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Energy-Minimizing Configurations in Nonlinear Elasticity

We discuss applications of the calculus of variations to nonlinear elasticity, and certain related issues. We confine attention to problems in $n > 1$ space dimensions. (A comprehensive account of one-dimensional problems has been given by Antman [3].)

Consider an elastic body occupying in a reference configuration a bounded domain $\Omega \subset \boldsymbol{R}^n$. We assume for ease of exposition that the body is *homogeneous*; i.e. it is composed of the same material at each point $x \in \Omega$. In a typical deformation X: $\Omega \to \boldsymbol{R}^n$ the *total stored-energy* of the body is given by the functional

$$E(x) = \int_{\Omega} W\big(x\nabla(X)\big)\,dX \tag{1}$$

where W denotes the *stored-energy function* of the material. Let $M^{n\times n}$ denote the space of real $n \times n$ matrices. We suppose that W: $M^{n\times n} \to \overline{\boldsymbol{R}}$ is continuous and bounded below, and that $W(A) = \infty$ if and only if $\det A \leqslant 0$. (The last requirement is imposed with the intention of making it energetically impossible to compress part of the body to zero volume or to change its orientation.) We suppose that the body is subjected to external body forces with potential $\Psi(X, x)$ per unit volume and for simplicity we consider the case when Ψ: $\boldsymbol{R}^n \times \boldsymbol{R}^n \to \boldsymbol{R}$ is continuous and bounded below. We consider a mixed displacement zero traction boundary value problem, in which it is required that

$$x(X) = \bar{x}(X), \qquad \text{a.e. } X \in \partial\Omega_1, \tag{2}$$

while the remainder $x(\partial\Omega \setminus \partial\Omega_1)$ of the boundary is traction-free. In (2) $\partial\Omega_1$ denotes a measurable subset of the boundary $\partial\Omega$ of Ω (which we assume to be strongly Lipschitz) with positive (n −1)-dimensional measure, and $\bar{x}$: $\partial\Omega_1 \to \mathbf{R}^n$ is a given measurable function. (More general conservative boundary problems are considered in Ball [4].) We define the total energy functional $I(x)$ by

$$I(x) \stackrel{\text{def}}{=} E(x) + \int_\Omega \Psi\big(X, x(X)\big)\,dX. \tag{3}$$

Corresponding to (2) we consider the set

$$\mathscr{A} = \{x \in W^{1,1}(\Omega; \mathbf{R}^n)\colon x \text{ satisfies (2)}, \ I(x) < \infty\} \tag{4}$$

and pose the following problem.

Problem. Does $I(x)$ attain an absolute minimum on $\mathscr{A}$?

In the case when $\partial\Omega_1 = \partial\Omega$ (pure displacement problem) a necessary condition that I attains an absolute minimum for every smooth Ψ and $\bar{x}$ is that W be $W^{1,1}$-*quasiconvex* (Ball and Murat [9]), i.e.

$$\int_D W\big(A + \nabla\varphi(X)\big)\,dX \geqslant (\operatorname{meas} D)\,W(A) \tag{5}$$

for any bounded open subset $D \subset \mathbf{R}^n$ with meas $\partial D = 0$, for all $A \in M^{n\times n}$ and for all $\varphi \in W_0^{1,1}(D; \mathbf{R}^n)$. The weaker condition that (5) hold for all $\varphi \in W_0^{1,\infty}(D; \mathbf{R}^n)$ was introduced by Morrey [16] and termed by him *quasiconvexity*; it implies in particular that for $W \in C^2(M_+^{n\times n})$ the *Legendre–Hadamard* (or *ellipticity*) *condition*

$$\frac{\partial^2 W(A)}{\partial A^i_\alpha \partial A^j_\beta}\, a^i b_\alpha a^j b_\beta \geqslant 0 \tag{6}$$

for all $A \in M_+^{n\times n}$ and all $a, b \in \mathbf{R}^n$ holds. $W^{1,1}$-quasiconvexity is also a necessary condition for sequential weak lower semicontinuity of $E(\cdot)$ in $W^{1,1}(\Omega; \mathbf{R}^n)$.

An example of a set of sufficient conditions ensuring that I attains its minimum is given by the following result. Of course the hypotheses imply that W is $W^{1,1}$-quasiconvex.

Theorem 1. *Let $n = 3$. Suppose that W is polyconvex, i.e. there exists a convex function g: $M^{3\times 3} \times M^{3\times 3} \times (0, \infty) \to \mathbf{R}$ such that*

$$W(A) = g(A, \operatorname{adj} A, \det A) \quad \textit{for all } A \in M_+^{3\times 3}, \tag{7}$$

where $M_+^{3\times 3} = \{A \in M^{3\times 3}\colon \det A > 0\}$. Suppose further that

$$W(A) \geqslant C + D(|A|^p + |\operatorname{adj} A|^q) \quad \textit{for all } A \in M_+^{3\times 3}, \tag{8}$$

where $D > 0$, C are constants, $p \geqslant 2$ and $q \geqslant p/(p-1)$. Then if $\mathscr{A}$ is nonempty I attains its absolute minimum on $\mathscr{A}$, and the minimiser x satisfies $\det \nabla x(x) > 0$ *a.e.* $X \in \Omega$.

Theorem 1 is proved in [9]; it is a slight refinement of earlier results of Ball [4, 5] and Ball, Currie and Olver [8]. The proof uses the direct method of the calculus of variations, the weak continuity properties of Jacobians (Reshetnyak [18, 20], Ball [4], Ball, Currie and Olver [8]), and an idea of Reshetnyak [19]. For some related semicontinuity theorems see Acerbi and Fusco [2] and Acerbi, Buttazzo and Fusco [1]. For pure displacement boundary value problems with appropriate boundary data it can be shown (Ball [7]) that under stronger growth conditions on W the minimiser x is a homeomorphism, so that interpenetration of matter does not occur. An analogous version of Theorem 1 holds for incompressible materials, all deformations of which satisfy the pointwise constraint $\det \nabla x(X) = 1$ a.e. $X \in \Omega$.

The stored-energy function W is said to be *isotropic* if

$$W(A) = \Phi(v_1, v_2, v_3) \quad \textit{for all } A \in M_+^{3\times 3} \tag{9}$$

for some symmetric function Φ of the principal stretches $v_i = v_i(A)$, that is the eigenvalues of $(A^T A)^{1/2}$. Following essentially the work of Ogden [17] on stored-energy functions appropriate for natural rubbers we consider the case

$$\Phi(v_1, v_2, v_3) = \sum_{i=1}^{M} c_i(v_1^{\alpha_i} + v_2^{\alpha_i} + v_3^{\alpha_i} - 3) + \sum_{j=1}^{N} d_j\big((v_2 v_3)^{\beta_j} + (v_3 v_1)^{\beta_j} + (v_1 v_2)^{\beta_j} - 3\big) + h(v_1 v_2 v_3), \tag{10}$$

where $M \geqslant 1$, $N \geqslant 1$, $c_i > 0$, $d_j > 0$, $\alpha_1 \geqslant \ldots \geqslant \alpha_M \geqslant 1$, $\beta_1 \geqslant \ldots \geqslant \beta_N \geqslant 1$ and h is convex, bounded below, with $\lim_{t\to 0+} h(t) = \infty$. Note that $v_1 v_2 v_3 = \det A$. Then the hypotheses of Theorem 1 hold provided $\alpha_1 \geqslant 2$ and $\beta_1 \geqslant \alpha_1/(\alpha_1 - 1)$; a special case is the Mooney–Rivlin material, for which $\alpha_1 = \beta_1 = 2$. For the function

$$\Phi(v_1, v_2, v_3) = c(v_1^\alpha + v_2^\alpha + v_3^\alpha - 3) + h(v_1 v_2 v_3), \tag{11}$$

with $c > 0$ and h as above, the hypotheses of Theorem 1 hold provided $\alpha \geqslant 3$.

A modification of the Saint Venant–Kirchhoff constitutive law satisfying (7), (8) has been proposed by Ciarlet and Geymonat [10].

There are physically interesting stored-energy functions W which are not $W^{1,1}$-quasiconvex and in particular do not satisfy the hypotheses of Theorem 1. We distinguish two ways in which this may occur. The first is when W fails to be quasiconvex, i.e. fails to satisfy (5); this case corresponds to materials which may change phase (see Ericksen [13], James [14, 15]). An example is furnished by an *elastic fluid*, for which

$$W(A) = h(\det A). \tag{12}$$

In this case (5), (6) and (7) are equivalent and are satisfied if and only if h is convex. For a van der Waals fluid, for example, h is not convex. Results proved by Acerbi and Fusco [2] and Dacorogna [12], for integrands taking finite values only, suggest that under strong growth conditions on W any minimizing sequence for $I(\cdot)$ has a subsequence converging weakly in $W^{1,1}(\Omega; \boldsymbol{R}^n)$ to a minimizer for the "relaxed problem" obtained by replacing W by its lower quasiconvex envelope. The corresponding result for an elastic fluid is proved in Dacorogna [11].

A second way in which W can fail to be $W^{1,1}$-quasiconvex is due to its growth properties for large $|A|$. As an example, consider the isotropic stored-energy function (11) with $c > 0, h \in C^3(0, \infty)$, $h'' > 0$ and $\lim_{t\to 0+} h(t) = \lim_{t\to\infty} \dfrac{h(t)}{t} = \infty$. If $a \geqslant 1$ this function satisfies (7), and if $a > 1$ it is *strongly elliptic*, i.e. (6) holds with strict inequality if a, b are non zero. However, if $1 < a < 3$ W is not $W^{1,1}$-quasiconvex. This can be proved by choosing $D = B \overset{\text{def}}{=} \{X \in \boldsymbol{R}^3\colon |X| < 1\}$, $A = \lambda 1$, and showing that for sufficiently large $\lambda > 0$ one can violate (5) with an appropriate discontinuous radial function φ. The problem of minimizing $E(\cdot)$ among radial deformations

$$x(X) = \frac{r(R)}{R} X, \qquad R = |X|, \tag{13}$$

subject to appropriate displacement or traction boundary data is considered in Ball [7]. For example, for the stored-energy function (11) under the above conditions with $1 < a < 3$ it is shown that for any $\lambda > 0$ the absolute minimum of $E(\cdot)$ among radial deformations (13) satisfying $r(0) \geqslant 0$, $r(R) \geqslant 0$ and $r(1) = \lambda$ is attained. Furthermore, there exists a critical value λ_c such that for $\lambda \leqslant \lambda_c$ the minimizer is trivial and given by $r(R) = \lambda R$ (i.e. $x = \lambda X$), whilst for $\lambda > \lambda_c$ the minimizer satisfies $r(0) > 0$, so that a cavity forms at the origin. The nontrivial minimizers are discontinuous weak solutions of the full set of 3-dimensional Euler–Lagrange equations for $E(\cdot)$. The reader is referred to [7] for analogous results for more general compressible and incompressible materials, for a discussion of the relevance of discontinuous minimizers to the phenomenon of internal rupture of rubber, and for comments concerning the relationship of the analysis to the literature on discontinuous solutions to nonlinear elliptic systems.

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