Proceedings of the Royal Society of Edinburgh, **114A**, 367–379, *1990*

Lower semicontinuity of multiple integrals and the Biting Lemma

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(MS received 4 October 1989)

Synopsis

Weak lower semicontinuity theorems in the sense of Chacon's Biting Lemma are proved for multiple integrals of the calculus of variations. A general weak lower semicontinuity result is deduced for integrands which are a composition of convex and quasiconvex functions. The "biting" weak limit of the corresponding integrands is characterised via the Young measure, and related to the weak * limit in the sense of measures. Finally, an example is given which shows that the Young measure corresponding to a general sequence of gradients may not have an integral representation of the type valid in the periodic case.

Introduction

A well-known weak lower semicontinuity theorem in the multi-dimensional calculus of variations is the following:

THEOREM A (Acerbi and Fusco [**1**]). *Let $\Omega \subset R^n$ be bounded and open, and let*

$$I(u) = \int_\Omega f(x, u, Du)\, dx, \quad u \in W^{1,p}(\Omega; R^N),$$

where $1 \leqq p < \infty$, and where $f: \Omega \times R^N \times M^{N\times n} \to R$ satisfies
(i) *f is a Carathéodory function;*
(ii) *$0 \leqq f(x, u, P) \leqq a(x) + C(|u|^p + |P|^p)$, for every $x \in R^n$, $u \in R^N$ and $P \in M^{N\times n}$, where $C > 0$ and $a(\cdot) \in L^1(\Omega)$;*
(iii) *f is quasiconvex in P.*
Let $u^{(j)} \rightharpoonup u$ in $W^{1,p}(\Omega; R^N)$. Then

$$I(u) \leqq \liminf_{j\to\infty} I(u^{(j)}) \tag{0.1}$$

(*for the relevant definitions and notation see Section* 1.)

The main result of this paper (Theorem 2.1) generalises Theorem A by replacing the inequalities in (ii) by the weaker condition

(ii)′ $$|f(x, u, P)| \leqq a(x) + C(|u|^p + |P|^p).$$

Thus, for example, we can cover the cases when (a) $p \geqq 2$ and $f = Q(P)$ is quadratic and quasiconvex (equivalently, quadratic and rank-one convex), (b) $f = J_\alpha(P)$, where $J_\alpha(P)$ is some rth order minor of P and $p \geqq r$. However, in case (b) it is known that for $p = r$ the conclusion (0.1) does not hold ([**9**, Counterexample 7.3]), and thus (0.1) is replaced in Theorem 2.1 by a slightly weaker assertion (see (0.2)). To describe this we draw attention to the following result, known as Chacon's Biting Lemma. For proofs and more general statements of the lemma, see [**13, 24, 10**].

THEOREM B (Biting Lemma). *Let $\Omega \subset R^n$ be bounded measurable, and let $f^{(j)}$ be a bounded sequence in $L^1(\Omega)$. Then there exist a function $f \in L^1(\Omega)$, a subsequence $f^{(\nu)}$ of $f^{(j)}$, and a nonincreasing sequence of measurable subsets $E_k \subset \Omega$ with $\lim_{k\to\infty} \operatorname{meas}(E_k) = 0$, such that*

$$f^{(\nu)} \rightharpoonup f \quad \text{in } L^1(\Omega \backslash E_k),$$

as $\nu \to \infty$ for each fixed k.

The results E_k which are removed (bitten) from Ω are associated with possible concentrations of the sequence $f^{(j)}$.

If $u^{(j)} \rightharpoonup u$ in $W^{1,p}(\Omega; R^N)$, then under hypothesis (ii) the sequence

$$f^{(j)}(x) := f(x, u^{(j)}(x), Du^{(j)}(x))$$

is bounded in $L^1(\Omega)$, so that Theorem B applies. The conclusion of Theorem 2.1 is then that

$$\int_{\Omega\backslash E_k} f(x, u, Du)\, dx \leqq \liminf_{\nu\to\infty} \int_{\Omega\backslash E_k} f(x, u^{(\nu)}, Du^{(\nu)})\, dx \quad \text{for all } k. \tag{0.2}$$

The proof of Theorem 2.1 is based on Theorems A and B and the identification of the limit in Theorem B via the Young measure $(\nu_x)_{x\in\Omega}$ corresponding to $Du^{(\nu)}$. In fact, it is shown that for each k,

$$f(x, u^{(\nu)}, Du^{(\nu)}) \rightharpoonup l(x) := \langle \nu_x, f(x, u(x), \cdot)\rangle \quad \text{in } L^1(\Omega\backslash E_k),$$

and that

$$l(x) \geqq f(x, u(x), Du(x)) \quad \text{for almost every } x \in \Omega.$$

Since $f^{(\nu)}$ is bounded in $L^1(\Omega)$, we may extract a further subsequence, again denoted $u^{(\nu)}$, such that

$$f(x, u^{(\nu)}, Du^{(\nu)}) \overset{*}{\rightharpoonup} \mu \quad \text{in } \mathcal{M}(\Omega),$$

where $\mathcal{M}(\Omega)$ denotes the space of measures on Ω. We make some remarks concerning the relationship between the measure μ and l, and in particular (Theorem 2.4) give a sufficient condition that $\mu \geqq l$.

As an application of Theorem 2.1 we recover (Corollary 2.3) a result of Zhang [**29**] concerning the biting weak continuity of Jacobians in the critical case. As a further application we prove in Theorem 2.6 a lower semicontinuity theorem of classical type for integrands of the form

$$g(x, f_1(x, u, Du), \ldots, f_M(x, u, Du)),$$

where g is nonnegative and is a convex nondecreasing function of its last M arguments, and where the functions f_i satisfy the hypotheses of Theorem 2.1.

In Section 3 we make some remarks concerning possible extensions of our results to the framework of the theory of compensated compactness. We also disprove a plausible conjecture concerning the Young measure $(\nu_x)_{x\in\Omega}$ corresponding to a sequence of gradients $Du^{(j)}$; in particular, following an idea of Ball and Murat [11], we exhibit such a Young measure which is independent of x but is not realisable as the Young measure of any sequence $D\phi(jx)$ corresponding to a function $\phi \in W^{1,1}_{\text{loc}}(R^n; R^N)$ with $D\phi$ periodic with respect to a given cube.

1. Notation and preliminaries

Throughout the rest of this paper Ω denotes a bounded open subset of R^n. We denote by $M^{N\times n} \cong R^{nN}$ the space of real $N\times n$ matrices, with norm $|P| = (\text{tr}\, P^TP)^{\frac{1}{2}}$. We write $C_0(\Omega)$ for the space of continuous functions $\phi: \Omega \to R$ having compact support in Ω, and define $C_0^1(\Omega) = C^1(\Omega) \cap C_0(\Omega)$. If $1 \leqq p \leqq \infty$, we denote by $L^p(\Omega; R^N)$ the Banach space of mappings $u: \Omega \to R^N$, $u = (u_1, \ldots, u_N)$, such that $u_i \in L^p(\Omega)$ for each i, with norm $\|u\|_{L^p(\Omega;R^N)} = \sum_{i=1}^N \|u_i\|_{L^p(\Omega)}$. Similarly, we denote by $W^{1,p}(\Omega; R^N)$ the usual Sobolev space of mappings $u \in L^p(\Omega; R^N)$ all of whose distributional derivatives $\partial u_i/\partial x_j = u_{i,j}$, $1 \leqq i \leqq N$, $1 \leqq j \leqq n$, belong to $L^p(\Omega)$. $W^{1,p}(\Omega; R^N)$ is a Banach space under the norm

$$\|u\|_{W^{1,p}(\Omega;R^N)} = \|u\|_{L^p(\Omega;R^N)} + \|Du\|_{L^p(\Omega;M^{N\times n})},$$

where $Du = (u_{i,j})$. We sometimes write $H^1(\Omega; R^N) = W^{1,2}(\Omega; R^N)$, $H^{-1}(\Omega; R^N) =$ dual space of $H_0^1(\Omega; R^N)$, where $H_0^1(\Omega; R^N)$ is the closure of $C_0^1(\Omega; \mathbb{R}^N)$ in the topology of $H^1(\Omega; R^N)$ and $C^1(\Omega; \mathbb{R}^N) = C_0^1(\Omega)^N$.

Weak and weak $*$ convergence of sequences are written $\rightharpoonup$ and $\overset{*}{\rightharpoonup}$, respectively. In the case of the space $\mathcal{M}(\Omega)$ of Radon measures on Ω we say that a sequence $\mu^{(j)} \overset{*}{\rightharpoonup} \mu$ in $\mathcal{M}(\Omega)$ if and only if

$$\langle \mu^{(j)}, \phi \rangle \to \langle \mu, \phi \rangle \quad \text{for all } \phi \in C_0(\Omega).$$

We also use the following special notation, motivated by Theorem B.

DEFINITION 1.1. A bounded sequence $f^{(j)}$ in $L^1(\Omega)$ *converges weakly in the sense of the Biting Lemma* to a function $f \in L^1(\Omega)$, written

$$f^{(j)} \overset{b}{\rightharpoonup} f \quad \text{in } \Omega,$$

provided there exists a sequence E_k of measurable subsets of Ω, satisfying $\lim_{k\to\infty}$ meas $(E_k) = 0$, such that for each k

$$f^{(j)} \rightharpoonup f \quad \text{in } L^1(\Omega \backslash E_k).$$

DEFINITION 1.2. A function $f: \Omega \times R^N \times R^s \to \bar{R}$ is a *Carathéodory function* if
(i) $f(\cdot, u, a)$ is measurable for every $u \in R^N$, $a \in R^s$,
(ii) $f(x, \cdot, \cdot)$ is continuous for almost every $x \in \Omega$.

DEFINITION 1.3. (see [19, 4, 5, 8]) A function $f: M^{N\times n} \to R$ is *quasiconvex* if

$$\int_U f(P + D\phi(x))\, dx \geqq f(P) \text{ meas } (U)$$

for every $P \in M^{N\times n}$, $\phi \in C_0^1(U; R^N)$, and every open bounded subset $U \subset R^n$. A Carathéodory function $f: \Omega \times R^N \times M^{N\times n} \to R$ is *quasiconvex in P* if there exists a subset $I \subset \Omega$ with meas $(I) = 0$ such that $f(x, u, \cdot)$ is quasiconvex for all $x \in \Omega \backslash I$, $u \in R^N$.

We make use of the following result (see **[16]**):

PROPOSITION 1.4. *Let $f: \Omega \times R^N \times R^s \to R$ be a Carathéodory function, and let $u^{(j)}: \Omega \to R^N$, $v^{(j)}: \Omega \to R^s$ be sequences of measurable functions satisfying*

$$u^{(j)} \to u \quad \text{in measure}, \quad v^{(j)} \rightharpoonup v \quad \text{in } L^1(\Omega; R^s).$$

Then $f(\cdot, u^{(j)}(\cdot), v^{(j)}(\cdot)) - f(\cdot, u(\cdot), v^{(j)}(\cdot))$ converges to zero in measure.

We also use the following theorem concerning the existence and properties of Young measures. For results in a more general context and proofs the reader is referred to **[12, 2, 7]**.

THEOREM 1.5. *Let $z^{(j)}$ be a bounded sequence in $L^1(\Omega; R^s)$. Then there exist a subsequence $z^{(\nu)}$ of $z^{(j)}$ and a family $(\nu_x)_{x\in\Omega}$ of probability measures on R^s, depending measurably on $x \in \Omega$, such that for any measurable subset $A \subset \Omega$,*

$$f(\cdot, z^{(\nu)}) \rightharpoonup \langle \nu_x, f(x, \cdot)\rangle \quad \text{in } L^1(A)$$

for every Carathéodory function $f: \Omega \times R^s \to R$ such that $f(\cdot, z^{(\nu)})$ is sequentially weakly relatively compact in $L^1(A)$.

2. Biting lower semicontinuity theorems

The following lower semicontinuity theorem is the main result of this paper.

THEOREM 2.1. *Let $1 \leqq p < \infty$ and let $f: \Omega \times R^N \times M^{N\times n} \to R$ satisfy*

(i) *f is a Carathéodory function;*

(ii) *$|f(x, u, P)| \leqq a(x) + C(|u|^p + |P|^p)$, for almost every $x \in \Omega$, $u \in R^N$, $P \in M^{N\times n}$, where C is a nonnegative constant and $a(\cdot) \in L^1(\Omega)$;*

(iii) *f is quasiconvex in P.*

Then given any sequence $u^{(j)} \rightharpoonup u$ in $W^{1,p}(\Omega; R^N)$, there exist a subsequence $u^{(\nu)}$ and a family $(\nu_x)_{x\in\Omega}$ of probability measures on $M^{N\times n}$, depending measurably on x, such that

$$f(x, u^{(\nu)}, Du^{(\nu)}) \xrightarrow{b} l(x) := \langle \nu_x, f(x, u(x), \cdot)\rangle \quad \text{in } \Omega \tag{2.1}$$

and

$$\langle \nu_x, f(x, u(x), \cdot)\rangle \geqq f(x, u(x), Du(x)) \quad \text{almost everywhere in } \Omega. \tag{2.2}$$

Moreover, if E_k denotes the sequence of measurable subsets of Ω in Definition 1.1 corresponding to (2.1) and satisfying $\lim_{k\to\infty}$ meas $(E_k) = 0$, then for each fixed k,

$$\int_{\Omega\backslash E_k} f(x, u(x), Du(x))\, dx \leqq \liminf_{\nu\to\infty} \int_{\Omega\backslash E_k} f(x, u^{(\nu)}(x), Du^{(\nu)}(x))\, dx. \tag{2.3}$$

To prove Theorem 2.1 we need the following lemma:

LEMMA 2.2. *Let* $f:\Omega\times R^N\times M^{N\times n}\to R$ *satisfy assumptions* (i), (ii) *of Theorem* 2.1. *If* $u^{(j)}\rightharpoonup u$ *in* $W^{1,p}(\Omega; R^N)$, *there exists a subsequence* $u^{(\nu)}$ *such that*

$$f(x, u^{(\nu)}, Du^{(\nu)}) \xrightarrow{b} \langle \nu_x, f(x, u(x), \cdot)\rangle = \int_{M^{N\times n}} f(x, u(x), P)\, d\nu_x(P) \text{ in } \Omega,$$

where $(\nu_x)_{x\in\Omega}$ *is the family of Young measures corresponding to the subsequence* $Du^{(\nu)}$.

Proof. By assumption (ii), $f^{(j)}:=f(x, u^{(j)}, Du^{(j)})$ and $\tilde f^{(j)}:=f(x, u, Du^{(j)})$ are bounded in $L^1(\Omega)$. Hence by Theorem B there exists a subsequence $u^{(\nu)}$ such that

$$f^{(\nu)} \xrightarrow{b} \chi, \quad \tilde f^{(\nu)} \xrightarrow{b} \tilde\chi \quad \text{in } \Omega \tag{2.4}$$

for some χ, $\tilde\chi\in L^1(\Omega)$. By the compactness of the embedding $W^{1,p}(\Omega)\subset L^p_{loc}(\Omega)$ we may and shall assume that

$$u^{(\nu)}\to u \quad \text{almost everywhere in } \Omega. \tag{2.5}$$

By (2.5) and Proposition 1.4,

$$f^{(\nu)}-\tilde f^{(\nu)}\to 0 \quad \text{in measure.} \tag{2.6}$$

But it is well-known (and easily proved using Egorov's theorem) that if a sequence $h^{(\nu)}\rightharpoonup h$ in $L^1(A)$, and $h^{(\nu)}\to H$ in measure in A, where $A\subset R^n$ is measurable, then $h=H$ almost everywhere in A. Applying this result to the sequence $h^{(\nu)}:=f^{(\nu)}-\tilde f^{(\nu)}$, we deduce from (2.4) and (2.6) that $\chi=\tilde\chi$ almost everywhere. But $g(x, P):=f(x, u(x), P)$ is a Carathéodory function and $\tilde f^{(\nu)}(x)=g(x, Du^{(\nu)}(x))$ is weakly relatively compact in $L^1(\Omega\backslash E_k)$ for each k, where the E_k are the sets corresponding to $\tilde f^{(\nu)}$ in Theorem B. Hence by Theorem 1.5

$$\tilde\chi(x)=\langle \nu_x, f(x, u(x), \cdot)\rangle \quad \text{almost everywhere,}$$

for which the desired result follows. □

Proof of Theorem 2.1. For $m=1, 2, \ldots$, let $g_m(t)=\max(t, -m)$, $t\in R$. Then g_m is convex and monotone, so that $g_m\circ f$ is still quasiconvex in P. Also

$$-m\leqq g_m(f(x, u, P))\leqq a(x)+C(|u|^p+|P|^p), \tag{2.7}$$

for all $x\in\Omega$, $u\in R^N$, $P\in M^{N\times n}$. Therefore by Lemma 2.2 applied to f, $|f|$ and $g_m\circ f$ there exists a subsequence $u^{(\nu)}$ of $u^{(j)}$ such that

$$f(x, u^{(\nu)}, Du^{(\nu)}) \xrightarrow{b} \langle \nu_x, f(x, u(x), \cdot)\rangle \quad \text{in } \Omega, \tag{2.8}$$

$$|f(x, u^{(\nu)}, Du^{(\nu)})| \xrightarrow{b} \langle \nu_x, |f(x, u(x), \cdot)|\rangle \quad \text{in } \Omega, \tag{2.9}$$

$$g_m(f(x, u^{(\nu)}, Du^{(\nu)})) \xrightarrow{b} \langle \nu_x, g_m(f(x, u(x), \cdot))\rangle \quad \text{in } \Omega, \tag{2.10}$$

as $\nu\to\infty$, for each fixed m. Let E_k, $E_{m,k}$ be the sequences of measurable subsets of Ω in Definition 1.1 corresponding to (2.8), (2.10) respectively, and denote by $\chi_{m,k}$ the characteristic function of $\Omega\backslash E_{m,k}$. Let $\phi\in L^\infty(\Omega)$, $0\leqq\phi(x)\leqq 1$ for almost every $x\in\Omega$. Let

$$g_{m,k}(x, u, P)=\phi(x)\chi_{m,k}(x)g_m(f(x, u, P)).$$

Then $g_{m,k}+m$ is quasiconvex in P and so by (2.7) and Theorem A we have

$$\liminf_{\nu\to\infty}\int_\Omega g_{m,k}(x, u^{(\nu)}, Du^{(\nu)})\,dx \geqq \int_\Omega g_{m,k}(x, u, Du)\,dx,$$

that is

$$\liminf_{\nu\to\infty}\int_{\Omega\setminus E_{m,k}} \phi(x)g_m(f(x, u^{(\nu)}, Du^{(\nu)}))\,dx \geqq \int_{\Omega\setminus E_{m,k}} \phi(x)g_m(f(x, u, Du))\,dx. \quad (2.11)$$

By (2.10) the left-hand side of (2.11) equals

$$\int_{\Omega\setminus E_{m,k}} \phi(x)\langle \nu_x, g_m(f(x, u(x), \cdot))\rangle\,dx. \quad (2.12)$$

Since ϕ is arbitrary with $0\leqq\phi\leqq 1$, and since $\lim_{k\to\infty}\text{meas}\,(E_{m,k})=0$, it follows from (2.11), (2.12) that

$$\langle \nu_x, g_m(f(x, u(x), \cdot))\rangle \geqq g_m(f(x, u(x), Du(x))) \quad \text{almost everywhere in } \Omega. \quad (2.13)$$

But by (2.9),

$$\langle \nu_x, |f(x, u(x), \cdot)|\rangle < \infty \quad \text{almost everywhere in } \Omega,$$

and since $|g_m(t)|\leqq|t|$ for all t,

$$|g_m(f(x, u(x), P))| \leqq |f(x, u(x), P)|,$$

for all $x\in\Omega$, $P\in M^{N\times n}$. Passing to the limit $m\to\infty$ in (2.13) using Lebesgue's dominated convergence theorem, we therefore obtain (2.2), from which (2.3) follows immediately. □

Notice that Theorem 2.1, which we have deduced using Theorem A, in turn implies Theorem A. This follows easily from (2.3) using the fact that $f\geqq 0$.

We can apply Theorem 2.1 to give a different proof of the following weak continuity result of Zhang for the determinants of mappings in $W^{1,n}(\Omega, R^n)$. (For further developments see [**20**, **15**].)

Corollary 2.3 (Zhang [**29**, Corollary 2.2]). *Let $n\geqq 2$ and $u^{(j)}\rightharpoonup u$ in $W^{1,n}(\Omega, R^n)$. Then there exists a subsequence $u^{(\nu)}$ of $u^{(j)}$ such that*

$$\det Du^{(\nu)} \xrightarrow{b} \det Du \quad \textit{in } \Omega.$$

Proof. Since $\det P$ is a null Lagrangian (cf. [**4**, **5**]), $\pm\det P$ are quasiconvex, so that the hypotheses of Theorem 2.1 are satisfied with $p=n$. Hence

$$\langle \nu_x, \det P\rangle = \det Du(x) \quad \text{almost everywhere in } \Omega. \quad \square$$

Under the hypotheses of Theorem 2.1, the sequence $f(x, u^{(\nu)}, Du^{(\nu)})$ is bounded in $L^1(\Omega)$. We may therefore extract a further subsequence, again denoted $u^{(\nu)}$, such that as $\nu\to\infty$

$$f(x, u^{(\nu)}, Du^{(\nu)}) \overset{*}{\rightharpoonup} \mu \quad \text{in } \mathcal{M}(\Omega).$$

We also have, by Theorem 2.1, that

$$f(x, u^{(\nu)}, Du^{(\nu)}) \xrightarrow{b} l \quad \text{in } \Omega.$$

As noted by Ball and Murat [**10**], for an arbitrary L^1-bounded sequence there is in general no relation between the weak $*$ limit in the sense of measures and that given by the Biting Lemma. In the present context, however, a relation can be established under the further hypothesis that f is bounded below by an affine function of the vector $J(P) \in R^{s(n,N)}$ consisting of all the minors $J_\alpha(P)$ of $P \in M^{N\times n}$ of orders $1 \leq r \leq \min(n, N)$. A key idea for the proof of the following result was suggested to us by F. Murat.

THEOREM 2.4. *Let* $p \geqq \min(n, N)$. *Suppose that f, in addition to the hypotheses of Theorem* 2.1, *satisfies*

$$f(x, u, P) - a \,.\, J(P) - b \geqq 0, \tag{2.14}$$

for all $x \in \Omega$, $u \in R^N$, $P \in M^{N\times n}$, *where* $a \in R^{s(n,N)}$ *and* $b \in R$. *Then* $\mu \geqq l$.

Proof. Let $\phi \in C_0(\Omega)$, $\phi \geqq 0$, and let E_k be as in Theorem 2.1. Since $\pm J(P)$ also satisfies the hypotheses of Theorem 2.1, by the same proof as in Corollary 2.3 we may assume that

$$J_\alpha(Du^{(\nu)}) \rightharpoonup J_\alpha(Du) \quad \text{in } L^1(\Omega\backslash E_k) \tag{2.15}$$

for every k and α. Also, by [**22**], for every α

$$J_\alpha(Du^{(\nu)}) \xrightarrow{*} J_\alpha(Du) \quad \text{in } \mathcal{M}(\Omega). \tag{2.16}$$

From (2.14) we have that

$$\int_\Omega [f(x, u^{(\nu)}, Du^{(\nu)}) - a \cdot J(Du^{(\nu)}) - b]\phi \, dx$$
$$\geqq \int_{\Omega\backslash E_k} [f(x, u^{(\nu)}, Du^{(\nu)}) - a \cdot J(Du^{(\nu)}) - b]\phi \, dx \tag{2.17}$$

for every k. Passing to the limit $\nu \to \infty$ in (2.17) and using (2.15), (2.16), we deduce that

$$\langle \mu, \phi \rangle - \int_\Omega [a \cdot J(Du) + b]\phi \, dx \geqq \int_{\Omega\backslash E_k} [l - a \cdot J(Du) - b]\phi \, dx.$$

Letting $k \to \infty$ and using the arbitrariness of ϕ, we obtain the result. □

If $f = f(P)$ is polyconvex, that is

$$f(P) = g(J(P)),$$

where $g\colon R^{s(n,N)} \to R$ is convex, then (2.14) follows from the inequality

$$g(J(P)) \geqq g(0) + \Delta \cdot J(p),$$

where $\Delta \in \partial g(0)$.

EXAMPLE 2.5. *Let* $n = N \geqq 2$. The quadratic function

$$f(P) = |P|^2 - (\operatorname{tr} P)^2$$

is used to prove optimal bounds in homogenisation theory ([27, 21]). The identity

$$f(Du) = \tfrac{1}{2}(u_{i,j} - u_{j,i})(u_{i,j} - u_{j,i}) + (u_i u_{j,i})_{,j} - (u_i u_{j,j})_{,i} \tag{2.18}$$

for smooth u shows that f is polyconvex. (In (2.18) the usual summation convention is in force.) Note that f is unbounded below.

Let $u^{(j)} \rightharpoonup u$ in $W^{1,2}(\Omega; R^n)$, and let $u^{(\nu)}$ be a subsequence such that

$$f(Du^{(\nu)}) \overset{*}{\rightharpoonup} \mu \quad \text{in } \mathcal{M}(\Omega),$$
$$f(Du^{(\nu)}) \overset{b}{\rightharpoonup} l \quad \text{in } \Omega.$$

A direct application of polyconvexity, or of the basic lemma of compensated compactness (see Section 3) implies only that the measure μ satisfies

$$\mu \geqq f(Du) \quad \text{in the sense of measures.}$$

However, Theorem 2.1 implies that

$$l \geqq f(Du) \quad \text{for almost every } x \in \Omega,$$

and Theorem 2.4 shows that this is potentially a stronger result. In fact if $0 \in \Omega$ and we let $u_1^\varepsilon = \varepsilon^{1-n/2}\phi(x/\varepsilon)$, $u_2^\varepsilon = \ldots = u_n^\varepsilon = 0$, where $\phi \in C_0^\infty(R^n)$, $\phi \not\equiv 0$, then it is easily checked that as $\varepsilon \to 0$

$$u^\varepsilon \rightharpoonup 0 \quad \text{in } W^{1,2}(\Omega; R^n),$$
$$Du^\varepsilon \to 0 \quad \text{uniformly outside any neighbourhood of } 0,$$

so that $l = 0$. But if $\psi \in C_0^\infty(\Omega)$ then

$$\lim_{\varepsilon \to 0} \int_\Omega f(Du^\varepsilon)\psi \, dx = \left(\int_{R^n} \sum_{i=2}^n \phi_{,i}^2 \, dy \right) \psi(0),$$

so that $\mu = (\int_{R^n} \sum_{i=2}^n \phi_{,i}^2 \, dy)\delta_0 \neq l$. (Of course the same argument shows that $\mu \neq l$ in the simpler case $f(P) = |P|^2$.)

The following is a general lower semicontinuity result for variational integrals:

THEOREM 2.6. *Let*

$$I(u) = \int_\Omega g(x, f_1(x, u, Du), \ldots, f_M(x, u, Du)) \, dx,$$

where $g: \Omega \times R^M \to [0, +\infty]$ *is a Carathéodory function such that for* $x \in \Omega$ *almost everywhere,*

(i) $a \mapsto g(x, a)$ *is convex for* $a = (a_1, \ldots, a_M) \in R^M$,

(ii) $g(x, a)$ *is nondecreasing in* $a_j \in I_j$ *for* $1 \leqq j \leqq M$, *where* $I_j \subset R$ *is a closed interval (finite or infinite), and where for* $1 \leqq j \leqq M$,

(iii) $f_j: \Omega \times R^N \times M^{N \times n} \to I_j$,

(iv) f_j *satisfies the hypotheses of Theorem* 2.1 *for some* p *independent of* j.

Let $u^{(j)} \rightharpoonup u$ *in* $W^{1,p}(\Omega; R^N)$. *Then*

$$I(u) \leqq \liminf_{j \to \infty} I(u^{(j)}).$$

Proof. Assume that $\alpha = \liminf_{j\to\infty} I(u^{(j)}) < \infty$. Then by Theorem 2.1 there exists a subsequence $u^{(\nu)}$ of $u^{(j)}$ such that

$$\lim_{\nu\to\infty} I(u^{(\nu)}) = \alpha, \tag{2.19}$$

$$f_j(x, u^{(\nu)}, Du^{(\nu)}) \xrightarrow{b} l_j(x) \geqq f_j(x, u, Du), \quad 1 \leqq j \leqq M, \tag{2.20}$$

where we may assume that the sets E_k corresponding to (2.20) are independent of j. Since $g(x, .)$ is convex and nonnegative, a standard lower semicontinuity theorem (see [**17**, **14**, **2**]) implies that

$$\begin{aligned}\alpha &\geqq \liminf_{\nu\to\infty} \int_{\Omega\setminus E_k} g(x, f_1(x, u^{(\nu)}, Du^{(\nu)}), \dots, f_M(x, u^{(\nu)}, Du^{(\nu)}))\, dx \\ &\geqq \int_{\Omega\setminus E_k} g(x, l_1(x), \dots, l_m(x))\, dx \\ &\geqq \int_{\Omega\setminus E_k} g(x, f_1(x, u, Du), \dots, f_M(x, u, Du))\, dx,\end{aligned}$$

where we have used hypotheses (ii), (iii) and (2.20). Letting $k\to\infty$, we deduce that $I(u) \leqq \alpha$, which by (2.19) gives the result. □

Remark 2.7. The use of the Biting Lemma to prove lower semicontinuity theorems follows Balder [**2**] and Lin [**18**].

EXAMPLE 2.8. Theorem 2.6 can be applied to the case of polyconvex integrands by setting $M = s(n, N)$, $f_j(x, u, P) = J_j(P)$, provided $p \geqq \min(n, N)$. This recovers as a special case the improvement by Zhang [**29**, Theorem 3.1] of a result of Ball and Murat [**9**, Theorem 6.1].

3. Remarks on Young measures and compensated compactness

It would be of interest to prove a version of Theorem 2.1 in the framework of compensated compactness. It does not seem obvious how to do this even in the quadratic case. To make this more precise, suppose that

(i) $u^{(l)} \to u$ in $L^2(\Omega; R^N)$ as $l\to\infty$,

(ii) $\sum_{j,k} a_{ijk}\, \partial u_j^{(l)}/\partial x_k \in$ compact set (for the strong topology) of $H^{-1}_{loc}(\Omega)$, for $i = 1, \dots, q$. Here the a_{ijk} are constant coefficients.

Define

$$\mathcal{V} = \left\{(\lambda, \xi) \in R^N \times R^n : \sum_{j,k} a_{ijk}\lambda_j\xi_k = 0, i = 1, \dots, q\right\}$$

$$\Lambda = \{\lambda \in R^N : \text{there exists } \xi \neq 0 \text{ with } (\lambda, \xi) \in \mathcal{V}\}.$$

Let Q be a quadratic form on R^N such that

$$Q(\lambda) \geqq 0 \quad \text{for all } \lambda \in \Lambda. \tag{3.1}$$

Since $Q(u^{(l)})$ is bounded in $L^1(\Omega)$, we can extract a subsequence $u^{(\nu)}$ of $u^{(l)}$ such that

$$Q(u^{(\nu)}) \overset{*}{\rightharpoonup} \mu \quad \text{in } \mathcal{M}(\Omega), \tag{3.2}$$

$$Q(u^{(\nu)}) \overset{b}{\rightharpoonup} l \quad \text{in } \Omega, \tag{3.3}$$

where by Theorem 1.5

$$l(x) = \langle \nu_x, Q(\cdot) \rangle \quad \text{for almost every } x \in \Omega. \tag{3.4}$$

The fundamental theorem of compensated compactness ([**25**, **26**, **21**]) says that the measure μ satisfies

$$\mu \geqq Q(u), \tag{3.5}$$

so that in particular (applying the theorem to $\pm Q$), if $Q(\lambda) = 0$ for all $\lambda \in \Lambda$ then

$$\mu = Q(u). \tag{3.6}$$

The analogue of Theorem 2.1 is the following

CONJECTURE 3.1. *Under the above hypotheses*

$$l \geqq Q(u) \quad \textit{almost everywhere in } \Omega. \tag{3.7}$$

Theorem 2.1 is a stronger statement of this conjecture in the gradient case, that is when $u^{(l)} = Dv^{(l)}$. For general coefficients a_{ijk} the proof of Theorem 2.1 does not apply, and the conjecture remains unproved. Another result consistent with the conjecture is that of Zhang [**29**, Theorem 2.6], namely that if

$$u^{(j)} \rightharpoonup u, \quad v^{(j)} \rightharpoonup v \qquad \text{in } L^2(\Omega; R^n),$$

$$\text{curl } u^{(j)} \in \text{compact set of } H^{-1}(\Omega; R^{n^2}),$$

$$\text{div } v^{(j)} \in \text{compact set of } H^{-1}(\Omega),$$

then there exist subsequences $u^{(\nu)}$, $v^{(\nu)}$ such that

$$u^{(\nu)} \cdot v^{(\nu)} \overset{b}{\rightharpoonup} u \cdot v \quad \text{in } \Omega.$$

Another open question is whether the limits in (3.2), (3.3) satisfy the inequality

$$\mu \geqq l \quad \text{in } \mathcal{M}(\Omega). \tag{3.8}$$

This question is not completely resolved even in the case of gradients, since there exist by the results of Terpstra [**28**] and Serre [**23**] (see also [**6**]) quadratic functions $Q(P)$, $P \in M^{N \times n}$, $\min(n, N) \geqq 3$, which are quasiconvex but not polyconvex, so that Theorem 2.4 does not apply.

Finally, we highlight the complicated structure of the Young measure $(\nu_x)_{x \in \Omega}$ corresponding to an arbitrary sequence of gradients by disproving a conjecture which would have led to simple proofs of lower semicontinuity theorems. The motivation for the conjecture is the following. Suppose for simplicity that $Q \subset R^n$ is an open cube, and that $\phi \in W^{1,p}_{loc}(R^n; R^N)$ with $D\phi$ Q-periodic. Define $u^{(j)}(x) = j^{-1}\phi(jx)$, $x \in \Omega$. It is well-known (see, for example, [**26**] and [**9**, Lemma A.1]) that if $f \in C_0(M^{N \times n})$ then

$$f(Du^{(j)}) \overset{*}{\rightharpoonup} \frac{1}{\text{meas}(Q)} \int_Q f(D\phi(y))\, dy \quad \text{in } L^\infty(\Omega).$$

Hence the Young measure $\nu = \nu_x$ corresponding to $Du^{(j)}$ is independent of x and given by

$$\langle \nu, f\rangle = \frac{1}{\text{meas}\,(Q)} \int_Q f(D\phi(y))\,dy, \quad f \in C_0(M^{N\times n}). \tag{3.9}$$

The conjecture is that the Young measure $(\nu_x)_{x\in\Omega}$ corresponding to an *arbitrary* weakly convergent sequence $u^{(j)}$ in $W^{1,p}(\Omega; R^N)$ (after extraction of a suitable subsequence) has a similar representation

$$\langle \nu_x, f\rangle = \frac{1}{\text{meas}\,(Q)} \int_Q f(D_y\phi(x, y))\,dy, \quad f \in C_0(M^{N\times n}), \tag{3.10}$$

for some function $\phi(x, y)$ satisfying $\phi(x, \cdot) \in W^{1,p}(\Omega; R^N)$ for almost every x. Were this true for a ϕ that was, for example, Q-periodic in y, then the quasiconvexity condition (cf. [**8**])

$$\frac{1}{\text{meas}\,(Q)} \int_Q f(x_0, u_0, D\psi(y))\,dy \geqq f\left(x_0, u_0, \frac{1}{\text{meas}\,(Q)} \int_Q D\psi(y)\,dy\right) \tag{3.11}$$

for all $x_0 \in \Omega$, $u_0 \in R^N$ and Q-periodic $\psi \in W^{1,p}_{loc}(R^n; R^N)$ would imply under appropriate growth hypotheses that

$$\langle \nu_x, f(x, u(x), \cdot)\rangle \geqq f(x, u(x), \langle \nu_x, P\rangle) = f(x, u(x), Du(x)),$$

where $u^{(\nu)} \rightharpoonup u$ in $W^{1,p}(\Omega; R^N)$, which is the result (2.2) of Theorem 2.1.

We disprove the conjecture by considering the sequence $u^{(j)}: \Omega \to R^2$, given by

$$\left.\begin{aligned} u_1^{(j)}(x_1, x_2) &= j^{-1} \sin(jx_1), \\ u_2^{(j)}(x_1, x_2) &= j^{-2} \cos(jx_1)\sin(j^2 x_2), \end{aligned}\right\} \tag{3.12}$$

which is of the type considered by Ball and Murat [**11**]. Since

$$Du^{(j)}(x_1, x_2) = \begin{pmatrix} \cos(jx_1) & 0 \\ -j^{-1}\sin(jx_1)\sin(j^2x_2) & \cos(jx_1)\cos(j^2x_2) \end{pmatrix}, \tag{3.13}$$

it follows that $u^{(j)}$ is bounded in $W^{1,\infty}(\Omega; R^2)$. Let $(\nu_x)_{x\in\Omega}$ be the Young measure corresponding to $Du^{(j)}$. It is easy to see that ν_x is independent of x; in fact, it suffices for this to check that $M(Du^{(j)})$ has a weak$*$ limit in $L^\infty(\Omega)$ that is independent of x for any monomial $M(A)$, and this is easily verified using test functions of the form $\theta(x_1)\chi(x_2)$. Then we have

PROPOSITION 3.2. *Let $\nu = \nu_x$ be the Young measure corresponding to the sequence of gradients $Du^{(j)}$ given by (3.13). Then there is no function $\phi \in W^{1,1}(Q; R^2)$, $Q = (0, 1)^2$, such that*

$$\langle \nu, f\rangle = \int_Q f(D\phi(x))\,dx \quad \textit{for all } f \in C_0(M^{2\times 2}). \tag{3.14}$$

In particular, there is no $\phi \in W^{1,1}_{loc}(R^2; R^2)$ whose gradient is Q-periodic such that $D\phi(jx)$ has the same Young measure ν as $Du^{(j)}$.

Proof. Since $Du^{(j)}$ is uniformly bounded, supp ν_x is bounded. Hence, by the monotone convergence theorem, it follows from (3.14) that $f(D\phi) \in L^1(Q)$ and

$$\langle \nu, f \rangle = \int_Q f(D\phi(x))\,dx, \tag{3.15}$$

for all $f \in C(M^{2\times 2})$

In (3.15) we first choose $f(A) = g(A_{21})$, $g \in C(R)$. Then since

$$\langle \nu, f \rangle = \lim_{j\to\infty} \int_Q f(Du^{(j)})\,dx, \tag{3.16}$$

we deduce that

$$g(0) = \int_Q g(\phi_{2,1}(x))\,dx$$

for all $g \in C(R)$, from which it follows easily that $\phi_{2,1} = 0$ in Q. Similarly $\phi_{1,2} = 0$ in Q, so that

$$\phi_1 = \phi_1(x_1), \quad \phi_2 = \phi_2(x_2). \tag{3.17}$$

Next we choose $f(A) = (A_{11})^2$. Then from (3.15)–(3.17) we obtain

$$\int_0^1 (\phi_1')^2\,dx_1 = \lim_{j\to\infty} \int_0^1 \cos^2(jx_1)\,dx_1 = \tfrac{1}{2}. \tag{3.18}$$

Choosing $f(A) = (A_{22})^2$, we obtain

$$\int_0^1 (\phi_2')^2\,dx_2 = \lim_{j\to\infty} \left(\int_0^1 \cos^2(jx_1)\,dx_1 \int_0^1 \cos^2(j^2x_2)\,dx_2 \right) = \tfrac{1}{4}. \tag{3.19}$$

Then, choosing $f(A) = (A_{11}A_{22})^2$, we get

$$\int_0^1 (\phi_1')^2\,dx_1 \int_0^1 (\phi_2')^2\,dx_2 = \lim_{j\to\infty} \left(\int_0^1 \cos^4(jx_1)\,dx_1 \int_0^1 \cos^2(j^2x_2)\,dx_2 \right) = \tfrac{3}{16}. \tag{3.20}$$

Combining (3.18)–(3.20) gives the desired contradiction. □

Acknowledgment

We thank F. Murat for some very helpful discussions. The research of K. Z. was supported by a grant from the British Council. The paper was completed while J. M. B. was visiting the Université de Paris, Dauphine, and he would like to thank the members of CEREMADE for their invitation and hospitality.

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(*Issued* 25 *May* 1990)