in Banach space of the form (2.2) with A bounded have been discussed by Massera and Schäeffer [25]. The corresponding saddle point theory is described by Daleckiĭ and Kreĭn [11].

Let η be a continuous, real-valued, nondecreasing function on $[0, \infty)$ with $\eta(0) = 0$. Let $f : X \to X$ be a continuous (nonlinear) operator such that

(i) $f(0) = 0$

(ii) $\|f(\varphi) - f(\psi)\| \leq \eta(r)\|\varphi - \psi\|$, $\|\varphi\|$, $\|\psi\| \leq r$

Consider the equation

$$(2.5) \qquad w(t) = T(t)w(0) + \int_0^t T(t-s)f(w(s))ds \, .$$

The integral in (2.5) is a Bochner integral in the Banach space X. For many ODE in a Banach space with not necessarily bounded operator coefficients, an equivalent 'variations of constants' form such as (2.5) exists (see the introduction). Solutions to such equations which satisfy (2.5) are termed 'mild solutions' by Browder [6]. We consider only continuous solutions to (2.5) for $t \geq 0$.

Lemma 2.1. Let $\tau > 0$ and continuous functions $w : [0, \tau] \to X$, $y : [-\tau, 0] \to X$ be such that

$$(2.6) \qquad y(t) = w(t+\tau) \quad \text{for all } t \in [-\tau, 0] \, .$$

If w satisfies (2.5) in $[0, \tau]$ then y satisfies

$$(2.7) \qquad T(-t)y(t) = y(0) + \int_0^t T(-s)f(y(s))ds \quad \text{for all } t \in [-\tau, 0].$$

Conversely, if (BU) holds and y satisfies (2.7), then w satisfies (2.5) in $[0, \tau]$.

Proof. The first half of the lemma is a straightforward calculation, using the fact that bounded linear operators may be brought inside the integral sign (Hille and Phillips [19]). So let y satisfy (2.7). It is easy to show that

$$(2.8) \quad T(\tau-t)[w(t) - T(t)w(0) - \int_0^t T(t-s)f(w(s))ds] = 0 \quad t \in [0, \tau].$$

Then y satisfies (2.5) in $[0, \tau]$ by (BU). □

By a solution of (2.5) in an interval $[-\tau, 0]$, $\tau > 0$, we mean a function $y : [-\tau, 0] \to x$ such that $w(\cdot)$ given by (2.6) is a solution to (2.5) in $[0, \tau]$. The lemma gives a sufficient condition for such a y to be a solution.

Following, for example, Chafee [8] and Hale [17] define for $\lambda > 0$

$$
(2.9) \qquad \begin{aligned} f_\lambda(\varphi) &= f(\varphi) &&\text{if } \|\varphi\| \leq \lambda \\ &= f\left(\frac{\lambda\varphi}{\|\varphi\|}\right) &&\text{if } \|\varphi\| > \lambda . \end{aligned}
$$

The properties of f and η ensure the existence of a continuous, nondecreasing function $\nu(\lambda)$, $\lambda \geq 0$, $\nu(0) = 0$, such that

$$
(2.10) \qquad \begin{aligned} &\|f_\lambda(\varphi)\| \leq \nu(\lambda)\lambda, \quad \|f_\lambda(\varphi) - f_\lambda(\psi)\| \leq \nu(\lambda)\|\varphi - \psi\| \\ &\qquad\qquad \text{for all } \varphi, \psi \in X . \end{aligned}
$$

In order to study (2.5) locally we investigate the global behaviour of the equation

$$
(2.11) \qquad w(t) = T(t)w(0) + \int_0^t T(t-s)f_\lambda(w(s))ds .
$$

Lemma 2.2. For $\lambda > 0$ sufficiently small and $\varphi \in X$ there exists a unique continuous solution $w(t)$, $w(0) = \varphi$, to (2.11).

Proof. Let $T > 0$. Define G to be the set of continuous functions $g : [0, T] \to x$ with $g(0) = \varphi$. Let $\delta > 0$

and define $\|g\|_\delta = \sup_{t\in[0,T]} e^{-\delta t}\|g(t)\|$. G forms a Banach space under $\|\ \|_\delta$. For $g \in G$, $t \in [0, T]$ define

$$(Pg)(t) = T(t)\varphi + \int_0^t T(t-s)f_\lambda(g(s))ds \ . \tag{2.12}$$

By writing

$$\int_0^t T(t-s)f_\lambda(g(s))ds = \int_0^t T(s)f_\lambda(g(t-s))ds \ , \tag{2.13}$$

and using (2.3), (2.10) it is easy to show that $P : G \to G$. For $g_1, g_2 \in G$, $t \in [0, T]$, it follows from (2.3), (2.10) that

$$\begin{aligned} &e^{-\delta t}\|(Pg_1)(t) - (Pg_2)(t)\| \leq \\ &\leq \int_0^T M\nu(\lambda)e^{(\mu-\delta)(t-s)}\|g_1 - g_2\|_\delta ds \ . \end{aligned} \tag{2.14}$$

Hence if $\delta < \mu$ and λ small enough P is a contraction on G and hence has a unique fixed point w. This completes the proof. □

<u>Remarks</u>: See Chu and Diaz [9] for extensions of this method of proof, which is due to Bielecki. Lemma 2.2 shows that existence and uniqueness holds for (2.5) in a neighborhood of zero.

The open (respectively closed) ball with centre a, radius ϵ, in a Banach space is denoted $B_\epsilon(a, Y)$ (respectively $\overline{B}_\epsilon(a, Y)$). Let $B_\epsilon(Y) \equiv B_\epsilon(0, Y)$.

A subset K of X is said to be <u>locally (+)-invariant</u> if there exists $\epsilon > 0$ such that for any $\varphi \in B_\epsilon(X) \cap K$

(i) for sufficiently small $t > 0$ a solution $w(t)$ of (2.5) exists with $w(0) = \varphi$,

(ii) if for $\tau > 0$ $w(t)$ exists and belongs to $B_\epsilon(X)$ for all $t \epsilon [0, \tau]$, then $w(t) \epsilon K$ for all $t \epsilon [0,\tau]$.

Local (-)-invariance is defined by changing > to < in (i), (ii).

Suppose that a Banach space Y is decomposed as $Y = \pi_1 Y \oplus \pi_2 Y$ for continuous linear operators π_1, π_2. Then a subset S of Y containing y_0 is said to be tangent to $\pi_2 Y$ at y_0 if $\|\pi_1(y-y_0)\| / \|\pi_2(y-y_0)\| \to 0$ as $y \to y_0$ in S.

Theorem 2.1. For sufficiently small $\delta > 0$ there exist sets

$$S^* = \{\varphi \epsilon X : \|\varphi_- + \varphi_0\| < \delta,\ \varphi_+ = p^*(\varphi_- + \varphi_0)\},$$

$$U^* = \{\varphi \epsilon X : \|\varphi_0 + \varphi_+\| < \delta,\ \varphi_- = q^*(\varphi_0 + \varphi_+)\},$$

termed the centre-stable, centre-unstable manifolds respectively, where p^*, q^* are Lipschitz functions defined on $B_\delta(\pi_- X \oplus \pi_0 X)$, $B_\delta(\pi_0 X \oplus \pi_+ X)$. S^* is locally (+)-invariant, while if (BU) holds U^* is locally (-)-invariant. Any solution of (2.5) which exists and remains in $B_\delta(x)$ for $t \geq 0$ lies on S^*, and any solution of (2.5) which exists and remains in $B_\delta(X)$ for $t \leq 0$ lies on U^*. Furthermore S^* is tangent to $\pi_- X \oplus \pi_0 X$ at zero.

Proof. First (compare Kelley [21]) we prove that for λ sufficiently small a centre-stable manifold exists for (2.11) defined by p^*_λ, which we show to be the unique solution for $t \geq 0$ of the system

$$w_-(t) = T(t)\varphi_- + \int_0^t T(t-s)\pi_- f_\lambda(w_-(s) + w_0(s) + $$
$$+ p_\lambda^*(w_-(s) + w_0(s)))ds,$$

(2.15)
$$w_0(t) = T(t)\varphi_0 + \int_0^t T(t-s)\pi_0 f_\lambda(w_-(s) + w_0(s) + $$
$$+ p_\lambda^*(w_-(s) + w_0(s)))ds,$$

$$p_\lambda^*(\varphi_- + \varphi_0) = \int_\infty^0 T(-s)\pi_+ f_\lambda(w_-(s) + w_0(s) + $$
$$+ p_\lambda(w_-(s) + w_0(s)))ds \ .$$

In (2.15) $w_-(.)$ and $w_0(.)$ are unique. Sometimes we make the dependence on φ_-, φ_0 explicit by writing $w_-(\varphi_- + \varphi_0, t)$, $w_0(\varphi_- + \varphi_0, t)$. Define

$$\mathfrak{X} = \{\chi : \pi_- X \oplus \pi_0 X \to \pi_+ X : \|\chi(\varphi_- + \varphi_0) - \chi(\psi_- + \psi_0)\| \leq$$
$$\leq \|\varphi_- + \varphi_0 - \psi_- - \psi_0\|, \text{ any } \varphi, \psi \in X,$$
$$\chi(0) = 0, \quad \|\chi\| \equiv \sup_{\varphi \in X} \|\chi(\varphi_- + \varphi_0)\| < \infty\} \ .$$

$\mathfrak{X}$ is a complete metric space with metric $d(\chi_1, \chi_2) = \|\chi_1 - \chi_2\|$. For $\chi \in \mathfrak{X}$ consider the system

$$w_-^\chi(t) = T(t)\varphi_- + \int_0^t T(t-s)\pi_- f_\lambda(w_-^\chi(s) + w_0^\chi(s) + \chi(w_-^\chi(s) + w_0^\chi(s)))ds,$$

(2.16)

$$w_0^\chi(t) = T(t)\varphi_0 + \int_0^t T(t-s)\pi_0 f_\lambda(w_-^\chi(s) + w_0^\chi(s) + \chi(w_-^\chi(s) + w_0^\chi(s)))ds .$$

Using the method of Lemma 2.2 one can prove that for given φ_-, φ_0 (2.16) has a unique continuous solution $w_-^\chi(t)$, $w_0^\chi(t)$ defined for $t \geq 0$.

Let $\tilde{\varphi}_- + \tilde{\varphi}_0 \in \pi_- X \oplus \pi_0 X$. For $t \geq 0$ let $\theta(t) = \|w_-^\chi(\varphi_- + \varphi_0, t) + w_0^\chi(\varphi_- + \varphi_0, t) - w_-^\chi(\tilde{\varphi}_- + \tilde{\varphi}_0, t) - w_0^\chi(\tilde{\varphi}_- + \tilde{\varphi}_0, t)\|$. Then from (2.4) and (2.10),

$$\theta(t) \leq K(\|\varphi_- - \tilde{\varphi}_-\| + e^{a_0 t}\|\varphi_0 - \tilde{\varphi}_0\|) + 2K\nu(\lambda)\int_0^t [e^{-a_-(t-s)} + e^{a_0(t-s)}]\theta(s)ds$$

$$\leq K(\|\varphi_- - \tilde{\varphi}_-\| + \|\varphi_0 - \tilde{\varphi}_0\|)e^{a_0 t} + 4K\nu(\lambda)\int_0^t e^{a_0(t-s)}\theta(s)ds.$$

Thus, by Gronwall's lemma,

$$\theta(t) \leq K\|\varphi_- + \varphi_0 - \tilde{\varphi} - \tilde{\varphi}_0\|e^{(a_0 + 4K\nu(\lambda))t}, \quad t \geq 0 . \tag{2.17}$$

Let $\chi, \psi \in \mathfrak{X}$, and for $t \geq 0$ and fixed $\varphi_- + \varphi_0$ let $\theta_1(t) = \|w_-^\chi(t) + w_0^\chi(t) - w_-^\psi(t) - w_0^\psi(t)\|$. Then

$$\theta_1(t) \le \int_0^t 2K\nu(\lambda)e^{a_0(t-s)}[\theta_1(s) + \|\chi(w_-^\chi(s)+w_0^\chi(s))$$

$$-\psi(w_-^\psi(s)+w_0^\psi(s)\|]ds$$

$$\le \int_0^t 2K\nu(\lambda)e^{a_0(t-s)}[\theta_1(s) + \|\chi(w_-^\chi(s)+w_0^\chi(s)) -$$

$$-\chi(w_-^\psi(s)+w_0^\psi(s))\| + \|\chi-\psi\|]ds$$

$$\le \frac{2K\nu(\lambda)}{a_0}\|\chi-\psi\| + \int_0^t 4K\nu(\lambda)e^{a_0(t-s)}\theta_1(s)ds,$$

so that, by Gronwall's lemma,

$$\theta_1(t) \le \frac{2K\nu(\lambda)}{a_0}\|\chi-\psi\|e^{(a_0+4K\nu(\lambda))t}, \quad t \ge 0 . \tag{2.18}$$

Now define the transformation Q on $\mathfrak{X}$ by

$$(Q\chi)(\varphi_- + \varphi_0) = \int_\infty^0 T(-s)\pi_+ f_\lambda(w_-^\chi(s) + w_0^\chi(s) + \chi(w_-^\chi(s) + w_0^\chi(s)))ds . \tag{2.19}$$

From (2.4), (2.10) it follows that $\|Q\chi\| < \infty$. Next, using (2.17),

$$\|(Q\chi)(\varphi_- + \varphi_0) - (Q\chi)(\tilde\varphi_- + \tilde\varphi_0)\| \le \le 2K^2\nu(\lambda)\|\varphi_- + \varphi_0 - \tilde\varphi_- - \tilde\varphi_0\|\int_0^\infty e^{(a_0-a_+ + 4K\nu(\lambda))t}dt. \tag{2.20}$$

Since $a_0 < a_+$ it follows that for λ sufficiently small $Q : \mathfrak{X} \to \mathfrak{X}$. But if $\chi, \psi \in \mathfrak{X}$

$$\|(Q\chi)(\varphi_- + \varphi_0) - (Q\psi)(\varphi_- + \varphi_0)\| \le$$

$$\le \int_0^\infty K\nu(\lambda)e^{-a_+ s}[2\theta_1(s) + \|\chi - \psi\|]ds$$

$$\le K\nu(\lambda)[\frac{1}{a_+} + \frac{2K\nu(\lambda)}{a_0}(a_+ - a_0 - 4K\nu(\lambda))^{-1}]\|\chi - \psi\|,$$

where we have used (2.18). Thus $Q : \mathfrak{X} \to \mathfrak{X}$ is a contraction for λ sufficiently small. Hence there is a unique fixed point $p^*_\lambda \in \mathfrak{X}$ satisfying (2.15). It is easy to prove that $w_-(t) + w_0(t) + p^*_\lambda(w_-(t) + w_0(t))$ defines a solution of (2.11) for $t \ge 0$. Furthermore $\|p^*_\lambda(w_-(t) + w_0(t))\|$ is bounded.

Next we need the following lemma, which is based on ideas of Hale [17, compare his Lemma 3.1, Theorem 3.2].

Lemma 2.3. Let x, y be two solutions of (2.11) which exist for $t \ge 0$ and are such that

$$x_-(0) + x_0(0) = y_-(0) + y_0(0) . \tag{2.21}$$

Then for λ sufficiently small there exists $\alpha > 0$ such that

$$\|x_+(0) - y_+(0)\| \le Ke^{-\alpha t}\|x_+(t) - y_+(t)\| \quad \text{for all } t \ge 0. \tag{2.22}$$

In particular, if a solution $w(t)$ of (2.11) exists for $t \ge 0$ and $\|w_+(t)\|$ is bounded, then $w_+(0) = p^*_\lambda(w_-(0) + w_0(0))$.

Proof. The last statement in the lemma follows immediately from (2.22). To prove (2.22) note that x satisfies for $t \geq 0$

$$(2.23i)\quad x_-(t) = T(t)x_-(0) + \int_0^t T(t-s)\pi_- f_\lambda(x_-(s)+x_0(s)+x_+(s))ds,$$

$$(2.23ii)\quad x_0(t) = T(t)x_0(0) + \int_0^t T(t-s)\pi_0 f_\lambda(x_-(s)+x_0(s)+x_+(s))ds,$$

$$(2.23iii)\quad T(-t)x_+(t) = x_+(0) + \int_0^t T(-s)\pi_+ f_\lambda(x_-(s)+x_0(s)+x_+(s))ds,$$

and similarly for y. Let $k = K\nu(\lambda)$, $k' = \dfrac{2k^2}{a_+ - a_0 - 2k}$,
$h(t) = \|x_-(t) - y_-(t)\| + \|x_0(t) - y_0(t)\|$, $g(t) = \|x_+(t) - y_+(t)\|$.
Then from (2.23i, ii) and their counterparts for y

$$(2.24)\qquad h(t) \leq 2k \int_0^t e^{a_0(t-s)}[h(s) + g(s)]ds,\ t \geq 0,$$

from which follows the estimate

$$(2.25)\qquad h(t) \leq 2k \int_0^t e^{(2k+a_0)(t-s)} g(s)ds,\ t \geq 0\ .$$

But from (2.23iii) and its counterpart for y,

$$(2.26)\qquad g(0) \leq Ke^{-a_+t}g(t) + k\int_0^t e^{-a_+s}[h(s) + g(s)]ds,\ t \geq 0.$$

Thus, using (2.25),

$$(2.27)\qquad g(0) \leq Ke^{-a_+t}g(t) + (k+k')\int_0^t e^{-a_+s}g(s)ds\ .$$

Letting $\psi(s) = g(t-s)$, $0 \leq s \leq t$, and applying Gronwall's lemma to the resulting form of (2.27) we obtain

$$\psi(t) \leq K\psi(0)e^{-(a_+ - k - k')t}, \tag{2.28}$$

which gives (2.22) for λ sufficiently small. □

If we now take $\delta = \lambda$, then since (2.5) and (2.11) coincide in $B_\delta(X)$ the existence of S^* is ensured with $p^* = p^*_\delta$. S^* is clearly locally (+)-invariant, while any solution existing and remaining in $B_\delta(X)$ for $t \geq 0$ lies on S^* by Lemma 2.3. The tangency of S^* to $\pi_- X \oplus \pi_0 X$ at zero is a consequence of (2.20) with $\tilde{\varphi}_- + \tilde{\varphi}_0 = 0$.

Similarly we may construct U^* through q^*_λ, the unique solution of the system

$$q^*_\lambda(\varphi_0 + \varphi_+) = \int_\infty^0 T(-s)\pi_- f_\lambda(q^*_\lambda(w_0(s) + w_+(s)) + w_0(s) + w_+(s))ds, \tag{2.29i}$$

$$w_0(t) = T(t)\varphi_0 + \int_0^t T(t-s)\pi_0 f_\lambda(q^*_\lambda(w_0(s) + w_+(s)) + w_0(s) + w_+(s))ds, \tag{2.29ii}$$

$$w_+(t) = T(t)\varphi_+ + \int_0^t T(t-s)\pi_+ f_\lambda(q^*_\lambda(w_0(s) + w_+(s)) + w_0(s) + w_+(s))ds, \tag{2.29iii}$$

for $t \leq 0$. Let $y(t) = q^*_\lambda(w_0(t) + w_+(t)) + w_0(t) + w_+(t)$. Then

$$(2.30) \qquad T(-t)y(t) = y(0) + \int_0^t T(-s)f_\lambda(y(s))ds, \quad t \leq 0 .$$

From Lemma 2.1 and (2.30) follows the local (-)-invariance of U^* when (BU) holds. The other assertions concerning U^* are proved in a similar way to those for S^*. □

<u>Theorem 2.2.</u> Given ϵ, $\min(a_-, a_+) > \epsilon > 0$, for sufficiently small $\delta > 0$ there exist sets

$$S = \{\varphi \in B_\delta(X) : \|\varphi_-\| < \delta/2K, \ \varphi_0 + \varphi_+ = p(\varphi_-)\},$$

$$U = \{\varphi \in B_\delta(X) : \|\varphi_+\| < \delta/2K, \ \varphi_- + \varphi_0 = q(\varphi_+)\},$$

termed the <u>stable and unstable manifolds</u> respectively, where p, q are Lipschitz functions defined for $\|\varphi_-\| < \delta/2K$, $\|\varphi_+\| < \delta/2K$. If $\varphi \in S$ then a unique solution $w(t)$ of (2.5) with $w(0) = \varphi$ exists for $t \geq 0$ and

$$(2.31) \qquad \|w(t)\| \leq 2Ke^{-(a_- -\epsilon)t}\|w_-(0)\|, \quad t \geq 0 .$$

If (BU) holds and $\varphi \in U$ then a solution $w(t)$ of (2.5) with $w(0) = \varphi$ exists for $t \leq 0$ and

$$(2.32) \qquad \|w(t)\| \leq 2Ke^{(a_+ -\epsilon)t}\|w_+(0)\|, \quad t \leq 0 .$$

Furthermore, S, U are tangent at zero to π_-X, π_+X respectively.

Proof. We just treat S, the proof for U being analogous. Throughout we assume $\lambda > 0$ chosen sufficiently small. We solve for p_λ the system

$$(2.33)\quad \begin{aligned} w_-(t) &= T(t)\varphi_- + \int_0^t T(t-s)\pi_- f_\lambda(w_-(s) + p_\lambda(w_-(s)))ds, \quad t \geq 0 \\ p_\lambda(\varphi_-) &= \int_\infty^0 T(-s)(\pi_0 + \pi_+)f_\lambda(w_-(s) + p_\lambda(w_-(s)))ds . \end{aligned}$$

Define $G = \{h : \pi_- X \to \pi_0 X \oplus \pi_+ X,\ \|h(\varphi_-) - h(\psi_-)\| \leq \leq \|\varphi_- - \psi_-\|$ for all $\varphi, \psi \in X$, $h(0) = 0\}$. G is a complete metric space with metric $\rho(h_1, h_2) \equiv \sup_{\substack{\varphi \in X \\ \varphi \neq 0}} \frac{\|h_1(\varphi_-) - h_2(\varphi_-)\|}{\|\varphi_-\|}$.

By using methods similar to those of Theorem 1.1 it is easy to establish for $h \in G$ the existence of a unique $w_-^h(\varphi_-, t)$ satisfying

$$(2.34)\quad \begin{aligned} w_-^h(\varphi_-, t) = T(t)\varphi_- + \int_0^t T(t-s)\pi_- f_\lambda(w_-^h(\varphi_-, s) + \\ + h(w_-^h(\varphi_-, s)))ds, \quad t \geq 0, \end{aligned}$$

and the estimates

$$(2.35)\quad \|w_-^h(\varphi_-, t) - w_-^h(\tilde{\varphi}_-, t)\| \leq K\|\varphi_- - \tilde{\varphi}_-\| e^{-(a_- - 2K\nu(\lambda))t},$$

$$(2.36)\quad \|w_-^{h_1}(\varphi_-, t) - w_-^{h_2}(\varphi_-, t)\| \leq \frac{K}{2}\|\varphi_-\| \rho(h_1, h_2) e^{-(a_- - 4K\nu(\lambda))t}.$$

Define the transformation P on G through

$$(2.37)\qquad \begin{aligned}(Ph)(\varphi_-) = \int_{\infty}^{0} T(-s)(\pi_0 + \pi_+) f_\lambda (w_-^h(\varphi_-, s) + \\ + h(w_-^h(\varphi_-, s)))ds\ .\end{aligned}$$

Using (2.35), (2.36) it is now simple to prove that $P: G \to G$ and is a contraction. For brevity we omit the details. The unique fixed point p_λ satisfies (2.33), where $w_-(t) = w_-^{p_\lambda}(\varphi_-, t)$. The estimate (2.31) follows from (2.35), and the other assertions concerning S are proved in an analogous way to those of Theorem 2.1. □

<u>Corollary</u>: Suppose $\pi_0 \equiv 0$. Any solution $w(t)$ of (2.5) which exists for $t \geq 0$, remains in $B_\delta(X)$, and satisfies $\|w_-(0)\| < \delta/2k$ lies on S. Any solution $w(t)$ of (2.5) which exists for $t \leq 0$, remains in $B_\delta(X)$ and satisfies $\|w_+(0)\| < \epsilon/2k$ lies on U.

<u>Proof</u>. When $\pi_0 \equiv 0$ the functions p_λ^*, p_λ constructed in Theorems 2.1, 2.2 are identical. □

<u>Remarks</u>: Theorem 2.2 may also be proved using the method given by Urabe [33, p. 77]. We could also prove the existence of a 'centre' manifold (see Kelley [21]). In some situations the stability of this manifold is known to govern the stability of the zero solution (Hale [17], Kelley [22]).

3. An Eigenvalue Problem in Hilbert Space

In order to state the main result of this section, Theorem 3.3, we need some preliminary discussion. Let H be a real infinite-dimensional Hilbert space with inner product (,) and corresponding norm $\| \ \|$. Throughout, A and B denote linear operators defined on dense linear subsets M_A and M_B of H.

A is <u>symmetric</u> iff $(Au, v) = (u, Av)$ for all $u, v \in M_A$. A is <u>semi-bounded below</u> iff there is a constant k with $(Au, u) \geq k \|u\|^2$ for all $u \in M_A$. A is <u>positive definite</u> iff A is semi-bounded below with k positive.

The following result is due to Friedrichs (see Dunford and Schwartz [14, Vol. 2], Mikhlin [26, 27] or Riesz and Sz.-Nagy [30]) and is used in the proof of the existence of a self-adjoint extension to a semi-bounded symmetric operator.

<u>Theorem 3.1.</u> Let A be symmetric and positive definite. Define for $u, v \in M_A$, $[u, v]_A = (Au, v)$. Then M_A can be completed by means of elements of H to form a Hilbert space H_A with inner product $[\ ,\ ]_A$ and corresponding norm $\|\ \|_A$.

A subset S of a Hilbert space V whose finite linear combinations are dense in V is called a <u>basis</u> of V.

The proof of the next theorem is given by Mikhlin [26].

<u>Theorem 3.2.</u> Let A and B by symmetric and positive definite with $M_A \subset M_B$. Suppose that the embedding $H_A \subset H_B$ holds and is compact. Consider the equation

$$Aw - \lambda Bw = 0, \tag{3.1}$$

in its weak form

$$[w, \varphi]_A - \lambda [w, \varphi]_B = 0 \text{ for all } \varphi \in H_A . \tag{3.2}$$

Then

(i) (3.2) has a countable set of real eigenvalues $\{\lambda_j\}$ and corresponding eigenvectors $\{w_j\}$, where $0 < \lambda_1 \leq \lambda_2 \leq \dots$, $\lambda_j \to \infty$ as $j \to \infty$.

(ii) $\{w_j\}$ is an orthogonal set in H_A, H_B and is a basis of the spaces H_A, H_B and H.

(3.3) (iii) $$\lambda_1 = \inf_{\substack{w \in H_A \\ \|w\|_B = 1}} \|w\|_A^2 = \frac{\|w_1\|_A^2}{\|w_1\|_B^2},$$

while if $j > 1$,

(3.4) $$\lambda_j = \inf_{\substack{w \in H_A \\ [w, w_i]_B = 0,\ 1 \leq i < j, \\ \|w\|_B = 1}} \|w\|_A^2 = \frac{\|w_j\|_A^2}{\|w_j\|_B^2}.$$

and (iv) (Minimax principle) For $k > 1$ and a set of elements $\{\psi_1, \dots, \psi_{k-1}\}$ in H_B define

$$\lambda(\psi_1, \dots, \psi_{k-1}) = \inf_{\substack{w \in H_A \\ \|w\|_B = 1 \\ [w, \psi_i]_B = 0,\ 1 \leq i \leq k}} \|w\|_A^2 .$$

Then λ_k equals the maximum of $\lambda(\psi_1, \dots, \psi_{k-1})$ taken over all subsets $\{\psi_1, \dots, \psi_{k-1}\}$ of H_B.

Applications of the following theorem are given in Lemma 5.1.

Theorem 3.3. Let the hypotheses of Theorem 3.2 hold and suppose that $w_j \in M_B$ for each j. Define $\lambda_0 = -\infty$. Let λ be a given number and let K be the unique positive integer such that $\lambda_{K-1} < \lambda \leq \lambda_K$. Then the equation

$$Ay - \lambda By = \mu y \tag{3.5}$$

in its weak form

$$[y, \varphi]_A - \lambda[y, \varphi]_B = \mu(y, \varphi) \text{ for all } \varphi \in H_A, \tag{3.6}$$

has a set of eigenvalues $\{\mu_i\}$ and corresponding eigenvectors $\{y_i\} \subset H_A$ such that

(i) precisely K-1 of the μ_i are negative

and (ii) $\{y_i\}$ is a basis of the spaces H and H_A.

Remark. If A, B are self-adjoint with domains M_A, M_B, and if the operator $A-\lambda_j B$ is self-adjoint, then it follows (Mikhlin [27]) that $w_j \in M_B$.

Proof of Theorem 3.3. Let $\{w_j\}$ be normalized by $\|w_j\|_B = 1$. Define $L \equiv A - \lambda B$. We begin by showing that L is semi-bounded below on M_A. Let $u \in M_A$ with $\|u\| = 1$. Then $u = \sum_{r=1}^{\infty} u_r w_r$ in H_A, where $u_r = [u, w_r]_B$, and, if $K > 1$,

$$(Lu, u) = \sum_{r=1}^{\infty} (\lambda_r - \lambda) u_r^2 \geq \sum_{r=1}^{K-1} u_r^2 (\lambda_r - \lambda) \geq -(\lambda - \lambda_{K-1}) \sum_{r=1}^{K-1} u_r^2 .$$

But since $\{w_j\} \subset M_B$, $|u_r| = |(u, Bw_r)| \leq \|Bw_r\|$, and hence $(Lu, u) \geq -(\lambda-\lambda_{K-1})\sum_{r=1}^{K-1} \|Bw_r\|^2$. Hence L is semi-bounded below on M_A. (The proof is similar if $K = 1$.) Since we can apply the same argument to the operator $\frac{1}{2}A - \lambda B$, there exists a constant $c > 0$ such that

$$L_1 \equiv L + cI \tag{3.7}$$

is positive definite on M_A, and in fact satisfies

$$c' \|u\|_A^2 \geq (L_1 u, u) \geq \frac{1}{2} \|u\|_A^2 \quad \text{for all } u \in M_A, \tag{3.8}$$

for some constant $c' > 0$. The left-hand side of (3.8) follows from the positive definiteness of B and Theorem 3.2 (iii). Setting $M_{L_1} = M_A$ it is now clear that $H_{L_1} = H_A$ and that the injection $H_{L_1} \subset H$ is compact. The existence of $\{\mu_i\}$ and $\{y_i\}$ satisfying (3.6) and (ii) follows from Theorem 3.2.

From the definition of L_1 and from Theorem 3.2 (iv) the following minimax principle holds:

$$\tag{3.9} \begin{cases} \mu_k = \max\limits_{\{\psi_1,\ldots,\psi_{k-1}\}\subset H} \mu(\psi_1,\ldots,\psi_{k-1}), \text{ where } k>1 \text{ and} \\ \mu(\psi_1,\ldots,\psi_{k-1}) = \min\limits_{\substack{(u,\psi_i)=0 \\ 1\leq i\leq k-1 \\ u\in H_A,\ \|u\|=1}} \|u\|_A^2 - \lambda\|u\|_B^2 . \end{cases}$$

To prove (i) note that if $K = 1$ then $\mu_1 \geq 0$, while if $K = 2$ then $\mu_1 < 0$ (this follows from the characterization of μ_1 as a minimum: see Theorem 3.2 (iii)). We next show that if $K > 2$ then $\mu_{K-1} < 0$. Choose $u = \sum_{r=1}^{K-1} u_r w_r$, where

$u_r = [u, w_r]_B$. These are K-2 equations for the K-1 unknowns $u_1, \ldots, u_{K-1}$. Therefore there exists a solution with the corresponding u satisfying $\|u\| = 1$ and

$$\|u\|_A^2 - \lambda \|u\|_B^2 = \sum_{r=1}^{K-1} (\lambda_r - \lambda) u_r^2 .$$

But

$$1 = \|u\| \leq \sum_{r=1}^{K-1} \|u_r w_r\| \leq \frac{1}{2} \sum_{r=1}^{K-1} (\theta u_r^2 + \frac{1}{\theta} \|w_r\|^2) ,$$

for any $\theta > 0$. Hence

$$\sum_{r=1}^{K-1} u_r^2 \geq \frac{2}{\theta} - \frac{1}{\theta^2} \sum_{r=1}^{K-1} \|w_r\|^2 .$$

By choosing θ large enough we can find δ such that

$$\sum_{r=1}^{K-1} u_r^2 \geq \delta > 0 .$$

Note that δ is independent of u. Hence

$$\|u\|_A^2 - \lambda \|u\|_B^2 \leq (\lambda_{K-1} - \lambda)\delta < 0,$$

and by (3.9) $\mu(\psi_1, \ldots, \psi_{K-2}) \leq (\lambda_{K-1} - \lambda)\delta < 0$ for any $\{\psi_1, \ldots, \psi_{K-2}\} \subset H$. Therefore $\mu_{K-1} < 0$.

We now show that, if $K > 1$, $\mu_K \geq 0$. Since $\{w_j\} \subset M_B$ and hence $\{Bw_j\} \subset H$,

$$\mu_K \geq \mu(Bw_1, \ldots, Bw_{K-1}) = \min_{\substack{u \in H_A \\ \|u\|=1 \\ (u, Bw_i)=0 \\ 1 \leq i \leq K-1}} \|u\|_A^2 - \lambda \|u\|_B^2 .$$

But for all u with $[u, w_i]_B = (u, Bw_i) = 0$, $1 \leq i \leq K-1$, the inequality

$$\|u\|_A^2 \geq \lambda_K \|u\|_B^2$$

holds (Theorem 3.2 (iii)). Since $\lambda \leq \lambda_K$ it follows that $\mu_K \geq 0$. Hence (i) holds. □

4. The Buckling Beam-Preliminaries

Let $\Omega =]0, \ell[$, $\ell > 0$. Denote by $(\, , \,)$ and $| \; |$ the inner product and norm in the Hilbert space $L^2 \equiv L^2(\Omega)$ of real-valued square integrable functions on Ω, so that $(f, g) = \int_0^\ell f(x)g(x)dx$. We shall use the Sobolev spaces $H^m \equiv H^m(\Omega)$, $H_0^m \equiv H_0^m(\Omega)$, the spaces $C^m(\Omega)$, $C^m(\overline{\Omega})$ of continuous functions, and the spaces $L^p(0, T; X)$, $\mathcal{D}'(0, T; X)$, where $1 \leq p \leq \infty$ and X is a Banach space. Definitions and properties of these spaces may be found, for example, in [2]. Define $G_1 = H_0^1 \cap H^2$. For brevity derivatives (in general distributional) are denoted by $\frac{\partial}{\partial t}(\,) = (\dot{\,})$ and $\frac{\partial}{\partial x}(\,) = (\,)'$. Weak convergence in a Banach space is written $\rightharpoonup$, while weak star convergence is written $\overset{*}{\rightharpoonup}$. Positive constants are denoted generically by C and C_i $(i = 1, 2, \ldots)$.

Throughout let V denote one of the spaces G_1, H_0^2. The cases $V = G_1$, $V = H_0^2$ correspond to beams with hinged and clamped ends respectively. V' denotes the dual of V. Let $\Sigma = V \times L^2$, which is a Hilbert space with norm given by $\|\{\psi, \chi\}\|^2 \equiv \alpha|\psi''|^2 + |\chi|^2$.

By a <u>reversible strong dynamical system</u> on a Banach space X we mean a function $\omega : \mathbb{R} \times X \to X$ satisfying

(i) $\omega^t : \varphi \to \omega(t, \varphi)$ is continuous for fixed $t \in \mathbb{R}$,

(ii) $\omega^\varphi : t \to \omega(t, \varphi)$ is continuous for fixed $\varphi \in X$,

(iii) $\omega(0, \varphi) = \varphi$ for all $\varphi \in X$,

(iv) $\omega(t+\tau, \varphi) = \omega(t, \omega(\tau, \varphi))$ for $t, \tau \in \mathbb{R}$, $\varphi \in X$.

If $\varphi \in X$ the <u>orbit</u> $O(\varphi)$ through φ is defined by $O(\varphi) = \bigcup_{t \in \mathbb{R}} \omega(t, \varphi)$.

The following existence theorem is proved in [3]:

<u>Theorem 4.1.</u> Let $T \in \mathbb{R}$, $T \neq 0$. If $\varphi = \{u_0, u_1\} \in \Sigma$ there exists a unique $u \equiv u(.\,, \varphi)$ with $\{u, \dot{u}\} \in L^\infty(0, T; \Sigma)$ and $u(0) = u_0$, $\dot{u}(0) = u_1$ such that u is a weak solution of (1.1) in the sense that

$$(\ddot{u}, \theta) + \alpha(u'', \theta'') - (\beta + k\,|u'|^2)(u'', \theta) + \delta(\dot{u}, \theta) = 0 \quad \text{for all } \theta \in V. \tag{4.1}$$

u satisfies the energy equation

$$E(t) + \delta \int_0^t |\dot{u}(s)|^2 ds = E(0), \tag{4.2}$$

where $E(t) = E(\{u(t), \dot{u}(t)\})$ and

$$E(\{\psi, \chi\}) \equiv \frac{1}{2}|\chi|^2 + \frac{\alpha}{2}|\psi''|^2 + \frac{\beta}{2}|\psi'|^2 + \frac{k}{4}|\psi'|^4. \tag{4.3}$$

If we set $\omega(t, \varphi) = \{u(t, \varphi), \dot{u}(t, \varphi)\}$ then ω is a reversible strong dynamical system on Σ.

Clearly E is nonincreasing along orbits of ω. Also E is (sequentially) weakly lower semicontinuous on Σ.

We shall need the following extra continuity properties of ω:

Lemma 4.1. Let $t_n \to t$ in $[0, T]$ as $n \to \infty$.

(i) If $\varphi_n \to \varphi$ in Σ then $\omega(t_n, \varphi_n) \to \omega(t, \varphi)$ in Σ.

(ii) If $\varphi_n \rightharpoonup \varphi$ in Σ then $\omega(t_n, \varphi_n) \rightharpoonup \omega(t, \varphi)$ in Σ.

Proof. Part (i) is immediate from the inequality

$$\|\omega(t_n, \varphi_n) - \omega(t, \varphi)\| \leq \|\omega(t_n, \varphi_n) - \omega(t_n, \varphi)\| + \|\omega(t_n, \varphi) - \omega(t, \varphi)\|,$$

and the inequality (see [3, Theorem 2])

$$\|\omega(t_n, \varphi_n) - \omega(t_n, \varphi)\| \leq \exp(Ct_n)\|\varphi_n - \varphi\| .$$

To prove (ii) let $t_n \to t$, $\varphi_n \rightharpoonup \varphi$. By (4.2), (4.3), $\|\omega(\cdot, \varphi_n)\|$ is bounded in $L^\infty(0, T; \Sigma)$. Therefore we may extract a subsequence $\omega(\cdot, \varphi_\mu)$ such that $u(\cdot, \varphi_\mu) \overset{*}{\rightharpoonup} u(\cdot, \varphi)$ in $L^\infty(0, T; V)$, $\dot{u}(\cdot, \varphi_\mu) \overset{*}{\rightharpoonup} \dot{u}(\cdot, \varphi)$ in $L^\infty(0, T; L^2)$ (see the proof of Theorem 5 in [3]) and thus, using (4.1), $\ddot{u}(\cdot, \varphi_\mu) \overset{*}{\rightharpoonup} \ddot{u}(\cdot, \varphi)$ in $L^\infty(0, T; V')$. Since $\{u(t_\mu, \varphi_\mu)\}$ is bounded in V we can assume that $u(t_\mu, \varphi_\mu) \rightharpoonup \chi$, say, in V. Thus $u(t_\mu, \varphi_\mu) \to \chi$ in L^2. But

$$(4.4) \qquad u(t_\mu, \varphi_\mu) - u(0, \varphi_\mu) = \int_0^{t_\mu} \dot{u}(\tau, \varphi_\mu)d\tau ,$$

and so for $\theta \in L^2$ we have

$$(4.5) \qquad \begin{aligned} (u(t_\mu, \varphi_\mu), \theta) - (u(0, \varphi_\mu), \theta) &= \int_0^t (\dot{u}(\tau, \varphi_\mu), \theta)d\tau + \\ &+ \int_t^{t_\mu} (\dot{u}(\tau, \varphi_\mu), \theta)d\tau, \quad t \in [0, T]. \end{aligned}$$

Let $\mu \to \infty$ in (4.5). The second integral tends to zero by the boundedness of $\dot{u}(\cdot, \varphi_\mu)$ in $L^\infty(0, T; L^2)$. Hence

$$(\chi, \theta) - (\varphi, \theta) = \int_0^t (\dot{u}(\tau, \varphi), \theta)d\tau = (u(t, \varphi), \theta) - (\varphi, \theta).$$

Hence $\chi = u(t, \varphi)$.

Now let $\psi \in V$. Since $(\ddot{u}(., \varphi_\mu), \psi) \in L^\infty(0, T)$ we can write

$$(4.6) \qquad (\dot{u}(t_\mu, \varphi_\mu), \psi) - (\dot{u}(0, \varphi_\mu), \psi) = \int_0^{t_\mu} (\ddot{u}(\tau, \varphi_\mu), \psi)d\tau$$

and deduce similarly that $\dot{u}(t_\mu, \varphi_\mu) \rightharpoonup \dot{u}(t, \varphi)$ in L^2. Thus $\omega(t_\mu, \varphi_\mu) \rightharpoonup \omega(t, \varphi)$ in Σ and hence the whole sequence converges. □

We now consider the equilibrium positions of (4.1). The following result is well-known:

Lemma 4.2. There exist eigenvalues $\{\lambda_j\}$ and corresponding eigenvectors $\{v_j\}$ for the problem

$$(4.7) \qquad \alpha(v'', \theta'') - \lambda(v', \theta') = 0 \quad \text{for all } \theta \in V.$$

$\{v_j\}$ is a basis of L^2, H_0^1 and V.

Proof. Let $H = L^2$, $A \equiv \frac{\alpha d^4}{dx^4}$ and $B = \frac{-d^2}{dx^2}$. If $V = G_1$ let $M_A = \{u \in C^\infty(\overline{\Omega});\ u = u'' = 0$ at $x = 0, \ell\}$, while if $V = H_0^2$ let $M_A = \{u \in C^\infty(\overline{\Omega}) : u = u' = 0$ at $x = 0, \ell\}$. Let $M_B = \{u \in C^\infty(\overline{\Omega}) : u = 0$ at $x = 0, \ell\}$. Then $H_A = V$ and $H_B = H_0^1$. The embedding $H_A \subset H_B$ is compact by the Rellich-Kondrachoff theorem. The result follows from Theorem 3.2. □

Any solution of (4.7) satisfies $\alpha v'''' + \lambda v'' = 0$. Thus if $V = G_1$, $v_j(x) = C_j \sin\left(\frac{j\pi x}{\ell}\right)$ and $\lambda_j = j^2\pi^2/\ell^2$. If $V = H_0^2$ expressions for λ_j, v_j can also easily be derived (see e.g. Timoshenko and Gere [32, p. 54]).

Let $v = \{v, 0\} \in \Sigma$ be an equilibrium state for ω. Then v satisfies (4.7) with $-\lambda = \beta_1 \equiv \beta + k|v'|^2$. Thus if $-\beta \le \lambda_1$ the only equilibrium state is $v = 0$. If $\lambda_n < -\beta \le \lambda_{n+1}$ there are $2n+1$ equilibrium states given by $v = 0$ and $v = \pm v_m$, $1 \le m \le n$, where we assume that v_m is normalized so that $\beta + k|v_m'|^2 = -\lambda_m$. The loads H, H_E mentioned in the introduction satisfy $H/H_E = -\beta/\lambda_1$.

5. The Linearized Equations and the Corresponding Exponential Decomposition

Let v be an equilibrium position. Setting $u = v + y$ in (1.1) we obtain formally

$$\ddot{y} + \alpha y'''' - \beta_1 y'' - 2k(v', y')v'' + \delta\dot{y} - k[2(v', y')y'' + |y'|^2 y'' + |y'|^2 v''] = 0. \tag{5.1}$$

The corresponding linearized equation is

$$\ddot{h} + \alpha h'''' - \beta_1 h'' - 2k(v', h')h'' + \delta\dot{h} = 0. \tag{5.2}$$

Theorem 5.1. Let $T \in \mathbb{R}$, $T \neq 0$. Let $\varphi = \{u_0, u_1\} \in \Sigma$. Then there exists a unique function h satisfying $h \in L^\infty(0, T; V)$, $\dot{h} \in L^\infty(0, T; L^2)$, the initial conditions $h(0) = u_0$, $\dot{h}(0) = u_1$ and equation (5.2) in the sense that

$$(5.3)\quad \begin{aligned} &(\ddot{h}, \psi) + \alpha(h'', \psi'') - \beta_1(h'', \psi) - 2k(v', h')(v'', \psi) + \\ &\quad + \delta(\dot{h}, \psi) = 0 \quad \text{for all } \psi \in V . \end{aligned}$$

Furthermore h satisfies the energy equation

$$(5.4)\qquad E_1(t) + \delta \int_0^t |\dot{h}(s)|^2 ds = E_1(0),$$

where $E_1(t) = E(\{h(t), \dot{h}(t)\})$ and

$$(5.5)\quad E_1(\{\psi, \chi\}) \equiv \frac{1}{2}|\chi|^2 + \frac{\alpha}{2}|\psi''|^2 + \frac{\beta_1}{2}|\psi'|^2 + k(v', \psi')^2.$$

If we let $T(t)\varphi = \{h(t), \dot{h}(t)\}$ then $\{T(t)\}$ $t \in \mathbb{R}$ is a strongly continuous group of linear operators on Σ.

Proof. This is omitted since it is a straightforward application of Lions' method similar to the proof of Theorem 4.1. □

In order to study the exponential decomposition induced by $\{T(t)\}$ we need two lemmas:

Lemma 5.1. Let $V = H_0^2$ and let $\{\lambda_j\}$, $\{v_j\}$ be as in Lemma 4.2 with $\lambda_0 = -\infty$.

(i) Let K be the unique nonnegative integer such that $\lambda_K < -\beta \leq \lambda_{K+1}$. Then there exist eigenvalues $\{\rho_j\}$ and corresponding eigenvectors

$\{r_j\}$ for the problem

$$(5.6)\qquad \begin{aligned} &\alpha r'''' - \beta r'' = \rho r \quad \text{in } \Omega, \\ &r = r' = 0 \quad \text{at } x = 0, \ell . \end{aligned}$$

$\{r_j\}$ is a basis of L^2 and H_0^2. Precisely K of the $\{\rho_j\}$ are negative.

(ii) Let m be a positive integer. There exist eigenvalues $\{\rho_{m,j}\}$ and corresponding eigenvectors $\{r_j\}$ for the problem

$$(5.7)\qquad \begin{aligned} &\alpha r_m'''' - 2k(r_m', v_m')v_m'' + \lambda_m r_m'' = \rho r_m \quad \text{in } \Omega \\ &r_m = r_m' = 0 \quad \text{at } x = 0, \ell . \end{aligned}$$

$\{r_{m,j}\}$ is a basis of L^2 and H_0^2. Precisely m-1 of the $\{\rho_{m,j}\}$ are negative and $\rho_{m,m}$ is positive.

Proof. We just prove (ii) as the proof of (i) is similar. Define A_m by

$$(5.8)\qquad A_m u = \alpha u'''' - 2k(u', v_m')v_m'' ,$$

with $M_{A_m} = \{u \in C^\infty(\overline{\Omega}) : u = u' = 0 \text{ at } x = 0, \ell\}$. Let $B_m \equiv -d^2/dx^2$ with $M_{B_m} = \{u \in C^\infty(\overline{\Omega}) : u = 0 \text{ at } x = 0, \ell\}$. If $u \in M_{A_m}$,

$$(5.9)\qquad (A_m u, u) = \alpha |u''|^2 + 2k(u', v_m')^2,$$

so that A_m is positive definite on M_{A_m}. From (5.9)

$$C_1 |u''|^2 \leq \|u\|^2_{A_m} \leq C_2 |u''|^2 \quad \text{for all } u \in M_{A_m} .$$

It follows that $H_{A_m} = H^2_0$. Clearly $H_{B_m} = H^1_0$. Consider the eigenvalue problem:

$$(5.10) \begin{cases} \text{to find } v \in H_{A_m} \text{ such that} \\ [v, \varphi]_{A_m} - \nu[v, \varphi]_{B_m} = 0 \quad \text{for all } \varphi \in H_{A_m} . \end{cases}$$

For each $j \geq 1$, v_j is an eigenfunction of (5.10) with eigenvalue $\nu_j = \lambda_j + 2k(v_j', v_m')$. Since $\{v_j\}$ is a basis of L^2 there are no other eigenfunctions of (5.10). Also it is clear that $\{v_j\} \subset M_{B_m}$. It follows from Theorem 3.3 that the $\{r_{m,j}\}$ exist and form a basis of L^2 and H^2_0, and that precisely $m-1$ of the $\rho_{m,j}$ are negative.

Finally, suppose $\rho_{m,m} = 0$. Then setting $r_m = r_{m,m}$ we have from (5.7) that

$$(5.11) \qquad \alpha r_m'''' - 2k(r_m', v_m')v_m'' + \lambda_m r_m'' = 0 .$$

Taking the inner product of (5.11) with v_m we see that $(r_m', v_m') = 0$. But then, by (5.11),

$$\alpha r_m'''' + \lambda_m r_m'' = 0$$

so that $r_m = Cv_m$ contradicting $(r_m', v_m') = 0$. Thus $\rho_{m,m} > 0$.
□

We remark that the conclusion $\rho_{m,m} > 0$ is connected with the absence of secondary bifurcation for the equilibrium problem for (1.1).

The proof of the next lemma is left to the reader.

Lemma 5.2. If $\sigma > 0$ and

$$\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} r \\ s \end{pmatrix},$$

where p, q, r, s are real, a, b, c, d complex numbers, then

$$\sigma p^2 + q^2 \leq (|a|^2 + \sigma |b|^2 + |c|^2/\sigma + |d|^2)(\sigma r^2 + s^2).$$

Theorem 5.2. An exponential decomposition holds for $\{T(t)\}$ with corresponding projection operators π_-, π_0, π_+ defined. The dimensions or codimensions of the corresponding invariant subspaces are given in Table 1.

		codim $\pi_- \Sigma$	dim $\pi_0 \Sigma$	dim $\pi_+ \Sigma$
$v=0$, $n \geq 0$	$\lambda_n < -\beta < \lambda_{n+1}$	n	0	n
	$-\beta = \lambda_{n+1}$	n+1	1	n
$v = \pm v_m$		m-1	0	m-1

Table 1

Given $\epsilon > 0$ there exists $K_\epsilon \geq 1$ such that, for all $\varphi \epsilon \Sigma$,

(5.12) $$\|T(t)\pi_-\varphi\| \leq K_\epsilon e^{-(b_- -\epsilon)t}\|\pi_-\varphi\|, \quad t \geq 0$$

(5.13) $$\|T(t)\pi_0\varphi\| = \|\pi_0\varphi\|, \quad t \epsilon \mathbb{R}, \quad \text{and}$$

(5.14) $$\|T(t)\pi_+\varphi\| \leq e^{b_+ t}\|\pi_+\varphi\|, \quad t \leq 0.$$

The exponents b_-, b_+ are given in Table 2.

	b_-	b_+
$v = 0$	$\mathrm{Re}\frac{1}{2}[\delta - \sqrt{\delta^2 - 4\rho_{N+1}}]$ where $N \geq 0$ is defined by $\lambda_N \leq -\beta < \lambda_{N+1}$	undefined if $-\beta \leq \lambda_1$; $= \frac{1}{2}[\sqrt{\delta^2 - 4\rho_n} - \delta]$ if $\lambda_n < -\beta \leq \lambda_{n+1}$, $n \geq 1$
$v = \pm v_m$	$\mathrm{Re}\frac{1}{2}[\delta - \sqrt{\delta^2 - 4\rho_{m,m}}]$	$\frac{1}{2}[\sqrt{\delta^2 - 4\rho_{m,m-1}} - \delta]$

Table 2

When $V = H_0^2$, $\{\rho_j\}$, $\{\rho_{m,j}\}$ are defined as in Lemma 5.1. If $V = G_1$ then

(5.15) $$\rho_j = \alpha(j\pi/\ell)^4 + \beta(j\pi/\ell)^2,$$

and

(5.16) $$\rho_{m,j} = \alpha(\pi/\ell)^4 j^2(j^2 - m^2), \quad j \neq m$$

(5.17) $$\rho_{j,j} = -2\rho_j .$$

Proof. We shall do the entire proof for the more difficult case $V = H_0^2$ and then indicate the modifications necessary if $V = G_1$. So let $V = H_0^2$ and $v = \pm v_m$, $1 \leq m \leq r$. Let $\{r_{m,j}\}$ be defined as in Lemma 5.1 and be normalized by $|r_{m,j}|^2 = 1$. Define U_m by

(5.18) $$U_m\psi \equiv \alpha\psi'''' - (\beta + k|v_m'|^2)\psi'' - 2k(v_m', \psi')v_m'' .$$

Since $\beta + k|v_m'|^2 = -\lambda_m$ it follows that $U_m r_{m,j} = \rho_{m,j} r_{m,j}$ for each j. Also if $c > -\rho_{m,1}$ then

(5.19) $$\|\psi\|_1^2 = (U_m\psi, \psi) + c|\psi|^2$$

defines an equivalent norm on H_0^2. Denote the corresponding inner product by $((\ ,\ ))$. Thus we can and shall norm Σ by $\|\{\psi, \theta\}\|^2 = \|\psi\|_1^2 + |\theta|^2$. Define the projection operators π_j by

(5.20) $$\pi_j\varphi \equiv \{u_{0j}r_{m,j}, u_{ij}r_{m,j}\}$$

for $\varphi = \{u_0, u_1\} \in \Sigma$, where $u_{0j} \equiv ((u_0, r_{m,j}))$, $u_j \equiv (u_1, r_{m,j})$. Note that

(5.21) $$\|\pi_j\varphi\|^2 = u_{0,j}^2(\rho_{m,j} + c) + u_{1j}^2$$

Since $\{r_{m,j}\}$ is an orthogonal basis of H_0^2 and L^2 we can write $\varphi = \sum_{j=1}^{\infty} \pi_j\varphi$ for any $\varphi \in \Sigma$, convergence holding in Σ. Setting $\psi = r_{m,j}$ in (5.3) we can write, using Theorem 5.1,

$$\pi_j T(t)\varphi = T(t)\pi_j\varphi = \{S_j(t)r_{m,j}, \dot{S}_j(t)r_{m,j}\}, \tag{5.22}$$

where

$$\ddot{S}_j + \delta\dot{S}_j + \rho_{m,j}S_j = 0. \tag{5.23}$$

The solution of (5.23) is

$$S_j(t) = A_j\exp(\nu_{j+}t) + B_j\exp(\nu_{j-}t)\ , \tag{5.24}$$

provided $\delta^2 \neq 4\rho_{m,j}$, where

$$\nu_{j\pm} = \frac{-\delta \pm \sqrt{\delta^2 - 4\rho_{m,j}}}{2} \tag{5.25}$$

and

$$S_j(t) = (C_j + D_j t)\exp(-\frac{\delta}{2}t), \tag{5.26}$$

if $\delta^2 = 4\rho_{m,j}$. A_j, B_j, C_j and D_j are arbitrary constants and

$$A_j = \frac{u_{ij} - \nu_{j-} u_{0j}}{\nu_{j+} - \nu_{j-}}, \quad B_j = \frac{\nu_{j+} u_{0j} - u_{ij}}{\nu_{j+} - \nu_{j-}},$$
(5.27)
$$C_j = u_{0j}, \quad D_j = u_{1j} + \frac{\delta}{2} u_{0j}.$$

Define $S_A = \{\varphi \in \Sigma : A_j = 0 \text{ all } j \text{ with } \rho_{m,j} \leq 0\}$.

Let U_A be the orthogonal complement of S_A in Σ, so that $\Sigma = U_A \oplus S_A$, and let π_+, π_- be the projection operators such that $U_A = \pi_+\Sigma$, $S_A = \pi_-\Sigma$. Since by Lemma 5.1 (ii) precisely m-1 of the $\{\rho_{m,j}\}$ are negative and $\rho_{m,m} > 0$, it follows that $\dim \pi_+\Sigma = \operatorname{codim} \pi_-\Sigma = m-1$. Clearly $\pi_\pm T(t) = T(t)\pi_\mp$ for all $t \in \mathbb{R}$. We now prove the exponential estimates for $\pi_\pm$. First note that it is sufficient to prove the following:

Given $\epsilon > 0$ there exists $K_\epsilon \geq 1$ such that for all j

(5.28) $$\|T(t)\pi_-\pi_j\varphi\| \leq K_\epsilon e^{-(b_- - \epsilon)t} \|\pi_-\pi_j\varphi\|, \quad t \geq 0,$$

and

(5.29) $$\|T(t)\pi_+\pi_j\varphi\| \leq e^{b_+ t} \|\pi_+\pi_j\varphi\|, \quad t \leq 0.$$

For it then follows from the orthogonality of $\{r_{m,j}\}$ that (5.12) and (5.14) hold when φ is of the form $\varphi = \sum_{j=1}^{N} \pi_j\varphi$. Then (5.12) and (5.14) follow by Theorem 5.1.

Consider first (5.29). Write $\nu_{j\pm} = \nu_\pm$ for simplicity. Then from (5.24), (5.27)

(5.30) $$T(t)\pi_+\pi_j\varphi = \exp(\nu_+ t)\pi_+\pi_j\varphi$$

and (5.29) follows.

Now suppose that $\varphi \in S_A$. Since $\rho_{m,j} \neq 0$ there are three possible cases

(a) $\rho_{m,j} > 0$, $\delta^2 \neq 4\rho_{m,j}$

(b) $\delta^2 = 4\rho_{m,j}$

(c) $\rho_{m,j} < 0$.

Case (a). It follows from (5.24), (5.27) that

$$\begin{pmatrix} S_j(t) \\ \dot{S}_j(t) \end{pmatrix} = \frac{1}{\nu_+ - \nu_-} [\exp(\nu_+ t)K_+ - \exp(\nu_- t)K_-] \begin{pmatrix} u_{0j} \\ u_{ij} \end{pmatrix}, \tag{5.31}$$

where $K_\pm = \begin{pmatrix} -\nu_\mp & 1 \\ -\nu_+\nu_- & \nu_\pm \end{pmatrix}$. Combining (5.21), (5.31) and Lemma 5.2 (with $\sigma = \rho_{m,j} + c$), and using the inequality

$$\sigma(p_1 + p_2)^2 + (q_1 + q_2)^2 \leq 2(\sigma p_1^2 + q_1^2 + \sigma p_2^2 + q_2^2),$$

we obtain, for $t \geq 0$,

$$\|T(t)\pi_j\varphi\|^2 \leq \frac{2}{|\nu_+ - \nu_-|^2} [\,|\exp(2\nu_+ t)| +$$

$$+ |\exp(2\nu_- t)|][\,|\nu_+|^2 + |\nu_-|^2 + \rho_{m,j} + c +$$

$$+ |\nu_+\nu_-|^2/(\rho_{m,j} + c)]\,\|\pi_j\varphi\|^2 \leq$$

$$\leq \frac{4e^{-2b_- t}}{|\delta^2 - 4\rho_{m,j}|}[\delta^2 + |\delta^2 - 4\rho_{m,j}| + \rho_{m,j} + c + \frac{\rho_{m,j}^2}{\rho_{m,j} + c}]\|\pi_j\varphi\|^2$$

$$= 4e^{-2b_- t}[\frac{\delta^2}{|\delta^2 - 4\rho_{m,j}|} + 1 + \frac{1 + c/\rho_{m,j}}{|\delta^2/\rho_{m,j} - 4|} + \frac{1}{(1/\rho_{m,j} + c)|\delta^2/\rho_{m,j} - 4|}]\|\pi_j\varphi\|^2.$$

Thus

$$(5.32)\qquad \|T(t)\pi_j\varphi\|^2 \leq Ce^{-2b_-t}\|\pi_j\varphi\|^2, \quad t \geq 0.$$

Case (b). In this case

$$(5.33)\qquad \begin{pmatrix} S_j(t) \\ \dot{S}_j(t) \end{pmatrix} = \exp(-\frac{\delta}{2}t)\begin{pmatrix} 1+\delta t/2 & t \\ -\delta^2 t/4 & 1-\delta t/2 \end{pmatrix}\begin{pmatrix} u_{0j} \\ u_{ij} \end{pmatrix},$$

and thus by Lemma 5.2, for $t \geq 0$,

$$\|T(t)\pi_j\varphi\|^2 \leq e^{-\delta t}\|\pi_j\varphi\|^2[(1+\delta t/2)^2 + (\frac{\delta^2}{4}+c)t^2 + \frac{\delta^4}{4(\delta^2+4c)} + (1-\frac{\delta t}{2})^2].$$

Thus

$$(5.34)\qquad \|T(t)\pi_j\varphi\|^2 \leq C_\epsilon^2 e^{-(\delta-2\epsilon)t}\|\pi_j\varphi\|^2, \quad t \geq 0.$$

Case (c). In this case

$$(5.35)\qquad T(t)\pi_j\varphi = \exp(\nu_-t)\pi_j\varphi$$

so that

$$(5.36) \qquad \|T(t)\pi_j\varphi\| \le e^{-b_-t}\|\pi_j\varphi\|, \quad t \ge 0.$$

Combining (5.32), (5.34) and (5.36) we obtain (5.28) and hence (5.12), (5.14).

Next consider the case $v = 0$. In the preceding analysis replace $\{\rho_{m,j}\}$ by $\{\rho_j\}$ and $\{r_{m,j}\}$ by $\{r_j\}$, where $\{\rho_j\}$ and $\{r_j\}$ are defined as in Lemma 5.1 (i) with $|r_j|^2 = 1$ each j. If $-\beta \neq \lambda_{n+1}$, any $n \ge 0$, then the analysis holds with the indicated modifications to codim $\pi_-\Sigma$, dim $\pi_+\Sigma$ and $b_\pm$. If $-\beta = \lambda_{n+1}$, some $n \ge 0$, then $\rho_{n+1} = \nu_{(n+1)+} = 0$ and $r_{n+1} = \pm v_{n+1}/|v_{n+1}|$. Define S_A as in the preceding proof and let Z_A be the subspace of Σ spanned by the element $\{r_{n+1}, 0\}$. Let U_A be the orthogonal complement of $Z_A \oplus S_A$ in Σ, and let π_-, π_0, π_+ be the projection operators such that $S_A = \pi_-\Sigma$, $Z_A = \pi_0\Sigma$, $U_A = \pi_+\Sigma$. It is clear from (5.24) that $T(t)\pi_0\varphi = \pi_0\varphi$, any $t \in \mathbb{R}$, and thus (5.13) holds. The proof now proceeds in a similar way to before.

Suppose now that $V = G_1$. Throughout in the above replace $r_{m,j}$ and r_j by $\sqrt{2/\ell}\, s_j$, where $s_j(x) \equiv \sin(j\pi x/\ell)$. $\{s_j\}$ is a basis of V and L^2 (see [2] or apply Theorem 3.2). Defining U_m as in (5.18) it is easy to see that

$$(5.37) \qquad U_m s_j = \rho_{m,j} s_j ,$$

where the $\rho_{m,j}$ are given by (5.16) and (5.17). It is clear that $\rho_{m,m} > 0$ while $\rho_{m,j} < 0$ if $j < m$. Also, if $\lambda_n < -\beta \le \le \lambda_{n+1}$ and

$$(5.38) \qquad \alpha s_j'''' - \beta s_j'' = \rho_j s_j,$$

then ρ_j is given by (5.15) and $\rho_j < 0$ for $j \le n$, $\rho_{n+1} \ge 0$. The proofs for any v are now straightforward adaptions of those for $V = H_0^2$. □

6. The Variation of Constants Formula

We can write (5.3) in the form

(6.1) $$\frac{d}{dt}(T(t)\varphi) = \begin{pmatrix} 0 & I \\ -B & -\delta I \end{pmatrix} T(t)\varphi,$$

where $B : V \to V'$ is defined by

(6.2) $$(B\theta)(\psi) = \alpha(\theta'', \psi'') - \beta_1(\theta'', \psi) - 2k(v', \theta')(v'', \psi) \quad \text{for } \theta, \psi \in V .$$

Let

(6.3) $$A \equiv \begin{pmatrix} 0 & I \\ -B & -\delta I \end{pmatrix} .$$

Then A is a continuous map of Σ into $L^2 \times V'$.

Lemma 6.1. Let $T > 0$ and let $g : [0, T] \to \Sigma$ be continuous. Define $G : [0, T] \to \Sigma$ by

(6.4) $$G(t) = \int_0^t T(t-s)g(s)ds .$$

Then

(6.5) $$\dot{G} = AG + g,$$

both sides being continuous functions from $[0, T]$ into $L^2 \times V'$.

Proof. Suppose, for example, that $V = H_0^2$ and $v = \pm v_m$, $1 \leq m \leq n$. We again norm Σ by $\|\{\psi, \theta\}\|^2 = \|\psi\|_1^2 + |\theta|^2$. Define for $t \in [0, T]$ the functions

$$g_r(t) = \sum_{j=1}^{r} \pi_j g(t) , \tag{6.6}$$

$$G_r(t) = \int_0^t T(t-s) g_r(s) ds, \tag{6.7}$$

$$F_r(t) = \|g - g_r\|^2(t) . \tag{6.8}$$

$g_r : [0, T] \to \Sigma$ is continuous. This implies by (5.12) and (5.14) that $G_r : [0, T] \to \Sigma$ is well-defined and continuous. Also, since there are explicit formulae for $T(t-s)g_r(s)$ (see the proof of Theorem 5.2), it is easy to verify that

$$\dot{G}_r = AG_r + g_r . \tag{6.9}$$

Now since $\{r_{m,j}\}$ is a basis of H_0^2 and L^2, $F_r(t)$ decreases to zero as $r \to \infty$ for fixed t. But each F_r is continuous on $[0, T]$. Hence by Dini's theorem (see e.g. Dieudonne [13, p. 129]), $F_r(t) \to 0$ uniformly in $[0, T]$ as $r \to \infty$. Therefore, for $t \in [0, T]$,

$$\|G_r(t) - G(t)\| = \left\|\int_0^t T(t-s)(g_r(s) - g(s)) ds\right\|$$

$$\leq C \sup_{s \in [0,T]} \|g_r(s) - g(s)\| ,$$

so that $\|G_r(t) - G(t)\| \to 0$ uniformly in $[0, T]$ as $r \to \infty$. Since $A : \Sigma \to L^2 \times V'$ is continuous, it follows that $AG_r \to AG$ in $L^\infty(0, T; L^2 \times V')$. Also $g_r \to g$ in $L^\infty(0, T; L^2 \times V')$. But $\dot{G}_r \to \dot{G}$ in $\mathcal{D}'(0, T; \Sigma)$. Hence by passing to the limit in (6.9) we obtain (6.5). The cases $v = 0$ and $V = G_1$ are treated similarly. □

Define $f : \Sigma \to \Sigma$ by

$$f(\{\theta, \chi\}) = \{0, k[2(v', \theta')\theta'' + |\theta'|^2\theta'' + |\theta'|^2 v'']\}. \tag{6.10}$$

<u>Lemma 6.2.</u> There exists a continuous, real-valued, nondecreasing function η on $[0, \infty)$ with $\eta(0) = 0$ such that

$$\|f(\varphi) - f(\psi)\| \leq \eta(r)\|\varphi - \psi\|, \quad \|\varphi\|, \|\psi\| \leq r. \tag{6.11}$$

<u>Proof</u>. Let $\varphi = \{\varphi_1, \varphi_2\}$ and $\psi = \{\psi_1, \psi_2\}$. Then

$$\|f(\varphi) - f(\psi)\| = k\,|2(v', \varphi_1')(\varphi_1'' - \psi_1'') - 2(v, \varphi_1'' - \psi_1'')\psi_1'' +$$

$$+ |\varphi_1'|^2(\varphi_1'' - \psi_1'') - (\varphi_1 + \psi_1, \varphi_1'' - \psi_1'')\psi_1'' -$$

$$- (\varphi_1 + \psi_1, \varphi_1'' - \psi_1'')v''\,| \leq C(r + r^2)\|\varphi - \psi\|$$

for $\|\varphi\|, \|\psi\| \leq r$. □

Theorem 6.1. Consider the equation

$$(6.12) \qquad w(t) = T(t)\varphi + \int_0^t T(t-s)f(w(s))ds \; .$$

Then w is a (continuous) solution of (6.12) if and only if $w = \{y, \dot{y}\}$, where $\{y, \dot{y}\}$ is the unique weak solution of (5.1) corresponding to initial data $w(0) = \varphi$.

Proof. Suppose $w : [0, T] \to \Sigma$, continuous, is a solution of (6.12). Clearly $w(0) = \varphi$. From (6.1)

$$\frac{d}{dt}(T(\cdot)\varphi) = AT(\cdot)\varphi \quad \text{belongs to } L^\infty(0, T; L^2 \times V').$$

Let $g(t) = f(w(t))$. By Lemma 6.1,

$$(6.13) \qquad \dot{w} = Aw - f(w(\cdot)) \;\text{ in }\; L^\infty(0, T; \; L^2 \times V')$$

If $w = \{y, z\}$ it follows that

$$(6.14) \qquad \dot{y} = z \;\text{ in }\; L^\infty(0, T; L^2),$$

$$(6.15) \qquad \dot{z} = -By - \delta z - k[\, 2(v', y')y'' + |y'|^2 y'' + |y'|^2 v''] \quad \text{in } L^\infty(0, T; V').$$

This means that $w = \{y, \dot{y}\}$ and that y is the unique weak solution of (5.1) satisfying $w(0) = \varphi$.

Conversely, suppose that $w = \{y, \dot{y}\}$ is the weak solution of (5.1) satisfying $w(0) = \varphi$. Define $\overline{w} : [0, T] \to \Sigma$ by

$$\bar{w}(t) = T(t)\varphi + \int_0^t T(t-s)f(w(s))ds . \tag{6.16}$$

By Lemma 6.1,

$$\dot{\bar{w}} = A\bar{w} + f(w(\cdot)) \text{ in } L^\infty(0, T; L^2 \times V') . \tag{6.17}$$

But w satisfies (6.13). Letting $u = \bar{w} - w$ we see that

$$\dot{u} = Au . \tag{6.18}$$

Since $u(0) = 0$ and solutions to the linear equation (6.18) are unique by Theorem 5.1 it follows that $u = 0$. Hence w satisfies (6.12). □

7. Application of the Saddle Point Analysis

Theorem 7.1. Let v be an equilibrium state for ω. Given $\epsilon > 0$ there exists $R = R(\epsilon) > 0$, $R \geq R' > 0$, and subsets of Σ

$$S^*(v, R) = \{v + \varphi : \|\varphi_- + \varphi_0\| < R, \ \varphi_+ = p^*(\varphi_- + \varphi_0)\},$$

$$U^*(v, R) = \{v + \varphi : \|\varphi_0 + \varphi_+\| < R, \ \varphi_- = q^*(\varphi_0 + \varphi_+)\},$$

$$S(v, R) = \{v + \varphi \in B_R(v, \Sigma) : \|\varphi_-\| < R/2K_\epsilon , \ \varphi_0 + \varphi_+ = p(\varphi_-)\},$$

$$U(v, R) = \{v + \varphi \in B_R(v, \Sigma) : \|\varphi_+\| < R/2K_\epsilon , \ \varphi_- + \varphi_0 = q(\varphi_+)\},$$

where p^*, q^*, p, q are Lipschitz functions defined on $B_R(\pi_-\Sigma \oplus \pi_0\Sigma)$, $B_R(\pi_0\Sigma \oplus \pi_+\Sigma)$, $B_{R/2K_\epsilon}(\pi_-\Sigma)$, $B_{R/2K_\epsilon}(\pi_+\Sigma)$ respectively. Any orbit of ω which remains in $B_{R'}(v, \Sigma)$ for $t \geq 0$ lies on $S^*(v, R)$ and any orbit of ω which remains in $B_{R'}(v, \Sigma)$ for $t \leq 0$ lies on $U^*(v, R)$.

If $\psi \in S(v, R)$ then

$$\|\omega(t, \psi) - v\| \leq 2K_\epsilon e^{-(b_- - \epsilon)t} \|\psi - v\|, \quad t \geq 0 . \tag{7.1}$$

If $\psi \in U(v, R)$ then

$$\|\omega(t, \psi) - v\| \leq 2K_\epsilon e^{(b_+ - \epsilon)t} \|\psi - v\|, \quad t \leq 0 . \tag{7.2}$$

S^*, U^*, S, U are tangent to $\pi_-\Sigma \oplus \pi_+\Sigma$, $\pi_0\Sigma \oplus \pi_+\Sigma$, $\pi_-\Sigma$, $\pi_+\Sigma$ respectively at v. If $v = \pm v_m$, $1 \leq m \leq n$, or if $v = 0$ with $-\beta \neq \lambda_{j+1}$ any $j \geq 0$, then S^* and S (respectively U^* and U) coincide in a sufficiently small neighborhood of v.

Proof. This is a direct application of Theorems 2.1, 2.2 (and its Corollary), 4.1, 5.1, 5.2, 6.1, and Lemma 6.2. □

It is clear from the theorem that, for example, the projection operator $\pi_- + \pi_0$ is a homeomorphism of $S^*(v,R) - v$ onto $B_R(\pi_-\Sigma \oplus \pi_0\Sigma)$. We therefore define codim $S^*(v, R) =$ codim $(\pi_-\Sigma \oplus \pi_0\Sigma)$, dim $U^*(v, R) =$ dim $(\pi_0\Sigma \oplus \pi_+\Sigma)$ and similarly for S, U. These dimensions and codimensions may be obtained from Table 1.

8. Stability and Instability of Motions of the Extensible Beam

We begin this section by defining some well known stability concepts, for an exhaustive treatment of which we refer the reader to the article by Knops and Wilkes [24].

The orbit $\mathcal{O}$ of ω is (positively) Lyapunov stable if for any $\varphi \in \mathcal{O}$ and $\epsilon > 0$ there exists $\delta > 0$ such that $\|\psi - \varphi\| < \delta$ implies $\|\omega(t, \psi) - \omega(t, \varphi)\| < \epsilon$ for all $t \geq 0$.

$\mathcal{O}$ is asymptotically stable if $\mathcal{O}$ is Lyapunov stable and if for any $\varphi \in \mathcal{O}$ there exists $\delta_1 > 0$ such that $\|\psi - \varphi\| < \delta_1$ implies $\|\omega(t, \psi) - \omega(t, \varphi)\| \to 0$ as $t \to \infty$.

$\mathcal{O}$ is Lyapunov unstable if it is not Lyapunov stable

Let v be an equilibrium state of ω. The region of attraction $A(v)$ of v is defined by

$$A(v) = \{\varphi \in \Sigma : \omega(t, \varphi) \to v \text{ in } \Sigma \text{ as } t \to \infty\}.$$

The region of backwards attraction $A_-(v)$ is defined by

$$A_-(v) = \{\varphi \in \Sigma : \omega(t, \varphi) \to v \text{ in } \Sigma \text{ as } t \to -\infty\}.$$

The following theorems are proved in [3]:

Theorem 8.1. If $\varphi \in \Sigma$ then $\omega(t, \varphi) \to v$ in Σ as $t \to \infty$, where v is an equilibrium state. Equivalently

$$\Sigma = A(0), \tag{8.1}$$

if $-\beta \leq \lambda_1$, and

$$\Sigma = A(0) \cup \bigcup_{i=1}^{n} [A(v_i) \cup A(-v_i)], \tag{8.2}$$

if $\lambda_n < -\beta \leq \lambda_{n+1}$.

Theorem 8.2. If $\lambda_1 < -\beta$ then v_1, $-v_1$ are interior points of $A(v_1)$, $A(-v_1)$ respectively.

In the case $\lambda_1 < -\beta$ it is clear from Theorems 8.1, 8.2 and the continuity of ω that $A(v_1)$, $A(-v_1)$ are arcwise-connected open sets.

We state without proof a lemma containing the main point of the proof of Theorem 8.1.

Lemma 8.1. Let $\varphi \in \Sigma$ be such that both $\|\omega(t, \varphi)\|$ and $\int_0^t |\dot{u}(s, \varphi)|^2 ds$ are bounded for $t \in [0, \infty)$ (respectively $t \in (-\infty, 0]$). Then $\omega(t, \varphi) \to v$, an equilibrium state, in Σ as $t \to \infty$ (respectively $t \to -\infty$).

Part (ii) of the next lemma supplements Theorem 7.1 by restricting the possible behaviour of any orbit in a neighbourhood of an equilibrium state, irrespective of whether or not it lies on one of the manifolds S^*, U^*, S, U.

Lemma 8.2. Let v be an equilibrium state.

(i) Suppose $\{y_n\} \subset \Sigma$, $t_n \geq 0$ are such that, as $n \to \infty$,

(a) $E(y_n) \to E(v)$ and (b) $\omega(t_n, y_n) \to v$ in Σ. Then $y_n \to v$ in Σ.

(ii) Given $\epsilon > 0$ there exists δ, $\epsilon > \delta > 0$ such that if $\|\varphi - v\| < \delta$ and $\|\omega(\tau, \varphi) - v\| < \delta$ for some $\tau \geq 0$, then $\|\omega(t, \varphi) - v\| < \epsilon$ for all $t \in [0, \tau]$.

(iii) Define for $\epsilon > 0$ the set $Y_\epsilon = \{y \in A(v) : \|y - v\| = \epsilon\}$. Then

$$\inf_{y \in Y_\epsilon} E(y) > E(v) . \tag{8.3}$$

Proof. To prove (i) let $\{y_n\}$, $t_n \geq 0$ satisfy (a) and (b), but $y_n \not\rightharpoonup v$ as $n \to \infty$. Then there exists $\epsilon > 0$, a subsequence $\{t_\mu\}$ of $\{t_n\}$, and sequences $\{s_\mu\}$, $\{z_\mu\}$ such that for each μ there holds $0 \leq s_\mu \leq t_\mu$, $z_\mu = \omega(s_\mu, y_\mu)$, $\|z_\mu - v\| = \epsilon$ and $\|\omega(t, z_\mu) - v\| \leq \epsilon$ for all $t \in [0, t_\mu - s_\mu]$. Without loss of generality we may suppose ϵ small enough for v to be the only equilibrium state in $\overline{B}_\epsilon(v, \Sigma)$.

Since $\{z_\mu\}$ is bounded we may extract a subsequence, again labelled $\{z_\mu\}$, such that $z_\mu \rightharpoonup z$, say, in Σ. We show that $z = v$. This is obvious if $\{t_\mu - s_\mu\}$ is bounded since we may suppose that $t_\mu - s_\mu \to T$, $T \geq 0$, so that $\omega(t_\mu - s_\mu, z_\mu) \rightharpoonup \omega(T, z)$ by Lemma 4.1 (ii); but $\omega(t_\mu - s_\mu, z_\mu) = \omega(t_\mu, y_\mu) \rightharpoonup v$, and hence $z = v$. If $\{t_\mu - s_\mu\}$ is unbounded we may suppose that $t_\mu - s_\mu \to \infty$. Since for any $t \geq 0$, $\omega(t, z_\mu) \rightharpoonup \omega(t, z)$ as $\mu \to \infty$, it follows that

$$\|\omega(t, z) - v\| \leq \liminf_{\mu \to \infty} \|\omega(t, z_\mu) - v\| \leq \epsilon. \tag{8.4}$$

By Theorem 8.1 $\omega(t, z) \to \overline{v}$, an equilibrium state, as $t \to \infty$. Thus $\|\overline{v} - v\| \leq \epsilon$ and hence $\overline{v} = v$. We deduce from (a) and the weak lower semicontinuity of E that for $t \geq 0$

$$\begin{aligned} E(v) &\leq E(\omega(t, z)) \leq \liminf_{\mu \to \infty} E(\omega(t, z_\mu)) \leq \liminf_{\mu \to \infty} E(z_\mu) \leq \\ &\leq \liminf_{\mu \to \infty} E(y_\mu) = E(v). \end{aligned} \tag{8.5}$$

It follows that $E(\omega(t, z)) = E(z)$ for all $t \geq 0$ and hence $z = v$.

Let $z_\mu = \{z_{1\mu}, z_{2\mu}\}$. Since $z_\mu \rightharpoonup v$, $|z_{1\mu}'|^2 \to |v'|^2$ as $\mu \to \infty$. Also $\liminf_{\mu \to \infty} E(z_\mu) = E(v)$. Thus

$$\liminf_{\mu \to \infty} [\frac{1}{2} \|z_\mu\|^2 + \frac{\beta}{2} |z'_{1\mu}|^2 + \frac{k}{4} |z'_{1\mu}|^4] =$$

$$= \frac{1}{2} \|v\|^2 + \frac{\beta}{2} |v'|^2 + \frac{k}{4} |v'|^4 ,$$

and so $\liminf_{\mu \to \infty} \|z_\mu\|^2 = \|v\|^2$. Thus a further subsequence z_μ tends strongly to v in Σ, which contradicts $\|z_\mu - v\| = \epsilon$. This proves (i).

Suppose (ii) is false. Then there exists $\epsilon > 0$ and sequences $\{\varphi_n\} \subset \Sigma$, $\tau_n \geq 0$ such that $\|\varphi_n - v\| < 1/n$, $\|\omega(\tau, \varphi_n) - v\| < 1/n$ and, for some $s_n \in [0, \tau_n]$, $\|\omega(s_n, \varphi_n) - v\| \geq \epsilon$. Let $y_n = \omega(s_n, \varphi_n)$, $t_n = \tau_n - s_n$. Then $E(\varphi_n) \geq E(y_n) \geq E(\omega(\tau_n, \varphi_n))$ and so, by the continuity of E, $E(y_n) \to E(v)$. Clearly $\omega(t_n, y_n) \to v$. Thus by (i) $y_n \to v$, contradicting $\|y_n - v\| \geq \epsilon$. Thus (ii) holds.

Suppose (iii) is false. Then there exist $\{y_n\} \subset Y_\epsilon$, $t_n \geq 0$ such that (a), (b) hold. Thus $y_n \to v$. This contradiction proves (iii). □

Remark: It is possible to prove that for $\epsilon > 0$ small enough $\inf_{y \in Y_\epsilon} E(y) \geq C\epsilon^2 + E(v)$. This improves (iii) but is unnecessary for our purposes. A tedious computational proof is given in [4].

Lemma 8.3. If v is an equilibrium state and $r > 0$ small enough

(8.6) $$A(v) \cap B_r(v, \Sigma) \subset S^*(v, R) \cap B_r(v, \Sigma),$$

(8.7) $$A_-(v) \cap B_r(v, \Sigma) \subset U^*(v, R) \cap B_r(v, \Sigma) .$$

If $v = \pm v_m$, $1 \leq m \leq n$, or if $v = 0$ with $-\beta \neq \lambda_{j+1}$ any $j \geq 0$, then for small enough $r > 0$

(8.8) $$A(v) \cap B_r(v, \Sigma) = S(v, R) \cap B_r(v, \Sigma),$$

(8.9) $$A_-(v) \cap B_r(v, \Sigma) = U(v, R) \cap B_r(v, \Sigma) .$$

Proof. By Lemma 8.2 (ii) there exists $r > 0$ such that $\varphi \in B_r(v, \Sigma)$ and $\omega(\tau, \varphi) \in B_r(v, \Sigma)$, $\tau \geq 0$, implies $\omega(t, \varphi) \in B_{R'}(v, \Sigma)$ for all $t \in [0, \tau]$. If $\varphi \in A(v) \cap B_r(v, \Sigma)$ there exists $T > 0$ such that $\omega(t, \varphi) \in B_r(v, \Sigma)$ for all $t \geq T$. Therefore $\omega(t, \varphi) \in B_{R'}(v, \Sigma)$ for all $t \geq 0$. Hence by Theorem 7.1 $\varphi \in S^*(v, R) \cap B_r(v, \Sigma)$ which establishes (8.6). The conclusion (8.7) is proved similarly, and (8.8), (8.9) are simple consequences of Theorem 7.1. □

In the following theorem '∂' denotes 'the boundary of'.

Theorem 8.3.

(i) If $-\beta \leq \lambda_1$ then any orbit converges to zero and is asymptotically stable.

(ii) Let $\lambda_1 < -\beta \leq \lambda_2$. Then

(8.10) $$\partial A(v_1) \cup \partial A(-v_1) = A(0).$$

(iii) $\lambda_n < -\beta \leq \lambda_{n+1}$, $n > 1$. Then

(8.11) $$\partial A(v_1) \cup \partial A(-v_1) = A(0) \cup \bigcup_{i=2}^{n} [A(v_i) \cup A(-v_i)].$$

(iv) Let $\lambda_1 < -\beta$. Any orbit in $A(v_1)$ or $A(-v_1)$ is asymptotically stable. If $v \neq v_1, -v_1$ then any orbit in $A(v)$ is Lyapunov unstable.

Proof. If $-\beta \leq \lambda_1$ then any orbit converges to zero by Theorem 8.1. Let $v = 0$ if $-\beta \leq \lambda_1$ and $v = \pm v_1$ if $\lambda_1 < -\beta$. We show that any orbit $\mathfrak{G}$ in $A(v)$ is asymptotically stable. Given $\epsilon > 0$ let $\delta > 0$ be given by Lemma 8.2 (ii). If $\varphi \in \mathfrak{G}$ there exists $T > 0$ such that $\|\omega(T,\varphi) - v\| < \delta/2$. Since $A(v)$ is open, by Lemma 4.1 (i) there exists $\delta_1 > 0$ such that $\|\psi - \varphi\| < \delta_1$ implies $\psi \in A(v)$, $\|\omega(T,\psi) - v\| < \delta$ and $\|\omega(t, \psi) - \omega(t, \varphi)\| < \epsilon$ for all $t \in [0, T]$. By Lemma 8.2 (ii) such an ψ satisfies $\|\omega(t, \psi) - v\| < \epsilon$ for all $t \geq T$, and hence $\|\omega(t, \psi) - \omega(t, \varphi)\| < 2\epsilon$ for all $t \geq 0$. This proves Lyapunov stability of $\mathfrak{G}$ and asymptotic stability follows.

Now let $\lambda_n < -\beta \leq \lambda_{n+1}$, $n > 1$. By (8.2) the l.h.s. of (8.11) is included in the r.h.s. We thus need the reverse inclusion. Let $v = \pm v_2$ and $\varphi \in A(v)$. Choose $r > 0$ small enough. There exists $T > 0$ such that $\omega(T,\varphi) \in A(v) \cap B_r(v, \Sigma)$, and so by Lemma 8.3 $\omega(T, \varphi) \in S(v, R) \cap B_r(v, \Sigma)$. Let $\pi_\pm$ be the projection operators associated with v. Consider for $|\epsilon_1|$ small enough $(\epsilon_1 \neq 0)$

$$\omega_{\epsilon_1} \equiv \omega(T, \varphi) + \epsilon_1 \chi , \tag{8.12}$$

where $\chi \neq 0$ is an element of $\pi_+\Sigma$. Then $\omega_{\epsilon_1} \in B_r(v, \Sigma)$ and $\pi_-(\omega_{\epsilon_1} - v) = \pi_-(\omega(T,\varphi) - v)$. But $\omega_{\epsilon_1} \neq \omega(T, \varphi)$ and hence $\omega_{\epsilon_1} \notin S(v, R)$. Therefore by Lemma 8.3 $\omega_{\epsilon_1} \notin A(v)$. Hence $\omega_{\epsilon_1} \in A(\overline{v})$ for some equilibrium state $\overline{v} \neq v$ with $E(\overline{v}) < E(\omega_{\epsilon_1})$. By choosing r small enough we can ensure that $E(\omega_{\epsilon_1}) < E(\pm v_3)$ (if $n > 2$) or $E(\omega_{\epsilon_1}) < E(0)$ (if $n = 2$). But if r is small enough, by the continuity of E and by Lemma 8.2 (iii) applied to $-v$ it is clear that $\overline{v} \neq -v$. Hence $\omega_{\epsilon_1} \in A(v_1) \cup A(-v_1)$. Let $\epsilon_1 \to 0$. Then $\omega(-T, \omega_{\epsilon_1}) \to \omega(-T, \omega(T, \varphi)) = \varphi$. Thus $\varphi \in \partial A(v_1) \cup \partial A(-v_1)$. This proves that

$$A(v_2) \cup A(-v_2) \subset \partial A(v_1) \cup \partial A(-v_1). \tag{8.13}$$

Applying the above argument to $\pm v_3$ (in case $n > 2$) we see that

$$A(v_3) \cup A(-v_3) \subset \partial A(v_1) \cup \partial A(-v_1) \cup \partial A(v_2) \cup \partial A(-v_2)$$

$$\subset \partial A(v_1) \cup \partial A(-v_1), \text{ by (8.13).}$$

The proof of (8.11) is completed inductively. At the final step we of course use (8.6) instead of (8.8). The proof of (8.10) is easier and is omitted. If $\lambda_1 < -\beta$ and $v \neq v_1, -v_1$ then (8.10), (8.11) clearly imply that any orbit in $A(v)$ is Lyapunov unstable. □

9. The Case of Three Equilibrium States

In this section we suppose that $\lambda_1 < -\beta < \lambda_2$ so that there are only three equilibrium states 0, v_1 and $-v_1$. By Theorems 5.2, 7.1

(9.1) $$\dim U(0, R) = \operatorname{codim} S(0, R) = 1,$$

(9.2) $$\dim U(\pm v_1, R) = \operatorname{codim} S(\pm v_1, R) = 0.$$

Let $\varphi_0 \neq 0$ belong to $A_-(0)$. Clearly $-\varphi_0 \in A_-(0)$. Since $\varphi_0 \notin A(0)$ we can without loss of generality suppose that $\varphi_0 \in A(v_1)$ so that $-\varphi_0 \in A(-v_1)$. Thus by (9.1) and (8.9),

(9.3) $$A(v_1) \cap A_-(0) = O(\varphi_0),$$

(9.4) $$A(-v_1) \cap A_-(0) = O(-\varphi_0).$$

There is thus just one orbit 'connecting' v_1 and 0, and just one orbit 'connecting' $-v_1$ and 0. For $V = G_1$ these orbits are of the form $\{T_1(t) \sin \frac{\pi x}{\ell}, \dot{T}_1(t) \sin \frac{\pi x}{\ell}\}$ and are shown in Figure 4 of Reiss and Matkowsky [29]. The situation is illustrated pictorially in Figure 1, in which $A(0) \cap \cap B_r(\Sigma)$ is shaded.

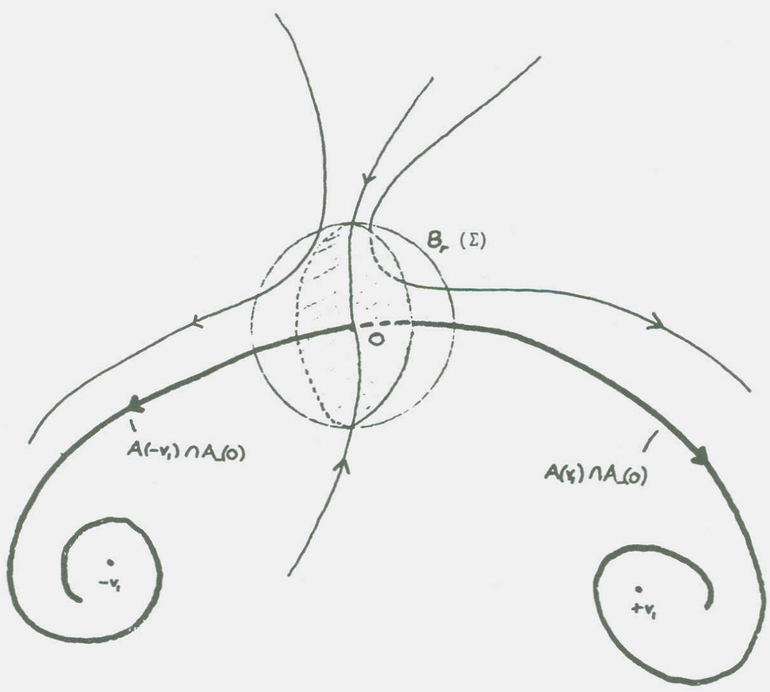

Figure 1

The figure suggests the physically appealing result

Theorem 9.1.

(9.5) $$\partial A(v_1) = \partial A(-v_1) = A(0)$$

Proof. Let $\pi_{\pm}$ be the projection operators associated with $v = 0$. Since $\dim \pi_+\Sigma = 1$ there exists $\chi \neq 0$ in Σ with $\pi_+\Sigma = \{a\chi : a \in \mathbb{R}\}$. Let $\psi \in A(0)$. Then there exists $\tau > 0$ such that $\omega(t, \psi) \in B_{r/2}(0, \Sigma) \cap S(0, R)$ for all $t \geq \tau$. Define $\psi_a(t) \equiv \omega(t, \psi) + a\chi$. If $a > 0$ is small enough $a\chi \in B_r(0, \Sigma)$ and $\psi_a(t) \in B_r(0, \Sigma)$ for $t \geq \tau$. Now $a\chi \notin A(0)$ since $\pi_-\chi = 0$, and so we can suppose without loss of generality that $a\chi \in A(v_1)$, which is an open, arcwise-connected set. Since $\pi_-\psi_a(t) = \pi_-\omega(t, \psi)$ it follows that $\psi_a(t) \notin S(0, R)$ for $t \geq \tau$. As $\psi_a(t) \to a\chi$ as $t \to \infty$ we can conclude that $\psi_a(\tau)$ and $a\chi$ are joined by an arc lying wholly in $A(v_1) \cup A(-v_1)$. Therefore $\psi_a(\tau) \in A(v_1)$. Since $-a\chi \in A(-v_1)$ we also have that $\psi_{-a}(\tau) \in A(-v_1)$. Hence $\omega(\tau, \psi) \in \partial A(v_1) \cap \partial A(-v_1)$. By backwards continuity $\psi \in \partial A(v_1) \cap \partial A(-v_1)$ and (9.5) follows. □

We now establish a very strong stability property of the connecting orbits.

Theorem 9.2. Given $\epsilon > 0$ there exists $\delta > 0$ such that if $\varphi \in B_\delta(0, \Sigma)$ then one of the following conditions holds:

(i) $\varphi \in A(0)$,

(ii) $\sup_{t \geq 0} \|\omega(t, \varphi) - \omega(t, \psi)\| < \epsilon$ for some $\psi \in A(v_1) \cap A_-(0)$,

(iii) $\sup_{t \leq 0} \|\omega(t, \varphi) - \omega(t, \theta)\| < \epsilon$ for some $\theta \in A(-v_1) \cap A_-(0)$.

Proof. It is enough to show, for instance, that if $\{\varphi_n\} \subset A(v_1)$, $\varphi_n \to 0$ in Σ as $n \to \infty$, then

$$\sup_{t \geq 0} \|\omega(t, \varphi_\mu) - \omega(t, \chi_\mu)\| \to 0 \text{ as } \mu \to \infty \tag{9.6}$$

for some $\{\chi_\mu\} \subset A(v_1) \cap A_-(0)$ and some subsequence $\{\varphi_\mu\}$ of $\{\varphi_n\}$.

Let such a sequence $\{\varphi_n\}$ be given. Since the weak topology on Σ is Hausdorff (Simmons [31]) there exist disjoint weakly open (and hence open) sets U_0, U_1 with $0 \in U_0$, $-v_1 \in U_0$ and $v_1 \in U_1$. We can suppose that $\{\varphi_n\} \subset U_0$. For each n there exists a largest time $t_n > 0$ such that $\omega(t_n, \varphi_n) \equiv \psi_n$ belongs to the complement of U_1 in Σ. Since $\{\psi_n\}$ is bounded we may extract a subsequence $\{\psi_\mu\}$ with $\psi_\mu \rightharpoonup \psi$, say, in Σ. Clearly $\psi \neq 0, v_1$. Hence $\{t_\mu\}$ is unbounded, and we can thus suppose $t_\mu \to \infty$ as $\mu \to \infty$. By Lemma 4.1 (ii) $\omega(t, \psi_\mu) \rightharpoonup \omega(t, \psi)$, $t \in \mathbb{R}$. Since $\omega(t, \psi)$ belongs to the weak closure of U_1 for $t \geq 0$, it follows that $\psi \in A(v_1)$. If $T > 0$, by weak convergence and Fatou's lemma we have that for $t \in \mathbb{R}$

$$\int_{-T}^{t} |\dot{u}(s, \psi)|^2 ds \leq \int_{-T}^{t} \liminf_{\mu\to\infty} |\dot{u}(s, \psi_\mu)|^2 ds \leq$$

$$\leq \liminf_{\mu\to\infty} \int_{-T}^{t} |\dot{u}(s, \psi_\mu)|^2 ds \leq \liminf_{\mu\to\infty} \int_{-t_\mu}^{t} |\dot{u}(s, \psi_\mu)|^2 ds = \tag{9.7}$$

$$= \liminf_{\mu\to\infty} [E(\phi_\mu) - E(\omega(t, \psi_\mu))] = -\limsup_{\mu\to\infty} E(\omega(t, \psi_\mu))$$

Hence $\int_{-\infty}^{0} |\dot{u}(s, \psi)|^2 ds$ exists and is finite. Since $\|\omega(t, \psi)\| \leq \liminf_{\mu \to\infty} \|\omega(t, \varphi_\mu)\|$, $t \in \mathbb{R}$, $O(\psi)$ is bounded. Thus by Lemma 8.1 $\omega(t, \psi) \rightharpoonup v$, an equilibrium state, as $t \to -\infty$.

Suppose for contradiction that $v = \pm v_1$. Define

$$H(t) \equiv |\dot{u}(t, \psi)|^2 + \alpha |u''(t, \psi)|^2 + \beta |u'(t, \psi)|^2 + k |u'(t, \psi)|^4 . \tag{9.8}$$

Then $H(t) = 2E(\omega(t, \psi)) + \frac{k}{2} |u'(t, \psi)|^2 \geq 2E(\psi) + \frac{k}{2} |u'(t, \psi)|^4$. Therefore $\liminf_{t \to -\infty} H(t) \geq 2E(\psi) + \frac{k}{2} |v_1'|^2 > 2E(v_1) + \frac{k}{2} |v_1'|^2 = 0$. However, it is easy to show by setting $\theta = u$ in (4.1) that $\int_\tau^0 H(s)ds$ is bounded independent of $\tau \leq 0$. This contradicts $\liminf_{t \to -\infty} H(t) > 0$. Hence $v = 0$. We now have that

$$0 = E(0) \leq \liminf_{t \to -\infty} E(\omega(t, \psi)) \leq \liminf_{t \to -\infty} \liminf_{\mu \to \infty} E(\omega(t, \psi_\mu)) \leq 0,$$

which implies that $E(\omega(t, \psi)) \to 0$ as $t \to -\infty$. Hence $\omega(t, \psi) \to 0$ as $t \to -\infty$ and $\psi \in A(v_1) \cap A_-(0)$.

From (9.7) we deduce

$$E(\omega(t, \psi)) \geq \limsup_{\mu \to \infty} E(\omega(t, \psi_\mu)), \quad t \in \mathbb{R}. \tag{9.9}$$

But

$$E(\omega(t, \psi)) \leq \liminf_{\mu \to \infty} E(\omega(t, \psi_\mu)), \quad t \in \mathbb{R}, \tag{9.10}$$

and thus $E(\omega(t, \psi_\mu)) \to E(\omega(t, \psi))$ each $t \in \mathbb{R}$. Therefore $\omega(t, \psi_\mu) \to \omega(t, \psi)$, $t \in \mathbb{R}$. In particular $\psi_\mu \to \psi$.

Given $\epsilon > 0$, by Lemma 8.2 (ii) there exists δ with $\epsilon/2 > \delta > 0$ such that if $\|\varphi\| < \delta$ and $\|\omega(\tau, \varphi)\| < \delta$, $\tau \geq 0$, then $\|\omega(t, \varphi)\| < \epsilon/2$ for all $t \in [0, \tau]$. Let $T > 0$ be large enough so that $\omega(-t, \psi) \in B_\delta(0, \Sigma)$ for $t \geq T$. Since $\omega(-T, \psi_\mu) \to \omega(-T, \psi)$, by Theorem 8.3 (iv) there exists N such that if $\mu > N$ then

$$\sup_{t \in [-T, \infty)} \|\omega(t, \psi_\mu) - \omega(t, \psi)\| < \epsilon. \tag{9.11}$$

Let N_1 be such that if $\mu > N_1$ then $\varphi_\mu \in B_\delta(0, \Sigma)$ and $\omega(-T, \psi_\mu) \in B_\delta(0, \Sigma)$. Then if $\mu > N_1$, $T < t_\mu$, by the choice of δ we have

$$(9.12) \qquad \sup_{t \in [-t_\mu, -T]} \|\omega(t, \psi_\mu) - \omega(t, \psi)\| < \epsilon .$$

Combining (9.11), (9.12), if $\mu > \max(N, N_1)$ then

$$(9.13) \qquad \sup_{t \in [-t_\mu, \infty)} \|\omega(t, \psi_\mu) - \omega(t, \psi)\| < \epsilon .$$

Let $\chi_\mu = \omega(-t_\mu, \psi)$. Then (9.6) follows. □

10. Global Structure of the Regions of Attraction and Backwards Attraction

Consider the case $\lambda_2 < -\beta < \lambda_{n+1}$. By Theorems 5.2, 7.1, dropping the dependence on R,

$$(10.1) \qquad \dim U(0) = \operatorname{codim} S(0) = n$$

$$(10.2) \qquad \dim U(\pm v_m) = \operatorname{codim} S(\pm v_m) = m-1, \quad 1 \leq m \leq n.$$

Clearly (10.2) holds for $-\beta = \lambda_{n+1}$ also. In particular $\dim U(\pm v_2) = 1$. Let $\psi_0 \in A_-(v_2)$ and let $\psi_0 \in A(v)$. Clearly $v = \pm v_1$. Since v_2 is antisymmetric and v is symmetric it follows that the reflection $\hat{\psi}_0$ of ψ_0 in $x = \ell/2$ belongs to $A_-(-v_2) \cap A(v)$, while $-\hat{\psi}_0 \in A_-(v_2) \cap A(-v)$. Since $-\psi_0 \in A_-(-v_2) \cap A(-v)$ it follows that (now we take $v = v_1$ without loss of generality)

$$A_-(v_2) \cap A(v_1) = O(\psi_0),$$

$$A_-(-v_2) \cap A(-v_1) = O(-\psi_0), \tag{10.3}$$

$$A_-(-v_2) \cap A(v_1) = O(\hat{\psi}_0),$$

$$A_-(v_2) \cap A(-v_1) = O(-\hat{\psi}_0) .$$

Hence there is just one orbit connecting, say, v_1 and v_2. A slight refinement of the proof of Theorem 9.1 now shows that

$$A(v_2) \cup A(-v_2) \subset \partial A(v_1) \cap \partial A(-v_1) . \tag{10.4}$$

Consider the case of hinged ends, when $V = G_1$. Let Y be a proper subset of the positive integers and define $S(Y) = \{s_j : j \in Y\}$, where $s_j(x) \equiv \sin(j\pi x/\ell)$. Let $G(Y)$, $L(Y)$ be the subspaces of G_1, L^2 respectively which are spanned by $S(Y)$. The existence and stability theory for (4.1) with $\Sigma = G_1 \times L^2$ now adapts to the case $\Sigma = G(Y) \times L(Y)$ in the obvious manner. (This is because the approximating solutions in the Galerkin method used to prove existence are actually solutions to (4.1) if the basis $\{s_j\}$ is used - no similar property holds for the clamped beam.)

First let $Y = \{j : 1 \leq j \leq n\}$. Then $G(Y) \times L(Y)$ is a 2n-dimensional space. The relations (10.1), (10.2) hold in both the cases $\Sigma = G_1 \times L^2$ and $\Sigma = G(Y) \times L(Y)$. Therefore the $A_-(v)$ are the same set in both cases, and thus all solutions connecting equilibrium states are composed of the first n modes. By considering sets $Y_{ij} = \{i, j\}$ we can establish the existence of orbits connecting any two equilibrium states. Considerations like these lead to the following conjecture (for $V = G_1$ or H_0^2) which is illustrated in Figure 2.

Conjecture. If v and $\bar{v}$ are two equilibrium states with v having energy m levels higher than $\bar{v}$ (e.g. $m = 3$ if $v = v_6$, $\bar{v} = -v_3$) then

$$A_-(v) \cap A(\bar{v}) \text{ is an } m\text{-dimensional manifold.}$$

Furthermore

$$\partial A(v_1) = \partial A(-v_1) = A(0) \cup \bigcup_{i=2}^{n} [A(v_i) \cup A(-v_i)] \; . \tag{10.5}$$

11. Concluding Remarks

Consider first the rate-of-decay estimates. In practice the only estimates of interest are those concerning convergence to a stable equilibrium state. By Theorems 5.2, 7.1 these are

$$\|\omega(t, \varphi) - v\| \leq Me^{-(b_- - \epsilon)t} \|\varphi - v\| \quad \text{for} \quad \varphi \in A(v), \tag{11.1}$$

where

$$b_- = \mathrm{Re}\, \frac{1}{2} [\delta - \sqrt{\delta^2 - 4\bar{\rho}}\,], \tag{11.2}$$

and

$$\bar{\rho} = \rho_1 \quad \text{if} \quad -\beta < \lambda_1, \; v = 0, \tag{11.3}$$

$$\bar{\rho} = \rho_{1,1} \quad \text{if} \quad \lambda_1 < -\beta, \; v = \pm v_1 \; . \tag{11.4}$$

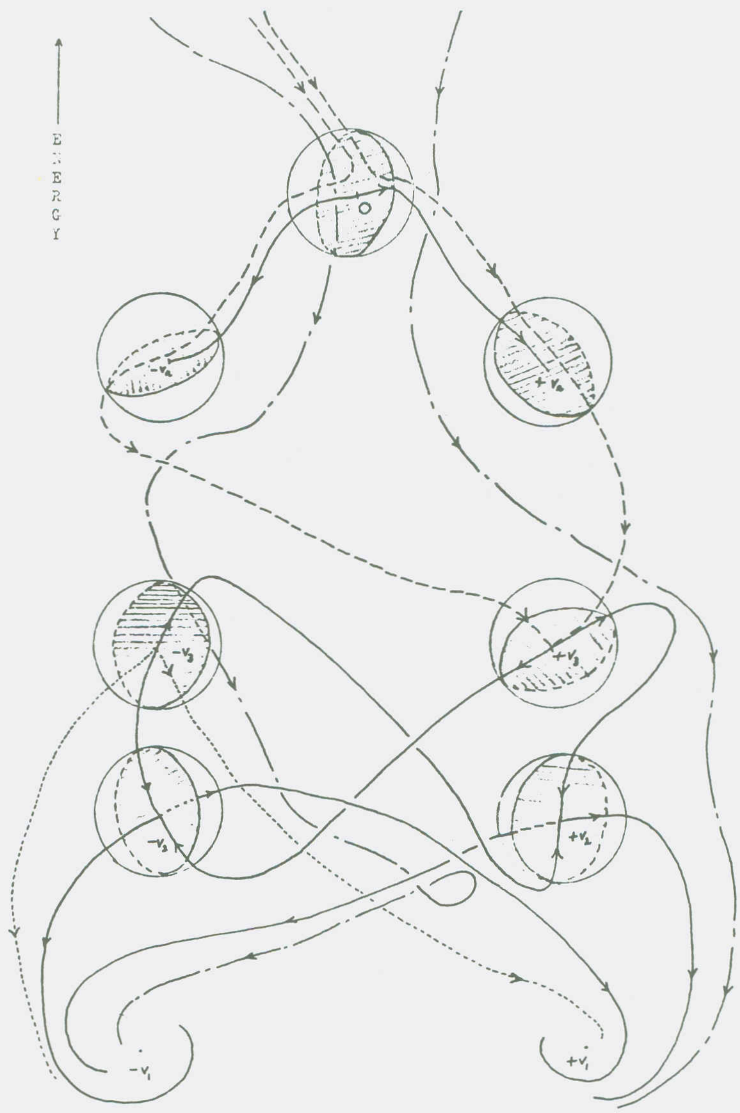

Figure 2

Key to Figure 2

Figure 2 is a schematic diagram of the Hilbert space Σ .

———>——— one-dimensional manifolds connecting equilibrium states

----->---- orbits in $A_{-}(-v_3) \cap A(\pm v_1)$

— — —>— — orbits in $A(v_3)$

—-—->-- orbits in $A(\pm v_1) \setminus \partial A(\pm v_1)$.

Arrows point in the direction of increasing time. Shown shaded are the intersections of $\partial A(\pm v_1)$ with the various $B_r(v, \Sigma)$. The 'energy' axis is not rigidly adhered to.

b_- attains a maximum of $\delta/2$ when $\delta = 2\sqrt{\rho}$. We therefore tentatively suggest that $\delta = 2\sqrt{\rho}$ gives optimal damping. The estimates (11.1) improve those obtained by Dickey in [12]. If $\beta = \lambda_1$, $v = 0$, then we know of no rate-of-decay estimates. If $V = G_1$ and $\omega(t, \varphi) = \{T_1(t)s_1, \dot{T}_1(t)s_1\}$ then $T_1(t)$ satisfies the ODE

$$\ddot{T}_1 + \delta\dot{T} + k_1 T^3 = 0 \tag{11.5}$$

for some $k_1 > 0$. We know of no rate-of-decay estimates even for this equation.

Finally consider the equation

$$\ddot{u} + \alpha u'''' - (\beta + k|u'|^2)u'' + \gamma\dot{u}'''' - \sigma(u', \dot{u}')u'' + \delta\dot{u} = 0 \tag{11.6}$$

where $\gamma, \sigma > 0$. The existence of dynamical systems generated by (11.6) was established in [3]. There are, however, difficulties involved in extending our analysis to (11.6). Consider, for example, the linearization of (11.6) about zero:

$$\ddot{u} + \alpha u'''' - \beta u'' + \gamma\dot{u}'''' + \delta\dot{u} = 0 . \tag{11.7}$$

Solving this equation under clamped end conditions leads to the nonlinear eigenvalue problem

$$\begin{gathered} \lambda^2 + \lambda(\gamma u'''' + \delta u) + \alpha u'''' - \beta u'' = 0 \\ u = u' = 0 \text{ at } x = 0, \ell . \end{gathered} \tag{11.8}$$

The corresponding problems for (1.1) are effectively linear. Some information about (11.8) would presumably be required in any proof that an exponential decomposition holds for (11.7). Were we to have such a proof then the saddle point analysis of section 2 would give local stability results for (11.6). It may be possible to tackle (11.8) by extending the work of Eisenfeld [15]. The absence of an obvious 'a priori' estimate for (11.6) suggests that the corresponding dynamical systems are not reversible. Thus global stability results would be harder to obtain than for (1.1). Quasi-reversibility techniques might be useful in this regard (see the conference proceedings [23] for references).

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