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Bifurcation and stability of homogeneous equilibrium configurations of an elastic body under dead-load tractions

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1. *Introduction*

In this paper we consider the equilibrium configurations of a homogeneous, incompressible, isotropic elastic body subjected to a uniform dead load surface traction of magnitude T whose direction is normal to the surface of the body in the reference configuration, and to no other forces. We concentrate on *homogeneous* equilibrium solutions, that is those for which the deformation gradient F is constant, and we study their bifurcations and stability (with respect to an appropriate static criterion) as T varies. Since it turns out that the equations for homogeneous equilibrium solutions, and the stability properties that we consider of these solutions, are independent of the shape of the body in the reference configuration, we can suppose if desired that this shape is a cube. (See Fig. 1·1.)

In (16) Rivlin studied the bifurcation of homogeneous equilibrium solutions of a cube of neo-Hookean material subjected to three pairs of equal and opposite uniformly distributed dead load surface tractions, and in particular he examined the case when all the tractions are equal as above. Rivlin restricted attention to *purely homogeneous* deformations, that is those of the form

$$x(X) = (v_1 X^1, v_2 X^2, v_3 X^3), \tag{1·1}$$

where the v_i are constants. (For the notation see Section 2.) For such deformations $F = \operatorname{diag}(v_1, v_2, v_3)$. In the case where the v_i are positive he showed that as T is increased the trivial solution $v_1 = v_2 = v_3 = 1$ ceases to minimize the total free energy locally within the class of purely homogeneous deformations, and that new local minima appear in which two of the v_i are equal, say

$$v_1 = v_2 > 1 > v_3 \quad (v_1 v_2 v_3 = 1). \tag{1·2}$$

These equilibrium solutions remain local minima for arbitrarily large T, and there

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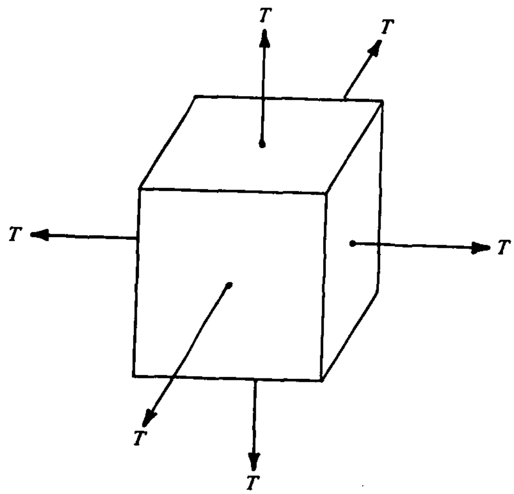

Fig. 1·1

are no other local minima. We shall call a solution satisfying (1·2) *plate-like*, describing the shape of the body for large deformations; in contrast, a solution satisfying

$$v_1 > 1 > v_2 = v_3 \quad (v_1 v_2 v_3 = 1), \tag{1·3}$$

will be called *rod-like*. In a later paper (17) Rivlin studied the stability of all the purely homogeneous equilibria with $v_i > 0$ with respect to arbitrary virtual displacements. (For related remarks see Hill (13).) With the exception of the trivial solution in the case $T < 0$, he concluded that all the above local minima are stable. (In fact, as we shall see, the nontrivial local minima are only neutrally stable.)

In Section 2 we describe some general results concerning homogeneous equilibrium solutions for an arbitrary isotropic incompressible material. We first show (Theorem 2·1) that if $T \neq 0$ then $x = FX$ is an equilibrium solution if and only if $F = Q\,\mathrm{diag}\,(v_1, v_2, v_3)\,Q^T$, where $Q \in O(3)$ and $(v_1 X^1, v_2 X^2, v_3 X^3)$ is a purely homogeneous equilibrium solution, a similar statement holding for $T = 0$. This reduces the problem to that of studying purely homogeneous equilibria. We then show (Theorem 2·2) how to determine the stability properties of a purely homogeneous equilibrium solution with respect to arbitrary virtual displacements, given its stability properties with respect to purely homogeneous deformations. Finally (Theorem 2·4), we give conditions under which the absolute minimizer of the total free energy among sufficiently regular deformations is homogeneous.

In Sections 3 and 4 we study the bifurcation of homogeneous equilibrium solutions from the point of view of singularity theory. An important role is played in the analysis by the equivariance of the governing equations under the symmetric group S_3. Although the methods are quite general, for ease of computation we restrict our discussion to the case of a separable stored-energy function

$$\Phi(v_1, v_2, v_3) = \phi(v_1) + \phi(v_2) + \phi(v_3). \tag{1·4}$$

The bifurcation from the trivial solution is then governed by the behaviour of $\phi(v)$ near $v = 1$. If

$$\phi(v) \sim \sum_{n=0}^{\infty} \frac{c_n}{n!}(v-1)^n, \tag{1·5}$$

is the Taylor expansion of ϕ at $v = 1$ then bifurcation occurs when $T = c_1 + c_2$. Unless

$$2c_2 + c_3 \simeq 0, \tag{1·6}$$

the nontrivial solutions all have two equal principal stretches v_i. However, in the critical case (1·6) a secondary bifurcation into solutions with all the v_i different may occur. The analysis applies in particular to the case of a Mooney–Rivlin material, for which

$$\Phi(v_1, v_2, v_3) = \frac{\mu}{2}(v_1^2 + v_2^2 + v_3^2 - 3) + \frac{\nu}{2}(v_1^{-2} + v_2^{-2} + v_3^{-2} - 3), \tag{1·7}$$

where $\mu > 0$ and $\nu \geqslant 0$ are constants. ($\nu = 0$ corresponds to a neo-Hookean material.) The critical case alluded to above then occurs when $k = \mu/\nu$ satisfies

$$k \simeq 3. \tag{1·8}$$

The analysis in Sections 3 and 4 is local; that is, it is restricted to a neighbourhood of the trivial solution and to values of T near the bifurcation point. However, in Section 5 we determine the global behaviour of purely homogeneous equilibrium solutions with $v_i > 0$ for a Mooney–Rivlin material and it turns out that no new bifurcation phenomena, not already observed in our local analysis, appear. (This is a particularly fortuitous instance of the principle that the global picture usually falls into place once the most singular local problems are well understood.) As soon as k is finite the plate-like solutions become unstable for sufficiently large T and then the only (neutrally) stable solutions are rod-like. The transition from the plate-like solutions to the rod-like solutions as T is increased is made via a secondary bifurcation involving neutrally stable solutions with all three stretches unequal. This behaviour persists for $3 < k < \infty$. As $k \to 3$ from above the solution branches with all stretches unequal coalesce into the single degenerate bifurcation point noted previously. If $k < 3$ there is a branch of neutrally stable rod-like solutions; the plate-like solutions are unstable for any T.

We conclude in Section 6 by discussing informally some further implications of singularity theory for this class of bifurcation problems.

The reader wishing to bypass singularity theory can read Section 2 and 5 directly, but will then need to perform some of the stability calculations in Section 5 explicitly.

2. *Homogeneous equilibrium solutions*

(i) *The governing equations*

We consider an elastic body of homogeneous isotropic incompressible material which occupies in a reference configuration a nonempty bounded domain $\Omega \subset \mathbb{R}^3$ having sufficiently regular (say strongly Lipschitz) boundary $\partial\Omega$. In a typical deformed configuration the particle having position X in the reference configuration is displaced to $x(X) = \{x^i(X)\}$. We write $F = \nabla x(X)$ for the deformation gradient; the principal stretches λ_i are the eigenvalues of $\sqrt{(F^T F)}$. The constraint of incompressibility takes the form

$$\det F = \lambda_1 \lambda_2 \lambda_3 = 1. \tag{2·1}$$

Since the material is isotropic the stored-energy function $W(F)$ can be expressed as a symmetric function $\Phi(\lambda_1, \lambda_2, \lambda_3)$ of the principal stretches. We suppose that W, Φ are C^1 functions defined on the sets $K = \{F \in M^{3\times 3} : \det F = 1\}$, $L = \{(\lambda_1, \lambda_2, \lambda_3) : \lambda_i > 0, \lambda_1\lambda_2\lambda_3 = 1\}$ respectively, where $M^{3\times 3}$ denotes the set of real 3×3 matrices. We extend W, Φ to neighbourhoods of K, L in $M^{3\times 3}$, $\mathbb{R}^3$ respectively in such a way that W, Φ remain C^1 and still satisfy

$$W(F) = \Phi(\lambda_1, \lambda_2, \lambda_3). \tag{2·2}$$

This extension, which is easily effected (cf. (3)), enables us to define derivatives of W, Φ which are not tangential to K, L respectively.

We suppose that the body is held in equilibrium under the sole action of surface tractions

$$t_R(X) = TN(X), \quad \text{a.e. } x \in \partial\Omega, \tag{2·3}$$

where t_R is the Piola–Kirchhoff stress vector, T is a given load, and $N = N(X)$ is the unit outward normal to $\partial\Omega$. The equilibrium equations are the Euler–Lagrange equations for the total free energy

$$I(x) = \int_\Omega [W(\nabla x(X)) - T \operatorname{tr} \nabla x(X)]\, dX, \tag{2·4}$$

subject to

$$\det \nabla x(X) = 1, \tag{2·5}$$

where the term

$$-\int_\Omega T \operatorname{tr} \nabla x \, dX = -\int_{\partial\Omega} t_R \cdot x \, dS$$

represents the work done by the surface tractions (2·3). Equivalently, they are the Euler–Lagrange equations for the unconstrained functional

$$\hat{I}(x) = \int_\Omega [W(\nabla x(X)) - T \operatorname{tr} \nabla x(X) - p(X)(\det \nabla x(X) - 1)]\, dX, \tag{2·6}$$

where the Lagrange multiplier $p(X)$ corresponds to a hydrostatic pressure. (The value of p depends on the way W is extended off K.)

A *homogeneous* deformation is one for which $\nabla x(X)$ is constant. If $x(X)$ is an equilibrium solution, so is $x(X) + a$ for any constant $a \in \mathbb{R}^3$. Thus in looking for homogeneous equilibrium solutions it suffices to consider deformations of the form

$$x(X) = FX, \tag{2·7}$$

where $F \in K$ is constant. For solutions of the form (2·7) the equilibrium equations take the well-known form

$$\frac{\partial W}{\partial F}(F) - T\mathbf{1} = pF^{-T}, \tag{2·8i}$$

$$\det F = 1, \tag{2·8ii}$$

where p is constant. Of course (2·8) are the equations for critical points of the functions

$$H(F) = W(F) - T\,\mathrm{tr}\,F, \quad \det F = 1, \tag{2·9}$$

and

$$\hat{H}(F) = W(F) - T\,\mathrm{tr}\,F - p(\det F - 1), \tag{2·10}$$

obtained by introducing (2·7) into (2·4), (2·6) respectively.

For a purely homogeneous deformation

$$x(X) = (v_1 X^1, v_2 X^2, v_3 X^3), \tag{2·10}$$

we have

$$F = \mathrm{diag}\,(v_1, v_2, v_3). \tag{2·11}$$

We consider first the case when all the v_i are positive (and thus equal to the principal stretches λ_i). By the isotropy of W

$$\frac{\partial W}{\partial F}(F) = \mathrm{diag}\,(\Phi_1, \Phi_2, \Phi_3), \tag{2·12}$$

where $\Phi_i = \partial\Phi(v_1, v_2, v_3)/\partial v_i$, and so the equilibrium equations (2·8) take the form

$$\Phi_i - T = p v_i^{-1} \quad (i = 1, 2, 3), \tag{2·13i}$$

$$v_1 v_2 v_3 = 1, \tag{2·13ii}$$

which are the equations for critical points of the functions

$$\Psi(v, T) = \Phi(v) - T \sum_{i=1}^{3} v_i, \quad v_1 v_2 v_3 = 1, \tag{2·14}$$

and

$$\hat{\Psi}(v, T) = \Phi(v) - T \sum_{i=1}^{3} v_i - p(v_1 v_2 v_3 - 1), \tag{2·15}$$

where we have set $v = (v_1, v_2, v_3)$.

If two of the v_i are negative, say $v_1 < 0$, $v_2 < 0$, $v_3 > 0$, then we set

$$\lambda = (-v_1, -v_2, v_3) = (\lambda_1, \lambda_2, \lambda_3),$$

and the equilibrium equations become

$$\begin{cases} \Phi_i(\lambda) + T = p\lambda_i^{-1} & (i = 1, 2), \\ \Phi_3(\lambda) - T = p\lambda_3^{-1}, \end{cases} \tag{2·16i}$$

$$\lambda_1 \lambda_2 \lambda_3 = 1. \tag{2·16ii}$$

(In this case the body undergoes a rotation through an angle π about the X^3-axis, so the problem is equivalent to that in which the v_i are all positive but two of the pairs of forces are reversed in orientation.)

(ii) *A characterization of homogeneous equilibrium solutions*

We now show how the study of the existence of homogeneous equilibrium solutions may be reduced to that for purely homogeneous ones.

THEOREM 2·1. *If* $T \neq 0$ *then the homogeneous deformation* $x = FX$ *is an equilibrium solution if and only if*

$$F = Q\,\mathrm{diag}\,(v_1, v_2, v_3)\,Q^T, \tag{2·17}$$

where $Q \in O(3)$ *and* (v_1X^1, v_2X^2, v_3X^3) *is a purely homogeneous equilibrium solution.*

If $T = 0$ *then* $x = FX$ *is an equilibrium solution if and only if*

$$F = Q\,\mathrm{diag}\,(\lambda_1, \lambda_2, \lambda_3)\,R, \tag{2·18}$$

where $Q, R \in O(3)$, $\lambda_i > 0$ *are the principal stretches of* F, *and* $(\lambda_1X^1, \lambda_2X^2, \lambda_3X^3)$ *is a purely homogeneous equilibrium solution.*

Proof. Let $x = FX$ be a homogeneous equilibrium solution. Using the polar decomposition theorem we can write F in the form

$$F = QDR, \tag{2·19}$$

where $D = \mathrm{diag}\,(\lambda_1, \lambda_2, \lambda_3)$ and $Q, R \in O(3)$. By (2·8i)

$$Q\,\mathrm{diag}\,(\Phi_1(\lambda), \Phi_2(\lambda), \Phi_3(\lambda))\,R - T\mathbf{1} = pQD^{-1}R. \tag{2·20}$$

If $T \neq 0$ then it follows from (2·20) that Q^TR^T is diagonal, and therefore *either* $RQ = \mathbf{1}$ *or* RQ is one of the matrices $\mathrm{diag}\,(1, -1, -1)$, $\mathrm{diag}\,(-1, 1, -1)$, $\mathrm{diag}\,(-1, -1, 1)$. In the first case $F = QDQ^T$ and (2·20) reduces to (2·13). In the second case we have, say, $R = \mathrm{diag}\,(-1, -1, 1)\,Q$, and (2·20) reduces to (2·16). If $T = 0$ then clearly (2·20) reduces to (2·13). ∎

Note that the set of homogeneous equilibrium solution is the same whatever be the shape of Ω.

(iii) *Stability*

We say that an equilibrium solution $x \in C^1(\bar{\Omega}; \mathbb{R}^3)$ is *stable* if it is a weak relative minimum of I, that is if for some $\epsilon > 0$

$$I(y) > I(x) \quad \text{whenever} \quad 0 < \|\nabla y - \nabla x\|_{C(\bar{\Omega};\, \mathbb{R}^9)} < \epsilon, \quad \det \nabla y \equiv 1,$$

and *neutrally stable* if for some $\epsilon > 0$

$$I(y) \geqslant I(x) \quad \text{whenever} \quad 0 < \|\nabla y - \nabla x\|_{C(\bar{\Omega};\, \mathbb{R}^9)} < \epsilon, \quad \det \nabla y \equiv 1,$$

but x is not stable. Otherwise we call x *unstable*. Koiter (11) has emphasized that it is far from obvious that an equilibrium solution which is stable according to our definition is Lyapunov stable in a suitable topology with respect to the equations of nonlinear elastodynamics. The choice in our definition of a seminorm depending only on deformation gradients reflects the invariance of I under rigid-body translations.

We first note that the invariance of I under the transformation $\nabla x \to Q\nabla xQ^T$ (resp. $\nabla x \to Q\nabla xR$) when $T \neq 0$ (resp. $T = 0$) implies that a homogeneous equilibrium solution $x = FX$ with F given by (2·17) (resp. (2·18)) is stable, neutrally stable or unstable according as the corresponding purely homogeneous equilibrium solution is stable, neutrally stable or unstable. Thus in studying the stability of homogeneous equilibrium solutions we may restrict attention to purely homogeneous solutions. In the next theorem we consider only the case when all the v_i are positive in (2·17), but analogous results hold for the case when two v_i are negative.

THEOREM 2·2. (*a*) *If* $T < 0$ *then the trivial solution* $x = X$ *is unstable. Let* $T \geqslant 0$.

Then $x = X$ is stable if $T > 0$ and $(1,1,1)$ is a strict local minimum of $\Psi(\cdot, T)$, neutrally stable if either $T > 0$ and $(1,1,1)$ is a nonstrict local minimum of $\Psi(\cdot, T)$ or $T = 0$ and $(1,1,1)$ is a strict or nonstrict local minimum of $\Psi(\cdot, T)$, and unstable otherwise.

(b) Let $T > 0$. If $(v_1, v_2, v_3) \neq (1,1,1)$, $v_i > 0$, is a strict or nonstrict local minimum of $\Psi(\cdot, T)$ then $(v_1 X^1, v_2 X^2, v_3 X^3)$ is neutrally stable; otherwise it is unstable.

Remark. If Φ satisfies the *Baker–Ericksen inequalities*

$$\frac{v_i \Phi_i - v_j \Phi_j}{v_i - v_j} > 0 \quad (v_i \neq v_j), \tag{2·21}$$

then it follows immediately from (2·13i) that there are no nontrivial purely homogeneous solutions with $v_i > 0$ in the case $T \leqslant 0$. The inequalities (2·21) are a consequence of strict rank 1 convexity of W (cf. (3)).

Proof of Theorem 2·2. (*a*) Let $T < 0$. To prove that $x = X$ is unstable it suffices to show that $F = \mathbf{1}$ is not a local minimum of H. Let $\Omega \in M^{3\times 3}$ be nonzero and skew-symmetric. Then $e^{\epsilon\Omega} \in SO(3)$ (in particular $\det e^{\epsilon\Omega} = 1$) and

$$\begin{aligned} H(e^{\epsilon\Omega}) &= W(e^{\epsilon\Omega}) - T \operatorname{tr} e^{\epsilon\Omega} \\ &= W(\mathbf{1}) - T \operatorname{tr}\left(\mathbf{1} + \epsilon\Omega + \frac{\epsilon^2}{2}\Omega^2 + O(\epsilon^3)\right) \\ &= H(\mathbf{1}) + \frac{\epsilon^2}{2} T \operatorname{tr}(\Omega\Omega^T) + O(\epsilon^3), \end{aligned}$$

as $\epsilon \to 0$, showing that $F = \mathbf{1}$ is not a local minimum.

Let $T \geqslant 0$ and suppose that $(1,1,1)$ is a strict local minimum of $\Psi(\cdot, T)$, i.e. there exists $\epsilon > 0$ such that $\Psi(\cdot, T) > \Psi((1,1,1), T)$ whenever $0 < \sum_{i=1}^{3} |\lambda_i - 1| < \epsilon$ and $\lambda_1\lambda_2\lambda_3 = 1$. Let $y \in C^1(\bar{\Omega})$ with $\det \nabla y \equiv 1$. Then

$$\nabla y(X) = Q(X) \operatorname{diag}(\lambda_1(X), \lambda_2(X), \lambda_3(X)) R(X), \tag{2·22}$$

where $Q(X), R(X) \in O(3)$ and the $\lambda_i(X)$ are the principal stretches of $\nabla y(X)$. If $\|\nabla y - \mathbf{1}\|_{C(\bar{\Omega};\, \mathbf{R}^3)}$ is sufficiently small, then

$$\sup_{x \in \Omega} \sum_{i=1}^{3} |\lambda_i(X) - 1| < \epsilon$$

and so

$$\begin{aligned} I(y) &= \int_\Omega (\Phi(\lambda(X)) - T \operatorname{tr}[Q(X) \operatorname{diag}(\lambda_1(X), \lambda_2(X), \lambda_3(X)) R(X)])\, dX \\ &= I(X) + \int_\Omega [\Psi(\lambda(X), T) - \Psi((1,1,1), T)]\, dX \\ &\quad + T \int_\Omega \sum_{i=1}^{3} [1 - (RQ)_{ii}(X)]\, \lambda_i(X)\, dX. \end{aligned}$$

Since $(RQ)_{ii}(X) \leqslant 1$ for $i = 1, 2, 3$, it follows that $I(y) \geqslant I(X)$, with equality if and only if $\lambda(X) \equiv (1,1,1)$ and either $T = 0$ or $(RQ)(X) = \mathbf{1}$ for all X. But if $R(X) = Q^T(X)$ for all X and $\lambda(X) \equiv (1,1,1)$ then $\nabla y(X) \equiv \mathbf{1}$; therefore $x = X$ is stable for $T > 0$. If $T = 0$ then $I(QX) = I(X)$ for any $Q \in SO(3)$ and so $x = X$ is only neutrally stable. The same argument shows that if $(1,1,1)$ is a nonstrict local minimum of $\Psi(\cdot, T)$ then $x = X$ is neutrally stable, and it is clear that otherwise $x = X$ is unstable.

(*b*) Let $D = \operatorname{diag}(v_1, v_2, v_3)$. Since $I(QDQ^TX) = I(DX)$ for any $Q \in SO(3)$, $x = DX$ can at most be neutrally stable. Let $y \in C^1(\bar{\Omega})$ with $\det \nabla y \equiv 1$, and let ∇y be given by (2·22). Then

$$I(y) - I(DX) = \int_\Omega [\Psi(\lambda(X), T) - \Psi(v, T)]\, dX + T \int_\Omega \sum_{i=1}^{3} [1 - (RQ)_{ii}(X)]\, \lambda_i(X)\, dX,$$

and the result follows as in (*a*). ∎

That nontrivial homogeneous equilibrium solutions are at most neutrally stable was observed by Adeleke (1) in an essentially equivalent problem. In fact the invariance of $I(x)$ under the transformation $\nabla x \to Q \nabla x Q^T$ implies that if $x \in C^1(\bar{\Omega}; \mathbb{R}^3)$ is a stable equilibrium solution then necessarily $x = X + a$ for some $a \in \mathbb{R}^3$. In particular it follows from part (*a*) of the theorem that there are no stable equilibrium solutions for $T < 0$. The instability of the trivial solution for $T < 0$ was established by Beatty (4).

In Section 5 we apply Theorem 2·2 to determine the stability of purely homogeneous equilibrium states with $v_i > 0$ for a Mooney–Rivlin material. The special case of a neo-Hookean material was discussed by Rivlin (17), but Theorem 2·2 shows that the nontrivial equilibrium solutions stated in (17) to be stable are in fact only neutrally stable. Let us analyse the method used in (17), but for a general isotropic material. Consider an equilibrium solution $x = F_0 X$ with corresponding hydrostatic pressure p_0; this solution is stable if and only if F_0 is a strict local minimum of $H(F)$ subject to $\Delta(F) \overset{\text{def}}{=} \det F - 1 = 0$. Then

$$\Delta^{(1)}(F_0)(G) \overset{\text{def}}{=} \frac{d}{d\tau} \Delta(F_0 + \tau G)|_{\tau=0} = \operatorname{tr}(F_0^{-T} G).$$

Let N denote the null space of $\Delta^{(1)}(F_0)$. By the implicit function theorem we can solve the equation

$$\Delta(F_0 + G + \tau(G) F_0^{-T}) = 0, \tag{2·23}$$

uniquely for $\tau(G) \in \mathbb{R}$ with $G \in N$ sufficiently small. Expanding (2·23) in a Taylor series we obtain $\tau(G) = O(|G|^2)$ and

$$\Delta^{(1)}(F_0)(G + \tau(G) F_0^{-T}) = -\frac{1}{2!} \Delta^{(2)}(F_0)[G, G] + O(|G|^3). \tag{2·24}$$

Since by (2·8)

$$H^{(1)}(F_0) = p_0 \Delta^{(1)}(F_0),$$

we obtain

$$H(F_0 + G + \tau(G) F_0^{-T}) - H(F_0) = \frac{1}{2!} (H^{(2)}(F_0) - p_0 \Delta^{(2)}(F_0))[G, G] + O(|G|^3). \tag{2·25}$$

It follows that if

$$(H^{(2)}(F_0) - p_0 \Delta^{(2)}(F_0))[G, G] > 0, \tag{2·26}$$

for all nonzero $G \in N$ then F_0 is a strict local minimum of H. For a neo-Hookean material Rivlin showed that for certain equilibria the expression in (2·26) is always non-negative; however his formulae show that it vanishes for some nonzero $G \in N$, and this precludes us from deducing stability. In such degenerate cases one might contemplate expanding (2·25) to higher order in G to decide whether F_0 is a local minimum. However, it is in general impossible to detect neutral stability by this procedure. The difficulties are illustrated by the trivial examples (*a*) e^{-1/x^2} and 0 have the same Taylor expansion at $x = 0$, and (*b*) the function $f(x, y) = (x^2 - y)^2$ is zero on the parabola $y = x^2$ but nevertheless a neighbourhood of the origin may be covered by a family of smooth curves (e.g. straight lines through the origin) along each of which f has a *strict* local minimum at the origin.

(iv) *Global minima*

PROPOSITION 2·3. *Suppose that*

$$\lim_{\substack{|v|\to\infty \\ v_1 v_2 v_3 = 1}} \frac{\Phi(v)}{|v|} = \infty. \tag{2·27}$$

Then $H(F)$ attains its minimum on $K = \{F : \det F = 1\}$.

Proof. Since

$$H(F) = |F| \left[\frac{W(F)}{|F|} - T\frac{\operatorname{tr} F}{|F|}\right]$$

it follows from (2·27) that H is bounded below on K and that any minimizing sequence F_j for H is bounded. The result follows. ▌

Of course any (even local) minimizer of $H(F)$ satisfies the equilibrium equations (2·8) for some p.

In general nonhomogeneous equilibrium solutions of the three-dimensional equilibrium equations will exist (for examples in the case $T = 0$ see Truesdell (23)). However a sufficiently smooth minimizer of I is in general homogeneous, as we now show.

THEOREM 2·4. *Let* (2·28) *hold. Suppose further that if $T \neq 0$ there are only finitely many purely homogeneous minimizers of H on K, while if $T = 0$ suppose that the only minimizer of $\Phi(v)$ on L is $(1, 1, 1)$ (as will be the case if the Baker–Ericksen inequalities* (2·21) *hold). Then the absolute minimum of I in $\{x \in C^1(\Omega; \mathbb{R}^3) : \det \nabla x = 1\}$ is attained, and all such minimizers are homogeneous.*

Proof. To minimize I one can do no better than to minimize its integrand pointwise. Thus if F minimizes H on K, $x = FX$ minimizes I. Conversely, if x minimizes I then $\nabla x(X)$ is a minimizer of H on K for all $X \in \Omega$. Hence if $T \neq 0$, by Theorem 2·1 we have

$$\nabla x(X) = Q(X)\, D(X)\, Q^T(X), \tag{2·29}$$

where $Q(X) \in O(3)$ and $D(X)$ is a purely homogeneous minimizer of H. Since the principal invariants of $\nabla x(X)$ are continuous functions of X and equal the elementary symmetric functions of the entries of the diagonal matrix $D(X)$, and since there are only finitely many possible values of these symmetric functions, it follows in particular that $\operatorname{tr} D(X)$ and $\operatorname{tr} D^2(X)$ are constants. The homogeneity of x now follows using a trick taken from Shield (21). Since $\nabla x(X)$ is symmetric, locally $x(X) = \nabla g(X)$ for some g. But then $\Delta x(X) = \nabla \Delta g(X) = \nabla \operatorname{tr} D(X) = 0$, so that x is harmonic (and in particular C^∞). Thus

$$\begin{aligned} 0 = \Delta(\operatorname{tr} D^2(X)) &= \Delta(x^i_{,\alpha} x^i_{,\alpha}) \\ &= x^i_{,\alpha\beta} x^i_{,\alpha\beta} + 2x^i_{,\alpha}(\Delta x^i)_{,\alpha}, \end{aligned}$$

and so $x^i_{,\alpha\beta} = 0$.

In the case $T = 0$ we have $\nabla x(X) = Q(X)\,1R(X) \in SO(3)$ for all $X \in \Omega$. It is well known that this implies that x is homogeneous; for the convenience of the reader we give a proof of this under our regularity assumption.

LEMMA 2·5. *Let $x \in C^1(\Omega; \mathbb{R}^3)$ be such that $\nabla x(X) \in SO(3)$ for all $X \in \Omega$. Then $\nabla x(X)$ is constant.*

Proof. It suffices to prove that $\nabla x(X)$ is constant in a small neighbourhood of a typical point, say $0 \in \Omega$. Since $\det \nabla x(0) \neq 0$, by the inverse function theorem x maps a neighbourhood N of 0 homeomorphically onto a neighbourhood N_1 of $x(0)$. Denoting the inverse function by $X(\cdot)$ we have that

$$\nabla x(X)^T\,\nabla x(X) = 1 \quad \text{for all} \quad X \in N, \tag{2·30}$$

$$\nabla X(x)^T\,\nabla X(x) = 1 \quad \text{for all} \quad X \in N_1. \tag{2·31}$$

Let $X, Y \in N$. Then by (2·30)

$$\begin{aligned} |x(X) - x(Y)| &= \left|\int_0^1 \nabla x(tX + (1-t)\,Y)\cdot(X-Y)\,dt\right| \\ &\leqslant \int_0^1 |\nabla x(tX+(1-t)\,Y)\cdot(X-Y)|\,dt \\ &= |X-Y|. \end{aligned}$$

The opposite inequality follows by applying the same argument to $X(\cdot)$ using (2·31). Hence x is an isometry, so that ∇x is constant. ∎

We note that Theorem 2·1–2·6 have obvious analogues valid for compressible materials.

3. *Bifurcations and symmetry*

(i) *Reformulation of the equilibrium equations*

Our aim is to discuss the bifurcations of solutions to the equilibrium equations (2·13). It turns out to be convenient to introduce new coordinates

$$w = (w_1, w_2, w_3); \quad w_i = \log v_i, \tag{3·1}$$

so that the incompressibility constraint (2·13ii) becomes a linear one,

$$w_1 + w_2 + w_3 = 0. \tag{3·2}$$

The equilibrium equations (2·13i) now take the form

$$h(w, T) = p, \tag{3·3}$$

where $h = (h_1, h_2, h_3)$ and

$$h_i(w, T) = e^{w_i}(\Phi_i(e^{w_1}, e^{w_2}, e^{w_3}) - T). \tag{3·4}$$

Let P denote the orthogonal projection of $\mathbb{R}^3$ onto the plane Γ defined by (3·2); thus

$$Pz = z - \frac{1}{3}\sum_{i=1}^{3} z_i\,(1,1,1). \tag{3·5}$$

Clearly (3·3) holds for some p if and only if w satisfies

$$Ph(w, T) = 0. \tag{3·6}$$

That $\Psi(\cdot, T)$ defined by (2·14) is invariant with respect to permutation of its arguments induces a corresponding invariance property of (3·3) which we now describe. For $\pi \in S_3$, $w \in \mathbb{R}^3$ we let

$$\gamma_\pi w = (w_{\pi(1)}, w_{\pi(2)}, w_{\pi(3)}). \tag{3·7}$$

Then $h(\cdot, T)$ is *equivariant* with respect to (3·7) in the sense that

$$h(\gamma_\pi w, T) = \gamma_\pi h(w, T) \quad \text{for all} \quad \pi \in S_3, w \in \mathbb{R}^3, \tag{3·8}$$

for any fixed T.

We now identify Γ with the complex plane $\mathbb{C}$ by means of the linear isomorphism S defined by $Sw = x + iy$, where

$$x = \tfrac{1}{2}w_1, \quad y = \frac{1}{2\sqrt{3}}(w_2 - w_3). \tag{3·9}$$

Correspondingly, we define

$$\tilde{\gamma}_\pi = S\gamma_\pi S^{-1}, \quad \pi \in S_3. \tag{3·10}$$

The reader may check that

$$\tilde{\gamma}_{(23)} z = \bar{z} \tag{3·11 i}$$

$$\tilde{\gamma}_{(123)} z = e^{-2\pi i/3} z, \tag{3·11 ii}$$

where $\bar{z}$ is the complex conjugate of z, (23) is a simple interchange permutation, and (123) is the cyclic permutation. In other words, the action of S_3 in our complex coordinates is the standard two dimensional, irreducible representation of the group. Similarly, we define $\tilde{h}(\cdot, T): \mathbb{C} \to \mathbb{C}$ by

$$\tilde{h}(z, T) = SPh(S^{-1}z, T). \tag{3·12}$$

Then (3·6) holds if and only if $\tilde{h}(Sw, T) = 0$, and by (3·5), (3·7), (3·8), (3·10) we have that

$$\begin{aligned}\tilde{h}(\tilde{\gamma}_\pi z, T) &= SPh(\gamma_\pi S^{-1}z, T)\\ &= SP\gamma_\pi h(S^{-1}z, T)\\ &= S\gamma_\pi Ph(S^{-1}z, T)\\ &= \tilde{\gamma}_\pi \tilde{h}(z, T),\end{aligned}$$

so that $\tilde{h}(\cdot, T)$ is equivariant with respect to the action of S_3 on $\mathbb{C}$.

(ii) *Review of* (10)

We have just shown that the equilibrium equations (2·13) can be written in the form

$$g(z, \lambda) = 0, \tag{3·13}$$

where $g: \mathbb{C} \times \mathbb{R} \to \mathbb{C}$ is equivariant. (In (3·13) we have set $g(z, \lambda) = \tilde{h}(z, \lambda + T_0)$, anticipating the change of variable $\lambda = T - T_0$ where T_0 is a bifurcation point.) In this subsection we recall the results from singularity theory in (10) concerning such equations and their bifurcations. Proofs are given only when it helps the exposition.

PROPOSITION 3·1. *If $g: \mathbb{C} \to \mathbb{C}$ is an equivariant C^∞ mapping, then there exist smooth real valued coefficients a and b such that*

$$g(z) = a(|z|^2, \mathrm{Rl}\, z^3)\, z + b(|z|^2, \mathrm{Rl}\, z^3)\, \bar{z}^2. \tag{3·14}$$

Table 3·1.

	Coefficients for $\tilde{Z}_n(z)$	
n	a	b
1	1	0
2	0	1
3	$3\|z\|^2$	0
4	$2\,\mathrm{Rl}\,z^3$	$3\|z\|^2$
5	$9\|z\|^4$	$2\,\mathrm{Rl}\,z^3$

Note that $|z|^2$ and $\mathrm{Rl}\,z^3$ are invariant under the action (3·11) of S_3 on $\mathbb{C}$; indeed by a theorem of Schwartz (20) the most general invariant function can be expressed as a function of $|z|^2$ and $\mathrm{Rl}\,z^3$ – precisely the form of the coefficients in (3·14). In (3·13) the mapping g also depends on the bifurcation parameter λ, and Proposition 3·1 has an obvious analogue in this case, with the coefficients a and b in (3·14) depending smoothly on λ.

As an illustration of the above proposition (with applications in § 4) consider for any positive integer n the equivariant map $Z_n: \mathbb{R}^3 \to \mathbb{R}^3$ given by

(3·15) $$Z_n(w) = (w_1^n, w_2^n, w_3^n).$$

Let $\tilde{Z}_n: \mathbb{C} \to \mathbb{C}$ be the corresponding equivariant map defined as in (3·12). Thus

(3·16) $$\tilde{Z}_n(z) = \tfrac{1}{6}(2w_1^n - w_2^n - w_3^n) + \frac{i}{2\sqrt{3}}(w_2^n - w_3^n),$$

where $z = x + iy$ and

(3·17) $$w_1 = 2x, \quad w_2 = -x + \sqrt{3}y, \quad w_3 = -x - \sqrt{3}y.$$

By the proposition $\tilde{Z}_n(z)$ must be expressible in the form (3·14). In Table 3·1 we have given the coefficients which occur for the cases $n = 1, \ldots, 5$, leaving for the reader the simple calculations needed to check this information.

Singularity theory methods concentrate on qualitative properties of the solution set (or bifurcation diagram) of an equation such as (3·13), where by qualitative properties we mean those properties not affected by changes of coordinates. Specifically, we shall call two equivariant C^∞ mappings $g, h: \mathbb{C} \times \mathbb{R} \to \mathbb{C}$ *equivariantly contact equivalent*, or more simply *equivalent*, if there exists a family of invertible matrices $S_{z\lambda}$ and a diffeomorphism of $\mathbb{C} \times \mathbb{R}$ of the form $(\Theta(z,\lambda), \Lambda(\lambda))$ such that

(3·18) $$h(z,\lambda) = S_{z\lambda}\, g(\Theta(z,\lambda), \Lambda(\lambda)),$$

where $S_{z\lambda}$ and Θ preserve the symmetry – see (9) or (10) for the exact requirement. (*Remark*: One explicit property of matrices $S_{z\lambda}$ which preserve the symmetry will be needed below: namely when $z = 0$, $S_{0\lambda} = \alpha(\lambda)\,1$ for some real valued function $\alpha(\lambda)$. See (10) for the proof.) We also require that $\Lambda'(\lambda) > 0$. Multiplication by an invertible matrix does not affect the zero set of an equation; thus if (3·18) is satisfied, the solution set of h is mapped onto that of g by the diffeomorphism (Θ, Λ). Note that this diffeomorphism maps every plane $\{\lambda = \text{const.}\}$ into another such plane. It should be understood that singularity theory methods are local, and (3·18) is only required to hold in a neighbourhood of the point under study; however, as illustrated by this paper, the local methods often go surprisingly far in clarifying the global picture.

One of the basic techniques of singularity theory involves the so-called normal form. A normal form theorem describes a class of bifurcation problems which, after an appropriate equivalence transformation, may be written in a specific, explicit form, usually chosen for its ease of solution. We give two such theorems here, the first corresponding to the generic case, the second covering a degenerate case which occurs, for example, when $\mu/\nu = 3$ for a Mooney–Rivlin material. For both these theorems, let $g: \mathbb{C} \times \mathbb{R} \to \mathbb{C}$ be an equivariant C^∞ mapping, and let a, b be the coefficients which occur in (3·14). We study g in a neighbourhood of the origin in $\mathbb{C} \times \mathbb{R}$; as regards λ this choice is just a matter of convenience, but for z it is quite important, since the origin is a point of maximal symmetry.

THEOREM 3·2. *If $a(0) = 0$, $b(0) \neq 0$ and $(\partial a/\partial\lambda)(0) \neq 0$, then g is equivalent to*

(3·19) $$N(z, \lambda) = \bar{z}^2 \pm \lambda z,$$

where

$$\operatorname{sign}(\lambda z) = \operatorname{sign}\left(b(0)\frac{\partial a}{\partial \lambda}(0)\right).$$

Any small perturbation of g is equivalent to g itself.

Thus the solution set of an equation satisfying the hypotheses of the theorem may be mapped onto the solution set of

(3·20) $$z^2 \pm \lambda z = 0,$$

by an appropriate diffeomorphism, at least near the origin. We emphasize that there are no higher order terms in (3·19) – the point of the theorem is that (locally) the higher order terms may be transformed away by an appropriate change of coordinates. The reader may easily verify that the solution set of (3·20) consists of four lines in 3-space, the line $z = 0$ corresponding to the trivial solution and three lines of nontrivial solutions

(3·21) $$z = \mp\lambda, \quad z = \mp e^{2\pi i/3}\lambda, \quad z = \mp e^{-2\pi i/3}\lambda.$$

This is sketched in Fig. 3·2. Note that the general point in the plane has 6 images when the group S_3 acts on it, but points on the lines

(3·22) $$\arg z = 0, \pi/3, 2\pi/3 \pmod{\pi},$$

have only three images, because they have higher symmetry. In terms of the identification (3·9) of the plane Γ with $\mathbb{C}$, (3·22) corresponds to points w for which $w_i = w_j$ for some $i \neq j$. Thus all solutions (3·21) possess this partial symmetry. Also in Fig. 3·2 we have indicated the (linear) stability of the various solution branches when sign (λz) is minus: $++$ for both eigenvalues positive (Convention: positive eigenvalues are stable), $+-$ for one positive and one negative eigenvalue, $--$ for both eigenvalues negative. In our application these labels correspond to stability with respect to purely homogeneous deformations. In general stability lies outside the scope of singularity theory, but it was determined for these special cases by ad hoc methods in §5 of (10). As to interpreting the hypotheses of Theorem 3·2, if $a(0) \neq 0$ then equation (3·13) is nonsingular and no bifurcation occurs. The condition $b(0) \neq 0$ requires that quadratic terms be present in g, and $(\partial a/\partial\lambda)(0) \neq 0$ requires that an eigenvalue cross the imaginary

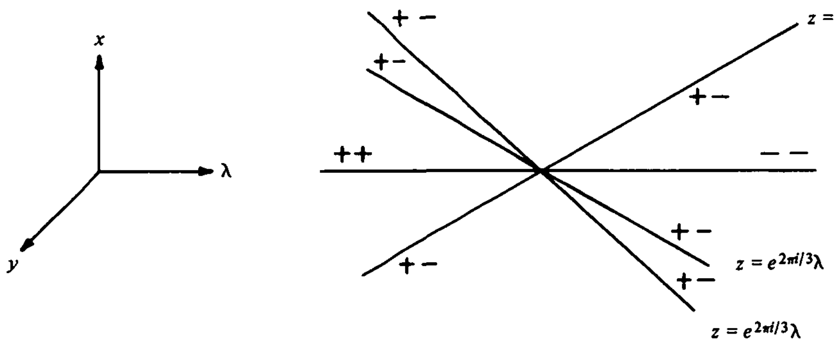

Fig. 3·2

axis with nonzero speed as λ varies. In the next theorem we suppose that $b(0) = 0$. To formulate appropriate hypotheses, let us write

$$a = A|z|^2 + B\mathrm{Rl}\, z^3 + \alpha\lambda + \text{h.o.t.} \tag{3·23i}$$

$$b = C|z|^2 + D\mathrm{Rl}\, z^3 + \beta\lambda + \text{h.o.t.} \tag{3·23ii}$$

where the higher order terms are $O(\lambda^2 + |\lambda|\,|z|^2 + |z|^4)$. This notation automatically incorporates the hypothesis $a(0) = b(0) = 0$, and we add the nondegeneracy conditions

$$A \neq 0, \quad \alpha \neq 0, \quad A\beta - C\alpha \neq 0, \quad AD - BC \neq 0. \tag{3·24}$$

Theorem 3·3. *If* (3·24) *is satisfied, then g is equivalent to*

$$N(z,\lambda) = (|z|^2 \pm \lambda) z + (\pm |z|^2 + \Delta\, Rl\, z^3)\bar{z}^2, \tag{3·25}$$

where

$$\operatorname{sign}(\lambda z) = \operatorname{sign}(A\alpha), \operatorname{sign}(|z|^2\bar{z}^2) = \operatorname{sign}[(C\alpha - A\beta)A\alpha],$$

and

$$\Delta = (AD - BC)\alpha^2/(A\beta - C\alpha)^2. \tag{3·26}$$

A universal unfolding of N is provided by the two parameter unfolding

$$F(z,\lambda,\delta,\epsilon) = (|z|^2 \pm \lambda) z + (\pm |z|^2 + (\Delta + \delta) Rl\, z^3 + \epsilon)\bar{z}^2. \tag{3·27}$$

The technical characterization of a universal unfolding is given in (9), but the essential property is that given any smooth perturbation p, $g + \tau p$ is equivalent to $F(\cdot\,;\delta,\epsilon)$ for some choice of δ and ϵ. In symbols, there exist smooth functions $\delta(\tau)$, $\epsilon(\tau)$ for small τ, with $\delta(0) = \epsilon(0) = 0$, such that

$$g + \tau p \sim F(\cdot\,;\delta(\tau),\epsilon(\tau)).$$

We shall see below that of the two parameters in the universal unfolding, δ plays an entirely subordinate role. (In the language of singularity theory, δ is a modal parameter.) Indeed, if we were to consider C^0 equivalence rather than C^∞ equivalence, δ would not be needed at all. Any two versions of (3·25) with different values of Δ, *but having the same sign*, may be related as in (3·18), the definition of equivalence, where S, Θ, and Λ are Lipschitz continuous. However the C^∞ structure is finer, and one needs to allow Δ to vary to guarantee the existence of a C^∞ solution of (3·18). Unfortunately it seems that C^0-equivalence cannot be studied directly – one must work through C^∞ equivalence.

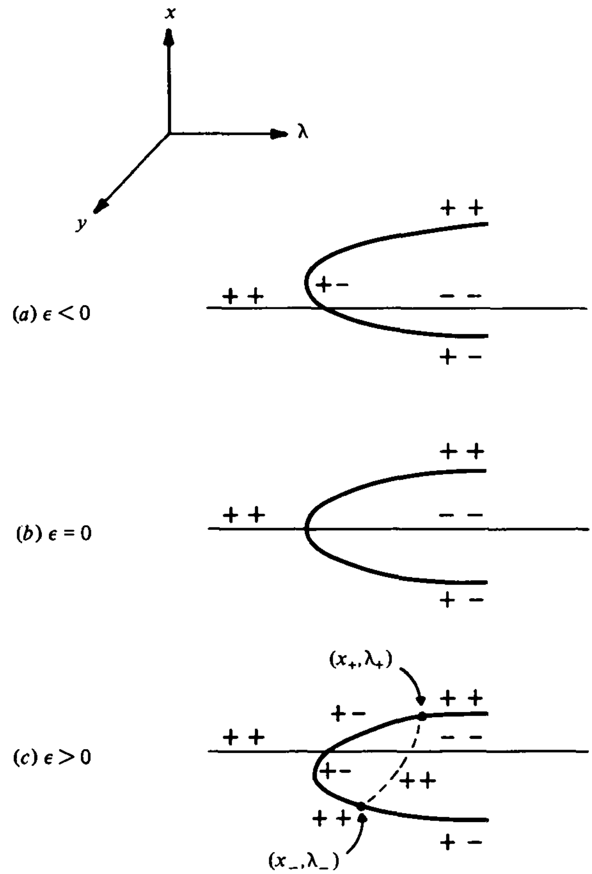

Fig. 3·3

The zero set of (3·27) is determined in (10), but for the reader's convenience we sketch the calculation here. For simplicity we consider only the situation where $\Delta > 0$ and the minus sign is chosen at both open positions in (3·27); this is the case which occurs for a Mooney–Rivlin material. We distinguish three possible ways in which (3·27) can vanish:

(1) $z = 0$,

(2) $z \neq 0$, but $z, \bar{z}^2$ are real multiples of one another,

(3) $z \neq 0$, and $z, \bar{z}^2$ are not real multiples of one another.

Case (1) requires no comment. In case (2), note that z and $\bar{z}^2$ are real multiples of one another if and only if z^3 is real; i.e. if and only if (3·22) is satisfied. Thus solutions found in case (2) have partial symmetry. To find the solutions explicitly it suffices to consider z real, the other two cases being obtainable by rotation. On setting $z = x$ and substituting into (3·27) we find

$$\lambda = x^2 - x^3 + (\Delta + \delta)\, x^4 + \epsilon x. \tag{3·28}$$

This solution branch is shown in Fig. 3·3 in the three cases $\epsilon < 0$, $\epsilon = 0$, and $\epsilon > 0$. The parameter δ has little effect on (3·28), provided $\Delta + \delta$ remains positive. We have *not* shown the two other solution branches which occur in case (2), but the reader should keep their existence in mind.

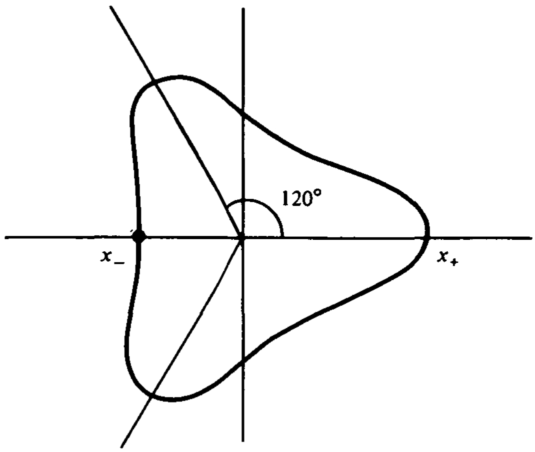

Fig. 3·4

In case (3), (3·27) can vanish only if both coefficients a and b vanish. The equation $b = 0$ defines a curve in the complex plane which is a distorted circle as sketched in Fig. 3·4 if $\epsilon > 0$, the origin if $\epsilon = 0$, or empty if $\epsilon < 0$. (As our theory is local we only consider solutions close to the origin.) Setting $a = 0$ defines λ as a function of z along this curve, and we have included this solution branch in Fig. 3·3 as a dashed curve – somewhat symbolically, as it must be remembered that these curves are invariant under the action of S_3 and also meet the two solution branches arising in case (2) but not shown in the figure.

The most noteworthy feature of these diagrams is the secondary bifurcation which occurs at the points labelled (x_+, λ_+) and (x_-, λ_-) in Fig. 3·3(*c*). For our stability analysis below we want to establish that

(3·29) $$\lambda_+ > \lambda_-.$$

On solving $b = 0$ we find that

$$x_\pm = \pm\sqrt{\epsilon} + \frac{(\Delta+\delta)\epsilon}{2} + O(\epsilon^{3/2}),$$

and (3·29) follows from the relation $\lambda = |z|^2$ satisfied on the solution branch of case (3). We have included the signs of the two eigenvalues of the Jacobian in the figure. The Jacobian is a 2×2 matrix with real entries obtained from the normal form by regarding the latter as two real equations in two real unknowns. In general one might suspect that the contact equivalences (3·18) might change the sign of these eigenvalues; we showed in (10) by *ad hoc* arguments that in fact this does not occur.

In case (1) the Jacobian of (3·27) is $-\lambda\mathbf{1}$, stable for $\lambda < 0$ and purely unstable for $\lambda > 0$. In case (2) the eigenvalues of the Jacobian are readily computable, since the Jacobian is diagonal. In case (3) the stability of the branches is most conveniently determined by the principle of exchange of stability, at least near the point of secondary bifurcation. (Herein lies the relevance of (3·29).) In fact it is shown in (10) that this branch is stable along its entire length.

4. *Local analysis*

(i) *Bifurcations of equilibrium solutions for a class of stored-energy functions*

The principal task of this section is to analyse equation (2·13) with the methods reviewed in §3. To keep the calculations reasonably simple we restrict attention to the case when Φ has the separable form

$$\Phi(v_1, v_2, v_3) = \phi(v_1) + \phi(v_2) + \phi(v_3). \tag{4·1}$$

There is experimental evidence (cf. Jones and Treloar (14)) that (4·1) is a good assumption for rubber. We suppose that $\phi(v)$ is a C^∞ function in the neighbourhood of $v = 1$, and that the Taylor expansion of ϕ about $v = 1$ is given by

$$\phi(v) \sim \sum_{n=0}^{\infty} \frac{c_n}{n!}(v-1)^n. \tag{4·2}$$

The corresponding function $\tilde{h}(z, T)$ defined by (3·12) has Taylor expansion

$$\tilde{h}(z, T) \sim \sum_{n=1}^{\infty} \frac{a_n}{n!} \tilde{Z}_n(z), \tag{4·3}$$

and it is easily verified that

$$\begin{cases} a_1 = c_1 + c_2 - T, \quad a_2 = c_1 + 3c_2 + c_3 - T, \quad a_3 = c_1 + 7c_2 + 6c_3 + c_4 - T, \\ a_4 = c_1 + 15c_2 + 25c_3 + 10c_4 + c_5 - T, \quad a_5 = c_1 + 31c_2 + 90c_3 + 65c_4 + 15c_5 + c_6 - T. \end{cases} \tag{4·4}$$

Since $\tilde{h}(\cdot, T)$ is equivariant it can be expressed in the form (3·14) for some coefficients $a(|z|^2, \mathrm{Rl}\, z^3, T)$ and $b(|z|^2, \mathrm{Rl}\, z^3, T)$. Referring to Table 3·1 we see from (4·3) that

$$a(0, 0, T) = c_1 + c_2 - T. \tag{4·5i}$$

$$b(0, 0, T) = \tfrac{1}{2}[c_1 + 3c_2 + c_3 - T]. \tag{4·5ii}$$

Bifurcation occurs when (4·5i) vanishes; i.e. when $T = T_0$, where

$$T_0 = c_1 + c_2. \tag{4·6}$$

The bifurcation which occurs for $T = T_0$ will be covered by Theorem 3·2 unless (4·5ii) also vanishes, that is, unless

$$2c_2 + c_3 = 0. \tag{4·7}$$

We concentrate primarily on the bifurcation problem which occurs when (4·7) is satisfied at least approximately, using Theorem 3·3 for the analysis. Let us evaluate the various coefficients in (3·23) when (4·7) is satisfied. Using Table 3·1 we may write out the first five terms of (4·3) explicitly:

$$\begin{aligned} -\lambda z - \frac{\lambda}{2}\bar{z}^2 + \tfrac{1}{2}(3c_3 + c_4)|z|^2 z + \frac{1}{4!}(18c_3 + 10c_4 + c_5)[2(Rl\, z^3) z + 3|z|^2 \bar{z}^2] \\ + \frac{1}{5!}(75c_3 + 65c_4 + 15c_5 + c_6)[9|z|^4 z + 2(Rl\, z^3)\bar{z}^2], \end{aligned} \tag{4·8}$$

where T_0 is given by (4·6) and $\lambda = T - T_0$. In computing (4·8) we have evaluated the coefficients of $\tilde{Z}_3$, $\tilde{Z}_4$, $\tilde{Z}_5$ at $T = T_0$ and neglected the corresponding terms $\lambda \tilde{Z}_n$, as these derivatives do not occur in (3·23); similarly, terms with $n > 5$ do not contribute. On comparison of (3·23) and (4·8) we find that

$$\begin{cases} A = \tfrac{1}{2}(3c_3 + c_4), \quad B = \tfrac{1}{12}(18c_3 + 10c_4 + c_5), \quad \alpha = -1 \\ C = \tfrac{1}{8}(18c_3 + 10c_4 + c_5), \quad D = \tfrac{1}{60}(75c_3 + 65c_4 + 15c_5 + c_6), \quad \beta = -\tfrac{1}{2}. \end{cases} \tag{4·9}$$

Let us now suppose, for example, that

$$3c_3+c_4 > 0, \quad 12c_3+8c_4+c_5 < 0, \quad AD-BC > 0. \tag{4·10}$$

Then the nondegeneracy conditions (3·24) hold, $\Delta > 0$, and the signs at the open positions in (3·25) are both negative. Therefore when $2c_2+c_3 = 0$ the solution set of $\tilde{h}(z,T) = 0$ near the bifurcation point has the form of Fig. 3·3*b*. Note that the stable branch in this figure occurs for $x > 0$, which corresponds to a rod-like deformation.

Moreover when $2c_2+c_3$ is nonzero but small, it follows from Theorem 3·3 that $\tilde{h}(z,T)$ is equivalent to (3·27). From (4·6), $b(0,0,T_0)$ is positive or negative according as $2c_2+c_3 > 0$ or $2c_2+c_3 < 0$ respectively. It follows from the unfolding theorem, as may be seen from (10), that $b(0,0,T_0) = \alpha\epsilon+O(\epsilon^2)$, where α is a positive number, provided that $b(0,0,T_0) = c_2+\frac{1}{2}c_3$ is small enough. Thus the bifurcation diagram of $\tilde{h}(z,T) = 0$ has the form of Fig. 3·3*a* or *c* according as $2c_2+c_3 < 0$ or $2c_2+c_3 > 0$ respectively. As before $x > 0$ is associated with a rod-like deformation, $x < 0$ with a plate-like one.

For a Mooney–Rivlin material (1·7) we have

$$c_1 = \mu-\nu, \quad c_2 = \mu+3\nu, \quad c_3 = -12\nu, \quad c_4 = 60\nu, \quad c_5 = -360\nu, \quad c_6 = 2520\nu,$$

and hence

$$T_0 = 2(\mu+\nu), \tag{4·11}$$

(a formula first obtained by Hill (12)), while the degenerate bifurcation occurs when $\mu = 3\nu$. Note that the conditions (4·10) hold when $\nu \neq 0$, and that the secondary bifurcations illustrated in Fig. 3·3*c* occur for $\mu > 3\nu$.

5. *Global analysis for a Mooney–Rivlin material*

In this section we solve (2·13) directly for the special case of a Mooney–Rivlin material with stored-energy function Φ given by (1·7). Substituting (1·7) into (2·13) we obtain the equations

$$\mu v_i^2-\nu v_i^{-2}-Tv_i = p \quad (i = 1,2,3), \tag{5·1 i}$$

$$v_1v_2v_3 = 1. \tag{5·1 ii}$$

As in the local analysis of the preceding section, solutions of (5·1) fall into three cases:

(1) the trivial solution $v_1 = v_2 = v_3 = 1$,
(2) solutions with two principal stretches equal,
(3) solutions with all three stretches unequal.

Case (1) requires no analysis, so we proceed to case (2). Let us suppose that

$$v_1 \neq v_2 = v_3. \tag{5·2}$$

We eliminate the pressure from (5·1) by subtracting the $i = 2$ component of (5·1 i) from the $i = 1$ component, and dividing by v_1-v_2; this yields

$$T = (v_1+v_2)(\mu+\nu v_3^2). \tag{5·3}$$

When (5·2) is included this becomes

$$T = (v_1+v_1^{-\frac{1}{2}})(\mu+\nu v_1^{-1}). \tag{5·4}$$

In other words, recalling the S_3 symmetry, we find three solution branches under case (2), (5·4) being the defining relationship for a typical branch. Note that $T = T(v_1)$ is

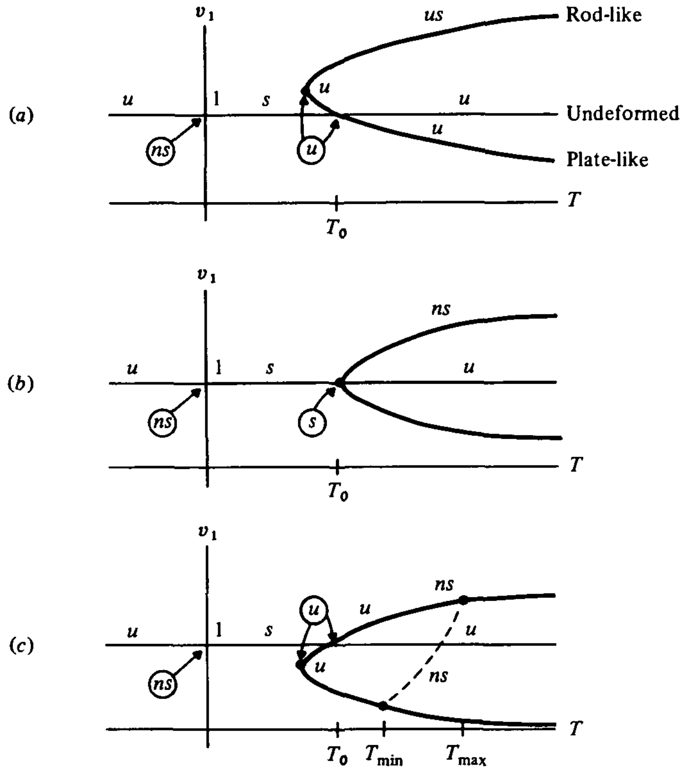

Fig. 5·1

strictly convex and tends to ∞ as v_1 tends to either zero or infinity. The branch defined by (5·4) crosses the trivial solution $v_i = 1$ at $T = 2(\mu+\nu)$, the bifurcation point noted in (4·11). Moreover, differentiating (5·4) we get

$$\left.\frac{dT}{dv_1}\right|_{v_1=1} = \tfrac{1}{2}(\mu-3\nu).$$

This gives rise to three qualitatively different pictures, depending on the sign of $\mu-3\nu$, as sketched in Fig. 5·1.

In case (3), we solve (5·1) as follows. Permute the indices in (5·3) cyclically, subtract the result from (5·3), and divide by $v_1 - v_3$; this yields the equation

(5·5) $$\nu(v_1v_2+v_2v_3+v_3v_1) = \mu.$$

Hence if $\nu = 0$ there are no solutions in case (3). If $\nu > 0$ we set $k = \mu\nu^{-1}$, $u_i = 1/v_i$, and (5·5) then becomes

(5·6) $$u_1+u_2+u_3 = k.$$

Of course we still have the relationship

(5·7) $$u_1u_2u_3 = 1.$$

The convex surface (5·7) intersects the plane (5·6) in a smooth, convex, closed curve if $k > 3$, at the single point $(1, 1, 1)$ if $k = 3$, and not at all if $k < 3$. Only the first case requires further comment. In Fig. 5·2 we have indicated the plane defined by (5·6) and its intersection with (5·7) when $k > 3$. To complete the bifurcation diagram

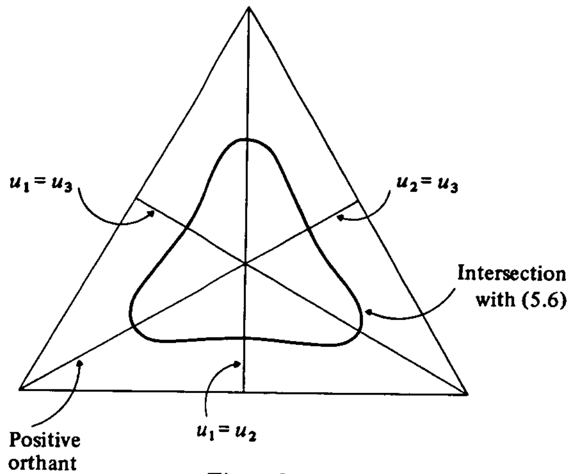

Fig. 5·2

(Fig. 5·1) we need to know the behaviour of T, computed from (5·3), along the intersection curve. Recall that by symmetry T is invariant with respect to the action of S_3; in particular T is invariant under reflection across the three lines $u_i = u_j$ $(i \neq j)$ in the figure. Thus the directional derivative of T along the curve must vanish at the intersection of the curve with each of the lines $u_i = u_j$. These considerations show that T restricted to the curve must have at least 6 critical points. We claim in fact that T has precisely 6, that T is monotone between any adjacent two of the 6 points shown in the figure. We prove the claim by direct computation. Let us parametrize the right half of the intersection curve in Fig. 5·2 by u_3. The parametrization becomes singular near the critical points at the ends of the interval, but there are at least two critical points of T in the interior of the interval. From (5·3) we have

$$T = \nu(u_1^{-1}+u_2^{-1})(k+u_3^{-2}) = \nu u_3(k-u_3)(k+u_3^{-2}). \tag{5·8}$$

Differentiating (5·8) we find

$$\frac{dT}{du_3} = \nu k u_3^{-2}(ku_3^2-2u_3^3-1). \tag{5·9}$$

Equation (5·9) has 3 real zeros, one of which is negative (to be ignored) and two that are positive. The two positive zeros have been accounted for above, and there are no others. This proves the claim.

Let us call the two positive zeros of (5·9) u_* and u^*, where

$$0 < u_* < 1 < \frac{k}{3} < u^*.$$

Since d^2T/du_3^2 is positive (resp. negative) at u_* (resp. u^*), we see that T has a local minimum, denoted $T_{\min}$ (resp. maximum, $T_{\max}$), at $u_3 = u_*$ (resp. u^*). This information enables us to complete Fig. 5·1(*c*). Note that we use the same convention as in §3 of including only one of the three symmetric branches and showing the asymmetric branch as connecting this branch to itself. For the reader's convenience in verifying this figure we remark that at the secondary bifurcation at $T = T_{\min}$ we have

$$u_2 = u_3 = u_* < 1, \quad u_1 = u_*^{-2} > 1;$$

recalling that $u_i = 1/v_i$, we see that

$$v_1 < 1 < v_2 = v_3,$$

a plate-like solution. In the figure we have shown

$$T_{\min} > T_0. \tag{5·10}$$

This inequality follows for k near 3 by the local analysis of § 3 and for k near ∞ since the whole secondary bifurcation branch recedes to infinity in this limit. We offer the following proof of (5·10) in general. From (5·9)

$$T_{\min} = \nu u_*(k-u_*)(k+u_*^{-2}). \tag{5·11}$$

Since $T_0 = 2(\mu+\nu)$, (5·10) is equivalent to

$$-ku_*^3 + k^2u_*^2 - (2k+3)u_* + k > 0. \tag{5·12}$$

Since u_* is a root of (5·9), (5·12) holds if and only if

$$k^2u_*^2 - 2(2k+3)u_* + 3k > 0. \tag{5·13}$$

We consider (5·13) as a quadratic in u_*; its minimum value is

$$k^{-2}(k-3)(3k^2+5k+3),$$

which is positive when $k > 3$. This proves (5·10).

We have also included stability labels in Fig. 5·1, s = stable, ns = neutrally stable, u = unstable. To verify these labels we first determine the stability with respect to purely homogeneous deformations. The result is that an equilibrium solution v is a strict local minimum of $\Psi(\cdot, T)$ if and only if it has the label s in Fig. 3·3; otherwise it is unstable. (We temporarily ignore the critical cases occurring at the bifurcation and limit points.) This follows either by direct computation, or by combining the local analysis of §§ 3, 4 with the following result due to Berger (5): for a variational problem, a solution branch can undergo a change of stability, as a parameter varies, only at a bifurcation point. In other words, any change of stability of a smooth solution branch must be accompanied by the appearance of new solutions. Thus we may restrict our attention to the neighbourhoods of the bifurcation points in Fig. 5·1, and the stability of the noncritical equilibrium solutions now follows immediately from Theorem 2·2. To determine the stability in the critical cases it is simplest to note that by the same proof as in Proposition 2·3 there is always an absolute minimizer of $\Psi(v, T)$. Also it is not possible for all the critical points of $\Psi(v, T)$ to be local minima. Hence (i) the nontrivial solutions corresponding to the limit points in Fig. 5·1 (a), (c) are unstable, (ii) at the primary bifurcation point $T = T_0$ the trivial solution is unstable when $\mu \neq 3\nu$, stable if $\mu = 3\nu$, (iii) the solutions at the secondary bifurcation points are neutrally stable.

Some caution should be attached to the interpretation of our stability results on account of a remark of Sawyers (17), who showed for a cube of neo-Hookean material that the nontrivial neutrally stable purely homogeneous solutions with $v_i > 0$ may become unstable when one of the pairs of equal and opposite surface tractions is slightly perturbed. Of course this behaviour is only to be expected in a case of neutral stability; a small desymmetrization of the problem will typically result in the continuum of equilibria $x = Q\,\mathrm{diag}\,(v_1, v_2, v_3)\,Q^TX$, $Q \in O(3)$, splitting up into a discrete number of neighbouring equilibria, some stable and some unstable.

As a numerical example we take the case $k = 7\cdot 1$. This was the value obtained by Rivlin (15) in his experiment on torsion of a cylinder of pure gum rubber. From (4·11), (5·9), (5·11) we find that

$$T_0 = 2\cdot 3\mu, \quad T_{\min} = 5\cdot 3\mu, \quad T_{\max} = 13\cdot 0\mu,$$

that at $T = T_{\min}$,

$$v_1 = 0\cdot 16, \quad v_2 = v_3 = 2\cdot 51,$$

and that at $T = T_{\max}$,

$$v_1 = 12\cdot 31, \quad v_2 = v_3 = 0\cdot 28.$$

When assessing the significance of such numerical results one should bear in mind inter alia that (i) it would be difficult experimentally to apply uniform dead load tractions to the entire surface (ii) the Mooney–Rivlin stored-energy function is not very accurate for large strains (iii) the bifurcations at $T = T_0$, $T_{\min}$, $T_{\max}$ may be preceded by other bifurcations into nonhomogeneous deformations or by rupture instabilities (perhaps of the type discussed in (3, 7)); certainly in the above example one would be unlikely to reach a stretch $v_1 = 12\cdot 31$ on account of rupture.

6. *On the robustness of the observed behaviour*

As the reader will have noted, Theorems 3·2 and 3·3 are quite general and could be applied to give information about homogeneous equilibrium solutions for a general isotropic material not necessarily having the form (4·1). It is perhaps useful to summarize the conclusions of these theorems in less technical language. Because of symmetry $v_1 = v_2 = v_3 = 1$ is always a critical point of $\Psi(v, T)$. Also because of symmetry, there is only one free coefficient in the quadratic terms of the Taylor expansion of Ψ at this critical point, and one in the cubic terms. Bifurcation occurs when, as T increases, the quadratic terms change sign, changing a local minimum of Ψ into a local maximum. The content of Theorem 3·2 is that, provided there are nonzero cubic terms in Ψ at the bifurcation point, the local structure of the resulting bifurcation is completely determined, up to diffeomorphism. Fig. 3·2 provides a representative picture. When the cubic terms vanish Theorem 3·3 is applicable, assuming that the nondegeneracy conditions (3·24) are satisfied. The full range of phenomena pictured in Fig. 3·3 will occur if the given problem is embedded in a one parameter family of problems such that the cubic terms at the bifurcation point change sign as the parameter is varied. Moreover, the behaviour of the one parameter family of bifurcation problems is robust with respect to perturbation – a perturbed one parameter family of bifurcation problems will run through exactly the same sequence of bifurcation diagrams as the parameter varies, differing only by appropriate diffeomorphisms. In particular, the cubic terms contributed by the two parts of (1·7) have opposite signs, and the degenerate case $\mu = 3\nu$ occurs when they are exactly balanced. (*Warning*: $T(v_1 + v_2 + v_3)$ also contributes cubic terms.)

The conclusions of the preceding paragraph were purely local, restricted to an appropriate neighbourhood of the trivial solution $v_1 = v_2 = v_3 = 1$. However it is also possible to extract quasi-global conclusions using singularity theory methods. As an example, consider an arbitrary (smooth, symmetry preserving) perturbation of $\Phi(v)$ given by (1·7), say $\Phi(v)+\epsilon P(v,\epsilon)$, and the corresponding equilibrium equations

$$\frac{\partial \Phi}{\partial v_i}+\epsilon\frac{\partial P}{\partial v_i}-T=\frac{p}{v_i} \quad (i=1,2,3), \tag{6·1i}$$

$$v_1v_2v_3=1. \tag{6·1ii}$$

For any bounded set

$$\mathcal{O}=\{(v,T):0<v_i<B, 0<T<C\},$$

there is a symmetry preserving diffeomorphism of $\mathcal{O}$ which maps the solutions of (5·1) in $\mathcal{O}$ onto the solutions of (6·1) in $\mathcal{O}$, provided $\mathcal{O}$ is sufficiently small. Of course there is a trade-off between the size of $\mathcal{O}$ and the maximum allowable ϵ, and some care is required near the boundary of $\mathcal{O}$, where the perturbation may push a solution of (5·1) across $\partial\mathcal{O}$. We do not give a precise formulation or proof of this result here, referring the reader instead to Theorem 2·15 of (8), an analogous result whose proof uses the same techniques. The essential point is that all the bifurcation phenomena depicted in Fig. 5·1 – the transcritical primary bifurcations in cases (*a*) and (*c*), the limit points, the secondary bifurcations – are stable with respect to symmetry preserving perturbations, except the degenerate primary bifurcation in case (*b*) when $\mu/\nu=3$, and, as discussed above, this phenomenon has codimension 1 and is stable within a one parameter family of bifurcation problems.

Another type of perturbation which can be treated is that in which the material is allowed to be slightly compressible. This is because, as shown by Rivlin (18), allowing a slight degree of compressibility in an incompressible material is in fact a regular perturbation. In our notation the essence of Rivlin's argument may be summarized as follows. Consider a compressible isotropic material with stored-energy function

$$\Phi(v)=\Phi_0(v)+\frac{(\tau-1)^2}{2\epsilon}\Phi_2(v,\epsilon), \tag{6·2}$$

where $\tau=v_1v_2v_3$ and $\epsilon\to 0$ describes the incompressible limit. We suppose that Φ_2 is positive and bounded away from zero. The equilibrium equations for purely homogeneous deformations with $v_i>0$ for this material are

$$\frac{\partial \Phi}{\partial v_i}=T \quad (i=1,2,3). \tag{6·3}$$

On writing out (6·3) we obtain the equations

$$\frac{\partial \Phi_0}{\partial v_i}+\frac{\tau-1}{\epsilon}\Phi_2\frac{\tau}{v_i}+\frac{(\tau-1)^2}{\epsilon}\frac{\partial \Phi_2}{\partial v_i}=T \quad (i=1,2,3). \tag{6·4}$$

Let us introduce a new variable $p=((\tau-1)/\epsilon)\Phi_2$. We may then augment (6·4) to a 4×4 system of equations

$$\frac{\partial \Phi_0}{\partial v_i}+\frac{\epsilon p^2}{\Phi_2^2}\frac{\partial \Phi_2}{\partial v_i}-T=p\frac{\tau}{v_i} \quad (i=1,2,3), \tag{6·5i}$$

$$\tau-1=-\frac{\epsilon p}{\Phi_2}.$$

When $\epsilon=0$, (6·5) yields the equations describing a critical point of $\Phi_0-T(v_1+v_2+v_3)$ in the surface $v_1v_2v_3=1$. In §3 we reduced these equations when $\epsilon=0$ to a pair of equations in two real unknowns and the bifurcation parameter T. In other words, we used two of the equations to solve for p and τ; the remaining two equations contributed the bifurcation equations. By the implicit function theorem this same reduction can be done for ϵ nonzero, and the result is an equivariant bifurcation problem

$$g(z, \lambda, \epsilon) = 0,$$

where $g: \mathbb{C} \times \mathbb{R} \times \mathbb{R} \to \mathbb{C}$. In short, we get a smooth unfolding of the original problem. Thus it follows from the unfolding theorem that the same bifurcation phenomena occur.

There is an extensive literature concerning the bifurcations, near a stress-free state, of solutions to the three-dimensional traction problem for a compressible material (cf. (21, 6)). For some remarks on the relationship between strict rank 1 convexity and homogeneous equilibrium solutions in the compressible traction problem see (2).

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REFERENCES

(1) ADELEKE, S. A. Stability of some states of plane deformation. *Arch. Rat. Mech. Anal.* **72** (1980), 243–263.
(2) BALL, J. M. Strict convexity, strong ellipticity, and regularity in the calculus of variations. *Math. Proc. Cambridge Philos. Soc.* **87** (1980), 501–513.
(3) BALL, J. M. Discontinuous equilibrium solutions and cavitation in nonlinear elasticity. *Phil. Trans. Roy. Soc. London A* **306** (1982), 557–611.
(4) BEATTY, M. F. A theory of elastic stability for incompressible hyperelastic bodies. *Int. J. Solids Struct.* **3** (1967), 23–37.
(5) BERGER, M. S. Bifurcation theory and the type numbers of Marston Morse. *Proc. Nat. Acad. Sci. U.S.A.* **69** (1972), 1737–8.
(6) CHILLINGWORTH, D. R. J., MARSDEN, J. E. and WAN, Y. H. Symmetry and bifurcation in three dimensional elasticity: Part I. *Arch. Rat. Mech. Anal.* **80** (1983), 295–331.
(7) GENT, A. N. and LINDLEY, P. B. Internal rupture of bonded rubber cylinders in tension. *Proc. Roy. Soc. London Ser. A* **249** (1958), 195–205.
(8) GOLUBITSKY, M. and SCHAEFFER, D. A theory for imperfect bifurcation theory via singularity theory. *Comm. Pure Appl. Math.* **32** (1979), 21–98.
(9) GOLUBITSKY, M. and SCHAEFFER, D. Imperfect bifurcation in the presence of symmetry. *Comm. Math. Phys.* **67** (1979), 205–232.
(10) GOLUBITSKY, M. and SCHAEFFER, D. Bifurcations with $O(3)$ symmetry, including applications to the Benard problem. *Comm. Pure Appl. Math.* (in the Press).
(11) KOITER, W. T. A basic open problem in the theory of elastic stability. In *Applications of Functional Analysis to Problems in Mechanics*, Springer Lecture Notes in Mathematics, vol. 503, pp. 366–373.
(12) HILL, R. On uniqueness and stability in the theory of finite elastic strain. *J. Mech. Phys. Solids* **5** (1957), 229–241.
(13) HILL, R. Eigenmodal deformations in elastic/plastic continua. *J. Mech. Phys. Solids* **15** (1967), 371–386.
(14) JONES, D. F. and TRELOAR, L. R. G. The properties of rubber in pure homogeneous strain. *J. Phys. D (Appl. Phys.)* **8** (1975), 1285–1304.
(15) RIVLIN, R. S. Torsion of a rubber cylinder. *J. Appl. Phys.* **18** (1947), 444–449.
(16) RIVLIN, R. S. Large elastic deformations of isotropic materials. II. Some uniqueness theorems for pure, homogeneous, demormations. *Phil. Trans. Roy. Soc. London Series A* **240** (1948), 491–508.
(17) RIVLIN, R. S. Stability of pure homogeneous deformations of an elastic cube under dead loading. *Quart. Appl. Math.* **32** (1974), 265–271.

(18) RIVLIN, R. S. Some research directions in finite elasticity. *Rheo. Acta* **16** (1977), 101–112.

(19) SAWYERS, K. N. Stability of an elastic cube under dead loading: two equal forces. *Int. J. Non-Linear Mechanics* **11** (1976), 11–23.

(20) SCHWARTZ, G. Smooth functions invariant under the action of a compact Lie group. *Topology* **14** (1975), 63–68.

(21) SHIELD, R. T. Deformations possible in every compressible, isotropic, perfectly elastic material. *J. Elasticity* **1** (1971), 91–92.

(22) STOPPELLI, F. Sull'existenza di soluzioni delli equazioni dell'elastostatica isoterma nel case di sollectizioni dotate di assi di equilibrie. *Richerche Mat.* **6** (1957), 261–287; **7** (1958), 71–101, 138–152.

(23) TRUESDELL, C. A. Some challenges offered to analysis by rational thermomechanics. In *Contemporary developments in continuum mechanics and partial differential equations*, ed. G. M. de La Penha and L. A. Medeiros (North-Holland, Amsterdam, 1978).