EXISTENCE OF SOLUTIONS IN FINITE ELASTICITY

J. M. BALL

Department of Mathematics, Heriot-Watt University, Riccarton, Currie, Edinburgh EH14 4AS, Scotland

(Received September 25, 1980)

Very few exact solutions are known to static and dynamic problems of finite elasticity, particularly in the case when the material is compressible. General theorems on existence of solutions provide reassurance that the theory is mathematically sound; for example it is important to understand whether or not solutions of the basic equations have singularities consistent with assumptions used in deriving the equations. But there are several other, equally important, reasons for studying questions of existence of solutions. One such is the establishment of convergence properties for numerical methods in elasticity (in this connection it should be noted that finite-difference schemes for certain partial differential equations may converge to solutions of *different* equations). Experience with other partial differential equations has also taught us that existence theorems are an essential prerequisite for the study of various qualitative properties of solutions (for example, bifurcation, stability and asymptotic behaviour). In a broader context, we today face problems in elasticity similar to unsolved questions in other branches of mechanics and physics, and the unifying nature of the theory of partial differential equations can thus lead us to hope, as has been the case in the past, that advances in elasticity will lead to corresponding progress in other fields. Here, however, we concentrate on a more specific reason for proving, or attempting to prove, existence theorems in elasticity, namely that it leads to information concerning the relationship between constitutive hypotheses (i.e. assumptions on the stored-energy function, or stress–strain law of the material) and smoothness properties of solutions. We will, as much as possible, avoid introducing

Proceedings of the IUTAM Symposium on Finite Elasticity, Lehigh University, August 10–15, 1980. General lecture.

Carlson, D.E. and Shield, R.T. (eds.)
Proceedings of the IUTAM Symposium on Finite Elasticity
ISBN 90 247 2629 8

technicalities from analysis, referring for these to various articles which it is the main purpose of this paper to summarize.

We confine attention mainly to static problems. In general the problem of proving the existence and uniqueness of solutions for initial-boundary value problems of elastodynamics is much harder, and the theory is far from complete even in one dimension. Part of the difficulty in the dynamic case is that solutions which are initially smooth may develop shocks (cf. MacCamy and Mizel [1], Klainerman and Majda [2], John [3]). Short-time existence of smooth solutions for smooth initial data for the case of an elastic body occupying all space has been established by Hughes, Kato and Marsden [4] under a strong ellipticity assumption (see (13) below) on the stored-energy function, but little, if anything, is known about global existence of weak solutions in three dimensions. For the static case we consider only conservative problems, which can be discussed within the framework of the calculus of variations. We do not discuss local existence theorems (cf. Stoppelli [5], van Buren [6], Marsden [7], and Valent [8]) in which solutions nearby a known solution are proved to exist when the data for the known solution is changed slightly. The applicability of such local theorems is severely restricted because there are few trivial solutions around which to perturb, and also by the fact that so far the methods have not been applied successfully to mixed boundary conditions.

We begin by giving two one-dimensional examples in which no absolute minimum for the total energy exists. The total energy $I(x)$ is given by

$$I(x)=\int_0^1 [W(X,x'(X))+\Phi(X,x(X))]dX, \tag{1}$$

where $x(X)$ denotes the deformed position of a particle with position X, $0\leqq X\leqq 1$, in a reference configuration, $W(X,p)$ is the stored-energy function, $\Phi(X,x)$ the body-force potential, and where $x'(X)=dx(X)/dX$. [Throughout this article we consider deformations $x(\cdot)$ whose derivatives may possess discontinuities. All derivatives of $x(\cdot)$ are to be understood in the sense of distributions (cf. Adams [9]), but while this technicality is important for the statements of the results to be correct, it can be safely ignored at a first reading.] Consider the problem of minimizing (1) subject to the boundary conditions

$$x(0)=0, \qquad x(1)=\alpha>0. \tag{2}$$

Formally, a minimizing function $x(\cdot)$ for (1) subject to (2) will satisfy the Euler–Lagrange, or equilibrium equation,

$$\frac{d}{dX}W_p(X,x'(X))=\Phi_x(X,x(X)), \qquad 0<X<1, \tag{3}$$

where subscripts denote partial derivatives. $W_p(X,\cdot)$ is the stress–strain function of the material at X.

EXAMPLE 1 (cf. [10]).

$$W(X,p)=\frac{(p-1)^2(p-2)^2}{p}+\frac{3}{2}X^2p,$$

$$\Phi(X,x)=x^2-\frac{3}{2}, \qquad \alpha=\frac{3}{2}.$$

In this case, an integration by parts shows that

$$I(x)=\int_0^1\left[\frac{(x'-1)^2(x'-2)^2}{x'}+\left(x-\frac{3X}{2}\right)^2\right]dX. \tag{4}$$

We are interested only in deformations $x(\cdot)$ for which $x'(X)>0$, so that interpenetration of matter does not occur and the orientation of the specimen is preserved; the behaviour $W(X,p)\to\infty$ as $p\to 0+$, corresponding to the requirement that infinite energy is required to compress a finite length of the specimen to zero length, will hopefully ensure this. For such functions $x(\cdot)$ the integrand in (4) is strictly positive, and so $I(x)>0$. But $I(x)$ may be made arbitrarily small by choosing for x a zig-zag line with slopes alternately 1 and 2, approximating closely the line $x=3X/2$. In fact, by rounding off the corners of such a zig-zag line one can show that *smooth* functions $x(\cdot)$ satisfying (2) exist which make I arbitrarily small. Thus *I does not attain its minimum* subject to (2). Examples of this type are classical in the calculus of variations, seem first to have been discussed by Weierstrass, and are described in the books of Bolza [11] and Young [12]. They occur whenever $W(X,p)$ is not a strictly convex function of p, that is, whenever stress does not increase with strain. When $W(X,p)\to\infty$ as $p\to 0+$ the construction of such examples is possible only when either W or Φ depends explicitly on X (cf. [10]). Although a minimizer for I does not exist, one can think of there being a 'generalized function' (cf. Young [12]), which takes its place. This object consists of the line $x=\frac{3}{2}X$ with a superimposed 'infinitesimal zig-zag' of slopes 1 and 2.

EXAMPLE 2.

$$I(x)=\int_0^1\left[\frac{1}{x'}+x'+x\right]dX. \tag{5}$$

In this case the strictly convex function $W(x')$ behaves linearly for large x'.

We can think of the problem of minimizing I as corresponding to a vertical bar acted on by gravity, whose lower end $X = 0$ is fixed, and whose upper end is raised to a height $x = \alpha$. Elementary calculations show that if $\alpha \leqq 2$ there is a unique minimizer $x_\alpha(\cdot)$ for I which is smooth for $\alpha > 2$ ($x_2'(\cdot)$ has a singularity at $X = 1$). But if $\alpha > 2$ no minimizer exists. However, in this case minimizing sequences for I tend to a 'generalized function' which consists of $x_2(X)$ for $0 \leqq X < 2$ and the vertical line $\{X = 1, 2 \leqq x \leqq \alpha\}$. (W. Noll has remarked to me that an exactly analogous calculation holds for isentropic equilibrium of a polytropic gas under gravity, this model predicting a finite height for the atmosphere.)

In both Examples 1 and 2, no minimizer that is a bona fide function exists, though in each case there is a corresponding 'generalized minimizer'. By assuming that $W(X, \cdot)$ is strictly convex and that $W(X, p)/p \to \infty$ as $p \to \infty$ one can prove the existence of a bona fide minimizer and show that under certain conditions it is smooth. For the details and precise hypotheses of such results see Antman [13] and the references therein, and [10]. Antman treats more general models of rods. For studies of one-dimensional elasticity and viscoelasticity when $W(X, \cdot)$ is not convex (as in Example 1) see Ericksen [14], James [15, 16], Slemrod [17], Andrews [18], and Andrews and Ball [19].

Turning now to three-dimensional elastostatics, we make the simplifying assumptions, for ease of exposition, that the material is homogeneous and that there is no body force.

The problem we consider is to minimize

$$I(x) = \int_\Omega W(\nabla x(X))dX \tag{6}$$

subject to

$$x(X) = \bar{x}(X) \qquad \text{for } X \in \partial\Omega_1, \tag{7}$$

where $\bar{x}$ is a given function. In (6), (7), $x(X)$ denotes the deformed position of a particle occupying the position X in a reference configuration Ω, which we assume to be a bounded, open subset of three-dimensional space with sufficiently smooth boundary $\partial\Omega$. We suppose that $\partial\Omega$ is composed of two disjoint portions $\partial\Omega_1$ and $\partial\Omega_2$, where $\partial\Omega_1$ has positive surface area. $W(F)$ is the stored-energy function of the material. If $x(\cdot)$ minimizes (6) subject to (7) then, formally,

$$\frac{d}{d\varepsilon} I(x + \varepsilon v)\Big|_{\varepsilon=0} = 0$$

for any smooth function v which vanishes on $\partial\Omega_1$; that is

$$\int_\Omega \frac{\partial W}{\partial F^i_\alpha}(\nabla x(X)) \frac{\partial v^i}{\partial X^\alpha}(X)dX = 0. \tag{8}$$

If (8) holds for all such v we say that x is a *weak solution* of the mixed boundary value problem in which the displacement of $\partial\Omega_1$ is specified by (7) and $\partial\Omega_2$ is traction-free. The reason for this terminology is that if x is a smooth weak solution then integrating (8) by parts and using the arbitrariness of v leads to the equilibrium equations

$$\frac{\partial}{\partial X^\alpha}\frac{\partial W}{\partial F^i_\alpha}(\nabla x(X)) = 0, \qquad X \in \Omega, \tag{9}$$

and the natural boundary condition

$$\frac{\partial W}{\partial F^i_\alpha} N_\alpha = 0, \qquad X \in \partial\Omega_2, \tag{10}$$

where $N = N(X)$ denotes the outward normal to $\partial\Omega$ at X. Let $M^{3\times3}$ denote the set of real 3×3 matrices, and $M^{3\times3}_+$ consist of those $F \in M^{3\times3}$ with $\det F > 0$. We introduce the following hypotheses on W:

(H1) $W: M^{3\times3}_+ \to \mathbb{R}$ is *polyconvex*, i.e. there exists a convex function $g: M^{3\times3} \times M^{3\times3} \times (0,\infty) \to \mathbb{R}$ such that

$$W(F) = g(F, \operatorname{adj} F, \det F) \qquad \text{for all } F \in M^{3\times3}_+,$$

where $\operatorname{adj} F$ denotes the (transposed) matrix of cofactors of F.

(H2) There exist constants $C > 0$, $p \geqq 2$, $q \geqq p/(p-1)$, $r > 1$, $s > 0$ such that

$$W(F) \geqq C(1 + |F|^p + |\operatorname{adj} F|^q + (\det F)^r + (\det F)^{-s})$$

for all $F \in M^{3\times3}_+$.

THEOREM 1. *Let* (H1), (H2) *hold, and suppose that*

$$\mathscr{A} \overset{\text{def}}{=} \{x: \bar{\Omega} \to \mathbb{R}^3 : \det \nabla x(X) > 0 \text{ a.e.}, I(x) < \infty, x_{|\partial\Omega_1} = \bar{x}_{|\partial\Omega_1}\}$$

is nonempty. Then I attains its minimum on $\mathscr{A}$.

(The abbreviation 'a.e.' stands for 'almost everywhere,' meaning 'except possibly on a set of zero volume.')

The theorem is a slight refinement due to Ball, Currie and Olver [20] of a result proved in [21, 22]. These papers can be consulted for a slightly more precise definition of $\mathscr{A}$ and for the detailed proofs. Before discussing the meaning of (H1) and (H2), we show that, at least, these hypotheses are satisfied by some reasonable models of natural rubber. To this end we consider an isotropic material, i.e. one for which

$$W(F)=\Phi(v_1,v_2,v_3)$$

is a symmetric function Φ of the principal stretches $v_i>0$. We suppose that

$$\Phi=\sum_{i=1}^{M}a_i(v_1^{\alpha_i}+v_2^{\alpha_i}+v_3^{\alpha_i}-3)$$
$$+\sum_{j=1}^{N}b_j((v_2v_3)^{\beta_j}+(v_3v_1)^{\beta_j}+(v_1v_2)^{\beta_j}-3)+h(v_1v_2v_3).$$

Then (H1), (H2) are satisfied (cf. [21, 22]) if

$$\alpha_1>\cdots>\alpha_M\geqq 1,\quad \beta_1>\cdots>\beta_N\geqq 1,\quad a_i>0,\quad b_i>0,\quad \alpha_1\geqq 2,$$

$$\beta_1\geqq\frac{\alpha_1}{\alpha_1-1},\quad h:(0,\infty)\to\mathbb{R}\text{ is convex},$$

$$h(\delta)\geqq k(\delta^r+\delta^{-s}),\qquad\text{for all }\delta>0,$$

and

$$k>0,\quad r>1,\quad s>0.$$

This class of stored-energy functions is a slight modification of a class introduced by Ogden [23]. We remark that the verification of (H1), and other such convexity conditions, is somewhat simpler when $W(F)$ is expressed in terms of principal stretches than when expressed in terms of the principal invariants of $B=FF^T$.

For incompressible materials a corresponding version of Theorem 1 holds (cf. Ball [21, 22]) and is satisfied by the above material with $h\equiv 0$. Thus we obtain existence for a Mooney–Rivlin material ($M=N=1$, $\alpha_1=\beta_1=2$), but for a single term stored-energy function

$$\Phi=\mu(v_1^\alpha+v_2^\alpha+v_3^\alpha-3),$$

$\mu>0$, we obtain existence only if $\alpha\geqq 3$. The Neo-Hookean material ($\alpha=2$) is not covered by the theorem.

The polyconvexity condition (H1) is a kind of convexity condition, but it *does not imply that* $W(F)$ *is convex* (as the example $W(F)=\det F$ shows). It is well known that convexity of $W(F)$ is unacceptable physically. There is no direct physical interpretation of polyconvexity known to me, but it is a natural mathematical condition, and, as we shall see below, implies other inequalities having an interpretation in terms of material stability. The mathematical significance of the arguments of the function g in the definition of polyconvexity can be described in several equivalent ways, one of which is the following.

PROPOSITION. *The Euler–Lagrange equations for*

$$\int_\Omega \psi(\nabla x(X))dX$$

are satisfied identically for all smooth functions $x(X)$ *if and only if*

$$\psi(F)=A+A_i^\alpha F_\alpha^i+B_\alpha^i(\operatorname{adj}F)_i^\alpha+C\det F,$$

where the coefficients are constants.

The earliest proof of (a more general version of) this proposition known to me is that of Landers [24]. For other references and equivalent properties of ψ see [20–22].

To describe the relationship of polyconvexity to other inequalities we make the definitions

DEFINITION 1 (Morrey [25]). W is *quasiconvex* if

$$\int_D W(F+\nabla\phi(Y))dY\geqq\int_D W(F)dY=(\operatorname{vol}D)\times W(F)$$

for all bounded, open sets $D\subset\mathbb{R}^3$, all $F\in M_+^{3\times 3}$, and all smooth functions ϕ vanishing on the boundary of D and such that $F+\nabla\phi(Y)\in M_+^{3\times 3}$ for all $Y\in D$.

DEFINITION 2. W is *rank* 1 *convex* if

$$W(tF+(1-t)G)\leqq tW(F)+(1-t)W(G)\tag{11}$$

whenever $0<t<1$ and $F-G$ has rank 1 (i.e. $F-G=\lambda\otimes\mu\neq 0$; where $(\lambda\otimes\mu)_\alpha^i=\lambda^i\mu_\alpha$ and $a,b\in\mathbb{R}^3$). W is *strictly rank* 1 *convex* if the inequality in (11) is strict.

(Note that the properties of the determinant imply that if $F,G\in M_+^{3\times 3}$ with $\operatorname{rank}(F-G)=1$ then $tF+(1-t)G\in M_+^{3\times 3}$ whenever $0\leqq t\leqq 1$, so that $W(tF+(1-t)G)$ is defined.)

By differentiating twice with respect to t it is easily shown that when W is twice continuously differentiable rank 1 convexity is equivalent to the *Legendre–Hadamard condition*

$$\frac{\partial^2 W}{\partial F_\alpha^i\partial F_\beta^j}(F)\lambda^i\mu_\alpha\lambda^j\mu_\beta\geqq 0\qquad\text{for all }\lambda,\mu\in\mathbb{R}^3,\tag{12}$$

and that the *strong-ellipticity condition*

$$(13) \qquad \frac{\partial^2 W}{\partial F^i_\alpha \partial F^j_\beta}(F)\lambda^i\mu_\alpha\lambda^j\mu_\beta > 0 \qquad \text{whenever } \lambda, \mu \in \mathbb{R}^3 \text{ are nonzero,}$$

implies that W is strictly rank 1 convex.

We can now state the implications

$$W \text{ polyconvex} \Rightarrow W \text{ quasiconvex} \Rightarrow W \text{ rank 1 convex.}$$

The proofs of these implications and further discussion can be found in [20–22].

To understand the relevance of matrices of rank 1 in elasticity, consider a plane in the reference configuration with normal μ. Then a function x exists such that $\nabla x = F =$ const. above the plane, $\nabla x = G =$ const. below the plane, and x is continuous across the plane if and only if $F - G = \lambda \otimes \mu$ for some $\lambda \in \mathbb{R}^3$. Furthermore, such a function x is a weak equilibrium solution if and only if

$$\frac{\partial W}{\partial F^i_\alpha}(F)\mu_\alpha = \frac{\partial W}{\partial F^i_\alpha}(G)\mu_\alpha, \qquad i = 1, 2, 3,$$

which expresses the balance of forces acting on the plane. Suppose now that there exists a natural state $F_0 \in M^{3\times3}_+$, so that

$$W(F) \geqq W(F_0) \qquad \text{for all } F \in M^{3\times3}_+.$$

Then we have the following result.

THEOREM 2 ([26]). *Let W be continuously differentiable. Then W is strictly rank* 1 *convex if and only if all weak equilibrium solutions of the above type are trivial* (*i.e.* $F = G$).

Thus strict rank 1 convexity is *necessary* for the smoothness of all weak solutions. Nontrivial piecewise affine weak solutions can occur in the twinning of elastic crystals; and, as has been discussed by Ericksen [27], the stored-energy functions for such crystals are not rank 1 convex. For other work on finite elasticity where rank 1 convexity is not assumed see Knowles and Sternberg [28].

A natural question is now whether strict rank 1 convexity is sufficient for the smoothness of weak equilibrium solutions (presupposing, of course, that W and the other data in the problem are smooth). The answer is negative, and this brings us to a discussion of the growth hypothesis (H2). Note first that (H2) implies that $W(F) \to \infty$ as $\det F \to 0+$. (We could equally well suppose that $W(F) \to \infty$ as $\det F \to b > 0$, and a modified version of Theorem 1 holds.) If F is confined to a bounded region of $M^{3\times3}_+$ having positive distance from the surface $\{\det F = 0\}$, then by adding a suitable constant to W we see that the

inequality in (H2) imposes no restriction on W at all. Indeed if we add to the properties of the set $\mathcal{A}$ of admissible deformations the constraints

$$0 < k \leqq \det \nabla x(X), \qquad \bar{W}(\nabla x(X)) \leqq l, \qquad \text{a.e.,}$$

where k, l are constants and $\bar{W}$ is polyconvex, then Theorem 1 holds without assuming (H2) and with $W(F)$ defined only for F satisfying the above constraints. (However, in this case a minimizer x cannot be expected to be an equilibrium solution since in general, for some values of X, $\nabla x(X)$ will lie on the boundary of the constraint set; i.e. $\det \nabla x(X) = k$ or $\bar{W}(\nabla x(X)) = l$.) Thus the growth condition (H2) restricts W only for arbitrarily small $\det F$ or arbitrarily large $|F|$, whereas for real materials one expects such F to be outside the range where elasticity is a good model. Nevertheless the behaviour of $W(F)$ for small $\det F$ and large $|F|$ has important consequences for the existence and smoothness of solutions within the context of pure elasticity theory; we have already seen that this is the case in one dimension. Quantitative estimates in terms of the growth behaviour of W and the magnitude of the boundary data of, for example, the size of the set of points X where $\nabla x(X)$ lies outside a given range might lead to a better understanding of this situation.

Note that (H1) allows, but (H2) excludes, the case of an elastic fluid, for which $W(F) = h(\det F)$; in this case W is rank 1 convex if and only if h is convex. A study of the equilibrium problem for elastic fluids, formulated in material coordinates, and concentrating on the case when h is not convex, can be found in Dacorogna [30].

A sufficient condition for (H2) to hold is that

$$(14) \qquad W(F) \geqq C_1(1 + |F|^p + (\det F)^{-s}) \qquad \text{for all } F \in M^{3\times3}_+$$

where $C_1 > 0$, $p > 3$ and $s > 0$; this follows from the facts that $\operatorname{adj} F$ and $\det F$ are quadratic and cubic functions of F respectively. A simple calculation (cf. [22, 29]) shows that a cube of infinitesimally small side ε can be deformed by means of a homogeneous deformation into a parallelepiped having a given finite diameter, with total stored-energy bounded above independently of ε, if and only if

$$(15) \qquad \frac{W(F)}{|F|^3} \not\to \infty \qquad \text{as } |F| \to \infty.$$

(Note that (14) and (15) cannot hold simultaneously.) When (15) holds the above calculation suggests the possibility of the existence of weak solutions in which voids or cavities form where none is present in the reference configuration. Such weak solutions exist, and a detailed study of some of them in both the case of compressible and incompressible materials will appear in [31]. Here we give an informal argument exhibiting such solutions in the incompressible case.

We consider radial deformations

$$x(X) = \frac{r(R)}{R} X, \qquad |X| = R, \quad |x| = r$$

of an incompressible, isotropic material which occupies in the reference configuration the unit ball $\{|X| < 1\}$. We suppose the boundary of the deformed body to be subjected to uniform radially outward dead load tractions of constant magnitude P. The only kinematically possible deformations have the form

(16) $$r^3 = R^3 + A^3,$$

A representing the radius of the cavity formed in the interior of the body. The total energy I can be expressed in terms of A using (16). Thus

(17) $$E(A) \stackrel{\text{def}}{=} \frac{I(A)}{4\pi} = \int_0^1 R^2 \Phi\left(r', \frac{r}{R}, \frac{r}{R}\right) dR - P(1 + A^3)^{1/3},$$

where the last term represents the potential energy due to the surface tractions. Substituting $v = r/R$ we obtain

$$E(A) = A^3 \int_{(1+A^3)^{1/3}}^{\infty} \frac{v^2}{(v^3 - 1)^2} \Phi(v^{-2}, v, v) dv - P(1 + A^3)^{1/3},$$

and thus

(18) $$E'(A) = A^2 \left[\int_{(1+A^3)^{1/3}}^{\infty} \frac{1}{v^3 - 1} \frac{d\Phi}{dv} dv - \frac{P}{(1 + A^3)^{2/3}} \right].$$

The condition for equilibrium is that $E'(A) = 0$. Thus there are two possibilities; either $A = 0$ and $r = R$, or

(19) $$P = (1 + A^3)^{2/3} \int_{(1+A^3)^{1/3}}^{\infty} \frac{1}{v^3 - 1} \frac{d\Phi}{dv} dv.$$

The integrals in (17), (18) may or may not converge, depending on the growth properties of Φ. In the special case

(20) $$\Phi(v_1, v_2, v_3) = \mu(v_1^\alpha + v_2^\alpha + v_3^\alpha - 3),$$

where $\mu > 0$, these integrals converge provided $-\frac{3}{2} < \alpha < 3$. Note that by (19) the solution with the cavity bifurcates from the trivial solution at the critical traction

(21) $$P_{\text{crit}} = \int_1^{\infty} \frac{1}{v^3 - 1} \frac{d\Phi}{dv} dv.$$

In the case of the Neo-Hookean material, given by (20) with $\alpha = 2$, we obtain $P_{\text{crit}} = 5\mu$ and in this case it is not hard to show that the solution with the cavity minimizes $I(A)$ and is stable, while the trivial solution $r = R$ is unstable for $P > P_{\text{crit}}$. The calculation leading to (21) gives the same result as that of Gent and Lindley [32] in their study of internal rupture of rubber under tension. The reason for this is discussed in [31].

We next turn to the question of whether the energy minimizers whose existence are guaranteed by Theorem 1 are invertible, so that interpenetration of matter does not occur. The following result for a pure displacement boundary-value problem is proved in [33].

THEOREM 3. *Let $\partial\Omega_2$ be empty. Let* (H1), (H2) *hold and suppose further that $p > 3$, $q > 3$, $s > 2q/(q-3)$. Let $\bar{x} \in \mathcal{A}$ be one-to-one in Ω and suppose that $\bar{x}(\Omega)$ satisfies the cone condition. Then the minimizer x is a homeomorphism (continuous with a continuous inverse) of Ω onto $\bar{x}(\Omega)$.*

The cone condition is a mild restriction on the irregularity of the boundary of $\bar{x}(\Omega)$ (cf. [32] for details).

We conclude with a list of three open problems related to the subject matter discussed here.

1. Are the minimizers whose existence are established in Theorem 1 weak solutions of the equilibrium equations?
2. Are they smooth (provided W and the boundary data are smooth)?
3. Does a version of Theorem 1 hold with a weakened version of (H2) allowing for the possibility of cavitation?

REFERENCES

1. R. C. MacCamy and V. J. Mizel, *Existence and non-existence in the large of solutions of quasilinear wave equations*, Arch. Rat. Mech. Anal. **25** (1967), 299–320.
2. S. Klainerman and A. Majda, *Formation of singularities for wave equations including the nonlinear vibrating string*, Comm. Pure and Appl. Math. **33** (1980), 241–263.
3. F. John, article in this volume.
4. T. J. R. Hughes, T. Kato and J. E. Marsden, *Well-posed quasi-linear second order hyperbolic systems with applications to nonlinear elasticity and general relativity*, Arch. Rat. Mech. Anal. **63** (1977), 273–294.
5. F. Stoppelli, *Un teorema di esistenza e di unicità relativo allé equazioni dell' elastostatica isoterma per deformazioni finite*, Ricerche Matematica **3** (1954), 247–267.
6. W. van Buren, *On the existence and uniqueness of solutions to boundary value problems in finite elasticity*, Thesis, Department of Mathematics, Carnegie–Mellon University, 1968. Research Report 68-ID7-MEKMA-RI, Westinghouse Research Laboratories, Pittsburgh, Pa. 1968.
7. J. E. Marsden and T. J. R. Hughes, *Topics in the mathematical foundations of elasticity*, in *Nonlinear Analysis and Mechanics*, Heriot–Watt Symposium, Vol. II (R. J. Knops, ed.), Pitman, London, 1978.
8. T. Valent, article in this volume.

9. R. A. Adams, *Sobolev Spaces*, Academic Press, New York, 1975.

10. J. M. Ball, *Remarques sur l'existence et la régularité des solutions d'élastostatique nonlinéaire*, in *Recent contributions to nonlinear partial differential equations* (H. Berestycki and H. Brezis, ed.), Pitman, Boston, 1981.

11. O. Bolza, *Lectures on the Calculus of Variations*, 3rd ed., reprinted Chelsea, New York, 1973.

12. L. C. Young, *Lectures on the Calculus of Variations and Optimal Control Theory*, W. B. Saunders, Philadelphia, 1969.

13. S. S. Antman, *Ordinary differential equations of nonlinear elasticity II: Existence and regularity theory for conservative boundary value problems*, Arch. Rat. Mech. Anal. **61** (1976), 353–393.

14. J. L. Ericksen, *Equilibrium of bars*, J. of Elasticity **5** (1975), 191–201.

15. R. D. James, *Co-existent phases in the one-dimensional static theory of elastic bars*, Arch. Rat. Mech. Anal. **72** (1979), 99–140.

16. R. D. James, *The propagation of phase boundaries in elastic bars*, Arch. Rat. Mech. Anal. **73** (1980), 125–158.

17. G. Andrews, *On the existence of solutions to the equation* $u_{tt} = u_{xxt} + \sigma(u_x)_x$, J. Differential Equations **35** (1980), 200–231.

18. M. Slemrod, *Inadmissibility of propagating phase boundaries according to certain viscosity criteria*, preprint.

19. G. Andrews and J. M. Ball, *Asymptotic behaviour and changes of phase in one-dimensional nonlinear viscoelasticity*, J. Differential Equations, to appear.

20. J. M. Ball, J. C. Currie and P. J. Olver, *Null Lagrangians, weak continuity, and variational problems of arbitrary order*, J. Functional Analysis, **41** (1981), 135–174.

21. J. M. Ball. *Convexity conditions and existence theorems in nonlinear elasticity*, Arch. Rat. Mech. Anal. **63** (1977), 337–403.

22. J. M. Ball, *Constitutive inequalities and existence theorems in nonlinear elastostatics*, in *Nonlinear Analysis and Mechanics*, Heriot–Watt Symposium Vol. I (R. J. Knops, ed.), Pitman, London, 1977.

23. R. W. Ogden, *Large deformation isotropic elasticity: On the correlation of theory and experiment for compressible rubberlike solids*, Proc. Roy. Soc. London **A328** (1972), 567–583.

24. A. W. Landers, *Invariant multiple integrals in the calculus of variations*, in *Contributions to the Calculus of Variations, 1938–1941*, Univ. Chicago Press, Chicago, 1942, pp. 184–189.

25. C. B. Morrey, *Quasi-convexity and the lower semicontinuity of multiple integrals*, Pacific J. Math. **2** (1952), 25–53.

26. J. M. Ball, *Strict convexity, strong ellipticity, and regularity in the calculus of variations*, Math. Proc. Camb. Phil. Soc. **87** (1980), 501–513.

27. J. L. Ericksen, *Special topics in elastostatics*, in *Advances in Applied Mechanics*, Vol. 17, Academic Press, New York, 1977.

28. J. K. Knowles and E. Sternberg, *On the failure of ellipticity of the equations for finite elastic plane strain*, Arch. Rat. Mech. Anal. **63** (1977), 321–326.

29. J. M. Ball, *Finite time blow-up in nonlinear problems*, in *Nonlinear Evolution Equations* (M. G. Crandall, ed.), Academic Press, New York, 1978.

30. B. Dacorogna, *A relaxation theorem and its application to the equilibrium of gases*, Arch. Rat. Mech. Anal., to appear.

31. J. M. Ball, *Discontinuous equilibrium solutions and cavitation in nonlinear elasticity*, to appear.

32. A. N. Gent and P. B. Lindley, *Internal rupture of bonded rubber cylinders in tension*, Proc. Roy. Soc. London, **A249** (1958), 195–205.

33. J. M. Ball, *Global invertibility of Sobolev functions and the interpenetration of matter*, Proc. Royal Soc. Edinburgh **88A** (1981), 315–328.