Proceedings of the Royal Society of Edinburgh, **108A**, 109–116, *1988*

Asymptotic behaviour of solutions to the Becker–Döring equations for arbitrary initial data

J. M. Ball and **J. Carr**

Department of Mathematics, Heriot-Watt University, Riccarton, Edinburgh EH14 4AS, Scotland

(MS received 12 June 1987)

Synopsis

The asymptotic behaviour of solutions to the Becker–Döring cluster equations is determined without the decay assumptions on the initial data made in **[3]**.

1. Introduction

The process of cluster formation has attracted considerable interest in many areas of pure and applied science **[1, 4, 5, 7]**. In a recent paper **[3]**, the mathematical theory of one of the models in this area was discussed, namely the Becker–Döring equations. If $c_r(t) \geqq 0$, $r = 1, 2, \ldots$, denotes the expected number of r-particle clusters per unit volume at time t, then the Becker–Döring equations are

$$\left.\begin{aligned} \dot{c}_r &= J_{r-1}(c) - J_r(c), \quad r \geqq 2 \\ \dot{c}_1 &= -J_1(c) - \sum_{r=1}^{\infty} J_r(c), \end{aligned}\right\} \tag{1.1}$$

where $c = (c_r)$, $J_r(c) = a_r c_1 c_r - b_{r+1} c_{r+1}$ and a_r, b_{r+1} are positive constants. The density of a solution of (1.1) is given by $\rho = \sum_{r=1}^{\infty} r c_r(t)$ and it is a conserved quantity.

One of the main results in **[3]** concerns the asymptotic behaviour of solutions of (1.1) for rapidly decaying initial data. It is proved that there is a critical density ρ_s such that if $0 \leqq \rho \leqq \rho_s$ there is a unique equilibrium solution with density ρ, while if $\rho > \rho_s$ there is no equilibrium with density ρ. If the initial density of a solution is denoted by ρ_0, then if $\rho_0 \leqq \rho_s$ the solution converges strongly in an appropriate sequence space X^+ to c^{ρ_0}, while if $\rho_0 > \rho_s$ the solution converges weak* to c^{ρ_s}. In the latter case the excess density $\rho_0 - \rho_s$ corresponds to the formation of larger and larger clusters as t increases, i.e. condensation. In this paper we introduce a technique which allows us to remove the restriction on the initial data. Since the ideas may be of use for other classes of equations we first present the method in an informal way in the hope that this reveals the motivation.

Let $T(t)$, $t \geqq 0$, be a semigroup on a metric space E and suppose that $\mathcal{V}$ is a continuous Lyapunov function for the flow $T(t)$. If $\phi \in E$ and the positive orbit $\mathcal{O}^+(\phi) \stackrel{\text{def}}{=} \bigcup_{t \geqq 0} T(t)\phi$ is relatively compact in E then (cf. **[6]**) the invariance of the ω-limit set $\omega(\phi) \stackrel{\text{def}}{=} \{\psi \in E : T(t_n)\phi \to \psi$ for some sequence $t_n \to \infty\}$ and the continuity of $\mathscr{V}$ can frequently be exploited to determine $\omega(\phi)$. However, the relative compactness of $\mathcal{O}^+(\phi)$ is often very difficult to prove and can be false. For example, our results will show that in the natural space $E = X^+$ the positive orbit of a solution to (1.1) is relatively compact if and only if the initial density ρ_0 satisfies $\rho_0 \leqq \rho_s$.

One way out of this impasse is to find another topology on E for which $T(t)$ retains the necessary continuity properties, $\mathscr{V}$ is still continuous and $\mathcal{O}^+(\phi)$ is relatively compact, and hence to obtain results on the behaviour of $T(t)\phi$ as $t \to \infty$ with respect to the new topology. This gives extra information which we could try to exploit to obtain relative compactness of appropriate orbits in the original metric. For example, this goal might be achieved by restricting the initial data to some $Y \subset E$. This was the method used in [**3**], the decay assumptions on the initial data allowing the construction of a supersolution which implies the relative compactness of orbits with $\rho_0 \leqq \rho_s$. Roughly speaking, the observation that enables us now to remove the restriction on the initial data is that, with respect to the new variables

$$x_n \overset{\text{def}}{=} \sum_{r=n}^{\infty} r c_r, \tag{1.2}$$

a supersolution can be constructed regardless of how slowly the initial data $x_n(0)$ tends to zero as $n \to \infty$. More precisely, we show first in Theorem 2 that there is a family of positive sequences $\lambda = (\lambda_n)$ satisfying $\lambda_n \to 0$ as $n \to \infty$ such that for any solution of (1.1) with initial density $\rho_0 < \rho_s$,

$$\sup_n \lambda_n^{-1} x_n(0) < \infty \quad \text{implies} \quad \sup_{t>0} \sup_n \lambda_n^{-1} x_n(t) < \infty. \tag{1.3}$$

Then, by means of a simple construction in Lemmas 3 and 4, we show that the sets $Y_\lambda = \{c \in X^+ \colon \sup_n \lambda_n^{-1} x_n < \infty\}$ cover X^+. Thus, given any solution with $\rho_0 < \rho_s$ we can find a corresponding supersolution guaranteeing relative compactness of the positive orbit.

The main result on asymptotic behaviour is given in Theorem 5. This theorem is in fact proved under hypotheses which are shown in [**3**] to guarantee the existence of solutions with arbitrary initial data in X^+, but which are not known to imply uniqueness. Even though our hypotheses may not, therefore, imply that (1.1) generates a semigroup on X^+, they do imply that it generates a "generalised flow" [**3**, Theorem 3.4, 3.5] and the strategy outlined above is unaffected. Further hypotheses guaranteeing uniqueness, and that (1.1) generates a semigroup, are given in [**3**, Theorems 3.6–3.8].

2. Preliminaries (see [3])

We first introduce some notation. Let

$$X = \{y = (y_r) \colon \|y\| < \infty\}, \quad \|y\| = \sum_{r=1}^{\infty} r\,|y_r|,$$

and let $X^+ = \{y \in X\colon \text{ each } y_r \geqq 0\}$. We shall also use another notion of convergence in X. We say that a sequence $\{y^j\}$ of elements of X converges weak* to $y \in X$ (symbolically $y^j \overset{*}{\rightharpoonup} y$) if (i) $\sup\{\|y^j\|: j=1,2,\ldots\}<\infty$ and (ii) $y_r^j \to y_r$ as $j \to \infty$, for each $r=1,2,\ldots$. We can express the weak* convergence as convergence in a metric space. For $\rho>0$, let $B_\rho=\{(y_r)\in X: \|y\|\leqq \rho\}$, $d(y,z)=\sum_{r=1}^{\infty}|y_r-z_r|$. Then (B_ρ, d) is a metric space. Clearly convergence in (B_ρ, d) is equivalent to weak* convergence and (B_ρ, d) is compact.

We shall make use of the following hypotheses concerning the coefficients in the Becker-Döring equations:

HYPOTHESIS 1. $a_r>0$, $b_r>0$ for all r.

HYPOTHESIS 2. Let $Q_1=1$ and $Q_{r+1}=(a_r/b_{r+1})Q_r$, $r\geqq 1$. Then $\lim_{r\to\infty} Q_r^{1/r}=z_s^{-1}$ exists with $0<z_s<\infty$.

HYPOTHESIS 3. $a_r=0(r/\ln r)$, $b_r=0(r/\ln r)$.

HYPOTHESIS 4. If $0\leqq z<z_s$, then $a_r z\leqq b_r$ for r sufficiently large.

The density of a solution of (1.1) is given by $\sum_{r=1}^{\infty} rc_r(t)$ and it is a conserved quantity. Thus we look for equilibrium solutions $c^\rho=(c_r^\rho)$ with $\rho=\sum_{r=1}^{\infty} rc_r^\rho$. From (1.1), $J_r(c^\rho)=0$ for all r so that $c_r^\rho=Q_r(c_1^\rho)^r$, where Q_r is defined in Hypothesis 2. In order to identify c_1^ρ, let $F(z)=\sum_{r=1}^{\infty} rQ_r z^r$. By Hypothesis 2 this series has a finite radius of convergence z_s. Let $\rho_s=\sup\{F(z): 0\leqq z\leqq z_s\}$. Then if $0\leqq\rho\leqq\rho_s$ and $\rho<\infty$, there is exactly one equilibrium given by $c_r=Q_r(z(\rho))^r$, where $z(\rho)$ is the unique root of $F(z)=\rho$. If $\rho_s<\rho<\infty$, then there is no equilibrium state with density ρ.

Let $V(c)=\sum_{r=1}^{\infty} c_r[\ln(c_r/Q_r)-1]$. Then V is a Lyapunov function for (1.1); that is, it is nonincreasing along solutions. For $0\leqq\rho\leqq\rho_s<\infty$ the equilibrium c^ρ is the unique minimiser of V on the set $X_\rho^+=\{c\in X^+: \|c\|=\rho\}$. Furthermore, every minimising sequence c^j of V on X_ρ^+ converges to c^ρ strongly in X. If $\rho_s<\rho<\infty$ and c^j is a minimising sequence of V on X_ρ^+, then c^j converges weak* to c^{ρ_s} in X but not strongly in X. (This variational structure occurs in other applications; see [**2**].) The aim of this paper is to show that if $c(t)$ is a solution of (1.1) with density ρ, then it is minimising for V on X^+ as $t\to\infty$, so that in particular if $0\leqq\rho_0\leqq\rho_s<\infty$, then $c(t)\to c^{\rho_0}$ strongly in X while for $\rho_0>\rho_s$, $c(t)\overset{*}{\rightharpoonup} c^{\rho_0}$ in X. The starting point for the analysis is the following result [**3**, Theorem 5.5]:

THEOREM 1. *Let Hypotheses* 1–3 *hold. Let* $c(t)$ *be a solution of* (1.1) *on* $[0,\infty)$ *and let* $\rho_0=\sum_{r=1}^{\infty} rc_r(0)$. *Then* $c(t)\overset{*}{\rightharpoonup} c^\rho$ *as* $t\to\infty$ *for some* ρ *with* $0\leqq\rho\leqq \min(\rho_0,\rho_s)$.

Thus to prove the desired result on asymptotic behaviour we have to show that if $\rho_0>\rho_s$ then $\rho=\rho_s$, while if $\rho_0\leqq\rho_s$ then $\rho=\rho_0$ and the convergence is strong.

Suppose that $\rho_0<\rho_s$. By Theorem 1, $c_1(t)\to z(\rho)$ as $t\to\infty$ for some $\rho<\rho_s$ with $F(z(\rho))=\rho$. Since $z(\rho)<z(\rho_s)=z_s$, there exist $T>0$ and $z<z_s$ such that

$$c_1(t)<z, \quad t\geqq T. \tag{2.1}$$

This global bound on $c_1(t)$ is the key to proving the desired compactness results.

Using (2.1), we proved in [**3**] that if $\lambda_n\to 0$ very rapidly, then $\sum_{n=1}^{\infty}\lambda_n^{-1}c_n(0)<\infty$

implies that $\sup_{t\geqq 0}\sum_{n=1}^{\infty}\lambda_n^{-1}c_n(t)<\infty$. Compactness then followed from the estimate

$$x_n(t)=\sum_{r=n}^{\infty} rc_r(t)\leqq \text{constant}\sum_{r=n}^{\infty} r\lambda_r. \tag{2.2}$$

Since this method is only useful if the series on the right-hand side of (2.2) is convergent, we work directly with the variable $x_n(t)$.

3. Results

The first result allows us to control $\lambda_n^{-1}x_n(t)$ for certain sequences (λ_n).

THEOREM 2. *Assume Hypothesis* 1 *holds and that* $a_r=O(r)$. *Let* $0<z<\infty$ *and suppose that* $a_rz\leqq b_r$ *for all* $r\geqq r_0\geqq 2$. *Let* (λ_r) *be a positive nonincreasing sequence such that* 7

$$\lambda_r-\lambda_{r+1}\geqq \mu_r(\lambda_{r-1}-\lambda_r)a_{r-1}z/b_r, \tag{3.1}$$

for all $r\geqq r_0$, *where*

$$\mu_r=\frac{r}{r-1}[1-za_r/rb_r]^{-1}. \tag{3.2}$$

Let $c_0\in X^+$ *and let* c *be a solution of* (1.1) *on* $[0,\infty)$, *with* $c(0)=c_0$ *satisfying* $c_1(t)<z$ *for all* $t\geqq 0$. *Suppose, further, that* c *is the only solution of* (1.1) *with initial data* c_0. *Then*

$$H(t)=\max\Big(\sup_{n\geqq r_0+1}\lambda_n^{-1}x_n(t),\ \lambda_{r_0}^{-1}\rho\Big)$$

is nonincreasing on $[0,\infty)$, *where* $\rho=\sum_{r=1}^{\infty} rc_r$.

Proof. Notice that if the solution c with initial data c_0 is unique, then so is that with initial data $c(\tau)$ for any $\tau\geqq 0$. Hence it suffices to prove that if $H(0)<\infty$ and $T>0$, then given any $\varepsilon>0$,

$$H(t)\leqq H(0)+\varepsilon \quad \text{for all} \quad t\in[0,T]. \tag{3.3}$$

Because of technical problems with the infinite dimensional system (1.1), we first prove a modified form of (3.3) for a finite dimensional approximation to (1.1). For $m\geqq 3$ define $c_0^{(m)}$ by $c_{or}^{(m)}=c_{or}$ if $1\leqq r\leqq m$, or $c_{or}^{(m)}=0$ if $r>m$, and let $c^{(m)}: [0,T]\to X^+$ be the unique solution of

$$\left.\begin{aligned} \dot c_r^{(m)}&=J_{r-1}(c^{(m)})-J_r(c^{(m)}),\quad 2\leqq r\leqq m-1\\ \dot c_1&=-J_1(c^{(m)})-\sum_{r=1}^{m-1}J_r(c^{(m)}),\quad \dot c_m^{(m)}=J_{m-1}(c^{(m)}),\\ c^{(m)}(0)&=c_0^{(m)},\quad c_r^{(m)}(t)=0,\quad r>m.\end{aligned}\right\} \tag{3.4}$$

Since $c_0^{(m)}\to c_0$ strongly in X as $m\to\infty$, by the uniqueness hypothesis and the proof of [**3**, Theorem 3.4], $c^{(m)}\to c$ in $C([0,T];X)$. In particular, there exists m_0 such that $c_1^{(m)}(t)<z$ for all $t\in[0,T]$ and $m\geqq m_0$. Fix $m\geqq m_0$ and set $y_n(t)=\sum_{r=n}^{\infty} rc_r^{(m)}(t)$, $g(t)=\sup_{n\geqq r_0+1}\lambda_n^{-1}y_n(t)$, $H^{(m)}(t)=\max(g(t),\lambda_{r_0}^{-1}\rho(c^{(m)}))$, where

$\rho(c^{(m)}) = y_1(t) = \text{constant}$. We first prove that, given any $\varepsilon > 0$,

$$H^{(m)}(t) \leqq H^{(m)}(0) + \varepsilon \quad \text{for all} \quad t \in [0, T]. \tag{3.5}$$

Suppose (3.5) is false. Since $H^{(m)}$ is continuous, there exists a least $s \in [0, T]$ such that $H^{(m)}(s) = K_\varepsilon := H^{(m)}(0) + \varepsilon$. Since $H^{(m)}(0) \geqq \lambda_{r_0}^{-1}\rho(c^{(m)})$, we have that $g(s) = K_\varepsilon$. Thus $\lambda_N^{-1} y_N(s) = K_\varepsilon$ for some minimal N with $m \geqq N \geqq r_0 + 1$ and

$$\lambda_{N-1}^{-1} y_{N-1}(s) < K_\varepsilon, \quad \lambda_{N+1}^{-1} y_{N+1}(s) \leqq K_\varepsilon, \quad \dot{y}_N(s) \geqq 0. \tag{3.6}$$

From the definition of y_N and (3.4)

$$\begin{aligned} \dot{y}_N &= \sum_{r=N}^{\infty} J_r(c^{(m)}) + NJ_{N-1}(c^{(m)}) - J_m(c^{(m)}) \\ &\leqq \sum_{r=N}^{\infty} J_r(c^{(m)}) + NJ_{N-1}(c^{(m)}) \\ &= \sum_{r=N+1}^{\infty} (a_r c_1^{(m)} - b_r) c_r^{(m)} + a_N c_1^{(m)} c_N^{(m)} + N(a_{N-1} c_1^{(m)} c_{N-1}^{(m)} - b_N c_N^{(m)}). \end{aligned}$$

By using $c_1^{(m)} < z$ and $a_r z \leqq b_r$ for $r \geqq r_0$ we thus have that

$$\begin{aligned} \dot{y}_N &\leqq a_N z c_N^{(m)} + N(a_{N-1} z c_{N-1}^{(m)} - b_N c_N^{(m)}) \\ &= N a_{N-1} z \left(\frac{y_{N-1} - y_N}{N-1} \right) - (N b_N - a_N z) \left(\frac{y_N - y_{N+1}}{N} \right) \end{aligned}$$

on $[0, T]$. Hence, using $N b_N \geqq a_N z$ and (3.6),

$$\begin{aligned} 0 \leqq \dot{y}_N(s) &< \frac{K_\varepsilon N b_N}{N-1} \left[(\lambda_{N-1} - \lambda_N) \frac{a_{N-1} z}{b_N} - \mu_N^{-1} (\lambda_N - \lambda_{N+1}) \right] \\ &\leqq 0. \end{aligned}$$

This contradiction proves (3.5). Since $H^{(m)}(0) \leqq H(0)$, it follows that

$$\lambda_n^{-1} \sum_{r=n}^{\infty} r c_r^{(m)}(t) \leqq H(0) + \varepsilon$$

for all $t \in [0, T]$, $n \geqq r_0 + 1$ and $m \geqq m_0$. Letting $m \to \infty$ and taking the supremum over $n \geqq r_0 + 1$ gives the result. □

Let $v_r > 0$ for $r \geqq r_0$ and define

$$S = \{\lambda = (\lambda_r) : \lambda_r \geqq \lambda_{r+1} \geqq 0 \text{ for all } r, \ \lambda_r - \lambda_{r+1} \geqq v_r(\lambda_{r-1} - \lambda_r) \quad \text{for all} \quad r \geqq r_0\}.$$

LEMMA 3. *S is closed under* (i) *addition,* (ii) *multiplication by a constant* $M \geqq 0$ *and* (iii) *taking infima.*

Proof. (i) and (ii) are obvious. To prove (iii), suppose that $\lambda_r = \inf_\alpha \lambda_r^{(\alpha)}$, and that

$$\lambda_r^{(\alpha)} \geqq \lambda_{r+1}^{(\alpha)} \geqq 0, \qquad \text{for all } r, \alpha, \tag{3.7}$$

$$\lambda_r^{(\alpha)} - \lambda_{r+1}^{(\alpha)} \geqq v_r(\lambda_{r-1}^{(\alpha)} - \lambda_r^{(\alpha)}), \quad \text{for all } \alpha, r \geqq r_0. \tag{3.8}$$

From (3.7) we have $\lambda_r^{(\alpha)} \geqq \lambda_{r+1} \geqq 0$ and hence $\lambda_r = \lambda_{r+1} \geqq 0$.
Writing (3.8) in the form

$$(1+\nu_r)\lambda_r^{(\alpha)} \geqq \lambda_{r+1}^{(\alpha)} + \nu_r \lambda_{r-1}^{(\alpha)}, \quad \text{for} \quad r \geqq r_0,$$

we have

$$(1+\nu_r)\lambda_r^{(\alpha)} \geqq \lambda_{r+1} + \nu_r \lambda_{r-1}, \quad \text{for} \quad r \geqq r_0,$$

and hence

$$(1+\nu_r)\lambda_r \geqq \lambda_{r+1} + \nu_r \lambda_{r-1}, \quad \text{for} \quad r \geqq r_0,$$

as required. □

LEMMA 4. *Suppose that there exists some positive sequence* $(\eta_r) \in S$ *satisfying* $\lim_{r\to\infty} \eta_r = 0$. *Let* (σ_r) *be any nonnegative sequence with* $\lim_{r\to\infty} \sigma_r = 0$. *Then there exists a strictly positive sequence* $\hat{\lambda} \in S$ *such that* $\hat{\lambda}_r \geqq \sigma_r$ *for all r and* $\lim_{r\to\infty} \hat{\lambda}_r = 0$.

Proof. Let $\bar{\sigma}_r \geqq \sigma_r$, $\bar{\sigma}_r > 0$, $\lim_{r\to\infty} \bar{\sigma}_r = 0$. Consider the set $\mathscr{S}$ of sequences $\lambda \in S$ satisfying $\lambda_r \geqq \bar{\sigma}_r$ for all r. The set $\mathscr{S}$ is nonempty since $\lambda_r \equiv K$ belongs to $\mathscr{S}$ for K sufficiently large. Define $\hat{\lambda}_r = \inf_{\lambda \in \mathscr{S}} \lambda_r$. Then $\hat{\lambda} \in S$ by Lemma 2 and $\hat{\lambda}_r \geqq \bar{\sigma}_r$ for all r. Suppose for contradiction that $\hat{\lambda}_r \not\to 0$ as $r \to \infty$. Then there exists $\varepsilon > 0$ such that $\hat{\lambda}_r \geqq \varepsilon$ for all r. Let m be such that $\bar{\sigma}_r \leqq \varepsilon/2$ for all $r \geqq m$ and consider the sequence $\mu_r^{(\beta)} = \varepsilon/2 + \beta\eta_r$. By Lemma 3, $\mu^{(\beta)} \in S$ for all $\beta \geqq 0$. Clearly, if β is sufficiently large we have $\mu_r^{(\beta)} \geqq \bar{\sigma}_r$ for all $r \geqq 0$. But then $\hat{\lambda}_r \leqq \mu_r^{(\beta)} < \varepsilon$ for r sufficiently large, a contradiction. □

THEOREM 5. *Let Hypotheses* 1–4 *hold. Suppose that* $c_0 \in X^+$ *and that c is the only solution of* (1.1) *on* $[0, \infty)$ *with initial data* c_0. *Let* $\rho_0 = \sum_{r=1}^\infty rc_r(0)$. *Then*

(i) *if* $0 \leqq \rho_0 \leqq \rho_s$, $c(t) \to c^{\rho_0}$ *strongly in X as* $t \to \infty$ *and*

$$\lim_{t\to\infty} V(c(t)) = V(c^{\rho_0}),$$

(ii) *if* $\rho_0 > \rho_s$, $c(t) \overset{*}{\rightharpoonup} c^{\rho_s}$ *as* $t \to \infty$, *and*

$$\lim_{t\to\infty} V(c(t)) = V(c^{\rho_s}) + (\rho_0 - \rho_s) \ln z_s.$$

Proof. We show that if $c(t) \overset{*}{\rightharpoonup} c^\rho$ as $t \to \infty$ for some $\rho < \rho_s$, then $c(t) \to c^\rho$ strongly, so that by density conservation $\rho = \rho_0$. The assertions in the theorem for $\rho_0 < \rho_s$ and $\rho_0 > \rho_s$ then follow immediately from Theorem 1. If $\rho_0 = \rho_s$, then we deduce that $\rho = \rho_s$ and the strong convergence follows from the fact that if a sequence $y^{(j)} \overset{*}{\rightharpoonup} y$ in X and $\|y^{(j)}\| \to \|y\|$ then $y_j \to y$ in X (cf. [**3**, Lemma 3.3]). The statements concerning $\lim_{t\to\infty} V(c(t))$ follow from the fact that Hypothesis 2 implies the sequential weak* continuity on X^+ of $\mathscr{V}(c) \overset{\text{def}}{=} V(c) - \ln z_s \sum_{r=1}^\infty rc_r$(cf. [**3**, Proposition 4.5]).

Without loss of generality we can suppose that $c_1(t) < z$ for some $z < z_s$ and all $t \geqq 0$. Let r_0 be such that $a_r z \leqq b_r$ for $r \geqq r_0$. Let $\sigma_n = \sum_{r=n}^{\infty} rc_{0r}$ so that $\sigma_n \to 0$ as $n \to \infty$. For $r \geqq r_0$ let $\nu_r = \mu_r a_{r-1} z / b_r$ where μ_r is given by (3.2) and define

$$\eta = \sum_{s=r}^{\infty} s\gamma_s,$$

where

$$\gamma_r = 1 \quad \text{for} \quad 0 \leqq r < r_0, \quad \text{and} \quad \gamma_r = \frac{a_{r-1}z}{b_r}(1 - za_r/rb_r)^{-1}\gamma_{r-1} \quad \text{for} \quad r \geqq r_0.$$

Now for $r \geqq r_0$,

$$\frac{r\gamma_r}{(r-1)\gamma_{r-1}} \leqq \frac{Q_r}{Q_{r-1}} z(1 - r^{-1})^{-2},$$

and so

$$r\gamma_r \leqq \frac{r^2}{(r_0 - 1)} \frac{Q_r z^r}{Q_{r_0-1} z^{r_0-1}}.$$

Since $\sum_{r=1}^{\infty} Q_r z_1^r$ is convergent for $z < z_1 < z_s$, it follows easily that $\gamma \in X$ and hence $\eta_r \to 0$ as $r \to \infty$. By construction $\eta \in S$. Let $\hat{\lambda}$ be the sequence given in Lemma 4. On applying Theorem 2, we deduce that

$$\sum_{r=n}^{\infty} rc_r(t) \leqq \hat{\lambda}_n \max(1, \lambda_{r_0}^{-1}\rho_0),$$

for all $n \geqq 1$ and $t \geqq 0$. Hence $\{c(t)\}$ is relatively compact in X and hence $c(t) \to c^{\rho_0}$ strongly in X as $t \to \infty$ as required.

Remark. Theorem 5 strengthens [**3**, Theorem 5.6], where, in addition to Hypotheses 1–3, it was assumed that

$$M \geqq b_{r+1} - a_r z_s, \quad b_r - a_r z_s \geqq 0 \tag{3.9}$$

for all r sufficiently large, where M is a constant, and that the decay hypothesis $\sum_{r=1}^{\infty} c_r(0)/Q_r z_s^r < \infty$ holds. We take this opportunity of pointing out, however, that the proof of [**3**, Theorem 5.6] works only in the case $\rho_s < \infty$, since the relative compactness of $\mathcal{O}^+(c)$ in X^+ requires the estimate $c_r(t) \leqq k_r$ for all $r \geqq 1$, $t \geqq 0$, where $\sum_{r=1}^{\infty} rk_r < \infty$. In the case $\rho_s = \infty$ the proof given in [**3**] works, replacing z_s by $z < z_s$ in both the hypotheses (3.9) and the proof.

References

1 M. Aizenman and T. A. Bak. Convergence to equilibrium in a system of reacting polymers. *Comm. Math. Phys.* **65** (1979), 203–230.

2 J. M. Ball. Loss of the constraint in convex variational problems. Volume for J.-L. Lions (to appear).

3 J. M. Ball, J. Carr and O. Penrose. The Becker–Döring cluster equations: basic properties and asymptotic behaviour of solutions. *Comm. Math. Phys.* **104** (1986), 657–692.

4 R. J. Cohen and G. B. Benedek. Equilibrium and kinetic theory of polymerization and the sol-gel transition. *J. Phys. Chem.* **86** (1982), 3696–3714.
5 R. Drake. In *Topics in current aerosol research. International reviews in aerosol physics and chemistry*, Vol. 2, eds. G. M. Hidy and J. R. Brock (Oxford: Pergamon Press 1972).
6 J. K. Hale. Dynamical systems and stability. *J. Math. Anal. Appl.* **26** (1969), 39–59.
7 E. M. Hendricks, M. H. Ernst and R. M. Ziff. Coagulation equations with gelation. *J. Statist. Phys.* **31** (1983), 519–563.

(*Issued* 3 *February* 1988)