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A version of the fundamental theorem
for Young measures

J.M. Ball

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UNIVERSITE PIERRE ET MARIE CURIE
CENTRE NATIONAL DE LA RECHERCHE SCIENTIFIQUE

Laboratoire d'Analyse Numérique
Tour 55-65 - 5ème étage
Université Pierre et Marie Curie
4, place Jussieu
75252 PARIS CEDEX 05 - France

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Résumé

La mesure de Young (ν_x) pour une suite $z^{(j)} : \Omega \longrightarrow \mathbb{R}^m$, où $\Omega \subset \mathbb{R}^n$ est mesurable, donne la distribution de probabilité asymptotique quand $j \longrightarrow \infty$ des valeurs de $z^{(j)}$ près de x , et donc caractérise la limite faible de $f(z^{(j)})$ pour toute fonction f continue. Dans cet article une version du théorème fondamental concernant l'existence des mesures de Young est démontrée sous des hypothèses moins restrictives que d'habitude sur $\{z^{(j)}\}$ et f .

Synopsis.

The Young measure (ν_x) corresponding to a given sequence of mappings $z^{(j)} : \Omega \longrightarrow \mathbb{R}^m$, where $\Omega \subset \mathbb{R}^n$ is measurable, gives the limiting probability distribution as $j \longrightarrow \infty$ of the values of $z^{(j)}$ near x , and thus characterizes the weak limit of $f(z^{(j)})$ for all continuous functions f . In this paper a version of the fundamental theorem concerning the existence of Young measures is proved under less restrictive hypotheses than usual on $\{z^{(j)}\}$ and f .

1. Introduction.

Let $\Omega \subset \mathbb{R}^n$ be measurable and let $z^{(j)}: \Omega \longrightarrow \mathbb{R}^m$ be a given sequence of functions. The fundamental theorem concerning Young measures asserts that under appropriate hypotheses there exists a subsequence $z^{(\mu)}$ of $z^{(j)}$ and a family of probability measures (ν_x) on \mathbb{R}^m (the *Young measure*) such that the weak limit of $f(z^{(\mu)})$ is given by the expectation

$$\langle \nu_x, f \rangle = \int_{\mathbb{R}^m} f(\lambda) d\nu_x(\lambda) \quad \text{a.e. } x \in \Omega, \quad (1)$$

for any continuous function $f: \mathbb{R}^m \longrightarrow \mathbb{R}$. The purpose of this note is to give a version of the theorem that seems in some respects to be sharper than other statements in the literature (in particular, as regards the hypotheses on the $z^{(j)}$ and f).

The Young measure (ν_x) can intuitively be thought of as giving the limiting probability distribution as $\mu \longrightarrow \infty$ of the values of $z^{(\mu)}$ near x . To be more precise, suppose that Ω is open and that $x \in \Omega$. Denote by $B(x, \delta)$ the open ball with centre x and radius $\delta > 0$. Keeping x, μ and δ fixed, let $\nu_{x, \delta}^{(\mu)}$ be the probability distribution of the values of $z^{(\mu)}(y)$ as y is chosen uniformly at random from $B(x, \delta)$. Then it is shown below that

$$\nu_x = \lim_{\delta \rightarrow 0} \lim_{\mu \rightarrow \infty} \nu_{x, \delta}^{(\mu)}, \quad (2)$$

where the convergence is weak* in the sense of measures.

The Young measure was introduced by L.C. Young (see [34, 35]) as a means of treating problems of the calculus of variations for which there does not exist a minimizer in a classical sense. Many applications and developments to the calculus of variations and optimal control theory have been made by MacShane [23], Gamkrelidze [19] and others. More abstract ideas in the same spirit are the integral currents of Federer & Fleming [18] and varifolds (see, for example, Allard [1]). The Young measure was developed as a tool for analysing nonlinear partial differential equations by Tartar [31], who suggested how it could help to prove existence theorems for nonlinear hyperbolic systems, a

programme he carried out for a single hyperbolic equation. These ideas were then significantly developed by DiPerna [12] to prove the existence of solutions for a system of two hyperbolic equations in one space dimension. Further results along these lines have been given by, for example, Dafermos [10], DiPerna [13], Rascle [26], Roytburd & Slemrod [27], and Serre [29]. Applications of the Young measure to variational problems of continuum mechanics have been made by Ball & Knowles [5], Ball & James [4], Chipot & Kinderlehrer [9], and Kinderlehrer [22]. In the last three papers cited, the application is to the description of the microstructure of crystals. (The order in which the limits are taken in (2) corresponds to the way in which the microstructure is experimentally observed; namely, a small region of diameter δ of the crystal is examined microscopically, but this region is typically larger than the length scale j^{-1} of the microstructure.)

The method used here to prove the fundamental theorem delivers (ν_x) directly via duality rather than by disintegration of a measure on a product space; the principal idea can be found in Castaing & Valadier [8], Warga [33], and recent descriptions appear in Capuzzo Dolcetta & Ishii [7], Slemrod & Roytburd[30]. An alternative direct construction is suggested in Tartar [32 pp268-9]. For the usual method see Berliocchi & Lasry [6], Tartar [31] and Balakrishnan [2]. A Young measure corresponding to a bounded sequence in L^p capable of detecting some concentrations as well as oscillations has been introduced by DiPerna & Majda [14,15] as a tool for studying vortex dynamics.

2. The fundamental theorem for Young measures.

Our aim is to prove the following result:

Theorem

Let $\Omega \subset \mathbb{R}^n$ be Lebesgue measurable, let $K \subset \mathbb{R}^m$ be closed, and let $z^{(j)}: \Omega \longrightarrow \mathbb{R}^m$, $j = 1, 2, \dots$, be a sequence of Lebesgue measurable functions

satisfying $z^{(j)}(\cdot) \rightarrow K$ in measure as $j \rightarrow \infty$, i.e. given any open neighbourhood U of K in \mathbb{R}^m

$$\lim_{j \rightarrow \infty} \text{meas} \{x \in \Omega : z^{(j)}(x) \notin U\} = 0.$$

Then there exists a subsequence $z^{(\mu)}$ of $z^{(j)}$ and a family (ν_x) , $x \in \Omega$, of positive measures on \mathbb{R}^m , depending measurably on x , such that

$$(i) \quad \|\nu_x\|_{\mathbb{M}} := \int_{\mathbb{R}^m} d\nu_x \leq 1 \quad \text{for a.e. } x \in \Omega,$$

$$(ii) \quad \text{supp } \nu_x \subset K \quad \text{for a.e. } x \in \Omega, \text{ and}$$

$$(iii) \quad f(z^{(\mu)}) \xrightarrow{*} \langle \nu_x, f \rangle = \int_{\mathbb{R}^m} f(\lambda) d\nu_x(\lambda)$$

in $L^\infty(\Omega)$ for each continuous function $f : \mathbb{R}^m \rightarrow \mathbb{R}$ satisfying

$$\lim_{|\lambda| \rightarrow \infty} f(\lambda) = 0.$$

Suppose further that $\{z^{(\mu)}\}$ satisfies the boundedness condition

$$\limsup_{k \rightarrow \infty} \sup_{\mu} \text{meas} \{x \in \Omega \cap B_R : |z^{(\mu)}(x)| \geq k\} = 0, \quad (3)$$

for every $R > 0$, where $B_R = B(0, R)$. Then $\|\nu_x\|_{\mathbb{M}} = 1$ for a.e. $x \in \Omega$ (i.e. ν_x is a probability measure), and given any measurable subset A of Ω

$$f(z^{(\mu)}) \xrightarrow{*} \langle \nu_x, f \rangle \quad \text{in } L^1(A) \quad (4)$$

for any continuous function $f : \mathbb{R}^m \rightarrow \mathbb{R}$ such that $\{f(z^{(\mu)})\}$ is sequentially weakly relatively compact in $L^1(A)$.

Remarks

1. The condition (3) is very weak, and is equivalent to the following: given any $R > 0$ there exists a continuous nondecreasing function $g_R : [0, \infty) \rightarrow \mathbb{R}$, with $\lim_{t \rightarrow \infty} g_R(t) = \infty$, such that

$$\sup_{\mu} \int_{\Omega \cap B_R} g_R(|z^{(\mu)}(x)|) dx < \infty. \quad (5)$$

In fact suppose that (5) holds. Then, since g_R is nondecreasing,

$$\sup_{\mu} \text{meas} \{x \in \Omega \cap B_R : |z^{(\mu)}(x)| \geq t\} \cdot g_R(t) \leq \sup_{\mu} \int_{\Omega \cap B_R} g_R(|z^{(\mu)}(x)|) dx.$$

Since $\lim_{t \rightarrow \infty} g_R(t) = \infty$, we obtain (3).

Conversely, if (3) holds, we may choose $0 < t_j < t_{j+1}$, $j=1,2,\dots$, so that

$$\sup_{\mu} \text{meas} \{x \in \Omega \cap B_R : |z^{(\mu)}(x)| \geq t_j\} \leq j^{-3},$$

and let

$$\bar{g}_R(t) = \begin{cases} 0 & \text{if } t \in [0, t_1), \\ j & \text{if } t \in [t_j, t_{j+1}). \end{cases}$$

Then

$$\begin{aligned} \sup_{\mu} \int_{\Omega \cap B_R} \bar{g}_R(|z^{(\mu)}(x)|) dx &= \sup_{\mu} \sum_{j=1}^{\infty} j \text{meas} \{x \in \Omega \cap B_R : t_{j+1} > |z^{(\mu)}(x)| > t_j\} \\ &\leq \sum_{j=1}^{\infty} j^{-2} < \infty. \end{aligned}$$

Choosing a suitable continuous $g_R \leq \bar{g}_R$ we thus obtain (5).

2. If the functions $z^{(j)}$ are uniformly bounded in $L^\infty(\Omega; \mathbb{R}^m)$ then the functions $f(z^{(j)})$ are uniformly bounded in $L^\infty(\Omega)$ for any continuous $f: \mathbb{R}^m \rightarrow \mathbb{R}$. Hence by the theorem there is a family of probability measures (ν_x) and a subsequence $z^{(\mu)}$ such that

$$f(z^{(\mu)}) \xrightarrow{*} \langle \nu_x, f \rangle \quad \text{in } L^\infty(\Omega)$$

for all such f . In this way we recover the form of the theorem given by Tartar [31]. If Ω is bounded and if the $z^{(j)}$ are uniformly bounded in $L^p(\Omega; \mathbb{R}^m)$ for some p , $1 < p < \infty$, then we obtain from the theorem the existence of a family of probability measures (ν_x) and a subsequence $z^{(\mu)}$ such that

$$f(z^{(\mu)}) \longrightarrow \langle \nu_x, f \rangle \quad \text{in } L^r(\Omega) \quad (6)$$

for any continuous $f: \mathbb{R}^m \rightarrow \mathbb{R}$ satisfying

$$|f(\lambda)| \leq \text{const.} (1 + |\lambda|^q), \quad \lambda \in \mathbb{R}^m, \quad (7)$$

where $q > 0$ and $1 < r < p/q$ (see Schonbek [28]).

3. If A is bounded, the condition that $\{f(z^{(\mu)})\}$ be sequentially weakly relatively compact in $L^1(A)$ is satisfied if and only if

$$\sup_{\mu} \int_A \psi(|f(z^{(\mu)})|) dx < \infty$$

for some continuous function $\psi : [0, \infty) \longrightarrow \mathbb{R}$ with $\lim_{\lambda \rightarrow \infty} \psi(\lambda)/\lambda = \infty$ (de

la Vallée Poussin's criterion; cf. MacShane [24], Dellacherie & Meyer [11]).

4. As explained in the introduction, the Young measure (ν_x) can be thought of as the limiting probability distribution of the values of $z^{(\mu)}$ near the point x . In fact if Ω is open and $x \in \Omega$ then for $\delta > 0$ sufficiently small the open ball $B(x, \delta)$ with centre x and radius δ is contained in Ω , and the formula

$$\langle \nu_{x, \delta}^{(\mu)}, f \rangle = \int_{B(x, \delta)} f(z^{(\mu)}(y)) dy \quad (8)$$

defines a continuous linear form on continuous functions $f : \mathbb{R}^m \longrightarrow \mathbb{R}$ of compact support. (Here and below $\int_E (\cdot) dx$ denotes $(\text{meas } E)^{-1} \int (\cdot) dx$.) The Radon measure $\nu_{x, \delta}^{(\mu)}$ is a probability measure giving the distribution of the values of $z^{(\mu)}$ in $B(x, \delta)$, and can be written

$$\nu_{x, \delta}^{(\mu)} = \int_{B(x, \delta)} \delta_{z^{(\mu)}(y)} dy, \quad (9)$$

where δ_a denotes the Dirac mass at $a \in \mathbb{R}^m$. As $\mu \longrightarrow \infty$, $\nu_{x, \delta}^{(\mu)} \xrightarrow{*} \nu_{x, \delta}$ in the sense of measures, where by the theorem

$$\langle \nu_{x, \delta}^{(\mu)}, f \rangle = \int_{B(x, \delta)} \langle \nu_y, f \rangle dy, \quad (10)$$

that is,

$$\nu_{x, \delta} = \int_{B(x, \delta)} \nu_y dy. \quad (11)$$

By Lebesgue's differentiation theorem, for any fixed f

$$\langle \nu_{x, \delta}, f \rangle \longrightarrow \langle \nu_x, f \rangle \quad \text{as } \delta \longrightarrow 0 \text{ for a.e. } x \in \Omega.$$

Choosing a countable dense set of functions f we deduce easily that $\nu_{x, \delta} \xrightarrow{*} \nu_x$ in the sense of measures as $\delta \longrightarrow 0$ for a.e. $x \in \Omega$. The number $1 - \|\nu_x\|_{\mathbb{M}}$ represents the limiting proportion of the points in $B(x, \delta)$ at which $z^{(\mu)}$ becomes unbounded.

Proof of the theorem

We denote by $C_0(\mathbb{R}^m)$ the Banach space of continuous functions $f : \mathbb{R}^m \rightarrow \mathbb{R}$ satisfying $\lim_{|\lambda| \rightarrow \infty} f(\lambda)/\lambda = 0$, with the norm $\|f\|_{C_0} = \sup_{\lambda \in \mathbb{R}^m} |f(\lambda)|$. A well known form of the Riesz representation theorem (Hewitt & Stromberg [20 p364]) asserts that there is a isometric isomorphism between the dual space $C_0(\mathbb{R}^m)^*$ of $C_0(\mathbb{R}^m)$ and the Banach space $M(\mathbb{R}^m)$ of bounded Radon measures on \mathbb{R}^m obtained by associating with each $\nu \in M(\mathbb{R}^m)$ the linear form $f \mapsto \int_{\mathbb{R}^m} f(\lambda) d\nu$ on $C_0(\mathbb{R}^m)$. The norm on $M(\mathbb{R}^m)$ is given by $\|\nu\|_M = \int_{\mathbb{R}^m} d|\nu|$. We associate with $z^{(j)}$ the mapping $\nu^{(j)} : \Omega \rightarrow M(\mathbb{R}^m)$ defined by

$$\nu^{(j)}(x) = \delta_{z^{(j)}(x)}. \quad (12)$$

For each j , $\nu^{(j)}$ belongs to the space $L_w^\infty(\Omega; M(\mathbb{R}^m))$ of equivalence classes of weak* measurable mappings $\mu : \Omega \rightarrow M(\mathbb{R}^m)$ that are essentially bounded, i.e.

$$\|\mu\|_{\infty, M} := \text{ess sup}_{x \in \Omega} \|\mu(x)\|_M < \infty. \quad (13)$$

(We say that μ is weak* measurable if $\langle \mu(x), f \rangle$ is measurable with respect to x for every $f \in C_0(\mathbb{R}^m)$.) Under the norm $\|\cdot\|_{\infty, M}$, $L_w^\infty(\Omega; M(\mathbb{R}^m))$ is a Banach space. Since $C_0(\mathbb{R}^m)$ is separable, there is an isometric isomorphism between the dual space of $L^1(\Omega; C_0(\mathbb{R}^m))$ and $L_w^\infty(\Omega; M(\mathbb{R}^m))$ obtained by associating with each $\mu \in L_w^\infty(\Omega; M(\mathbb{R}^m))$ the linear form

$$\psi \mapsto \int_{\Omega} \langle \mu(x), \psi(x, \cdot) \rangle dx \quad (14)$$

on $L^1(\Omega; C_0(\mathbb{R}^m))$ (see Edwards [17 p588] A. & C. Ionescu Tulcea [21 p93], Meyer [25 p244]). But

$$\|\nu^{(j)}\|_{\infty, M} = 1 \quad \text{for all } j.$$

Since $C_0(\mathbb{R}^m)$ is separable, so is $L^1(\Omega; C_0(\mathbb{R}^m))$, and hence (cf. Dunford & Schwartz [16 pp 424-426]) there exists a subsequence $\nu^{(\mu)}$ of $\nu^{(j)}$ and an element $\nu = (\nu_x)$ of $L_w^\infty(\Omega; M(\mathbb{R}^m))$ such that $\nu^{(\mu)} \xrightarrow{*} \nu$ in $L_w^\infty(\Omega; M(\mathbb{R}^m))$. By (14) this implies that

$$\int_{\Omega} \psi(x, z^{(\mu)}(x)) dx \rightarrow \int_{\Omega} \langle \nu_x, \psi(x, \cdot) \rangle dx \quad (15)$$

as $\mu \rightarrow \infty$ for every $\psi \in L^1(\Omega; C_0(\mathbb{R}^m))$. In particular, taking $\psi(x, \lambda) = \phi(x)f(\lambda)$, where $\phi \in L^1(\Omega)$ and $f \in C_0(\mathbb{R}^m)$, we have that

$$f(z^{(\mu)}) \xrightarrow{*} \langle \nu_x, f \rangle \quad \text{in } L^\infty(\Omega) \quad (16)$$

for every $f \in C_0(\mathbb{R}^m)$, which is (iii). By weak* lower semicontinuity of the norm, $\|\nu\|_{\infty, M} \leq 1$, which is (i). To prove (ii), assume that $K \neq \mathbb{R}^m$ and let $f \in C_0^K(\mathbb{R}^m) := \{g \in C_0(\mathbb{R}^m) ; g|_K = 0\}$. Then, taking $U = \{z \in \mathbb{R}^m : |f(z)| < \varepsilon\}$, it follows from the hypothesis that $z^{(j)}(\cdot) \rightarrow K$ in measure that $f(z^{(\mu)}(\cdot)) \rightarrow 0$ in measure. Since f is bounded, we deduce that

$$\begin{aligned} \int_{\Omega} \phi(x) \langle \nu_x, f \rangle dx &= \lim_{\mu \rightarrow \infty} \int_{\Omega} \phi(x) f(z^{(\mu)}(x)) dx \\ &= 0 \end{aligned}$$

for every $\phi \in L^1(\Omega)$. Since $C_0(\mathbb{R}^m)$ is separable so is $C_0^K(\mathbb{R}^m)$, and it follows that for a.e. $x \in \Omega$ we have $\langle \nu_x, f \rangle = 0$ for all $f \in C_0^K(\mathbb{R}^m)$, i.e. $\text{supp } \nu_x \subset K$. Since $\nu^{(j)}(x) \geq 0$ a.e. we deduce similarly that $\nu_x \geq 0$ a.e..

Now suppose that (3) holds. Define $\vartheta^k \in C_0(\mathbb{R}^m)$ by

$$\vartheta^k(\lambda) = \begin{cases} 1 & \text{for } |\lambda| \leq k, \\ 1 + k - |\lambda| & \text{for } k \leq |\lambda| \leq k + 1, \\ 0 & \text{for } |\lambda| \geq k + 1. \end{cases} \quad (17)$$

Then if $E \subset \Omega$ is bounded and measurable

$$\begin{aligned} \lim_{\mu \rightarrow \infty} \int_E \vartheta^{(k)}(z^{(\mu)}(x)) dx &= \int_E \langle \nu_x, \vartheta^{(k)} \rangle dx \\ &\leq \int_E \|\nu_x\|_M dx. \end{aligned} \quad (18)$$

But

$$0 \leq \int_E (1 - \vartheta^{(k)}(z^{(\mu)}(x))) dx \leq \frac{\text{meas}\{x \in E : |z^{(\mu)}(x)| \geq k\}}{\text{meas } E},$$

so that letting $k \rightarrow \infty$ we get from (3) and (18) that

$$1 \leq \int_E \|\nu_x\|_M dx. \quad (19)$$

Since $\|\nu_x\|_M \leq 1$ a.e. and E is arbitrary, (19) implies that $\|\nu_x\|_M = 1$ a.e..

Suppose further that $f : \mathbb{R}^m \rightarrow \mathbb{R}$ is continuous and that $f(z^{(\mu)})$ is sequentially weakly relatively compact in $L^1(\Omega)$. Let $f^+ = \max(f, 0)$, $f^- = \max$

$(-f, 0)$, so that $f = f^+ - f^-$. By the Dunford-Pettis theorem (Dunford & Schwartz [16 p292]) $f^+(z^{(\mu)})$ and $f^-(z^{(\mu)})$ are both sequentially weakly relatively compact in $L^1(\Omega)$. Hence to prove (4) we can and will suppose that $f \geq 0$. It also clearly suffices to prove (4) for the case when A is bounded and $f(z^{(\mu)}) \rightarrow \chi$, say, in $L^1(A)$. Define $f^k \in C_0(\mathbb{R}^m)$ by $f^k = \vartheta^k f$, where ϑ^k is given by (17). Let $\phi \in L^\infty(A)$. We claim that

$$\int_A \phi f^k(z^{(\mu)}) dx \longrightarrow \int_A \phi f(z^{(\mu)}) dx \quad (20)$$

as $k \rightarrow \infty$, uniformly in μ . In fact

$$\left| \int_A \phi (f^k(z^{(\mu)}) - f(z^{(\mu)})) dx \right| \leq \text{const.} \int_{\{x \in A : |z^{(\mu)}(x)| \geq k\}} f(z^{(\mu)}) dx, \quad (21)$$

and, given $\varepsilon > 0$, by the Dunford-Pettis theorem there exists an $M > 0$ such that

$$\sup_{\mu} \int_{\{x \in A : f(z^{(\mu)}(x)) \geq M\}} f(z^{(\mu)}) dx \leq \varepsilon.$$

Hence by (3)

$$\int_{\{x \in A : |z^{(\mu)}(x)| \geq k\}} f(z^{(\mu)}) dx \leq \varepsilon + M \text{meas} \{x \in A : |z^{(\mu)}(x)| \geq k\} \leq 2\varepsilon,$$

for all μ if k is sufficiently large, which together with (21) gives (20).

Since also by (iii)

$$\lim_{\mu \rightarrow \infty} \int_A \phi f^k(z^{(\mu)}) dx = \int_A \phi \langle \nu_x, f^k \rangle dx \quad (22)$$

it follows that

$$\lim_{k \rightarrow \infty} \int_A \phi \langle \nu_x, f^k \rangle dx = \int_A \phi \chi dx. \quad (23)$$

Choosing $\phi \geq 0$ and noting that f^k is increasing, we deduce from the monotone convergence theorem that $\langle \nu_x, f \rangle = \chi$ a.e. in A as required.

□

Remarks on the proof

1. The reader is warned that the mappings $\nu^{(j)}$ do not in general belong to the

space $L^\infty(\Omega; M(\mathbb{R}^m))$ of essentially bounded strongly measurable mappings from Ω to $M(\mathbb{R}^m)$ even when the $z^{(j)}$ are smooth. This space cannot be identified with the dual of $L^1(\Omega; C_0(\mathbb{R}^m))$ via (14).

2. The same argument as in the proof shows that under the hypothesis (3), for any measurable $A \subset \Omega$,

$$f(\cdot, z^{(\mu)}) \longrightarrow \langle \nu_x, f(x, \cdot) \rangle \text{ in } L^1(A) \quad (24)$$

for every Carathéodory function $f : A \times \mathbb{R}^m \longrightarrow \mathbb{R}$ such that $\{f(\cdot, z^{(\mu)})\}$ is sequentially weakly relatively compact in $L^1(A)$.

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References

- [1] W.K. Allard, "On the first variation of a varifold", *Annals of Math.* **95** (1972) 417-491.
- [2] A.V. Balakrishnan, 'Applied Functional Analysis', Springer, 1976.
- [3] J.M. Ball, "Material instabilities and the calculus of variations", Proc. conference on "Phase transformations and material instabilities in solids", Mathematics Research Center, University of Wisconsin, Academic Press, Publication No. 52, (1984) 1-20.
- [4] J.M. Ball and R D James, "Fine phase mixtures as minimizers of energy", *Arch. Rat. Mech. Anal.*, **100** (1987) 13-52.
- [5] J.M. Ball and G. Knowles, unpublished work summarised in [3].
- [6] H. Berliocchi and J.M. Lasry, "Intégrales normales et mesures paramètres en calcul des variations", *Bull. Soc. Math. France* **101** (1973) 129-184.
- [7] I. Capuzzo Dolcetta and H. Ishii, "Approximate solutions of the Bellman equation of deterministic control theory", *Appl. Math. Optim.*, **11** (1984) 161-181.
- [8] C. Castaing and M. Valadier, 'Convex Analysis and Measurable Multifunctions', Springer Lecture Notes in Mathematics, Vol. **580**, 1977.
- [9] M. Chipot and D. Kinderlehrer, "Equilibrium configurations of crystals", *Arch. Rat. Mech. Anal.*, to appear.
- [10] C.M. Dafermos, to appear.
- [11] C. Dellacherie and P-A. Meyer, 'Probabilités et Potentiel', Hermann, 1975.

- [12] R. J. DiPerna, "Convergence of approximate solutions to conservation laws", Arch. Rat. Mech. Anal., **82** (1983) 27-70.
- [13] R. J. DiPerna, "Convergence of the viscosity method for isentropic gas dynamics", Comm. Math. Phys. **91** (1983) 1- 30.
- [14] R. J. DiPerna and A. J. Majda, "Oscillations and concentrations in weak solutions of the incompressible fluid equations", Comm. Math. Phys., **108** (1987) 667-689.
- [15] R. J. DiPerna and A. J. Majda, "Concentrations in regularizations for 2-D incompressible flow", Comm. Pure Appl. Math., **40** (1987) 301-345.
- [16] N. Dunford and J. T. Schwartz, 'Linear Operators', Part I, Interscience, 1967.
- [17] R. E. Edwards, 'Functional Analysis', Holt, Rinehart and Winston, 1965.
- [18] H. Federer and W. H. Fleming, "Normal and integral currents", Annals of Math., **72** (1960) 458-520.
- [19] R. V. Gamkrelidze, "On sliding optimal states", Dokl. Akad. Nauk. SSSR **143** (1962) 1243-1245 = Soviet Math. Doklady **3** (1962) 559-561.
- [20] E. Hewitt and K. Stromberg, 'Real and Abstract Analysis', Springer, 1965.
- [21] A. & C. Ionescu Tulcea, 'Topics in the Theory of Lifting', Springer, New York, 1969.
- [22] D. Kinderlehrer, "Remarks about equilibrium configurations of crystals", in 'Material Instabilities in Continuum Mechanics', ed. J. M. Ball, Oxford University Press, 1988, pp. 217-241.
- [23] E. J. MacShane, "Generalized curves", Duke Math. J., **6** (1940) 513-536.
- [24] E. J. MacShane, 'Integration', Princeton Univ. Press, 1947.
- [25] P.-A. Meyer, 'Probability and Potentials', Blaisdell, Waltham, 1966.
- [26] M. Rascle, "Un résultat de 'compacité par compensation' à coefficients variables. Application à l'élasticité non linéaire", C.R. Acad. Sci. Paris, **302** (1986) 311-314.
- [27] V. Roytburd & M. Slemrod, "An application of the method of compensated compactness to a problem in phase transitions", in 'Material Instabilities in Continuum Mechanics', ed. J. M. Ball, Oxford University Press, 1988, pp. 427-463.
- [28] M. E. Schonbek, "Convergence of solutions to nonlinear dispersive equations", Comm. in Partial Diff. Equations, **7** (1982) 959-1000.
- [29] D. Serre, "La compacité par compensation pour les systèmes hyperboliques non linéaires à une dimension d'espace", J. Math. Pure et Appl., **65** (1987) 423-468.
- [30] M. Slemrod and V. Roytburd, "Measure-valued solutions to a problem in dynamic phase transitions", in 'Contemporary Mathematics', Vol. **50**, Amer. Math. Soc., 1987.
- [31] L. Tartar, "Compensated compactness and applications to partial differential equations", in 'Nonlinear Analysis and Mechanics', Heriot-Watt Symposium, Vol. **IV**, Pitman Research Notes in Mathematics, 1979, pp. 136-192.
- [32] L. Tartar, "The compensated compactness method applied to systems of conservation laws", in 'Systems of Nonlinear Partial Differential Equations', ed. J. M. Ball, NATO ASI Series, Vol. **C111**, Reidel, 1982, pp. 263-285.
- [33] J. Warga, "Relaxed variational problems", J. Math. Anal. Appl., **4** (1962) 111-128.
- [34] L. C. Young, "Generalized curves and the existence of an attained absolute minimum in the calculus of variations", Comptes Rendus de la Société des Sciences et des Lettres de Varsovie, classe III, **30** (1937) 212-234.
- [35] L. C. Young, 'Lectures on the Calculus of Variations and Optimal Control Theory', Saunders, 1969 (reprinted by Chelsea, 1980).

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