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Regularity of quasiconvex envelopes

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Abstract. We prove that the quasiconvex envelope of a differentiable function which satisfies natural growth conditions at infinity is a C^1 function. Without the growth conditions the result fails in general. We also obtain results on higher regularity (in the sense of $C^{1,\alpha}_{\text{loc}}$) and similar results for other types of envelopes, including polyconvex and rank-1 convex envelopes.

1 Introduction and the main results

The existence of minimisers for multiple integrals

$$I(u) = \int_\Omega F(x, u, \nabla u)\, dx$$

defined on mappings $u : \Omega \subset \mathbb{R}^m \to \mathbb{R}^n$, is classically deduced by use of the direct method. In cases where the minimum of I is not attained, or, more generally, when minimising sequences develop oscillations in the gradient, it is often useful to consider the related relaxed problem. This consists in minimising the lower semicontinuous envelope of $I(u)$. If F is a Carathéodory integrand satisfying the growth conditions

$$c_1|\xi|^p - c_2 \le F(x, v, \xi) \le c_2(|\xi|^p + 1), \tag{1.1}$$

where $(x, v, \xi) \in \Omega \times \mathbb{R}^n \times \mathbb{R}^{n\times m}$ and $c_1,\ c_2 > 0,\ p > 1$, then the relaxation is the weak lower semicontinuous envelope $\overline{I}$ of I on the Sobolev space $W^{1,p}(\Omega, \mathbb{R}^n)$. By a general result (cf. [14], [1], [41]) $\overline{I}$ is again a multiple integral and

$$\overline{I}(u) = \int_\Omega \overline{F}(x, u, \nabla u)\, dx,$$

where the integrand $\overline{F}$ is obtained by taking for each (x, v) the quasiconvex envelope of $F(x, v, \cdot)$ (see Sect. 2 for terminology). The quasiconvex envelope is still relevant in connection with relaxation without the growth conditions (1.1), but the situation is more complicated (see [6], [32], [33] for examples, and [4], [15], [35] for systematic expositions and further references).

Assuming (1.1), the Euler-Lagrange system associated with the envelope $\overline{I}$ is formally given by

$$\text{(EL)} \qquad \operatorname{div}\nabla_\xi \overline{F}(x, u, \nabla u) = \nabla_u \overline{F}(x, u, \nabla u).$$

This naturally raises questions about differentiability properties of $\overline{F}$ which we address in this paper. The calculations involved in computing $\overline{F}$ explicitly for interesting examples of F are at best extremely involved. At present it has been done in only very few cases, see e.g. [28], [29], [31], [2], [21], [19] and [39]. The approach taken in this paper is rather abstract and is based on a representation formula for the quasiconvex envelope from [14] and elementary properties of separately convex functions.

It is not difficult to see that even if F is a C^∞ function satisfying (1.1), the envelope $\overline{F}$ is in general not differentiable with respect to u. We present a large class of such examples in Sect. 5 (in all dimensions $m, n \geq 1$). However, it is possible to show by adapting an argument from [43] (see also [17] and [18]) that, under quite general conditions, $\overline{F}$ has finite one-sided directional derivatives with respect to u everywhere. We intend to pursue this elsewhere. The situation for the ξ variables is different. Theorems A and B below imply that if F is differentiable with respect to ξ, then so is $\overline{F}$. More general versions of these theorems can be found in Sect. 3.

Theorem A *Suppose $f : \mathbb{R}^{n\times m} \to \mathbb{R}$ is differentiable and either that it satisfies for some $p \geq 0$ the growth condition*

$$\liminf_{\xi\to\infty} \frac{f(\xi)}{|\xi|^p} = \infty \text{ and } \limsup_{\xi\to\infty} \frac{f(\xi)}{|\xi|^{p+1}} < \infty, \tag{1.2}$$

or that it is bounded from below, locally Lipschitz and satisfies the growth condition

$$\frac{\nabla f(\xi)}{\max\{f(\xi), 1\}} \to 0 \text{ as } \xi \to \infty. \tag{1.3}$$

Then the quasiconvex envelope f^{qc} is a C^1 function.

The method of proof also yields a formula for the gradient ∇f^{qc} of the envelope in terms of ∇f and a sub-probability measure ν on the space of matrices $\mathbb{R}^{n\times m}$: $\nabla f^{qc} = \int \nabla f \, d\nu$. Under the growth conditions (1.1), ν can be interpreted as a minimising (gradient p-) Young measure, i.e., in the terminology of [27], ν is a homogeneous $W^{1,p}$-Young measure and

$f^{qc}(\overline{\nu}) = \int f\, d\nu$, where $\overline{\nu}$ denotes the centre of mass of ν. In the setting of nonlinear hyperelastostatics the formula for the gradient

$$\nabla f^{qc}(\overline{\nu}) = \int \nabla f\, d\nu, \quad \overline{\nu} := \langle \nu, \mathrm{id} \rangle,$$

then has the interpretation that the solution to the relaxed problem, i.e. the problem of minimising the effective elastic energy

$$\int_\Omega f^{qc}(\nabla u)\, dx$$

subject to prescribed boundary conditions, not only yields the effective local energy density (which is clear), but also the effective local stress. However, it should be noted that, in common with many other results involving quasiconvexity, our assumptions on the stored-energy function f are inconsistent with the natural requirement that $f(\xi) \to \infty$ as $\det \xi \to 0^+$. For stored-energy functions satisfying this requirement it is not obvious whether the representation (2.1) of f^{qc} holds, or whether $\int_\Omega f^{qc}(\nabla u)\, dx$ is the correct effective elastic energy.

Another consequence of Theorem A is that any minimiser $\overline{u}$ of a multiple integral $I(u)$ with $F(x, v, \xi) = f(\xi)$ satisfying (1.1) satisfies the (possibly degenerate) elliptic system (EL).

Without (1.2) or (1.3) the envelope may not be differentiable. In Theorem 5.1 we present a C^∞ function f satisfying

$$0 < \liminf_{\xi \to \infty} \frac{f(\xi)}{|\xi|} < \infty \text{ and } \limsup_{\xi \to \infty} \frac{f(\xi)}{|\xi|^2} < \infty,$$

and for which f^{qc} is not differentiable. Hence, the growth conditions (1.2) or (1.3) cannot be omitted in general. However, the conclusion of Theorem A holds under various other hypotheses (see Theorems 3.1 and 3.4). Theorem 5.1 also shows that neither the polyconvex envelope, f^{pc}, nor the rank-1 convex envelope, f^{rc}, will in general be differentiable without a growth assumption like (1.2) or (1.3). This contrasts with the case of convex envelopes where weaker growth conditions suffice (see e.g. [25], [24], [7] and [30]).

Quasiconvexification can have a smoothing effect. Indeed, f need not be differentiable for f^{qc} to be C^1. It suffices that the following condition is satisfied:

$$\forall \xi \quad \exists a : \limsup_{\eta \to 0} \frac{f(\xi + \eta) - f(\xi) - \langle a, \eta \rangle}{|\eta|} \leq 0. \tag{1.4}$$

When (1.4) holds we say that f is upper semidifferentiable. An important subclass of the upper semidifferentiable functions consists of the functions

$f(\xi) = \min\{f_1(\xi), \ldots, f_N(\xi)\}$, where $f_1, \ldots, f_N$ are differentiable. As a consequence of Theorem 3.1 and Lemma 3.2 we have the following theorem.

Theorem B *Suppose $f : \mathbb{R}^{n\times m} \to \mathbb{R}$ is continuous, upper semidifferentiable and that it satisfies* (1.2). *Then the quasiconvex envelope f^{qc} is a C^1 function.*

A slightly stronger result is obtained for the polyconvex envelope in Proposition 4.1.

In general, there is a natural limit to how smooth one can expect a quasiconvex envelope to be. The quasiconvex envelope of a C^∞ function (even of a real analytic function) is not in general of class C^2. This already happens on the level of convex envelopes for functions $f : \mathbb{R} \to \mathbb{R}$ and the mechanism is exactly the same for quasiconvex envelopes (see the remark after Proposition 3.7 in Sect. 3). However, it follows from Proposition 3.7 that if f is of class $C^{1,\alpha}_{\text{loc}}$, $0 < \alpha \leq 1$, if (1.2) or (1.3) and another condition (an estimate on the local Hölder constant of ∇f) holds, then f^{qc} is also of class $C^{1,\alpha}_{\text{loc}}$. Here we say that a function $f : \mathbb{R}^{n\times m} \to \mathbb{R}$ is of class $C^{1,\alpha}_{\text{loc}}$ if it is C^1 and if ∇f is locally α-Hölder continuous.

In Theorem 5.5 we exhibit an important class of, in general, non-differentiable functions to which our results apply. For a subset $K \subset \mathbb{R}^{n\times m}$ define $\text{dist}(\xi, K) := \inf\{|\xi - \eta| : \eta \in K\}$, where $|\cdot|$ denotes the Euclidean norm. The following result is contained in Theorem 5.5.

Theorem C *Suppose $K \subset \mathbb{R}^{n\times m}$ is compact and $1 < p < \infty$. If $f(\xi) := \text{dist}(\xi, K)^p$, then the quasiconvex envelope f^{qc} is of class $C^{1,1}_{\text{loc}}$ for $p > 2$ and of class $C^{1,p-1}$ for $p \leq 2$.*

In the scalar case the relation between existence of a minimiser (in a Sobolev space and in a class of Young measures) and the smoothness properties of the convex envelope of the integrand is well understood. See [12, 13], [24], and also [25], where the differentiability properties of the convex envelope is studied without specific reference to the attainment problem. Though we have not highlighted it, the results of our paper show that also in the multi-dimensional case differentiability of the quasiconvex envelope is a consequence of the existence of a Young measure minimiser. However, the converse implication remains to be established. The attainment problem for the multi-dimensional case has been studied in [2] and [16]. Recently, interesting results on the attainment problem for the scalar case have been established in [11], [42] and [44].

The paper is organized as follows. In Sect. 2 we recall the main definitions and state some preliminary results. Sect. 3 contains the precise statements of the main results together with their proofs. In Sect. 4 we consider the case of polyconvex envelopes and, very briefly, other types of envelopes too.

Sect. 5 contains examples showing the sharpness of some of the hypotheses and some non-trivial examples of functions with envelopes of class $C^{1,\alpha}_{\text{loc}}$.

2 Preliminaries

Throughout the paper we use standard Euclidean norms on the spaces $\mathbb{R}^m$, $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$. For example, if $\xi, \eta \in \mathbb{R}^{n\times m}$ are two matrices, then their inner product is denoted $\langle \xi, \eta\rangle$ and the norm of ξ is denoted $|\xi|$ $(= \sqrt{\langle \xi, \xi\rangle})$. Denote by $\text{Lip}_0([0,1]^m, \mathbb{R}^n)$ the space of Lipschitz mappings $u : [0,1]^m \to \mathbb{R}^n$ that vanish on the boundary of $[0,1]^m$.

For an extended real-valued function $f : \mathbb{R}^{n\times m} \to \mathbb{R} \cup \{\infty\}$ we use the following terminology:

• f is quasiconvex if for all $u \in \text{Lip}_0([0,1]^m, \mathbb{R}^n)$ and $\xi \in \mathbb{R}^{n\times m}$ the inequality

$$\int_{(0,1)^m} f(\xi + \nabla u(x))\, dx \geq f(\xi)$$

holds whenever the right-hand side is well-defined as a Lebesgue integral ($\pm\infty$ are allowed as values of the integral).

• f is polyconvex if there exists a convex function $F : \mathbb{R}^\tau \to \mathbb{R} \cup \{\infty\}$, such that $f(\xi) = F(M(\xi))$ for all $\xi \in \mathbb{R}^{n\times m}$, where $M(\xi)$ is the vector of all minors of ξ arranged in some fixed order and $\tau = \tau(m,n)$ is the number of such minors.

• f is rank-1 convex if for any $\xi_1, \xi_2 \in \mathbb{R}^{n\times m}$ with $\text{rank}(\xi_1 - \xi_2) = 1$ and each $t \in (0,1)$ the following inequality holds

$$tf(\xi_1) + (1-t)f(\xi_2) \geq f(t\xi_1 + (1-t)\xi_2).$$

• f is separately convex if for any $\xi_1, \xi_2 \in \mathbb{R}^{n\times m}$ for which $\xi_1 - \xi_2$ has only one non-zero entry, and each $t \in (0,1)$, the following inequality holds

$$tf(\xi_1) + (1-t)f(\xi_2) \geq f(t\xi_1 + (1-t)\xi_2).$$

For $S \subseteq \mathbb{R}^{n\times m}$, we say that $f : S \to \mathbb{R} \cup \{\infty\}$ is separately convex if the function obtained by extending f as ∞ outside S is separately convex.

It can be shown that polyconvexity implies quasiconvexity for upper semicontinuous extended real-valued functions. For real-valued functions it can be shown that quasiconvexity implies rank-1 convexity (see [22]). Finally, it is clear that rank-1 convexity implies separate convexity. In the special cases $m = 1$ or $n = 1$ the concepts coincide with ordinary convexity; in the multi-dimensional case $m, n > 1$, considered in this paper, none of the implications are in general reversible (for an overview see [15] and [35]).

The quasiconvex envelope of f is the extended real-valued function

$$f^{qc}(\xi) = \sup\{g(\xi) : g \text{ quasiconvex}, g \leq f\}.$$

Here we use the convention that $\sup \emptyset = -\infty$. It is not hard to show that [for a real-valued funtion f] either $f^{qc} \equiv -\infty$ or $f^{qc} > -\infty$ everywhere and f^{qc} is quasiconvex. The polyconvex, rank-1 convex and separately convex envelopes are defined similarly.

Proposition 2.1. *The quasiconvex envelope f^{qc} of a continuous function $f : \mathbb{R}^{n\times m} \to \mathbb{R}$ can be represented by the formula*

$$f^{qc}(\xi) = \inf\left\{ \int_{(0,1)^m} f(\xi + \nabla u(x))\, dx \, : \, u \in \mathrm{Lip}_0([0,1]^m, \mathbb{R}^n) \right\}. \tag{2.5}$$

Remark. It is an open problem whether the formula is valid if f is allowed to be extended real-valued.

The proof for the case where f is bounded from below by a null Lagrangian can be found in [14] (see also [15]) and a proof for the general case is contained in the appendix of [26]. For later use we reformulate this representation for the envelope in terms of measures. Let $\mathbf{Q}$ denote the set of probability measures ν on $\mathbb{R}^{n\times m}$ for which there exist $\xi \in \mathbb{R}^{n\times m}$, $u \in \mathrm{Lip}_0([0,1]^m, \mathbb{R}^n)$, such that

$$\langle \nu, h\rangle := \int_{(0,1)^m} h(\xi + \nabla u)\, dx, \quad h \in C_0^0(\mathbb{R}^{n\times m}),$$

where $C_0^0(\mathbb{R}^{n\times m})$ denotes the space of continuous real-valued functions on $\mathbb{R}^{n\times m}$ tending to zero at infinity. The formula (2.5) can then be rewritten as

$$f^{qc}(\xi) = \inf\left\{ \int_{\mathbb{R}^{n\times m}} f\, d\nu \, : \, \nu \in \mathbf{Q},\, \overline{\nu} = \xi \right\},$$

where $\overline{\nu} := \langle \nu, \mathrm{id}\rangle$ denotes the centre of mass of ν (see e.g. Chap. 2 of [37]).

The next result shows that the Lipschitz constant of a separately convex function can be estimated by its oscillation. This observation is well known (see [34] p. 112), however, it appears that the following version with an explicit constant is missing in the literature. Our argument is a variant of the one presented in [15]. In the statement $B(\xi_0, r)$ denotes the open ball with centre at ξ_0 and radius r, and we recall that the oscillation of f on the set S is $\mathrm{osc}(f; S) := \sup\{|f(\xi) - f(\eta)| : \xi, \eta \in S\}$.

Lemma 2.2. *If $f : B(\xi_0, 2r) \to \mathbb{R}$ is separately convex, then*

$$\mathrm{lip}(f; B(\xi_0, r)) \le \sqrt{mn}\, \frac{\mathrm{osc}(f; B(\xi_0, 2r))}{r}.$$

In particular, a separately convex function is locally Lipschitz on the interior of its effective domain.

Remark. A slight modification of the proof below yields for rank-1 convex f the inequality

$$\operatorname{lip}(f; B(\xi_0, r)) \leq \sqrt{N}\frac{\operatorname{osc}(f; B(\xi_0, 2r))}{r},$$

where $N = \min\{m, n\}$.

Proof. Let $c = \operatorname{osc}(f; B(\xi_0, 2r))/r$. We first show that if $\xi, \eta \in B(\xi_0, r)$, such that $\xi - \eta$ is proportional to one of the canonical basis vectors e_i of $\mathbb{R}^{n\times m}$, then

$$f(\eta) - f(\xi) \leq c|\eta - \xi|. \tag{2.6}$$

Choose ζ to be the intersection of $\partial B(\xi_0, 2r)$ with the ray from ξ through η. So $\eta \in \operatorname{conv}(\{\xi, \zeta\})$ and $|\xi - \zeta| \geq r$. Hence, convexity of f along this ray in the direction $\pm e_i$ yields as required

$$\frac{f(\eta) - f(\xi)}{|\eta - \xi|} \leq \frac{f(\zeta) - f(\xi)}{|\zeta - \xi|} \leq \frac{\operatorname{osc}(f; B(\xi_0, 2r))}{r} = c.$$

Next we show that any pair of points $\xi, \eta \in B(\xi_0, r)$ can be joined by a chain $\xi = \zeta_0, \zeta_1, \ldots, \zeta_{mn} = \eta$ in $B(\xi_0, r)$, such that $\zeta_i - \zeta_{i-1} = \langle \eta - \xi, e_{\sigma(i)}\rangle e_{\sigma(i)}$ for a suitable bijection σ of $\{1, \ldots, mn\}$. By translation we can assume that $\xi_0 = 0$. Put $\zeta_0 := \xi$. Define $I_+ = \{i : |\langle \xi, e_i\rangle| \geq |\langle \eta, e_i\rangle|\}$ and let σ be any bijection of $\{1, \ldots, mn\}$, such that $\sigma(i) \in I_+$ if (and only if) $i \leq \operatorname{card}(I_+)$. For $i = 1, \ldots, mn$ let

$$\zeta_i := \xi - \sum_{j=1}^{i} \langle \xi - \eta, e_{\sigma(j)}\rangle e_{\sigma(j)}.$$

Observe that, $\zeta_i - \zeta_{i-1} = \langle \xi - \eta, e_{\sigma(i)}\rangle e_{\sigma(i)}$,

$$|\zeta_i| \leq |\zeta_{i-1}| \leq |\xi| < r \text{ if } 1 \leq i \leq \operatorname{card}(I_+)$$

and

$$|\zeta_i| \leq |\zeta_{i+1}| \leq |\eta| < r \text{ if } \operatorname{card}(I_+) \leq i < mn.$$

Hence the chain $\zeta_0, \zeta_1, \ldots, \zeta_{mn}$ has the claimed properties. The proof is finished by repeated use of (2.6):

$$\begin{aligned}
|f(\xi) - f(\eta)| &\leq \sum_{i=1}^{mn} |f(\zeta_i) - f(\zeta_{i-1})| \\
&\leq \sum_{i=1}^{mn} c|\zeta_i - \zeta_{i-1}| = \sum_{i=1}^{mn} c|\langle \eta - \xi, e_{\sigma(i)}\rangle| \\
&\leq c(\sum_{i=1}^{mn} 1^2)^{1/2}(\sum_{i=1}^{mn} |\langle \eta - \xi, e_{\sigma(i)}\rangle|^2)^{1/2} = c\sqrt{mn}|\eta - \xi|.
\end{aligned}$$

□

Corollary 2.3. *Let $B \subset \mathbb{R}^{n\times m}$ be an open ball and $f : B \to \mathbb{R}$ be separately convex. Denote by $\mathcal{D} \subseteq B$ the set where f is differentiable. Then $\nabla f : \mathcal{D} \to \mathbb{R}^{n\times m}$ is continuous.*

Proof. Let $\xi_0 \in \mathcal{D}$ and consider $h(\xi) := f(\xi) - f(\xi_0) - \langle \nabla f(\xi_0), \xi - \xi_0\rangle$. The function h is separately convex and differentiable at each point of $\mathcal{D}$. Note that ∇f is continuous relative to $\mathcal{D}$ at ξ_0 if $\nabla h(\xi) \to 0$ as $\xi \to \xi_0$, $\xi \in \mathcal{D}$. Observe that

$$|\nabla h(\xi)| \leq \operatorname{lip}(h; B(\xi_0, r)) \quad \text{if } \xi \in \mathcal{D} \text{ and } |\xi - \xi_0| < r$$

and, since f is differentiable at ξ_0,

$$\operatorname{osc}(h; B(\xi_0, 2r))/r \to 0 \quad \text{as } r \to 0^+.$$

The proof is concluded by use of Lemma 2.2. □

Lemma 2.4. *Suppose $f : B(\xi_0, r) \to \mathbb{R}$ is separately convex and that $a : \mathbb{R}^{n\times m} \to \mathbb{R}$ is affine with $a(\xi_0) = f(\xi_0)$. Then*

$$\begin{aligned} &-\inf\{f(\xi) - a(\xi) : \xi \in B(\xi_0, r)\} \\ &\quad \leq (2^{mn} - 1)\sup\{f(\xi) - a(\xi) : \xi \in B(\xi_0, r)\}. \end{aligned}$$

Remarks. 1. The constant in the inequality is best possible if instead of the Euclidean norm we use the norm $|\xi|_\infty = \max\{|\xi_1|, \dots, |\xi_N|\}$. The function

$$f(\xi) = 1 - \Pi_{i=1}^N (1 - \xi_i),\ \xi = (\xi_1, \dots, \xi_N) \in \mathbb{R}^N,$$

is separately convex, $f(0, \dots, 0) = 0$ and

$$-\inf_{|\xi|_\infty \leq 1} f(\xi) = (2^N - 1) \sup_{|\xi|_\infty \leq 1} f(\xi).$$

2. A slight modification of the proof below yields for rank-1 convex f the inequality

$$\begin{aligned} &-\inf\{f(\xi) - a(\xi) : \xi \in B(\xi_0, r)\} \\ &\quad \leq (2^N - 1)\sup\{f(\xi) - a(\xi) : \xi \in B(\xi_0, r)\}, \end{aligned}$$

where $N = \min\{m, n\}$.

Proof. We can assume that $\xi_0 = 0$ and $a \equiv 0$. Let $t < r$ and take $\xi_1 \in \overline{B}(0, t) := \{\xi : |\xi| \leq t\}$, such that $f(\xi_1) = \inf f(B(0, t))$. Let $\xi_1, \xi_2, \dots, \xi_{2^{mn}}$ denote the orbit of ξ_1 under reflections in the coordinate hyperplanes (it is not assumed that the ξ_i's are distinct). By an induction argument using the separate convexity we get that

$$f(0) \leq \sum_{i=1}^{2^{mn}} 2^{-mn} f(\xi_i),$$

and since $\xi_i \in \overline{B}(0,t)$,

$$0 = 2^{mn} f(0) \leq \sum_{i=1}^{2^{mn}} f(\xi_i) \leq \inf f(\overline{B}(0,t)) + (2^{mn}-1) \sup f(\overline{B}(0,t)).$$

The inequality follows if we let $t \to r^-$. □

In the statement of the next result we make use of the following terminology. A function $f : B(\xi_0, r) \to \mathbb{R}$ is upper semidifferentiable at ξ_0 if there exists $a \in \mathbb{R}^{n\times m}$, such that

$$\limsup_{\eta\to 0} \frac{f(\xi_0+\eta) - f(\xi_0) - \langle a, \eta\rangle}{|\eta|} \leq 0. \tag{2.7}$$

The set of $a \in \mathbb{R}^{n\times m}$ that satisfy (2.7) is denoted by $\partial^* f(\xi_0)$.

Corollary 2.5. *Suppose $g : B(\xi_0, r) \to \mathbb{R}$ is separately convex, $f : B(\xi_0, r) \to \mathbb{R}$ is upper semidifferentiable at ξ_0, that $g \leq f$ on $B(\xi_0, r)$ and $g(\xi_0) = f(\xi_0)$. Then f and g are differentiable at ξ_0 and $\nabla f(\xi_0) = \nabla g(\xi_0)$.*

The proof is a straightforward application of Lemma 2.4 and is left to the interested reader.

3 Proof of main results

Theorem 3.1. *Suppose that $f : \mathbb{R}^{n\times m} \to \mathbb{R}$ is bounded from below, continuous and upper semidifferentiable. Assume furthermore that for each η*

$$\frac{f^{qc}(\xi+\eta) - f^{qc}(\xi)}{\max\{f(\xi), 1\}} \to 0 \text{ as } \xi \to \infty \tag{3.1}$$

and

$$\sup\left\{\frac{|f^{qc}(\xi+t\eta) - f^{qc}(\xi)|}{t\max\{f(\xi),1\}} : \xi \in \mathbb{R}^{n\times m}, t \in (0,1)\right\} < \infty. \tag{3.2}$$

Then f^{qc} is a C^1 function.

Before presenting the proof we give two conditions that are easy to verify and that imply the conditions (3.1) and (3.2).

Lemma 3.2. *Let $f : \mathbb{R}^{n\times m} \to \mathbb{R}$ be bounded from below. If for some $p \in [0, \infty)$,*

$$\liminf_{\xi\to\infty} \frac{f(\xi)}{|\xi|^p} > 0 \text{ and } \limsup_{\xi\to\infty} \frac{f(\xi)}{|\xi|^{p+1}} = 0 \tag{3.3}$$

or

$$\frac{f(\xi)}{|\xi|^p} \to \infty \text{ as } \xi \to \infty \text{ and } \limsup_{\xi\to\infty} \frac{f(\xi)}{|\xi|^{p+1}} < \infty, \tag{3.4}$$

then (3.1) *and* (3.2) *hold.*

Proof. We give the proof for the case where f satisfies (3.3); the other case is analogous. Since for c positive, $(c+1)\max\{f(\xi),1\} \geq f(\xi)+c$, we can assume, by adding a constant to f if necessary, that $f(\xi) \geq 1$ for all ξ. By virtue of (3.3) there exists $c > 0$, such that

$$f(\xi) \geq \max\{\frac{1}{c}|\xi|^p, 1\},$$

and for any $\varepsilon > 0$ there exists $c(\varepsilon) \geq 1$, such that

$$f^{qc}(\xi) \leq c(\varepsilon) + \varepsilon|\xi|^{p+1}.$$

Fix $\eta \in \mathbb{R}^{n\times m}$ and $t \in (0,1)$; clearly

$$|f^{qc}(\xi+t\eta) - f^{qc}(\xi)| \leq \text{lip}(f^{qc}; B(0,|\xi|+|\eta|))t|\eta|,$$

and by Lemma 2.2,

$$\text{lip}(f^{qc}; B(0,|\xi|+|\eta|)) \leq \sqrt{mn}\frac{\text{osc}(f^{qc}; B(0,2(|\xi|+|\eta|)))}{|\xi|+|\eta|}.$$

Because f^{qc} is positive we can estimate the oscillation simply by using the upper bound:

$$\text{osc}(f^{qc}; B(0,2(|\xi|+|\eta|))) \leq c(\varepsilon) + \varepsilon 2^{p+1}(|\xi|+|\eta|)^{p+1}.$$

Hence by use of the lower bound we get

$$\frac{|f^{qc}(\xi+t\eta) - f^{qc}(\xi)|}{tf(\xi)} \leq \sqrt{mn}\frac{c(\varepsilon) + \varepsilon 2^{p+1}(|\xi|+|\eta|)^{p+1}}{\max\{\frac{1}{c}|\xi|^p, 1\}(|\xi|+|\eta|)}|\eta|$$

and, as ε was arbitrary, this inequality implies (3.1) and (3.2). □

Proof of Theorem 3.1. In consideration of Corollary 2.3 it suffices to show that f^{qc} is differentiable. As above we can assume that $f(\xi) \geq 1$ for all ξ. The proof proceeds in three steps and relies on the following result about weak* convergence of measures.

Lemma 3.3. *Suppose that f, $g : \mathbb{R}^{n\times m} \to \mathbb{R}$ are continuous functions, that $f(\xi) \geq 1$ for all ξ and that $g(\xi)/f(\xi) \to 0$ as $\xi \to \infty$. If $\nu_j \overset{*}{\rightharpoonup} \nu$ in $C_0^0(\mathbb{R}^{n\times m})'$ and*

$$\sup_j \int f\, d\nu_j < \infty,$$

then g is ν-integrable and

$$\lim_{j\to\infty} \int g\, d\nu_j = \int g\, d\nu.$$

The proof of this lemma is standard and is omitted here.

1. Note that for each ξ_0, $\xi \mapsto f(\xi + \xi_0)$ satisfies the hypotheses of the theorem. Hence it suffices to show that f^{qc} is differentiable at 0. Referring to Proposition 2.1 we can take probability measures $\nu_j \in \mathbf{Q}$ with $\overline{\nu}_j = 0$ satisfying

$$\int f\, d\nu_j < f^{qc}(0) + \frac{1}{j}. \tag{3.5}$$

Extracting a subsequence (for convenience not relabelled) we can assume that $\nu_j \overset{*}{\rightharpoonup} \nu$ in $C_0^0(\mathbb{R}^{n\times m})'$. The measure ν is a sub-probability measure and because f is continuous and bounded from below we get that

$$\int f\, d\nu \leq \lim_{j\to\infty} \int f\, d\nu_j = f^{qc}(0) < \infty. \tag{3.6}$$

We remark that the measure ν might not have a centre of mass, and that if it does, then $\overline{\nu}$ is possibly different from 0.

2. Claim: $f = f^{qc}$ on $\mathrm{spt}(\nu)$, and hence by Corollary 2.5 f and f^{qc} are differentiable with the same derivative at each point of $\mathrm{spt}(\nu)$. Here $\mathrm{spt}(\nu)$ denotes the support of the measure ν, i.e. $\mathrm{spt}(\nu) := \{\xi : \nu(B(\xi, r)) > 0 \text{ for all } r > 0\}$.

Because f and f^{qc} are continuous it suffices to show that $f = f^{qc}$ ν-a.e. Since $f \geq f^{qc}$, only $f \leq f^{qc}$ ν-a.e. remains to be verified. For that purpose let $\varepsilon > 0$ and consider $E = \{\xi : f(\xi) - \varepsilon > f^{qc}(\xi)\}$. Because E is open, $\liminf_{j\to\infty} \nu_j(E) \geq \nu(E)$. We therefore have

$$f^{qc}(0) \leq \int f^{qc}\, d\nu_j \leq \int f\, d\nu_j - \varepsilon\nu_j(E),$$

and so

$$f^{qc}(0) \leq f^{qc}(0) - \varepsilon \limsup_{j\to\infty} \nu_j(E),$$

thus, $\nu(E) \leq \limsup_{j\to\infty} \nu_j(E) = 0$.

3. For fixed $t > 0$ and $\eta \in \mathbb{R}^{n\times m}$ we have by quasiconvexity of $\xi \mapsto f^{qc}(\xi + t\eta)$ that

$$f^{qc}(t\eta) \le \int f^{qc}(\xi + t\eta)\, d\nu_j(\xi),$$

and hence, by (3.5),

$$\Lambda_t(\eta) := \frac{f^{qc}(t\eta) - f^{qc}(0)}{t} \le \int \frac{f^{qc}(\xi + t\eta) - f^{qc}(\xi)}{t}\, d\nu_j(\xi) + \frac{1}{jt}.$$

In view of (3.1), (3.5) and the continuity of the integrands it follows from Lemma 3.3 that (t, η still fixed)

$$\int \frac{f^{qc}(\xi + t\eta) - f^{qc}(\xi)}{t}\, d\nu_j(\xi) \to \int \frac{f^{qc}(\xi + t\eta) - f^{qc}(\xi)}{t}\, d\nu(\xi),$$

and thus

$$\Lambda_t(\eta) \le \int \frac{f^{qc}(\xi + t\eta) - f^{qc}(\xi)}{t}\, d\nu(\xi).$$

By virtue of (3.2) there exists a constant $c = c(\eta)$, such that for all ξ and $0 < t < 1$

$$\left| \frac{f^{qc}(\xi + t\eta) - f^{qc}(\xi)}{t} \right| \le c f(\xi).$$

The right-hand side is ν-integrable by (3.6). From Step 2 we have for ν-a.e. ξ

$$\lim_{t\to 0+} \frac{f^{qc}(\xi + t\eta) - f^{qc}(\xi)}{t} = \langle \nabla f(\xi), \eta \rangle,$$

and therefore the dominated convergence theorem implies that $\langle \nabla f, \eta \rangle$ is ν-integrable and that

$$\Lambda(\eta) := \limsup_{t\to 0+} \Lambda_t(\eta) \le \int \langle \nabla f, \eta \rangle\, d\nu. \tag{3.7}$$

Using that $\langle \nabla f, \eta \rangle$ is ν-integrable for all η we infer that ∇f is ν-integrable.

Observe that Λ is positively homogeneous of degree 1 and, as a limit superior of separately convex functions, is separately convex. By Corollary 2.5 we therefore deduce that Λ is differentiable at 0 with $\nabla \Lambda(0) = \int \nabla f\, d\nu$ and together with the homogeneity this yields

$$\Lambda(\eta) = \left\langle \int \nabla f\, d\nu, \eta \right\rangle, \quad \forall \eta \in \mathbb{R}^{n\times m}. \tag{3.8}$$

We claim that as a consequence f^{qc} is upper semidifferentiable at 0, and hence, by Corollary 2.5, differentiable at 0. Indeed, otherwise we could find $\delta > 0$ and $\xi_j \to 0$, such that

$$\frac{f^{qc}(\xi_j) - f^{qc}(0) - \Lambda(\xi_j)}{|\xi_j|} > \delta.$$

Up to a subsequence we have that $\xi_j' := \xi_j/|\xi_j| \to \xi_\infty'$, and therefore by local Lipschitz continuity of f^{qc} we have for some constant L

$$\delta < \frac{f^{qc}(|\xi_j|\xi_\infty') - f^{qc}(0) - \Lambda(|\xi_j|\xi_\infty')}{|\xi_j|} + L|\xi_j' - \xi_\infty'| + \Lambda(\xi_j' - \xi_\infty')$$

and by (3.8) we get a contradiction as $j \to \infty$. This concludes the proof. □

Theorem 3.4. *Suppose $f : \mathbb{R}^{n\times m} \to \mathbb{R}$ is continuous, bounded from below and upper semidifferentiable. Assume furthermore that for each η*

$$\frac{f(\xi+\eta) - f(\xi)}{\max\{f(\xi), 1\}} \to 0 \text{ as } \xi \to \infty \tag{3.9}$$

and

$$\sup\left\{\frac{|f(\xi+t\eta) - f(\xi)|}{t\max\{f(\xi),1\}} : \xi \in \mathbb{R}^{n\times m},\ t \in (0,1)\right\} < \infty. \tag{3.10}$$

Then f^{qc} is a C^1 function.

To prove Theorem 3.4 we proceed as in the proof of Theorem 3.1. The only change is in Step 3 where we estimate the difference quotient $\Lambda_t(\eta)$ as

$$\Lambda_t(\eta) \le \int \frac{f(\xi+t\eta) - f(\xi)}{t}\, d\nu_j(\xi) + \frac{1}{jt}.$$

The next lemma shows how to verify the conditions (3.9) and (3.10).

Lemma 3.5. *Let $f : \mathbb{R}^{n\times m} \to \mathbb{R}$ be bounded from below, locally Lipschitz and differentiable. If*

$$\frac{\nabla f(\xi)}{\max\{f(\xi), 1\}} \to 0 \text{ as } \xi \to \infty, \tag{3.11}$$

then conditions (3.9) *and* (3.10) *hold.*

Proof. We can assume that $f(\xi) \ge 1$. For $\xi, \eta \in \mathbb{R}^{n\times m}$ and $t > 0$ define $g(s) := f(\xi + st\eta)$, $s \in \mathbb{R}$. Then g is differentiable, locally Lipschitz and if we let

$$h(r) := \sup_{|\xi|\ge r} \frac{|\nabla f(\xi)|}{f(\xi)},\ r \ge 0,$$

then $|g'(s)| \le h(|\xi + st\eta|)t|\eta|g(s)$. Thus by integration

$$\ln\frac{g(1)}{g(0)} \le \int_0^1 h(|\xi + st\eta|)t|\eta|\, ds.$$

Since h is decreasing we deduce

$$f(\xi + t\eta) \le e^{h(0)t|\eta|} f(\xi). \tag{3.12}$$

Now

$$|g(1) - g(0)| \le \int_0^1 |g'(s)|\, ds \le \int_0^1 h(|\xi + st\eta|)t|\eta| g(s)\, ds$$

and since $\max\{0, |\xi| - t|\eta|\} \le |\xi + st\eta|$ we get

$$|f(\xi + t\eta) - f(\xi)| \le h(\max\{0, |\xi| - t|\eta|\})t|\eta| \int_0^1 f(\xi + st\eta)\, ds.$$

Together with (3.12) this implies for $t \in (0, 1)$ and any η,

$$|f(\xi + t\eta) - f(\xi)| \le e^{h(0)t|\eta|} h(\max\{0, |\xi| - |\eta|\}) f(\xi) t|\eta|,$$

concluding the proof. □

Proposition 3.6. *Suppose that $f : \mathbb{R}^{n\times m} \to \mathbb{R}$ satisfies the conditions of either Theorem* 3.1 *or Theorem* 3.4. *Let $\xi_0 \in \mathbb{R}^{n\times m}$ and $\nu_j \in \mathbf{Q}$, $\overline{\nu}_j = \xi_0$ and*

$$\int f\, d\nu_j \to f^{qc}(\xi_0).$$

If $\nu_j \overset{*}{\rightharpoonup} \nu$ in $C_0^0(\mathbb{R}^{n\times m})'$, then ν is a sub-probability measure with the following properties:

$$f = f^{qc} \quad \textit{on } \operatorname{spt}(\nu),$$

$$f \textit{ is differentiable at each } \xi \in \operatorname{spt}(\nu) \textit{ with } \nabla f(\xi) = \nabla f^{qc}(\xi),$$

and

$$\nabla f^{qc}(\xi_0) = \int \nabla f\, d\nu.$$

Remark. The measure ν might not have a centre of mass, i.e. $\int |\xi|\, d\nu(\xi) = \infty$ is not excluded. Even if ν has a centre of mass it can be different from ξ_0. Furthermore, the inequality $f^{qc}(\xi_0) \ge \int f\, d\nu$ might be strict.

Proof. The claims follow as in the proof of Theorem 3.1. □

Proposition 3.7. *Assume that $f : \mathbb{R}^{n\times m} \to \mathbb{R}$ satisfies the conditions of Theorem* 3.1 *or of Theorem* 3.4, *and furthermore that for some $\alpha \in (0, 1]$ and $c > 0$ we have for each ξ and some $a \in \partial^* f(\xi)$*

$$f(\xi + \eta) - f(\xi) - \langle a, \eta\rangle \le c \max\{f(\xi), 1\}|\eta|^{1+\alpha} \tag{3.13}$$

for $|\eta| \le 1$. Then f^{qc} is a $C^{1,\alpha}_{\mathrm{loc}}$ function, and more precisely

$$|\nabla f^{qc}(\xi + \eta) - \nabla f^{qc}(\xi)| \le cc_1 \max\{f(\xi), 1\}|\eta|^{\alpha}$$

for $|\eta| \le 1/2$, where $c_1 = c_1(mn, \alpha)$.

Remarks. 1. The condition (3.13) is satisfied with $\alpha = 1$ for example if f is a C^2 function satisfying for some constant $c > 0$

$$|\nabla^2 f(\xi)| \leq c \max\{f(\xi), 1\}.$$

Further examples can be found in Sect. 5.

2. In general, for smooth functions $f : \mathbb{R} \to \mathbb{R}$ the convex envelope f^c is not of class C^2. For example any smooth function $f : \mathbb{R} \to \mathbb{R}$ satisfying $f = f^c$ outside $[-1, 1]$, $f \geq 0$, $= 0$ exactly at ± 1 and $f''(\pm 1) > 0$ provides such an example. The situation is the same in higher dimensions $f : \mathbb{R}^{n\times m} \to \mathbb{R}$. Indeed let $g : \mathbb{R} \to \mathbb{R}$ be a C^∞ function for which g^c is not C^2. Define $f(\xi) := g(\xi_{1,1})$, $\xi \in \mathbb{R}^{n\times m}$, where $\xi_{1,1}$ is the $(1, 1)$ entry of the matrix ξ. It is easily seen that $f^{qc}(\xi) = g^c(\xi_{1,1})$.

Proof. We only give the proof for the case where f satisfies the conditions (3.1) and (3.2) of Theorem 3.1. The proof in the other case is analogous.

It follows from Theorem 3.1 that f^{qc} is C^1 and we can assume that $f \geq 1$. The main step in proving that ∇f^{qc} is locally α-Hölder continuous consists in verifying the inequality:

$$|f^{qc}(\xi + \eta) - f^{qc}(\xi) - \langle \nabla f^{qc}(\xi), \eta\rangle| \leq (2^{mn} - 1) c f^{qc}(\xi)|\eta|^{1+\alpha}, \tag{3.14}$$

for all $\xi, \eta \in \mathbb{R}^{n\times m}$ with $|\eta| \leq 1$. To establish (3.14) we fix ξ_0 and proceed as in the proof of Theorem 3.1 to find a sub-probability measure ν with the following properties:

$$f = f^{qc} \nu\text{-a.e. (and hence } f \text{ is differentiable } \nu\text{-a.e.)},$$

$$\nabla f^{qc}(\xi_0) = \int \nabla f^{qc} \, d\nu,$$

$$\int f \, d\nu \leq f^{qc}(\xi_0)$$

and for all η

$$f^{qc}(\xi_0 + \eta) - f^{qc}(\xi_0) - \langle \nabla f^{qc}(\xi_0), \eta\rangle \leq \int [f^{qc}(\xi + \eta) - f^{qc}(\xi) - \langle \nabla f^{qc}(\xi), \eta\rangle] \, d\nu(\xi).$$

Now for ν-a.e. ξ, $f^{qc}(\xi + \eta) - f^{qc}(\xi) - \langle \nabla f^{qc}(\xi), \eta\rangle \leq f(\xi + \eta) - f(\xi) - \langle \nabla f(\xi), \eta\rangle$ and, invoking (3.13), the latter is estimated from above by $c f(\xi)|\eta|^{1+\alpha}$ for all ξ and $|\eta| \leq 1$. Collecting the above estimates we have for $|\eta| \leq 1$,

$$f^{qc}(\xi_0 + \eta) - f^{qc}(\xi_0) - \langle \nabla f^{qc}(\xi_0), \eta\rangle \leq c f^{qc}(\xi_0)|\eta|^{1+\alpha}$$

and as the left-hand side is a separately convex function of η we infer by use of Lemma 2.4 that

$$\inf_{|\delta|\le|\eta|}[f^{qc}(\xi_0+\delta)-f^{qc}(\xi_0)-\langle\nabla f^{qc}(\xi_0),\delta\rangle]\ge(1-2^{mn})cf^{qc}(\xi_0)|\eta|^{1+\alpha}.$$

These inequalities yield in combination (3.14). It is a standard result that (3.14) implies local α-Hölder continuity of ∇f^{qc}, however, in the present situation it is possible to give a simpler direct proof. For fixed ξ the function $h(\eta):=f^{qc}(\xi+\eta)-f^{qc}(\xi)-\langle\nabla f^{qc}(\xi),\eta\rangle$ is separately convex. By Lemma 2.2,

$$\mathrm{lip}(h;B(0,r))\le\sqrt{mn}\,\mathrm{osc}(h;B(0,2r))/r,$$

and by use of (3.14) we get for $r\in(0,1/2]$

$$\mathrm{osc}(h;B(0,2r))\le Ccf^{qc}(\xi)r^{1+\alpha},$$

where $C=2(2^{mn}-1)$. For $|\eta|\le 1/2$ we therefore obtain

$$|\nabla h(\eta)|\le\mathrm{lip}(h;B(0,|\eta|))\le Cc\sqrt{mn}2^{1+\alpha}f^{qc}(\xi)|\eta|^{\alpha},$$

and since $\nabla h(\eta)=\nabla f^{qc}(\xi+\eta)-\nabla f^{qc}(\xi)$ this inequality concludes the proof with $c_1=\sqrt{mn}2^{1+\alpha}C$. □

4 Other envelopes

It is not hard to extend the results in Sect. 2 on separately convex functions to functions that are convex in the directions of a general basis for $\mathbb{R}^{n\times m}$. The results in Sect. 3 can therefore be extended to envelopes corresponding to any notion of convexity for which the envelope can be expressed as an infimum over a class of probability measures (as in Proposition 2.1), and which imply convexity in the directions of a basis. These include separate convexity, rank-1 convexity and polyconvexity. They also include A-quasiconvexity (cf. [23] and the references therein) in the case where the cone for the corresponding directional convexity spans the space. Instead of formulating these results explicitly we prefer to focus on the case of polyconvex envelopes, where we can obtain a slightly stronger result.

The polyconvex envelope f^{pc} of an extended real-valued function $f:\mathbb{R}^{n\times m}\to\mathbb{R}\cup\{\infty\}$ is by definition the largest (extended real-valued) polyconvex function, which minorizes f (identically $-\infty$ if no such function exists). In symbols,

$$f^{pc}(\xi):=\sup\{g(\xi):g:\mathbb{R}^{n\times m}\to\mathbb{R}\cup\{\infty\}\text{ polyconvex, }g\le f\}.$$

Since polyconvex functions in particular are rank-1 convex it follows from Lemma 2.2 that f^{pc} is locally Lipschitz on the interior of its effective domain $\mathrm{dom}_e(f^{pc}):=\{\xi:f^{pc}(\xi)<\infty\}$. However, an extended real-valued polyconvex function need not be continuous or even semicontinuous everywhere.

It can be shown (cf. [9, Sect. 1.1 (6)]) that for all ξ,

$$f^{pc}(\xi) = \inf\left\{\sum_{i=1}^{\tau+1}\lambda_i f(\xi_i) : \lambda_i \geq 0, \sum_{i=1}^{\tau+1}\lambda_i = 1, \sum_{i=1}^{\tau+1}\lambda_i M(\xi_i) = M(\xi)\right\}, \tag{4.1}$$

where $M(\xi) \in \mathbb{R}^\tau$ is the vector of all minors of ξ arranged in some fixed order and where $\tau = \tau(m,n)$ is the number of such minors. The proof in [9] is given only for the case of real-valued functions, but it is not difficult to see that it also covers the case of extended real-valued functions (see also [15]).

Proposition 4.1. *Let $f : \mathbb{R}^{n\times m} \to \mathbb{R}\cup\{\infty\}$ be an extended real-valued continuous function and suppose that f is upper semidifferentiable where it is finite and that*

$$\frac{f(\xi)}{|\xi|^{p-1}} \to \infty \text{ as } \xi \to \infty, \tag{4.2}$$

where $p = \min\{m,n\}$ (and $m, n \geq 2$). Then the polyconvex envelope $f^{pc} : \mathbb{R}^{n\times m} \to \mathbb{R}\cup\{\infty\}$ is C^1 on the interior of its effective domain.

Proof. In view of Corollary 2.3 it suffices to show that f^{pc} is differentiable on the interior of its effective domain. We assume that the interior of the effective domain $\mathrm{dom}_e(f^{pc})$ of f^{pc} is non-empty and let ξ be a fixed point in this set. Referring to (4.1) we select a minimising sequence $\{(\lambda_i^{(k)}, \xi_i^{(k)})_{1\leq i\leq\tau+1}\}_{k=1}^{\infty}$, i.e.

$$\lambda_i^{(k)} \geq 0, \sum_{i=1}^{\tau+1}\lambda_i^{(k)} = 1, \sum_{i=1}^{\tau+1}\lambda_i^{(k)} M(\xi_i^{(k)}) = M(\xi)$$

and

$$\sum_{i=1}^{\tau+1}\lambda_i^{(k)} f(\xi_i^{(k)}) \to f^{pc}(\xi) \text{ as } k \to \infty.$$

Extracting a subsequence (for convenience not relabelled) we can assume that for each i, $\lambda_i^{(k)} \to \lambda_i \in [0,1]$ as $k \to \infty$. By (4.2), we may assume that

$$\sup_k \sum_{i=1}^{\tau+1}\lambda_i^{(k)}\theta(|\xi_i^{(k)}|^{p-1}) < \infty \tag{4.3}$$

for some convex, increasing function $\theta : [0,\infty) \to \mathbb{R}$ satisfying $\theta(t)/t \to \infty$ as $t \to \infty$. For later use we note that we, without loss of generality, can assume that $\theta(t) \le t^{p/(p-1)}$ for $t > 0$. In view of (4.3) we can find a subsequence (for convenience not relabelled), such that $\xi_i^{(k)} \to \xi_i$ as $k \to \infty$ for those i where $\lambda_i > 0$. For i such that $\lambda_i = 0$ we define $\xi_i = \xi$. Observe that (since $p \ge 2$)

$$\sum_{i=1}^{\tau+1} \lambda_i = 1 \text{ and } \sum_{i=1}^{\tau+1} \lambda_i \xi_i = \xi$$

and by the continuity of f

$$\sum_{i=1}^{\tau+1} \lambda_i f(\xi_i) < \infty,$$

where we use the convention $0 \cdot \infty = 0$.

Select a convex function $F : \mathbb{R}^\tau \to \mathbb{R} \cup \{\infty\}$, such that $f^{pc} = F \circ M$. For positive integers l define

$$F_l(X) = \sup_{|Y| \le l} \{\langle X, Y\rangle - F^\star(Y)\}, \quad X \in \mathbb{R}^\tau,$$

where $F^\star(X) := \sup_Y\{\langle X, Y\rangle - F(Y)\}$ denotes the polar of F. Then F_l are convex, $F_l(X) \le \min\{c_l(1 + |X|), F(X)\}$ for some constants c_l and $F_l(X) \nearrow F(X)$ as $l \nearrow \infty$ for X in the interior of $\mathrm{dom}_e(F)$, the effective domain of F (see [36]). Define $f_l(\zeta) = \max\{F_l(M(\zeta)), \theta(|\zeta|^{p-1})\}$; then f_l are polyconvex and

$$\theta(|\zeta|^{p-1}) \le f_l(\zeta) \le C_l(1 + |\zeta|^p)$$

for suitable constants C_l. We claim that $f_l(\zeta) \nearrow f^{pc}(\zeta)$ as $l \nearrow \infty$ for ζ in the interior of $\mathrm{dom}_e(f^{pc})$. To verify this we observe that $\mathrm{dom}_e(F)$ is convex; if therefore $B(\xi, r) \subset \mathrm{dom}_e(f^{pc})$, then the convex hull $\mathrm{co}[M(B(\xi, r))] \subset \mathrm{dom}_e(F)$. By virtue of [9, Sect. 3.2 (20)] the convex hull $\mathrm{co}[M(B(\xi, r))]$ is a neighbourhood of $M(\xi)$ in $\mathbb{R}^\tau$, hence the claim follows.

For $\eta \ne 0$ and $t > 0$ sufficiently small $\xi + t\eta$ belongs to the interior of $\mathrm{dom}_e(f^{pc})$. For such fixed t and η we have that

$$\frac{f^{pc}(\xi + t\eta) - f^{pc}(\xi)}{t} < \frac{f_l(\xi + t\eta) - f_l(\xi)}{t} + \varepsilon_l,$$

where $\varepsilon_l \to 0$ as $l \to \infty$. Since f_l is polyconvex and $f_l \le f$ we conclude for any k

$$\frac{f^{pc}(\xi + t\eta) - f^{pc}(\xi)}{t} - \varepsilon_l <$$

$$\sum_{i=1}^{\tau+1} \lambda_i^{(k)} \frac{f_l(\xi_i^{(k)} + t\eta) - f_l(\xi_i^{(k)})}{t} + \frac{1}{t} \sum_{i=1}^{\tau+1} \lambda_i^{(k)} (f(\xi_i^{(k)}) - f_l(\xi)).$$

Observe that Lemma 2.2 implies that for some constant $A = A(C_l)$

$$\frac{|f_l(\zeta + t\eta) - f_l(\zeta)|}{t} \leq A|\eta|(1 + |\zeta|^{p-1})$$

for all ζ. Hence by (4.3) we get as $k \to \infty$

$$\frac{f^{pc}(\xi + t\eta) - f^{pc}(\xi)}{t} - \varepsilon_l < \sum_{i=1}^{\tau+1} \lambda_i \frac{f_l(\xi_i + t\eta) - f_l(\xi_i)}{t} + \frac{1}{t}(f^{pc}(\xi) - f_l(\xi)).$$

Next let $l \to \infty$ to obtain

$$\frac{f^{pc}(\xi + t\eta) - f^{pc}(\xi)}{t} \leq \sum_{i=1}^{\tau+1} \lambda_i \frac{f^{pc}(\xi_i + t\eta) - f^{pc}(\xi_i)}{t}.$$

As in the proof of Theorem 3.1 (we identify ν_j with $\sum_{i=1}^{\tau+1} \lambda_i^{(j)} \delta_{\xi_i^{(j)}}$, where $\delta_{\xi_i^{(j)}}$ denotes the Dirac measure at $\xi_i^{(j)}$), for i such that $\lambda_i > 0$, we have that f is differentiable at ξ_i and that

$$\frac{f^{pc}(\xi_i + t\eta) - f^{pc}(\xi_i)}{t} \to \langle \nabla f(\xi_i), \eta \rangle \text{ as } t \to 0^+,$$

and hence with the convention $0 \cdot (\text{undefined}) = 0$ we have

$$\limsup_{t \to 0^+} \frac{f^{pc}(\xi + t\eta) - f^{pc}(\xi)}{t} \leq \sum_{i=1}^{\tau+1} \lambda_i \langle \nabla f(\xi_i), \eta \rangle.$$

The proof is finished as the proof of Theorem 3.1. □

5 Examples

The following example shows that the differentiability of a quasiconvex envelope might fail if the growth conditions of Theorems A and B are not satisfied. Compare in particular with Theorem A in the case where $p = 1$.

Theorem 5.1. *Suppose $m, n \geq 2$. There exists a C^∞ function $f : \mathbb{R}^{n \times m} \to \mathbb{R}$ satisfying the growth conditions*

$$\liminf_{\xi \to \infty} \frac{f(\xi)}{|\xi|} > 0 \quad \textit{and} \quad \limsup_{\xi \to \infty} \frac{f(\xi)}{|\xi|^2} < \infty,$$

and where the polyconvex, quasiconvex and rank-1 convex envelopes (resp. f^{pc}, f^{qc} and f^{rc}) are not differentiable.

The proof of this theorem relies on the following two elementary lemmas.

Lemma 5.2. *Let $\xi_1, \xi_2 \in \mathbb{R}^{2\times 2}$, $\xi_1 \neq \xi_2$ and define*

$$g(\xi) := \max\{\langle \xi_1, \xi\rangle, \langle \xi_2, \xi\rangle + \det\xi\},\ \xi \in \mathbb{R}^{2\times 2}.$$

Then g has the following properties:

(1) g is polyconvex,
(2) g is not differentiable at 0,
(3) for all $\xi \in \mathbb{R}^{2\times 2}$,

$$g(\xi) \leq \langle \xi_1, \xi\rangle + \frac{1}{2}|\xi + \mathrm{cof}(\xi_2 - \xi_1)|^2$$

and equality holds in particular for $\xi = -\mathrm{cof}(\xi_2 - \xi_1)$.

In (3) we write $\mathrm{cof}\xi$ for the cofactor matrix of ξ, i.e.

$$\mathrm{cof}\xi = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix} \text{ if } \xi = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Proof. It is clear that g is polyconvex. (2) is a consequence of

$$\left.\begin{array}{l} \langle \xi_1, \xi\rangle = 0 = \langle \xi_2, \xi\rangle + \det\xi \\ \nabla\langle \xi_1, \cdot\rangle = \xi_1 \neq \xi_2 = \nabla(\langle \xi_2, \cdot\rangle + \det(\cdot)) \end{array}\right\} \text{ at } \xi = 0.$$

As regards (3) we write $g(\xi) = \langle \xi_1, \xi\rangle + \max\{0, h(\xi)\}$, where $h(\xi) := \langle \xi_0, \xi\rangle + \det\xi$, $\xi_0 := \xi_2 - \xi_1$. Taylor expansion of h about $-\mathrm{cof}\xi_0$ yields $h(\xi) = \det(\xi + \mathrm{cof}\xi_0)$, and the desired inequality follows by use of Hadamard's inequality. A straightforward computation establishes the equality for $\xi = -\mathrm{cof}\xi_0$. □

Lemma 5.3. *Let ξ_1, ξ_2, g be as above and assume additionally that $\langle \xi_1, \mathrm{cof}(\xi_2 - \xi_1)\rangle > 0$. Define $\xi_3 := \mathrm{cof}(\xi_2 - \xi_1)/|\xi_2 - \xi_1|$ and*

$$S := \{\xi \in \mathbb{R}^{2\times 2} : |\xi - \langle \xi, \xi_3\rangle \xi_3| \leq \frac{1}{|\xi|}, \langle \xi, \xi_3\rangle \geq |\xi_1 - \xi_2|\}.$$

With $c := 2 + |\xi_1 - \xi_2|^{-2}$ we have for $\xi \in S$ that $g(\xi) \leq c + \langle \xi_1, \xi\rangle$.

Proof. Write $\xi = t\xi_3 + \eta$, where $t := \langle \xi, \xi_3\rangle$. If $\xi \in S$, then

$$|\xi_1 - \xi_2| \leq t \leq |\xi| \leq \sqrt{1 + t^2} \text{ and } |\eta| \leq \frac{1}{|\xi|} \leq \frac{1}{t} \leq \frac{1}{|\xi_1 - \xi_2|}.$$

Using these inequalities, Cauchy-Schwarz' inequality, the algebraic identities $\langle \mathrm{cof}\xi_3, \xi_3\rangle = 2\det\xi_3$ and $\det(a + b) = \det a + \langle \mathrm{cof} a, b\rangle + \det b$, we get for $\xi \in S$,

$$\begin{aligned} \langle \xi_2 - \xi_1, \xi\rangle + \det\xi &= |\xi_1 - \xi_2|\langle \mathrm{cof}\xi_3, \eta\rangle + \langle \mathrm{cof}\xi_3, \eta\rangle t + \det\eta \\ &\leq 1 + 1 + \frac{1}{2}|\xi_1 - \xi_2|^{-2} < c, \end{aligned}$$

and the claimed inequality is an easy consequence. □

Proof of Theorem 5.1. We first establish the theorem in the case $m = n = 2$. Let ξ_1, ξ_2, ξ_3, g and S be as in Lemmas 5.2 and 5.3. It is important to notice that we additionally can arrange that $\operatorname{rank}(\xi_2 - \xi_1) = 1$. For example this can be achieved by taking

$$\xi_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad \xi_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Take a cut-off function $\varphi \in C^\infty(\mathbb{R}^{2\times 2})$ satisfying: $0 \le \varphi \le 1$ and

$$\varphi(\xi) = \begin{cases} 1 & \text{if } \xi = \langle \xi, \xi_3 \rangle \xi_3 \text{ and } \langle \xi, \xi_3 \rangle \ge 2|\xi_1 - \xi_2| \\ 0 & \text{if } \xi \notin S. \end{cases}$$

Define

$$f(\xi) := \varphi(\xi)(c + \langle \xi_1, \xi \rangle) + (1 - \varphi(\xi)) \times \left(\langle \xi_1, \xi \rangle + \frac{1}{2}|\xi + \operatorname{cof}(\xi_2 - \xi_1)|^2 \right), \xi \in \mathbb{R}^{2\times 2}.$$

In view of Lemmas 5.2 and 5.3, $f(\xi) \ge g(\xi)$ for all $\xi \in \mathbb{R}^{2\times 2}$, and since $-\operatorname{cof}(\xi_2 - \xi_1) \notin S$, equality holds at $\xi = -\operatorname{cof}(\xi_2 - \xi_1)$. Next we verify that f satisfies the stated growth conditions. The upper bound is obvious. Concerning the lower bound we observe that for $\xi \in S$,

$$\langle \xi_1, \xi \rangle \ge \langle \xi_1, \xi_3 \rangle \sqrt{\max\{0, |\xi|^2 - 1\}} - \frac{|\xi_1|}{|\xi_1 - \xi_2|},$$

and for $\xi \in \mathbb{R}^{2\times 2}$,

$$\langle \xi_1, \xi \rangle + \frac{1}{2}|\xi + \operatorname{cof}(\xi_2 - \xi_1)|^2 \ge \frac{1}{2}|\xi|^2 - 2(|\xi_1| + |\xi_2|)|\xi| + \frac{1}{2}|\xi_1 - \xi_2|^2,$$

hence

$$\liminf_{\xi \to \infty} \frac{f(\xi)}{|\xi|} \ge \langle \xi_1, \xi_3 \rangle = \frac{\langle \xi_1, \operatorname{cof}(\xi_2 - \xi_1) \rangle}{|\xi_2 - \xi_1|} > 0.$$

We claim that f^{pc}, f^{qc} and f^{rc} are not differentiable at 0. Recall from Lemma 5.2 that g is not differentiable at 0 and hence by Corollary 2.5 not upper semidifferentiable either. The claim therefore follows if we can establish that

$$g \le f^{pc} \le f^{qc} \le f^{rc} \tag{5.1}$$

and

$$f^{rc}(0) = 0 = g(0). \tag{5.2}$$

The inequalities (5.1) are obvious. To see that (5.2) holds we note that the matrix $\zeta := -\mathrm{cof}(\xi_2 - \xi_1)$ has rank one and hence by rank-1 convexity that for $t > 1$,

$$f^{rc}(-\frac{1}{t}\zeta) \leq \frac{1}{t} f(-t\zeta) + \frac{t-1}{t} f(\zeta).$$

Passing to the limit $t \to \infty$ we get

$$f^{rc}(0) \leq -\langle \xi_1, \zeta \rangle + f(\zeta) = 0,$$

where the last equality follows from Lemma 5.2. This finishes the proof for the case $m = n = 2$. In case $\max\{m, n\} > 2$, let $P : \mathbb{R}^{n\times m} \to L$ denote orthogonal projection onto the subspace

$$L = \left\{ \begin{pmatrix} a & b & 0 & \dots & 0 \\ c & d & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ & \dots & \dots & \dots & \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix} : a, b, c, d \in \mathbb{R} \right\}.$$

If we define $F : \mathbb{R}^{n\times m} \to \mathbb{R}$ by $F(\xi) := f(P(\xi)) + |\xi - P(\xi)|^2$, then it follows from the above consideration that F has the claimed properties. □

The next result gives a class of smooth integrands $F = F(u, \xi)$ for which the envelopes $\overline{F}$ are not differentiable with respect to u.

Proposition 5.4. *Any C^∞ function $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfying $F(u, \xi) \geq -|u|$ for all (u, ξ) and*

$$F(u, 0) = -u, \ F(u, 1) = u, \ F(u, 2) = -u \text{ and } F(u, 3) = u$$

for all $|u| \leq 1$ has the property that $\overline{F}(u, \xi) := F(u, \cdot)^c|_\xi = -|u|$ for $|u| \leq 1$ and $1 \leq \xi \leq 2$. Hence $\overline{F}$ is not differentiable with respect to u.

Remark. An example of such a function is $F(u, \xi) = -u\cos(\pi\xi)$.

Proof. Clearly, $\overline{F}(u, \xi) \geq -|u|$. On the other side we have by convexity of $\overline{F}(u, \cdot)$ that $\overline{F}(u, \xi) \leq -|u|$ for $|u| \leq 1$ and $\xi \in [1, 2]$, and hence that $\overline{F}(u, \xi) = -|u|$ for $|u| \leq 1$, $\xi \in [1, 2]$. □

For the statement of the next theorem we introduce some additional notation. A continuous function $f : \mathbb{R}^{n\times m} \to \mathbb{R}$ satisfies growth condition G_p if

$$\limsup_{\xi\to\infty} \frac{|f(\xi)|}{|\xi|^p} < \infty, \quad \text{when } p \in [1, \infty),$$

and no condition is required when $p = \infty$. We denote by $\mathcal{P}_p$ the set of probability measures μ on $\mathbb{R}^{n\times m}$ that have a p^{th} moment, i.e.

$$\int |\xi|^p \, d\mu(\xi) < \infty \text{ when } p < \infty$$

and the support $\operatorname{spt}(\mu)$ is bounded when $p = \infty$. Let

$$\mathcal{M}_p^{pc} = \{\nu \in \mathcal{P}_p : \langle \nu, M \rangle = M(\overline{\nu}), \forall \text{ minors } M \text{ of order } \leq p\},$$
$$\mathcal{M}_p^{qc} = \{\nu \in \mathcal{P}_p : \langle \nu, f \rangle \geq f(\overline{\nu}), \forall \text{ quasiconvex } f \text{ satisfying } G_p\}$$
$$\mathcal{M}_p^{rc} = \{\nu \in \mathcal{P}_p : \langle \nu, f \rangle \geq f(\overline{\nu}), \forall \text{ rank-1 convex } f \text{ satisfying } G_p\}.$$

For a compact set $K \subset \mathbb{R}^{n\times m}$ and $1 < p < \infty$ define $f(\xi) := \operatorname{dist}(\xi, K)^p$, where $\operatorname{dist}(\xi, K) = \inf\{|\xi - \eta| : \eta \in K\}$.

Theorem 5.5. *The envelopes f^{pc}, f^{qc} and f^{rc} belong to $C^{1,1}_{\text{loc}}$ if $p > 2$ and to $C^{1,p-1}$ if $p \leq 2$. Furthermore, if $\nu \in \mathcal{M}_p^{qc}$ (resp. $\nu \in \mathcal{M}_p^{pc}$ or $\nu \in \mathcal{M}_p^{rc}$) is minimising, i.e.*

$$\int f\, d\nu = f^{qc}(\overline{\nu}), \quad (\textit{resp.} = f^{pc}(\overline{\nu}) \textit{ or } = f^{rc}(\overline{\nu}),)$$

then $f = f^{qc}$ on $\operatorname{spt}(\nu)$ (resp. $f = f^{pc}$ or $f = f^{rc}$) and f is differentiable at each point of $\operatorname{spt}(\nu)$.

Remarks. 1. The differentiability of f at ξ_0 is equivalent to the uniqueness of minimisers for

$$\min\{|\xi_0 - \eta| : \eta \in K\}.$$

Hence another way of expressing the last part of the proposition is that the metric projection onto K is unique on the support of a minimising measure ν. Recall that a closed set is convex if and only if the metric projection is everywhere single-valued.

2. In [45] it is shown that $f(\xi) = \operatorname{dist}(\xi, K)^2$ is rank-1 convex if and only if K is convex. The envelopes f^{rc}, f^{qc} and f^{pc} are therefore not in general powers of distance functions.

The result is an immediate consequence of Propositions 3.6, 3.7 (and their versions for f^{rc}, f^{pc}) and the following two lemmas.

Lemma 5.6. *Let $p \in (1, \infty)$. Then there exists a finite c_p, such that for all $\zeta, \eta \in \mathbb{R}^{n\times m}$ with $|\eta| \leq 1$,*

$$|\zeta + \eta|^p - |\zeta|^p - p\langle \zeta, \eta \rangle |\zeta|^{p-2} \leq \begin{cases} c_p |\eta|^p & \text{if } 1 < p \leq 2, \\ c_p(1 + |\zeta|^{p-2})|\eta|^2 & \text{if } p > 2. \end{cases}$$

Proof. Since $p > 1$ both sides of the claimed inequality are continuous, so we can assume $\zeta \neq 0$.

First consider the case $|\zeta| \leq 2|\eta|$. Then the left-hand side can be estimated from above by $(|\zeta| + |\eta|)^p - |\zeta|^p + p|\zeta|^{p-1}|\eta| \leq (3^p + p2^{p-1})|\eta|^p$

and since $|\eta| \le 1$ we get

$$|\zeta+\eta|^p - |\zeta|^p - p\langle\zeta,\eta\rangle|\zeta|^{p-2} \le \begin{cases} (3^p + p2^{p-1})|\eta|^p & \text{if } 1 < p \le 2, \\ (3^p + p2^{p-1})|\eta|^2 & \text{if } p > 2. \end{cases} \tag{5.3}$$

Next we assume $|\zeta| \ge 2|\eta|$. Then the function $h(t) := |\zeta + t\eta|^p$ is twice continuously differentiable on $[0,1]$ and the left-hand side of the claimed inequality equals

$$\int_0^1 (1-t)h''(t)\,dt,$$

where $h''(t) = p|\zeta + t\eta|^{p-4}((p-2)\langle\zeta + t\eta, \eta\rangle^2 + |\zeta + t\eta|^2|\eta|^2)$.

If $p > 2$ we get since $2|\eta| \le |\zeta|$, $h''(t) \le p(p-1)2^{p-2}|\zeta|^{p-2}|\eta|^2$. If $1 < p \le 2$ we get since $|\zeta| \ge 2|\eta|$, $h''(t) \le p|\zeta + t\eta|^{p-2}|\eta|^2 \le p(|\zeta| - |\eta|)^{p-2}|\eta|^2 \le p|\eta|^p$. Collecting the inequalities we obtain

$$|\zeta+\eta|^p - |\zeta|^p - p\langle\zeta,\eta\rangle|\zeta|^{p-2} \le \begin{cases} \frac{1}{2}p|\eta|^p & \text{if } 1 < p \le 2, \\ p(p-1)2^{p-3}|\zeta|^{p-2}|\eta|^2 & \text{if } p > 2. \end{cases} \tag{5.4}$$

The claimed inequality follows with $c_p = 3^p + p2^{p-1}$ by combination of (5.3) and (5.4). □

Lemma 5.7. *Let $K \subset \mathbb{R}^{n\times m}$ be a compact set and $p \in (1,\infty)$. If $f(\xi) :=$ $\operatorname{dist}(\xi, K)^p$, then f is upper semidifferentiable and there exists a constant $c = c(p, K)$, such that for each ξ there is $a \in \partial^* f(\xi)$ with the property*

$$f(\xi+\eta) - f(\xi) - \langle a,\eta\rangle \le \begin{cases} c|\eta|^p & \text{if } 1 < p \le 2, \\ c(1+|\xi|^{p-2})|\eta|^2 & \text{if } p > 2, \end{cases}$$

for all $|\eta| \le 1$.

Remark. The proof shows that the constant $c = c(p, K)$ can be taken independently of K if $p \in (1,2]$.

Proof. For a given matrix ξ let $\Phi(\xi) \in K$ denote a matrix, such that $|\xi - \Phi(\xi)| = \operatorname{dist}(\xi, K)$, and put

$$a = p|\xi - \Phi(\xi)|^{p-2}(\xi - \Phi(\xi)).$$

By Lemma 5.6 with $\zeta = \xi - \Phi(\xi)$ we have, noting that $f(\xi + \eta) \leq |\xi + \eta - \Phi(\xi)|^p$,

$$f(\xi+\eta) - f(\xi) - \langle a, \eta \rangle \leq \begin{cases} c_p|\eta|^p & \text{if } 1 < p \leq 2, \\ c_p(1 + |\xi - \Phi(\xi)|^{p-2})|\eta|^2 & \text{if } p > 2, \end{cases}$$

for $|\eta| \leq 1$. Because $\Phi(\xi) \in K$ and K is compact the desired inequality follows from this. □

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