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DISCONTINUOUS EQUILIBRIUM SOLUTIONS AND CAVITATION IN NONLINEAR ELASTICITY

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CONTENTS

A study is made of a class of singular solutions to the equations of nonlinear elastostatics in which a spherical cavity forms at the centre of a ball of isotropic material placed in tension by means of given surface tractions or displacements. The existence of such solutions depends on the growth properties of the stored-energy function W for large strains and is consistent with strong ellipticity of W.

Under appropriate hypotheses it is shown that a singular solution bifurcates from a trivial (homogeneous) solution at a critical value of the surface traction or displacement, at which the trivial solution becomes unstable. For incompressible materials both the singular solution and the critical surface traction are given explicitly, and the stability of all solutions with respect to radial motion is determined. For compressible materials the existence of singular solutions is proved for a class of strongly elliptic materials by means of the direct method of the calculus of variations, an important step in the analysis being to show that the only radial equilibrium solutions without cavities are homogeneous.

Work of Gent & Lindley (1958) shows that the critical surface tractions obtained agree with those observed in the internal rupture of rubber.

1. Introduction

In this paper we investigate a class of singular solutions to the equations of nonlinear elasticity in which a hole forms in the interior of an elastic body in a state of tension. In view of the terminology commonly used in the special case of an elastic fluid it is natural to refer to this phenomenon of hole formation as *cavitation.*

Consider a homogeneous elastic body having stored-energy function W with respect to a reference configuration in which the body occupies the open subset Ω of $\mathbb{R}^n$. (Here n is the space dimension, and we shall usually suppose that $n > 1$. By considering the case of general n we are able both to treat the cases $n = 2$ and $n = 3$ simultaneously, and to see explicitly how n enters various formulae.) In a typical deformation in which the particle at $X \in \Omega$ is displaced to $x(X) \in \mathbb{R}^n$ the total stored energy is given by

$$E(x) = \int_\Omega W(\nabla x(X))\,\mathrm{d}X. \tag{1.1}$$

The equilibrium equations of nonlinear elasticity with zero body force are the Euler–Lagrange equations for (1.1), namely

$$\frac{\partial}{\partial X^\alpha}\left[\frac{\partial W(\nabla x(X))}{\partial x^i{}_{,\alpha}}\right] = 0, \quad (i = 1, \ldots, n). \tag{1.2}$$

We say that $x(\cdot)$ is a *weak equilibrium solution* if (1.2) holds in the sense of distributions (for the precise definition see §4); this is a way of stating the principle of virtual work, and under quite general conditions (cf. Antman & Osborn 1979) it is equivalent to requiring that the resultant force on almost every sub-body vanishes.

In general a weak equilibrium solution may possess singularities, even if W is smooth. A central problem in nonlinear elasticity is to understand how various hypotheses on W affect the existence and nature of possible singularities of weak equilibrium solutions. For example, one possible singularity is that in which the deformation gradient ∇x jumps across a smooth $(n-1)$-dimensional surface, x being smooth elsewhere. It was shown in Ball (1980) that every weak equilibrium solution of this type is C^1 (i.e. the jump in ∇x across the surface is zero) if and only if W is *strictly rank* 1 *convex*. Strict rank 1 convexity is defined in §3; for present purposes it suffices to say that it is almost equivalent to the *strong ellipticity condition*

$$\frac{\partial^2 W(F)}{\partial F^i_\alpha \partial F^j_\beta}\, a^i b_\alpha a^j b_\beta > 0 \tag{1.3}$$

for all $F \in M^{n\times n}_+$ and all non-zero $a, b \in \mathbb{R}^n$, where $M^{n\times n}_+$ denotes the set of real $n \times n$ matrices with positive determinant. For elastic crystals symmetry considerations show that W cannot be strongly elliptic, and indeed in the phenomenon of twinning we observe singular solutions of the type described (for discussions of this see Ericksen (1977), Nadai (1950)).

The type of singularity that concerns us in connection with cavitation is that in which x is discontinuous at a point. Whether or not such a singularity can occur depends on the rate of growth of $W(F)$ for large $|F|$. For example, let us agree to call an elastic material *strong* if

$$W(F) \geqslant C(|F|^p+1) \quad \text{for all} \quad F \in M_+^{n\times n}, \tag{1.4}$$

for some $p > n$. Then any x with $E(x) < \infty$ belongs to the Sobolev space $W^{1,p}(\Omega; \mathbb{R}^n)$† and by the Sobolev imbedding theorem is continuous, so that no type of fracture can occur. (Implications of (1.4) for invertibility of $x(\cdot)$ are discussed in Ball (1981*a*).) A condition slightly weaker than (1.4) is that

$$W(F)/|F|^n \to \infty \quad \text{as} \quad |F| \to \infty. \tag{1.5}$$

It was shown in Ball (1977*b*, 1978) that (1.5) admits of the interpretation that it fails if and only if an infinitesimal n-cube of material may be homogeneously deformed with finite stored energy so as to have unit diameter. To find cavitation solutions of finite energy, then, we should look in the class of stored-energy functions for which (1.5) fails.

The solutions with cavities that we construct correspond to radial deformations of a ball of homogeneous isotropic material under prescribed radial displacements or surface forces. For $\Omega = \{X \in \mathbb{R}^n : |X| < 1\}$, $n > 1$, these deformations have the form

$$x(X) = \frac{r(R)}{R}\,X, \tag{1.6}$$

where $R = |X|$. Since the material is isotropic $W(F)$ can be expressed as a symmetric function $\Phi(v_1, \ldots, v_n)$ of the eigenvalues v_i of $(F^TF)^{\frac{1}{2}}$. For an incompressible material the only kinematically admissible deformations of the form (1.6) are given by

$$r^n = R^n + A^n, \tag{1.7}$$

where $A \geqslant 0$ is the cavity radius, and we may ask if $x(\cdot)$ given by (1.6) is then a weak equilibrium solution (for an incompressible material (1.2) must be modified to take account of the arbitrary hydrostatic pressure). For a compressible material r has to satisfy (cf. theorem 4.2) the radial equilibrium equation

$$\mathrm{d}(R^{n-1}\Phi_1)/\mathrm{d}R = (n-1)\,R^{n-2}\Phi_2, \tag{1.8}$$

where $\Phi_i = \partial\Phi(r'(R), r(R)/R, \ldots, r(R)/R)/\partial v_i$, and the problem is essentially to prove the existence of solutions to this equation such that $r(0) > 0$.

We now describe the principal results and implications of our analysis, starting with the incompressible case. We show (theorem 4.3) that if $A > 0$ then (1.7) generates a weak equilibrium solution if and only if

$$\frac{v^{n-1}}{(v^n-1)^2}\frac{\mathrm{d}\hat{\Phi}(v)}{\mathrm{d}v} \in L^1(\delta, \infty) \quad \text{for} \quad \delta > 1, \tag{1.9}$$

where $\hat{\Phi}(v) \overset{\text{def}}{=} \Phi(v^{1-n}, v, \ldots, v)$. We then show (§5.1) that (1.9) can hold for strongly elliptic stored-energy functions, and that then the deformation may or may not have finite energy.

† Provided $\nabla x(X)$ in (1.1) is interpreted in the sense of distributions, other interpretations might be relevant for a complete description of fracture.

In particular, *strong ellipticity does not imply the regularity of weak equilibrium solutions.* A complete analysis is given (§5.2) of the radial dead-load traction problem in which the radial component P of the Piola–Kirchhoff stress at $R = 1$ is specified. It is found that when (1.9) holds (1.7) generates a weak solution of the dead-load traction problem for *any* $A \geqslant 0$, and that to make the problem determinate it is necessary to specify in addition the value of the Cauchy stress $T(0) = \lim_{R\to 0+} T(R)$ on the cavity surface. With the choice $T(0) = 0$ (the natural boundary condition for the variational problem) the solutions with cavities ($A > 0$) bifurcate from the trivial solution $A = 0$ at a critical value P_{cr} of P given by

$$P_{\text{cr}} = \int_1^\infty \frac{1}{v^n - 1} \frac{\mathrm{d}\hat{\Phi}(v)}{\mathrm{d}v}\,\mathrm{d}v. \tag{1.10}$$

If the Baker–Ericksen inequalities hold† then the bifurcation at $P = P_{\text{cr}}$ is locally to the right. Similar conclusions are reached for the Cauchy traction problem in which the radial component P of the Cauchy stress at $R = 1$ is specified. The value of P_{cr} is the same, but if the Baker–Ericksen inequalities hold then the bifurcation is (globally) to the left. For both traction problems we determine (§5.4) the stability of the various equilibrium solutions with respect to radial motions satisfying the equations of nonlinear elastodynamics. This part of our analysis generalizes the classical theory of Lord Rayleigh (1917) for collapse of a bubble in an elastic fluid, and is our closest approach to the theory of cavitation in fluids. We study the initiation of cavitation, cavity collapse and rebounds; in particular we show that when $P > P_{\text{cr}}$ the initial value problem for $r(R, t)$ in which $r(R, 0) = R$, $\dot{r}(R, 0) = 0$ has two distinct solutions with the same energy, namely the trivial solution $r(R, t) = R$ and a solution corresponding to formation of a cavity from zero initial radius. The results are illustrated by specific examples, including that of a neo-Hookean material for which P_{cr} is evaluated and for which the bifurcation diagrams and dynamic phase portraits are given.

The analysis of cavitation for compressible materials is more difficult, since it involves studying the ordinary differential equation (1.8), which is singular as $R \to 0+$. We begin in §6 by giving various theorems guaranteeing that the only solutions of (1.8) with $r(0) = 0$ are trivial; i.e. they have the form $r(R) = \lambda R$ for some $\lambda > 0$. The hypotheses of these theorems are satisfied by many strongly elliptic stored-energy functions of practical interest, but we show by means of an example (having infinite energy) that strong ellipticity is not by itself sufficient for all solutions without holes to be trivial. The methods of §6 use a variety of devices from the theory of ordinary differential equations and the calculus of variations.

In §7 we study cavitation for a class of stored-energy functions of the form

$$\Phi(v_1, \ldots, v_n) = \sum_{i=1}^{n} \phi(v_i) + h(v_1 v_2 \ldots v_n), \tag{1.11}$$

where ϕ and h are convex functions satisfying suitable growth conditions. We consider first the displacement boundary value problem in which $r(1) = \lambda > 0$ is specified. We show using the direct method of the calculus of variations that a minimizer of E in the class of radial deformations exists for this problem (theorem 7.1), that is satisfies (1.8) for $R > 0$, and that if $r(0) > 0$ then the natural boundary condition $T(0) = 0$ holds (theorem 7.3). We then show (theorem 7.7) that for λ sufficiently large any minimizer satisfies $r(0) > 0$, but (theorem 7.8) that if λ corresponds to a compression then $r(R) = \lambda R$ is the unique minimizer. Similar results

† These inequalities are proved in proposition 3.1 to be a consequence of strict rank 1 convexity.

(theorems 7.2–7.8) hold for the dead-load traction problem. In §7.5 the bifurcation of equilibrium solutions with cavities from the trivial solution is analysed under additional hypotheses on ϕ. For the displacement boundary value problem an equation for the bifurcation point λ_{cr} is derived, and it is shown (theorem 7.9) that for $\lambda > \lambda_{cr}$ there is a unique radial minimizer r_λ of E with $r_\lambda(0) > 0$ and that r_λ is the only non-trivial radial equilibrium solution. An explicit formula is given relating r_λ and r_μ for $\lambda > \mu > \lambda_{cr}$. The trivial solution $r = \lambda R$ is stable (for the definition see §7.5) if $\lambda \leqslant \lambda_{cr}$ and unstable if $\lambda > \lambda_{cr}$, while if $\lambda > \lambda_{cr}$ then r_λ is stable. For the dead-load and Cauchy traction problems similar but less complete information is obtained; if $r = R$ is a natural state of the material and the Baker–Ericksen inequalities hold then the critical dead load P_{cr} exceeds the critical Cauchy traction P^*. The results are applied in §7.6 to the special case when $\phi(v) = \mu v^2$ (a compressible version of the neo-Hookean material) and when $n \geqslant 3$ the values of λ_{cr}, P_{cr} and P^* are given in a more explicit form. The results of §5 for the neo-Hookean material are obtained in the limit of zero compressibility. For $n = 2$ an equilibrium solution with $r(0) > 0$ having infinite energy is constructed; by proposition 7.11 such a solution necessarily satisfies $T(0) = -\infty$.

For an incompressible material the formula (1.10) for P_{cr} was first obtained in a somewhat different form and in a different context by Bishop *et al.* (1945). They conducted experiments on the indentation of copper by a cylindrical punch with a conical head, and observed that the load on the punch rose to a maximum as penetration increased. They gave arguments suggesting that the maximum punch load for a ductile material could be estimated in terms of the pressures that, when applied to the surface of a small cylindrical or spherical hole, will expand the hole indefinitely. They calculated these pressures under the assumption that the material was elasto-plastic, and incompressible in the plastic region. When the yield strength of the material is set equal to zero these pressures turn out to be the same as P_{cr} for $n = 2, 3$ respectively.

The formula (1.10) was also obtained for two special materials in a beautiful paper by Gent & Lindley (1958). Following earlier experimental work of Yerzley (1939), Gent & Lindley conducted experiments on internal rupture in rubber in which thin vulcanized rubber cylinders were bonded to plane metal end-pieces and placed in tension. Internal flaws appeared at a well defined, and comparatively small, tensile load. Gent & Lindley observed that for thin cylinders the stress in the centre of the test-piece is given approximately by a negative hydrostatic pressure equal to the applied tensile stress. They proposed a simple prescription for calculating the critical tensile stress P_c. Suppose that the test-piece initially contains an extremely small spherical cavity, 'sufficiently small for the region around it to be treated as an infinitely thick spherical shell'. Assuming the cavity surface to be stress-free, calculate the dead-load traction P_c at infinity needed to expand the cavity indefinitely. It turns out that $P_c = P_{cr}$; the reason for this is explained in §8. Further experiments along these lines are described and analysed by Lindsey (1967).

The equilibrium solutions with cavities described in this paper can be viewed as providing examples of irregular weak solutions to strongly elliptic systems. After work of De Giorgi (1968) on linear elliptic systems with bounded measurable coefficients, Giusti & Miranda (1968) and Maz'ya (1968) were the first to exhibit discontinuous solutions to strongly elliptic systems. Since then many other examples of irregular solutions have been given; good summaries of the literature are given by Giaquinta (1981) and Hildebrandt (1980). These examples do not apply directly to nonlinear elasticity, however, since they all assume that the integrand in (1.1) is convex in ∇x, a condition well known to be physically unacceptable for compressible materials (a summary of the reasons and references are given in Ball (1977*a*)); furthermore, they usually assume that the integrand depends explicitly on x (an exception is the example of Necas (1977), but in this case the dimensions are inappropriate for elasticity). Nevertheless, these examples partly motivated this paper on account of their resemblance to hole formation. An attempt to relate De Giorgi's counterexample to linear elasticity has been made by Podio Guidugli (1977).

The analysis of this paper depends crucially on the behaviour of $W(F)$ for arbitrarily large $|F|$; it would seem that any continuum theory of mechanics that attempts to model cavitation must make hypotheses on the material behaviour for arbitrarily large values of the deformation gradient. On the other hand, measurements designed to determine W are customarily made in statics prior to fracture, often in deformations supposed in advance to be homogeneous, when values of $|F|$ outside a finite range are not observed. Thus it is difficult to obtain evidence for what we should assume about $W(F)$ for large $|F|$. One should be cautious not to conclude too hastily that a material fails to be elastic outside the range of F observed in homogeneous static deformations, since the fact that a particular F is not observed may be due to an instability (such as cavitation!). In view of these uncertainties it is reassuring that a formula such as (1.10) is insensitive to changes in $W(F)$ for large F, provided it is assumed, say, that $|\mathrm{d}\hat{\Phi}(v)/\mathrm{d}v| \leqslant C(|v|^{\alpha}+1)$, where $C > 0$ and $\alpha < n-1$. In this connection Gent & Lindley noted that two different models of rubber used by them gave rise to similar values of P_{cr}.

Preliminary announcements of some of the results of this paper were made in Ball (1980, 1981*a–c*).

2. Notation

In this paper we work in n-dimensional real Euclidean space $\mathbb{R}^n$, $n \geqslant 1$. The interesting cases for applications are $n = 1$, 2 and 3. Let $M^{n\times n}$ denote the set of real $n\times n$ matrices, and write $M_+^{n\times n} = \{F \in M^{n\times n} : \det F > 0\}$. We denote by $\mathrm{SO}(n)$ the special orthogonal group on $\mathbb{R}^n$.

We make extensive use of standard results concerning Sobolev spaces (cf. Adams 1975). If $E \subset \mathbb{R}^n$ is open, $m \geqslant 1$, $1 \leqslant p \leqslant \infty$, we denote by $L^p(E; \mathbb{R}^m)$ the Banach space of (equivalence classes of) mappings $u: E \to \mathbb{R}^m$ such that $\|u\|_p < \infty$, where

$$\|u\|_p = \begin{cases} \left[\int_E |u(X)|^p \,\mathrm{d}X\right]^{1/p}, & 1 \leqslant p < \infty, \\ \operatorname*{ess\,sup}_{X\in E} |u(X)|, & p = \infty. \end{cases}$$

The corresponding Sobolev space $W^{1,p}(E; \mathbb{R}^m)$ is the Banach space consisting of those $u \in L^p(E; \mathbb{R}^m)$ such that $\|u\|_{1,p} < \infty$, where

$$\|u\|_{1,p} \stackrel{\text{def}}{=} \|u\|_p + \|\nabla u\|_p,$$

and where the gradient $\nabla u: E \to \mathbb{R}^{nm}$ is understood in the sense of distributions. We write $L^p(E) = L^p(E; \mathbb{R})$, $W^{1,p}(E) = W^{1,p}(E; \mathbb{R})$. If $r \in \{0, 1, \ldots\} \cup \{\infty\}$ then $C^r(E; \mathbb{R}^m)$ denotes the space of r-times continuously differentiable functions on E with values in $\mathbb{R}^m$. We define $C^r(\bar{E}; \mathbb{R}^m)$ to be the space of functions $u \in C^r(E; \mathbb{R}^m)$ which are restrictions of functions in $C^r(\mathbb{R}^n; \mathbb{R}^m)$, and $C_0^\infty(E; \mathbb{R}^m)$ to be the space of functions $u \in C^\infty(E; \mathbb{R}^m)$ with compact support contained in E. We write $C^r(E; \mathbb{R}) = C^r(E)$ etc.

We denote by $\mathscr{H}^r$ r-dimensional (Hausdorff) measure. We write S^{n-1} for the unit sphere $\{X \in \mathbb{R}^n : |X| = 1\}$. We use the notation $\bar{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$. We denote by C, C_i generic constants whose values may vary from line to line.

3. Stored-energy functions and isotropy

We are concerned with a homogeneous elastic body having *stored-energy function* $W: M_+^{n\times n} \to \mathbb{R}$. The function W is defined with respect to a fixed reference configuration in which the body occupies the bounded open subset $\Omega \subset \mathbb{R}^n$. The significance of W is that the total elastic energy stored in a deformation $x:\Omega \to \mathbb{R}^n$ is given by

$$E(x) = \int_\Omega W(\nabla x(X))\,\mathrm{d}X. \tag{3.1}$$

Frame-indifference requires that

$$W(QF) = W(F) \quad \text{for all} \quad F \in M_+^{n\times n},\ Q \in \mathrm{SO}(n). \tag{3.2}$$

We say that W is *isotropic* if in addition

$$W(F) = W(FQ) \quad \text{for all} \quad F \in M_+^{n\times n},\ Q \in \mathrm{SO}(n). \tag{3.3}$$

It is well known (see Truesdell & Noll 1965, pp. 28, 317) that W is isotropic if and only if there exists a symmetric function

$$\Phi: \mathbb{R}^n_{++} \to \mathbb{R}, \quad \mathbb{R}^n_{++} = \{c = (c_1, \ldots, c_n) \in \mathbb{R}^n : c_i > 0 \quad \text{for} \quad 1 \leqslant i \leqslant n\},$$

such that

$$W(F) = \Phi(v_1, \ldots, v_n) \quad \text{for all} \quad F \in M_+^{n\times n}, \tag{3.4}$$

where $v_1, \ldots, v_n$ denote the singular values (or principal stretches) of F (i.e. the eigenvalues of $(F^{\mathrm{T}}F)^{\frac{1}{2}}$). It is known (Ball 1982) that $W \in C^r(M_+^{n\times n})$ if and only if $\Phi \in C^r(\mathbb{R}^n_{++})$, $r = 0, 1, 2$ or ∞. We write $\Phi_{,i} \equiv \partial\Phi/\partial v_i$, etc.

The stored-energy function W is said to be *rank* 1 *convex* if and only if the inequality

$$W(tF + (1-t)G) \leqslant tW(F) + (1-t)W(G) \tag{3.5}$$

holds for all $t \in (0, 1)$ and all $F, G \in M_+^{n\times n}$ such that $F - G = a \otimes b \neq 0$ for some $a, b \in \mathbb{R}^n$, and *strictly rank* 1 *convex* if strict inequality holds in (3.5). If $W \in C^2(M_+^{n\times n})$ then W is rank 1 convex if and only if the *Legendre–Hadamard condition* holds, i.e.

$$\frac{\partial^2 W(F)}{\partial F^i_\alpha\, \partial F^j_\beta}\, a^i b_\alpha a^j b_\beta \geqslant 0 \tag{3.6}$$

for all $F \in M_+^{n\times n}$ and all $a, b \in \mathbb{R}^n$. If $W \in C^2(M_+^{n\times n})$ is *strongly elliptic*, i.e.

$$\frac{\partial^2 W(F)}{\partial F^i_\alpha\, \partial F^j_\beta}\, a^i b_\alpha a^j b_\beta > 0 \tag{3.7}$$

for all $F \in M_+^{n\times n}$ and all non-zero $a, b \in \mathbb{R}^n$, then W is strictly rank 1 convex, but the converse is false.

Proposition 3.1. *Let $W \in C^1(M_+^{n\times n})$ be strictly rank* 1 *convex and isotropic. Then* (*a*) (*the tension–extension property*) *$\Phi_{,i} = \Phi_{,i}(v_1, \ldots, v_n)$ is a strictly increasing function of v_i for fixed $v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n$*; (*b*) (*the Baker–Ericksen inequalities*) *if $i \neq j$ and $v_i \neq v_j$ then*

$$(v_i \Phi_{,i} - v_j \Phi_{,j})/(v_i - v_j) > 0. \tag{3.8}$$

Proof. If W is strictly rank 1 convex then $\mathrm{d}W(F+ta\otimes b)/\mathrm{d}t$ is a strictly increasing function of t whenever $a\otimes b\neq 0$. Set $F=\operatorname{diag}(v_1,\ldots,v_n)$, $a=b=e_i$, where $e_i=(\delta_{ij})_{j=1}^n$ is the ith basis vector. Then $W(F+ta\otimes b)=\Phi(v_1,\ldots,v_{i-1},v_i+t,v_{i+1},\ldots,v_n)$ so that (*a*) holds.

To prove (*b*) we assume without loss of generality that $i=1, j=2$. It is easily seen that for fixed $v_3,\ldots,v_n$ the function $\overline{W}(G)=\Phi(\lambda_1,\lambda_2,v_3,\ldots,v_n)$, where λ_1,λ_2 denote the singular values of G, is a strictly rank 1 convex function belonging to $C^1(M_+^{2\times 2})$. Hence we may and shall also assume that $n=2$. For given $u_1>0$, $u_2>0$, $u_1\neq u_2$, let

$$F(t)=\begin{bmatrix}u_1 & 0\\ t & u_2\end{bmatrix},\quad t>0.$$

Then
$$F^{\mathrm{T}}(t)F(t)=\begin{bmatrix}u_1^2+t^2 & tu_2\\ tu_2 & u_2^2\end{bmatrix},$$

and so the singular values $v_1(t)$, $v_2(t)$ of $F(t)$ satisfy

$$v_1(t)\,v_2(t)=u_1u_2,\quad v_1^2(t)+v_2^2(t)=u_1^2+u_2^2+t^2. \tag{3.9}$$

Hence
$$\begin{aligned}\mathrm{d}W(F(t))/\mathrm{d}t&=\Phi_{,1}\dot v_1(t)+\Phi_{,2}\dot v_2(t)\\ &=t[v_1(t)\,\Phi_{,1}-v_2(t)\,\Phi_{,2}]/[v_1^2(t)-v_2^2(t)].\end{aligned}$$

Since by hypothesis $\mathrm{d}W(F(t))/\mathrm{d}t$ is strictly increasing it follows that

$$[v_1(t)\,\Phi_{,1}-v_2(t)\,\Phi_{,2}]/[v_1^2(t)-v_2^2(t)]>0\quad\text{for}\quad t>0.$$

But given any $v_1>0$, $v_2>0$, $v_1\neq v_2$, we can find u_1, u_2 and $t>0$ such that (3.9) holds. Hence
$$(v_1\Phi_{,1}-v_2\Phi_{,2})/(v_1-v_2)>0$$
as required.

Note that in proposition 3.1 we do not assume that $W\in C^2(M_+^{n\times n})$, nor that W is strongly elliptic. Note also that the same proof establishes that if W is rank 1 convex then each $\Phi_{,i}$ is non-decreasing and the *weakened Baker–Ericksen inequalities*

$$(v_i\Phi_{,i}-v_j\Phi_{,j})/(v_i-v_j)\geqslant 0,\quad i\neq j,\quad v_i\neq v_j, \tag{3.10}$$

hold. For a proof that strong ellipticity implies (3.8) see Hayes (1969). For $n=2$ necessary and sufficient conditions for W to be strongly elliptic, in terms of Φ, have been given by Knowles & Sternberg (1978) (see also Aubert & Tahraoui 1980, Hill 1979).

The *Piola–Kirchhoff stress tensor* $T_{\mathrm{R}}(F)$ is defined for $W\in C^1(M_+^{n\times n})$ by

$$T_{\mathrm{R}}(F)=\partial W(F)/\partial F.$$

If $W\in C^1(M_+^{n\times n})$ is isotropic and if $F=\operatorname{diag}(v_1,\ldots,v_n)$, $v_i>0$, then

$$T_{\mathrm{R}}(F)=\operatorname{diag}(\Phi_{,1},\ldots,\Phi_{,n}),$$

where $\Phi_{,i}=\Phi_{,i}(v_1,\ldots,v_n)$.

The *Cauchy stress tensor* $T(F)$ is related to $T_{\mathrm{R}}(F)$ through the formula

$$T(F)=(\det F)^{-1}\,T_{\mathrm{R}}(F)F^{\mathrm{T}}.$$

We now consider *incompressible elasticity*, in which the deformation x is subjected to the pointwise constraint
$$\det\nabla x(X)=1.$$

In accordance with this constraint we consider a homogeneous incompressible material with stored-energy function $W: M_1^{n\times n}\to\mathbb{R}$, where $M_1^{n\times n}\stackrel{\text{def}}{=}\{F\in M^{n\times n}:\det F=1\}$. Any such W may be extended to the whole of $M_+^{n\times n}$; for example, we can set

$$\hat W(F)\stackrel{\text{def}}{=}W((\det F)^{-1/n}F),\quad F\in M_+^{n\times n}, \tag{3.11}$$

and then $W \in C^r(M_1^{n\times n})$ if and only if $\hat{W} \in C^r(M_+^{n\times n})$. We therefore assume from now on that $W: M_+^{n\times n} \to \mathbb{R}$. We say that W is *frame-indifferent* if

$$W(QF) = W(F) \quad \text{for all} \quad F \in M_1^{n\times n}, \quad Q \in \mathrm{SO}(n), \tag{3.12}$$

and *isotropic* if in addition

$$W(F) = W(FQ) \quad \text{for all} \quad F \in M_1^{n\times n}, \quad Q \in \mathrm{SO}(n). \tag{3.13}$$

It is well known that W is isotropic if and only if there exists a symmetric function

$$\Phi: K^n \to \mathbb{R}, \quad K^n \stackrel{\text{def}}{=} \{c = (c_1, \ldots, c_n) \in \mathbb{R}^n_{++}; \quad c_1 c_2 \ldots c_n = 1\}$$

such that
$$W(F) = \Phi(v_1, \ldots, v_n) \quad \text{for all} \quad F \in M_1^{n\times n}. \tag{3.14}$$

We may and shall also assume that Φ is defined on the whole of $\mathbb{R}^n_{++}$ in such a way that (3.14) holds for all $F \in M_+^{n\times n}$ (for example, $\hat{W}$ defined by (3.11) is isotropic in the sense of (3.3).)

We say that W is *rank* 1 *convex* if (3.5) holds for all $t \in (0, 1)$ and all $F, G \in M_1^{n\times n}$ such that $F - G = a \otimes b \neq 0$ for some $a, b \in \mathbb{R}^n$. The fact that the function $H \mapsto \det H$ is rank 1 affine (i.e. $\pm \det H$ are rank 1 convex) implies that any such a, b satisfy $\langle(\operatorname{adj} G)^{\mathrm{T}}, a \otimes b\rangle = 0$, where $\langle A, B\rangle \stackrel{\text{def}}{=} \operatorname{tr}(A^{\mathrm{T}}B)$ and $\operatorname{adj} G$ is the transposed matrix of cofactors of G, and that $tF + (1-t)G \in M_1^{n\times n}$ for $t \in (0, 1)$. Strict rank 1 convexity is defined analogously. If $W \in C^2(M_+^{n\times n})$ then W is rank 1 convex if and only if the Legendre–Hadamard condition holds in the form

$$\frac{\partial^2 W(F)}{\partial F^i_\alpha \, \partial F^j_\beta} a^i b_\alpha a^j b_\beta \geqslant 0 \tag{3.15}$$

for all $F \in M_1^{n\times n}$ and all $a, b \in \mathbb{R}^n$ such that $\langle(\operatorname{adj} F)^{\mathrm{T}}, a \otimes b\rangle = 0$. If $W \in C^2(M_+^{n\times n})$ is strongly elliptic, i.e. (3.7) holds for all $F \in M_1^{n\times n}$ and all non-zero $a, b \in \mathbb{R}^n$ such that $\langle(\operatorname{adj} F)^{\mathrm{T}}, a \otimes b\rangle = 0$, then W is strictly rank 1 convex, but the converse is false. The tension–extension property is not meaningful for incompressible materials, but the same proof as in proposition 3.1 shows that for an incompressible isotropic material the Baker–Ericksen inequalities (resp. (3.10)) hold if $W \in C^1(M_+^{n\times n})$ is strictly rank 1 convex (resp. rank 1 convex). Note that the Baker–Ericksen inequalities take the same form for any extension of Φ to $\mathbb{R}^n_{++}$, since $v_i \Phi_i - v_j \Phi_j$ is a surface derivative on K^n.

In incompressible elasticity the *Piola–Kirchhoff extra stress tensor* $T^*_{\mathrm{R}}(F)$ is defined for $W \in C^1(M_+^{n\times n})$ by

$$T^*_{\mathrm{R}}(F) = \partial W(F)/\partial F, \quad F \in M_1^{n\times n}.$$

The corresponding *Cauchy extra stress tensor* is given by

$$T^*(F) = T^*_{\mathrm{R}}(F) F^{\mathrm{T}}, \quad F \in M_1^{n\times n}.$$

The Cauchy stress tensor T is then given by

$$T = -p\mathbf{1} + T^*(F), \tag{3.16}$$

where p is an arbitrary hydrostatic pressure, and the Piola–Kirchhoff stress tensor T_{R} by

$$T_{\mathrm{R}} = -pF^{-\mathrm{T}} + T^*_{\mathrm{R}}(F). \tag{3.17}$$

The pressure p in (3.16), (3.17) depends on the choice of the extension of W to $M_+^{n\times n}$, but the form of (3.16), (3.17) is independent of the extension; to see the latter note that since

$$\partial(\det F)/\partial F = (\operatorname{adj} F)^{\mathrm{T}},$$

the expression
$$\langle \partial W(F)/\partial F, G\rangle, \quad F \in M_1^{n\times n},$$
depends only on surface derivatives of W on $M_1^{n\times n}$ whenever
$$\langle (\operatorname{adj} F)^{\mathrm{T}}, G\rangle = 0.$$
Now write
$$p = q + \frac{1}{n}\left\langle \frac{\partial W}{\partial F}, F\right\rangle$$
so that
$$T_{\mathrm{R}} = -q\mathbf{1} + \hat{T}_{\mathrm{R}}(F),$$
where
$$\hat{T}_{\mathrm{R}}(F) \stackrel{\text{def}}{=} \frac{\partial W}{\partial F} - \frac{1}{n}\left\langle \frac{\partial W}{\partial F}, F\right\rangle F^{-\mathrm{T}}.$$
But for any $H \in M^{n\times n}$,
$$\langle \hat{T}_{\mathrm{R}}(F), H\rangle = \left\langle \frac{\partial W}{\partial F}, H - \frac{1}{n}\langle F^{-\mathrm{T}}, H\rangle F\right\rangle.$$
Since
$$\left\langle (\operatorname{adj} F)^{\mathrm{T}}, H - \frac{1}{n}\langle F^{-\mathrm{T}}, H\rangle F\right\rangle = 0, \quad F \in M_1^{n\times n},$$
$\hat{T}_{\mathrm{R}}(F)$ depends only on surface derivatives.

4. Radial deformations and weak solutions

Let $B_\rho \stackrel{\text{def}}{=} \{X \in \mathbb{R}^n : |X| < \rho\}$, $B \stackrel{\text{def}}{=} B_1$. We consider radial deformations $x: B \to \mathbb{R}^n$ of the form
$$x(X) = \frac{r(R)}{R} X, \quad R = |X|. \tag{4.1}$$
Clearly $|r(R)| = |x(X)|$. We sometimes abbreviate (4.1) by writing $R \mapsto r(R)$.

Lemma 4.1. *Let $n > 1$. Let $1 \leqslant p < \infty$ and let x be given by (4.1). Then $x \in W^{1,p}(B; \mathbb{R}^n)$ if and only if $r(\cdot)$ is absolutely continuous on $(0, 1)$ and*
$$\int_0^1 R^{n-1}\left[|r'(R)|^p + \left|\frac{r(R)}{R}\right|^p\right] \mathrm{d}R < \infty. \tag{4.2}$$
In this case the weak derivatives of $x(\cdot)$ are given by
$$\nabla x(X) = \frac{r(R)}{R}\mathbf{1} + \frac{X \otimes X}{R^3}[Rr'(R) - r(R)], \quad \text{a.e. } X \in B. \tag{4.3}$$

Proof. Let $x \in W^{1,p}(B; \mathbb{R}^n)$. Since $|r(R)| = |x(X)|$,
$$\int_0^1 R^{n-1}|r(R)|^p \mathrm{d}R = C\int_B |x(X)|^p \mathrm{d}X < \infty,$$
and so $r \in L^p_{\mathrm{loc}}(0, 1)$. Extend x by zero outside B. Let $\rho(X)$ be a symmetric mollifier, i.e.
$$\rho(X) = \hat{\rho}(R),\ \rho \in C^\infty(\mathbb{R}^n),\ \int_{\mathbb{R}^n} \rho(X)\,\mathrm{d}X = 1,\ \rho \geqslant 0,\ \operatorname{supp}\rho \subset B,$$
and define $\rho_\epsilon(X) = \epsilon^{-n}\rho(X/\epsilon)$. Then $x_\epsilon \stackrel{\text{def}}{=} \rho_\epsilon * x$ is smooth and
$$x_\epsilon(X) = \int_{\mathbb{R}^n} \rho_\epsilon(X - Z)\frac{r(|Z|)}{|Z|} Z\,\mathrm{d}Z$$

is parallel to X, so that

$$x_\epsilon(X) = \frac{r_\epsilon(R)}{R} X, \quad X \neq 0,$$

for some function $r_\epsilon \in C^\infty(0, 1)$. Then

$$\nabla x_\epsilon(X) = \frac{r_\epsilon(R)}{R} \mathbf{1} + \frac{X \otimes X}{R^3} [Rr'_\epsilon(R) - r(R)], \tag{4.4}$$

and hence

$$|\nabla x_\epsilon(X) - \nabla x_\delta(X)|^p = \left\{[r'_\epsilon(R) - r'_\delta(R)]^2 + (n-1)\left[\frac{r_\epsilon(R) - r_\delta(R)}{R}\right]^2\right\}^{p/2}. \tag{4.5}$$

Let $0 < R_1 < R_2 < 1$. Since $x_\epsilon \to x$ in $W^{1,p}(B_{R_2} \backslash \bar{B}_{R_1}; \mathbb{R}^n)$,

$$\lim_{\epsilon, \delta \to 0} \int_{R_1}^{R_2} R^{n-1} \left\{[r'_\epsilon(R) - r'_\delta(R)]^2 + (n-1)\left[\frac{r_\epsilon(R) - r_\delta(R)}{R}\right]^2\right\}^{p/2} \mathrm{d}R = 0$$

and hence r_ϵ is a Cauchy sequence in $W^{1,p}(R_1, R_2)$. Hence $r_\epsilon \to r$ in $W^{1,p}(R_1, R_2)$ and r is absolutely continuous. Also

$$\int_{R_1}^{R_2} R^{n-1} \left\{[r'(R)]^2 + (n-1)\left[\frac{r(R)}{R}\right]^2\right\}^{p/2} \mathrm{d}R = C \int_{B_{R_2} \backslash B_{R_1}} |\nabla x(X)|^p \,\mathrm{d}X,$$

from which (4.2) follows. Passing to the limit in (4.4) we obtain (4.3).

Conversely, let $r(\cdot)$ be absolutely continuous on $(0, 1)$ and satisfy (4.2). Then clearly $x \in L^p(B; \mathbb{R}^n)$. Let ρ be a mollifier on $\mathbb{R}$, let $\rho_\epsilon(t) \equiv \epsilon^{-1}\rho(t/\epsilon)$, extend r by zero outside $(0, 1)$, and define $r_\epsilon = \rho_\epsilon * r$, $x_\epsilon(X) = [r_\epsilon(R)/R]X$. By (4.5) we see that ∇x_ϵ is a Cauchy sequence in $L^p(B_{R_2} \backslash \bar{B}_{R_1}; \mathbb{R}^{n^2})$ and hence that $x_\epsilon \to x$ in $W^{1,p}(B_{R_2} \backslash \bar{B}_{R_1}; \mathbb{R}^n)$. Furthermore

$$\int_B |\nabla x(X)|^p \,\mathrm{d}X < \infty.$$

To prove that $x \in W^{1,p}(B; \mathbb{R}^n)$ we must also check that the $x^i{}_{,\alpha}$ are the weak derivatives of x in B. To this end let h be a smooth function satisfying $h(R) \equiv 1$ for $R \geqslant 1$, $h(R) = 0$ for $R \leqslant \frac{1}{2}$. Let $\phi \in C_0^\infty(B)$ and define $\psi_\epsilon(X) \equiv h(R/\epsilon)\phi(X)$. Clearly $\psi_\epsilon \in C_0^\infty(B \backslash \bar{B}_{\epsilon/4})$ and thus

$$\int_B x^i \psi_{\epsilon,\alpha} \,\mathrm{d}X = -\int_B x^i{}_{,\alpha} \psi_\epsilon \,\mathrm{d}X, \tag{4.6}$$

where $f_{,\alpha}$ denotes $\partial f / \partial X^\alpha$. But

$$\left|\int_B x^i(\phi_{,\alpha} - \psi_{\epsilon,\alpha}) \,\mathrm{d}X\right| = \left|\int_{B_\epsilon} x^i \left[\phi_{,\alpha} - h\left(\frac{R}{\epsilon}\right)\phi_{,\alpha} - \epsilon^{-1}h'\left(\frac{R}{\epsilon}\right)\frac{X_\alpha}{R}\phi\right] \mathrm{d}X\right|$$

$$\leqslant C(1 + \epsilon^{-1}) \int_0^\epsilon R^{n-1} |r(R)| \,\mathrm{d}R$$

$$\leqslant C(1 + \epsilon) \int_0^\epsilon R^{n-2} |r(R)| \,\mathrm{d}R,$$

which tends to zero as $\epsilon \to 0$. Similarly

$$\lim_{\epsilon \to 0} \left|\int_B x^i{}_{,\alpha}(\phi - \psi_\epsilon) \,\mathrm{d}X\right| = 0.$$

Thus by (4.6)

$$\int_B x^i \phi_{,\alpha} \,\mathrm{d}X = -\int_B x^i{}_{,\alpha} \phi \,\mathrm{d}X$$

as required.

If $n = 1$, any radial deformation $x \in W^{1,p}(B)$ is continuous, so that $r(0) = 0$. For $n > 1$, however, lemma 4.1 guarantees the existence of radial deformations $x \in W^{1,p}(B; \mathbb{R}^n)$ with $r(0) > 0$. We now examine under what conditions radial deformations (in particular those with $r(0) > 0$) can be weak solutions of the equilibrium equations of nonlinear elasticity. Because we are not interested here in deformations corresponding to eversion† or in which there is interpenetration of matter we shall only consider radial deformations with $r(R) \geqslant 0$ for all $R \in [0, 1]$.

We consider a homogeneous isotropic elastic body occupying B in a reference configuration, and with corresponding stored-energy function $W \in C^1(M_+^{n\times n})$. We suppose that there are no applied body forces. We say that a deformation $x \in W^{1,1}(B; \mathbb{R}^n)$ is a *weak equilibrium solution* if $\det \nabla x(X) > 0$ a.e. $X \in B$, $\partial W(\nabla x(\cdot))/\partial F \in L^1(B; \mathbb{R}^{n^2})$, and

$$\int_B \frac{\partial W}{\partial x^i{}_{,\alpha}} \phi^i{}_{,\alpha} \,\mathrm{d}X = 0 \quad \text{for all} \quad \phi \in C_0^\infty(B; \mathbb{R}^n). \tag{4.7}$$

Our definition is that appropriate for a displacement boundary value problem; the necessary modifications for traction boundary conditions are discussed in §7.

THEOREM 4.2. *Let $n > 1$. Let $x \in W^{1,1}(B; \mathbb{R}^n)$, $x(X) = [r(R)/R]X$, be a radial deformation. Then x is a weak equilibrium solution if and only if $r'(R) > 0$ a.e. $R \in (0, 1)$, $R^{n-1}\Phi_1(R)$, $R^{n-1}\Phi_2(R) \in L^1(0, 1)$ and*

$$R^{n-1}\Phi_1(R) = (n-1)\int_1^R \rho^{n-2}\Phi_2(\rho)\,\mathrm{d}\rho + \text{const.}, \quad a.e.\ R \in (0, 1), \tag{4.8}$$

where
$$\Phi_i(R) \overset{\text{def}}{=} \Phi_{,i}(v_1(R), \ldots, v_n(R))$$
and
$$v_1(R) = r'(R), \quad v_j(R) = r(R)/R \quad for \quad j = 2, \ldots, n.$$

Proof. Let $S = \{X \in \mathbb{R}^n : \frac{1}{2} < |X| < \frac{3}{2}\}$. There is a one:one correspondence between functions $\phi \in C^\infty(B\backslash\{0\}; \mathbb{R}^n)$ and functions $\psi \in C^\infty((0, 1) \times S; \mathbb{R}^n)$ given by the formulae

$$\psi(R, Y) = \phi(RY/|Y|), \tag{4.9}$$

$$\phi(X) = \psi(|X|, X/|X|), \tag{4.10}$$

and we have that
$$\frac{\partial \phi^i}{\partial X^\alpha} = \frac{\partial \psi^i}{\partial R} Y_\alpha + \frac{1}{R}\frac{\partial \psi^i}{\partial Y^j}(\delta^j_\alpha - Y^j Y_\alpha), \tag{4.11}$$

where $R = |X|$, $Y = X/|X|$.

By lemma 4.1, or by formally setting $\phi = x$ in (4.11), we have that

$$x^i{}_{,\alpha} = r'(R)\, Y^i Y_\alpha + \frac{r(R)}{R}(\delta^i_\alpha - Y^i Y_\alpha).$$

Let $Q = Q(Y) \in \mathrm{SO}(n)$ with $Qe_1 = Y$, $e_1 \equiv (1, 0, \ldots, 0)$. Then

$$\begin{aligned} Q \operatorname{diag}(v_1(R), \ldots, v_n(R))\, Q^{\mathrm{T}} &= Qe_1 \otimes e_1 Q^{\mathrm{T}} r' + \frac{r}{R}\mathbf{1} - \frac{r}{R} Qe_1 \otimes e_1 Q^{\mathrm{T}} \\ &= \nabla x. \end{aligned}$$

In particular $\det \nabla x(X) = r'(r/R)^{n-1}$, and since $r \geqslant 0$ it follows that $\det \nabla x(X) > 0$ a.e. $X \in B$ if and only if $r'(R) > 0$ a.e. $R \in (0, 1)$, and in this case, by the isotropy of W,

$$\frac{\partial W(\nabla x)}{\partial x^i{}_{,\alpha}} = (Q \operatorname{diag}(\Phi_1(R), \ldots, \Phi_n(R))\, Q^{\mathrm{T}})^\alpha_i = \Phi_1 Y^i Y_\alpha + \Phi_2(\delta^i_\alpha - Y^i Y_\alpha).$$

† The eversion of thick spherical shells has been studied by Antman (1979).

Therefore
$$\frac{\partial W}{\partial x^i_{,\alpha}}\,\phi^i_{,\alpha} = \Phi_1 Y_i \frac{\partial \psi^i}{\partial R} + \frac{1}{R}\,\Phi_2 \frac{\partial \psi^i}{\partial Y^j}\,(\delta^j_i - Y^j Y_i). \tag{4.12}$$

Let x be a weak equilibrium solution. Choose $\psi(R, Y) = h(R)\,Y$ with $h \in C_0^\infty(0, 1)$. By (4.12),
$$\int_0^1 R^{n-1}\left[\Phi_1 h'(R) + (n-1)\,\Phi_2 \frac{h(R)}{R}\right]\mathrm{d}R = 0.$$

A standard argument now shows that $R^{n-1}\Phi_1$ is absolutely continuous and that (4.8) holds. Clearly $\partial W/\partial x^i_{,\alpha} \in L^1(B)$ implies that $R^{n-1}\Phi_1$, $R^{n-2}\Phi_2 \in L^1(0, 1)$.

Conversely, suppose that $R^{n-1}\Phi_1$, $R^{n-1}\Phi_2 \in L^1(0, 1)$ and that (4.8) holds. Let $\phi \in C_0^\infty(B; \mathbb{R}^n)$ and define ψ by (4.9). By (4.12)
$$\begin{aligned}\int_B \frac{\partial W}{\partial x^i_{,\alpha}}\,\phi^i_{,\alpha}\,\mathrm{d}X &= \lim_{\epsilon\to 0}\int_{S^{n-1}}\int_\epsilon^1 R^{n-1}\left[\Phi_1 Y_i \frac{\partial \psi^i}{\partial R} + \frac{1}{R}\,\Phi_2 \frac{\partial \psi^i}{\partial Y^j}\,(\delta^j_i - Y^j Y_i)\right]\mathrm{d}R\,\mathrm{d}\mathscr{H}^{n-1}Y \\ &= \lim_{\epsilon\to 0}\left[\int_{S^{n-1}}\int_\epsilon^1 R^{n-2}\Phi_2\left[\frac{\partial \psi^i}{\partial Y^j}\,(\delta^j_i - Y^j Y_i) - (n-1)\,\psi^i Y_i\right]\mathrm{d}R\,\mathrm{d}\mathscr{H}^{n-1}Y\right. \\ &\qquad \left. -\int_{S^{n-1}} \epsilon^{n-1}\Phi_1(\epsilon)\,\phi^i(\epsilon Y)\,Y_i\,\mathrm{d}\mathscr{H}^{n-1}Y\right].\end{aligned} \tag{4.13}$$

Let $h \in C_0^\infty(0, 1)$, $\theta \in C^\infty(\mathbb{R}^n; \mathbb{R}^n)$. Applying (4.11) to $\psi = h(R)\,\theta(Y)$ we obtain
$$\begin{aligned}0 = \int_B \frac{\partial \phi^i}{\partial X^i}\,\mathrm{d}X &= \int_{S^{n-1}}\int_0^1 R^{n-1}\left[h'(R)\,\theta^i(Y)\,Y_i + \frac{h(R)}{R}\,\frac{\partial \theta^i}{\partial Y^j}\,(\delta^j_i - Y^j Y_i)\right]\mathrm{d}R\,\mathrm{d}\mathscr{H}^{n-1}Y \\ &= \int_0^1\int_{S^{n-1}}\left[\frac{\partial \theta^i}{\partial Y^j}\,(\delta^j_i - Y^j Y_i) - (n-1)\,\theta^i Y_i\right]\mathrm{d}\mathscr{H}^{n-1}Y\,h(R)\,R^{n-2}\,\mathrm{d}R.\end{aligned}$$

Since h is arbitrary,
$$\int_{S^{n-1}}\left[\frac{\partial \theta^i}{\partial Y^j}\,(\delta^j_i - Y^j Y_i) - (n-1)\,\theta^i Y_i\right]\mathrm{d}\mathscr{H}^{n-1}Y = 0. \tag{4.14}$$

By (4.8),
$$\mathrm{d}[R^n\Phi_1(R)]/\mathrm{d}R = R^{n-1}[\Phi_1 + (n-1)\,\Phi_2].$$

Since $R^{n-1}\Phi_1$, $R^{n-1}\Phi_2$ are integrable,
$$\lim_{R\to 0+} R^n\Phi_1(R) = 0.$$

Thus by (4.13), (4.14),
$$\begin{aligned}\int_B \frac{\partial W}{\partial x^i_{,\alpha}}\,\phi^i_{,\alpha}\,\mathrm{d}X &= -\lim_{\epsilon\to 0}\epsilon^n\Phi_1(\epsilon)\int_{S^{n-1}}\left[\frac{\phi^i(0)}{\epsilon}\,Y_i + \phi^i_{,k}(0)\,Y^k Y_i + o(1)\right]\mathrm{d}\mathscr{H}^{n-1}Y \\ &= 0.\end{aligned}$$

We now consider an *incompressible* homogeneous isotropic elastic material with stored-energy function $W \in C^1(M_+^{n\times n})$ relative to B. Since for a radial deformation
$$\det \nabla x = r^{n-1}r'/R^{n-1} = 1,$$
the only possible such deformations are given by
$$r^n = R^n + k,$$
where k is a constant. As for a compressible material we shall assume that $r \geqslant 0$, so that $k = A^n$ for some $A \geqslant 0$. Thus
$$r(R) = (R^n + A^n)^{1/n}, \tag{4.15}$$
and $r(0) = A$. If $n > 1$ then lemma 4.1 shows that the corresponding radial deformation x belongs to $W^{1,p}(B; \mathbb{R}^n)$ whenever $1 \leqslant p < n$.

The equilibrium equations for our incompressible elastic body under zero body forces are the Euler–Lagrange equations for the functional

$$\int_B \{W(\nabla x(X)) - p(X)[\det \nabla x(X) - 1]\} \mathrm{d}X,$$

where the *pressure* $p(X)$ is a Lagrange multiplier corresponding to the constraint of incompressibility. By definition, then, a deformation $x \in W^{1,1}(B; \mathbb{R}^n)$ is a *weak equilibrium solution* with corresponding measurable pressure function $p(X)$ if $\det \nabla x = 1$ a.e. in B,

$$\partial W(\nabla x(X))/\partial x^i{}_{,\alpha} - p(X)\,(\operatorname{adj} \nabla x(X))^\alpha_i \in L^1(B; \mathbb{R}^{n^2}),$$

and
$$\int_B \left[\frac{\partial W}{\partial x^i{}_{,\alpha}} - p(X)\,(\operatorname{adj} \nabla x)^\alpha_i\right] \phi^i{}_{,\alpha} \mathrm{d}X = 0 \quad \text{for all} \quad \phi \in C_0^\infty(B; \mathbb{R}^n).$$

The modifications to this definition that are necessary for traction boundary conditions are discussed in §5. Notice that the radial deformation (4.15) with $A = 0$ is a weak equilibrium solution for any W and for any constant pressure p, since in this case $x = X$. For $A > 0$ we have the following result.

Notation. We write $\hat{\Phi}(v)$ for the value taken by $\Phi(v_1, \ldots, v_n)$ when $v_1 = v^{1-n}$ and $v_2 = \ldots = v_n = v$.

THEOREM 4.3. *Let $n > 1$. The radial deformation* (4.15) *with $A > 0$ is a weak equilibrium solution if and only if*

$$\frac{v^{n-1}}{(v^n - 1)^2} \frac{\mathrm{d}\hat{\Phi}(v)}{\mathrm{d}v} \in L^1(\delta, \infty) \quad \textit{for} \quad \delta > 1.$$

In this case
$$p(X) = \tilde{p}(R) = (R/r)^{n-1} \Phi_1(R) - T(R),$$

where $T(R)$ is the radial component of the Cauchy stress, given for $R > 0$ by

$$T(R) = \int_{[1+(A/R)^n]^{1/n}}^{(1+A^n)^{1/n}} \frac{1}{v^n - 1} \frac{\mathrm{d}\hat{\Phi}}{\mathrm{d}v} \mathrm{d}v + \text{const.} \tag{4.16}$$

Proof. We first show that $p(X)$ is unique up to a constant. If $p_1(X), p_2(X)$ are two possible pressures and $q(X) = p_1(X) - p_2(X)$ then

$$\int_B q(X)\,(\operatorname{adj} \nabla x)^\alpha_i \phi^i{}_{,\alpha} \mathrm{d}X = 0 \quad \text{for all} \quad \phi \in C_0^\infty(B; \mathbb{R}^n).$$

Let $C = B_{(1+A^n)^{1/n}} \backslash \bar{B}_A$. Then $x(\cdot)$ is a C^∞ measure-preserving diffeomorphism of $B\backslash\{0\}$ onto C. Hence

$$\int_C q(X(x)) \frac{\partial \phi^i(x)}{\partial x^i} \mathrm{d}x = 0 \quad \text{for all} \quad \phi \in C_0^\infty(C; \mathbb{R}^n),$$

which implies that q is constant.

Suppose $x(X) = [r(R)/R]X$, r given by (4.15) with $A > 0$, is a weak equilibrium solution with corresponding pressure $p(X)$. By the same argument as in theorem 4.2,

$$0 = \int_{S^{n-1}} \int_0^1 R^{n-1} \left\{\left[\Phi_1 - p(X)\left(\frac{r}{R}\right)^{n-1}\right] h'(R) + (n-1)\left[\Phi_2 - p(X)\, r'\left(\frac{r}{R}\right)^{n-2}\right] \frac{h(R)}{R}\right\} \mathrm{d}R\, \mathrm{d}\mathscr{H}^{n-1} Y$$
$$\text{for all} \quad h \in C_0^\infty(0, 1),$$

and hence

$$\int_0^1 R^{n-1}\left\{\left[\Phi_1-\tilde{p}(R)\left(\frac{r}{R}\right)^{n-1}\right]h'(R)+(n-1)\left[\Phi_2-\tilde{p}(R)\,r'\left(\frac{r}{R}\right)^{n-2}\right]\frac{h(R)}{R}\right\}\mathrm{d}R = 0$$

$$\text{for all}\quad h\in C_0^\infty(0,1),$$

where
$$\tilde{p}(R) \equiv \frac{1}{\mathscr{H}^{n-1}(S^{n-1})}\int_{S^{n-1}} p(RY)\,\mathrm{d}\mathscr{H}^{n-1}Y.$$

Define
$$T(R) = -\tilde{p}(R)+(R/r)^{n-1}\Phi_1(R).$$

Then $r^{n-1}T(R)$ is absolutely continuous for $0 < R < 1$ and

$$\frac{\mathrm{d}}{\mathrm{d}R}[r^{n-1}T(R)] = (n-1)R^{n-2}\left[\frac{R}{r}\,T(R)+\Phi_2-\left(\frac{R}{r}\right)^n\Phi_1\right]\quad \text{a.e. } R\in(0,1).$$

Hence T is absolutely continuous for $0 < R < 1$ and

$$\frac{\mathrm{d}T}{\mathrm{d}R} = \frac{(n-1)}{R}\left(\frac{R}{r}\right)^{n-1}\left[\Phi_2-\left(\frac{R}{r}\right)^n\Phi_1\right].$$

Integrating between 1 and R, and setting $r/R = v$, we obtain (4.16). The hypothesis that $\partial W/\partial x^i{}_{,\alpha}-p(X)\,(\operatorname{adj}\nabla x(X))^\alpha_i\in L^1(B)$ implies that

$$r^{n-1}T(R),\quad R^{n-1}\left[\Phi_2-\left(\frac{R}{r}\right)^n\Phi_1+\frac{R}{r}\,T(R)\right]\in L^1(0,1),$$

and hence
$$R^{n-1}\left[\Phi_2-\left(\frac{R}{r}\right)^n\Phi_1\right]\in L^1(0,1). \tag{4.17}$$

Setting $v = r/R$ we see that (4.17) holds if and only if

$$\frac{v^{n-1}}{(v^n-1)^2}\,\frac{\mathrm{d}\hat{\Phi}(v)}{\mathrm{d}v}\in L^1(\delta,\infty)\quad\text{for}\quad \delta > 1.$$

By the uniqueness of pressures it therefore suffices to show that if (4.16) and (4.17) hold then x is a weak solution with pressure $\tilde{p}(R)$. We make use of the following simple lemma (see for example Hardy *et al.* 1952, p. 169).

LEMMA 4.4. *If* $f\in L^1(0,1)$ *and* $F(R) = \int_1^R [f(s)/s]\,\mathrm{d}s+\text{const.}$, *then* $F\in L^1(0,1)$.

Proof.
$$\begin{aligned}\int_\epsilon^1 |F(R)|\,\mathrm{d}R &\leqslant C+\int_\epsilon^1\int_R^1\frac{|f(s)|}{s}\,\mathrm{d}s\,\mathrm{d}R\\ &= C-\epsilon\int_\epsilon^1\frac{|f(s)|}{s}\,\mathrm{d}s+\int_\epsilon^1|f(s)|\,\mathrm{d}s\\ &\leqslant C+\int_\epsilon^1|f(s)|\,\mathrm{d}s\end{aligned}$$

for any $\epsilon\in(0,1)$.

Applying the lemma with $F\equiv T, f = (n-1)(R/r)^{n-1}[\Phi_2-(R/r)^n\Phi_1]$, we see that if (4.16), (4.17) hold then $T\in L^1(0,1)$. Hence

$$r^{n-1}T(R),\quad R^{n-1}\left[\Phi_2-\left(\frac{R}{r}\right)^n\Phi_1+\frac{R}{r}\,T(R)\right]\in L^1(0,1)$$

and so $\partial W/\partial x^i{}_{,\alpha}-p(X)\,(\operatorname{adj}\nabla x(X))^\alpha_i\in L^1(B)$.

Proceeding as for compressible materials, we have that for $\phi \in C_0^\infty(B; \mathbb{R}^n)$, $\psi(R, Y) \equiv \phi(RY/|Y|)$,

$$\int_B \left[\frac{\partial W}{\partial x^i_{,\alpha}} - p(X)\,(\operatorname{adj} \nabla x)^\alpha_i\right] \phi^i_{,\alpha}\,\mathrm{d}X$$

$$= \lim_{\epsilon\to 0}\int_{S^{n-1}}\int_\epsilon^1 R^{n-1}\left\{\left[\Phi_1 - \tilde{p}(R)\left(\frac{r}{R}\right)^{n-1}\right] Y_i \frac{\partial \psi^i}{\partial R} + \frac{1}{R}\left[\Phi_2 - \tilde{p}(R)\,r'\left(\frac{r}{R}\right)^{n-2}\right]\frac{\partial \psi^i}{\partial Y^j}\,(\delta^j_i - Y^jY_i)\right\}\mathrm{d}R\,\mathrm{d}\mathscr{H}^{n-1}Y$$

$$= \lim_{\epsilon\to 0}\left\{\int_{S^{n-1}}\int_\epsilon^1 R^{n-2}\left[\Phi_2 - \tilde{p}(R)\,r'\left(\frac{r}{R}\right)^{n-2}\right]\left[\frac{\partial \psi^i}{\partial Y^j}\,(\delta^j_i - Y^jY_i) - (n-1)\,\psi^i Y_i\right]\mathrm{d}R\,\mathrm{d}\mathscr{H}^{n-1}Y - \int_{S^{n-1}} r(\epsilon)^{n-1}\,T(\epsilon)\,\phi^i(\epsilon Y)\,Y_i\,\mathrm{d}\mathscr{H}^{n-1}Y\right\}$$

$$= -\lim_{\epsilon\to 0}\int_{S^{n-1}} r(\epsilon)^{n-1}\,T(\epsilon)\,\phi^i(\epsilon Y)\,Y_i\,\mathrm{d}\mathscr{H}^{n-1}Y.$$

But
$$\frac{\mathrm{d}}{\mathrm{d}R}\,[RT(R)] = T(R) + (n-1)\left(\frac{R}{r}\right)^{n-1}\left[\Phi_2 - \left(\frac{R}{r}\right)^n \Phi_1\right],$$

and hence $\lim_{R\to 0+} RT(R)$ exists. Since $T \in L^1(0, 1)$, $RT(R) \to 0$ as $R \to 0+$. Therefore x is a weak solution.

5. Cavitation in isotropic incompressible elasticity

5.1. *Weak equilibrium solutions, stored energy, and stress*

Let $n > 1$. We adopt the assumptions and notation of §4 for an incompressible material, and assume in addition that W is bounded below. By theorem 4.3, the radial deformation with $r(R) = (R^n + A^n)^{1/n}$, $A > 0$, is a weak equilibrium solution if and only if

$$\frac{v^{n-1}}{(v^n-1)^2}\,\frac{\mathrm{d}\hat{\Phi}(v)}{\mathrm{d}v} \in L^1(\delta, \infty) \quad \text{for} \quad \delta > 1, \tag{5.1}$$

or, equivalently, setting $v = r/R$, if and only if

$$r^{n-1}R^{n+1}\Phi'(R) \in L^1(0, 1), \tag{5.2}$$

where $\Phi(R) \stackrel{\text{def}}{=} \Phi(r'(R), r(R)/R, \ldots)$, $\Phi'(R) = \mathrm{d}\Phi(R)/\mathrm{d}R$. Also, if $T(R)$ is defined by (4.16), $T(0) \stackrel{\text{def}}{=} \lim_{R\to 0+} T(R)$ exists if and only if

$$\int_\delta^\infty \frac{1}{v^n-1}\,\frac{\mathrm{d}\hat{\Phi}(v)}{\mathrm{d}v}\,\mathrm{d}v \stackrel{\text{def}}{=} \lim_{a\to\infty}\int_\delta^a \frac{1}{v^n-1}\,\frac{\mathrm{d}\hat{\Phi}(v)}{\mathrm{d}v}\,\mathrm{d}v \tag{5.3}$$

exists for $\delta > 1$, or, equivalently, if and only if

$$\lim_{\tau\to 0+}\int_\tau^1 R^n\Phi'(R)\,\mathrm{d}R \tag{5.4}$$

exists. The total stored energy of the deformation is

$$E(A) = \omega_n\int_0^1 R^{n-1}\Phi(R)\,\mathrm{d}R = \omega_n A^n \int_{(1+A^n)^{1/n}}^\infty \frac{v^{n-1}}{(v^n-1)^2}\,\hat{\Phi}(v)\,\mathrm{d}v, \tag{5.5}$$

where $\omega_n = \mathscr{H}^{n-1}(S^{n-1})$. We define $E(0) = (\omega_n/n)\,\Phi(1, 1, \ldots, 1)$.

Example 5.1. Let $\Phi(v_1, \ldots, v_n) = \mu(\sum_{i=1}^n v_i^\alpha - n)$, where $\mu > 0$, $\alpha \in \mathbb{R}$. Then $R \mapsto (R^n + A^n)^{1/n}$, $A > 0$, is a weak equilibrium solution if and only if

$$-1 - 2/(n-1) < \alpha < n+1. \tag{5.6}$$

The total stored energy $E(A)$ is finite if and only if $T(0)$ exists and is finite, and if and only if

$$-1 - 1/(n-1) < \alpha < n. \tag{5.7}$$

The inequalities (5.6), (5.7) follow simply from (5.1)–(5.5). Note that (5.7) is stronger than (5.6).

From example 5.1 we can immediately draw the following important conclusions:

(i) *there exist weak equilibrium solutions having infinite stored energy;*

(ii) *strong ellipticity does not imply the regularity of weak equilibrium solutions.*

Conclusion (ii) follows from the fact that if $\alpha > 1$ then

$$W(F) = \mu\left(\sum_{i=1}^n v_i^\alpha - n\right)$$

satisfies
$$\frac{\partial^2 W(F)}{\partial F_\beta^i \partial F_\gamma^j} G_\beta^i G_\gamma^j > 0 \quad \text{for all non-zero} \quad G \in M_+^{n \times n}, \tag{5.8}$$

and is thus strongly elliptic. The proof of (5.8) is a little tedious (see, for example, Ball 1982) except in the special case $\alpha = 2$, when $W(F) = \mu[\mathrm{tr}(F^{\mathrm{T}}F) - n]$.

Let $A > 0$. For general W we have the following implications.

Proposition 5.1.

(*a*) *Suppose $T(0)$ exists. Then $T(0)$ is finite if and only if $E(A) < \infty$, and then*

$$-\int_\delta^\infty \frac{1}{v^n - 1} \frac{\mathrm{d}\hat{\Phi}(v)}{\mathrm{d}v} \mathrm{d}v + n \int_\delta^\infty \frac{v^{n-1}}{(v^n-1)^2} \hat{\Phi}(v) \, \mathrm{d}v = \frac{1}{\delta^n - 1} \Phi(\delta^{1-n}, \delta, \ldots) \tag{5.9}$$

for any $\delta > 1$.

(*b*) *If the weakened Baker–Ericksen inequalities hold then $T(R)$ is a non-decreasing function of R. In particular $T(0)$ exists, so that part* (*a*) *applies. If the Baker–Ericksen inequalities hold then $T(R)$ is strictly increasing.*

Proof.

(*a*) For any $\tau > 0$ we have that

$$\int_\tau^1 R^n \Phi'(R) \, \mathrm{d}R + n \int_\tau^1 R^{n-1} \Phi(R) \, \mathrm{d}R = \Phi(1) - \tau^n \Phi(\tau). \tag{5.10}$$

Suppose $T(0)$ exists. If $T(0)$ is finite then since by assumption Φ is bounded below $\int_\tau^1 R^{n-1}\Phi(R)\,\mathrm{d}R$ is bounded above, and hence $E(A) < \infty$. Conversely, if $E(A) < \infty$ then $\lim_{\tau \to 0+} \tau^n \Phi(\tau)$ exists, and so $\lim_{\tau \to 0+} \tau^n \Phi(\tau) = 0$. Hence $T(0)$ is finite. Passing to the limit $\tau \to 0+$ in (5.10) and setting $v = r/R$ we obtain (5.9).

(*b*) By (4.16) we have that

$$T(R) = A^{-n} \int_R^1 \rho^n \Phi'(\rho) \, \mathrm{d}\rho + \text{const.}$$

Since
$$\Phi'(R) = -\frac{(n-1)}{r}\left(r'\Phi_1 - \frac{r}{R}\Phi_2\right)\left(r' - \frac{r}{R}\right),$$

the result follows immediately.

Remark. The connection between the Baker–Ericksen inequalities and monotonicity of $T(\cdot)$ has been observed by Knowles (1962).

5.2. *The dead-load traction problem*

We now consider the case when dead-load tractions are prescribed on ∂B, that is

$$T_R(X)X = P(X), \quad X \in S^{n-1}, \tag{5.11}$$

where $P: S^{n-1} \to \mathbb{R}^n$ is given. We say that $x \in W^{1,1}(B; \mathbb{R}^n)$ is a *weak solution to the traction boundary value problem* with corresponding measurable pressure $p(X)$ if $\det \nabla x = 1$ a.e. in B,

$$\frac{\partial W(\nabla x(X))}{\partial x^i_{,\alpha}} - p(X)\,(\operatorname{adj} \nabla x(X))^\alpha_i \in L^1(B; \mathbb{R}^{n^2}),$$

and
$$\int_B \left[\frac{\partial W}{\partial x^i_{,\alpha}} - p(X)\,(\operatorname{adj} \nabla x)^\alpha_i\right] \phi^i_{,\alpha}\,\mathrm{d}X - \int_{S^{n-1}} P(X)\cdot\phi(X)\,\mathrm{d}\mathscr{H}^{n-1}X = 0, \tag{5.12}$$

for all $\phi \in C^\infty(\bar{B}; \mathbb{R}^n)$, these equations being the Euler–Lagrange equations for the functional

$$\int_B \{W(\nabla x(X)) - p(X)\,[\det \nabla x(X) - 1]\}\,\mathrm{d}X - \int_{S^{n-1}} P(X)\cdot x(X)\,\mathrm{d}\mathscr{H}^{n-1}X.$$

We suppose that the prescribed traction $P(X)$ is radial, so that

$$P(X) = PX, \quad X \in S^{n-1}, \tag{5.13}$$

where $P \in \mathbb{R}$.

It is easily verified that the identity deformation $r(R) = R$ is a weak solution to the traction problem for any isotropic W with corresponding (constant) pressure

$$p(X) = \Phi_{,1}(1, 1, \ldots, 1) - P.$$

We call this deformation the *trivial solution*.

A similar proof to that of theorem 4.3 shows that the radial deformation

$$R \mapsto (R^n + A^n)^{1/n}, \quad A > 0, \tag{5.14}$$

is a weak solution to the traction problem if and only if it is a weak equilibrium solution with corresponding pressure

$$p(X) = \tilde{p}(R) = (R/r)^{n-1}\Phi_1(R) - T(R),$$

where
$$T(R) = \frac{P}{(1+A^n)^{(n-1)/n}} + \int_{[1+(A/R)^n]^{1/n}}^{(1+A^n)^{1/n}} \frac{1}{v^n - 1}\,\frac{\mathrm{d}\hat{\Phi}}{\mathrm{d}v}\,\mathrm{d}v. \tag{5.15}$$

Thus *whenever* (5.14) *is a weak equilibrium solution it is also a weak solution to the traction problem for any* P. So, for the traction problem, not only is strong ellipticity not a sufficient condition for regularity, but also there are in general *infinitely many* discontinuous weak solutions corresponding to a given P. The reason for this at first sight strange result is that the definition of a weak solution imposes no traction boundary condition on the cavity surface $r = A$, so that physically the problem is underdetermined. To eliminate this indeterminacy we seek solutions for which $T(0)$ has a given finite constant value. For reasons discussed shortly and in §8 we shall in fact require that

$$T(0) = 0. \tag{5.16}$$

(The case $T(0) = k \neq 0$ can be treated similarly.) So that it is possible to satisfy this condition we assume from now on that

$$\int_\delta^\infty \frac{1}{v^n - 1}\,\frac{\mathrm{d}\hat{\Phi}(v)}{\mathrm{d}v}\,\mathrm{d}v \overset{\text{def}}{=} \lim_{a\to\infty} \int_\delta^a \frac{1}{v^n - 1}\,\frac{\mathrm{d}\hat{\Phi}(v)}{\mathrm{d}v}\,\mathrm{d}v$$

exists and is finite for $\delta > 1$. By proposition 5.1 this implies that $E(A) < \infty$ for any $A > 0$. By (5.15) the possible values of A such that (5.16) is satisfied are the roots of the equation

$$P = f(A), \tag{5.17}$$

where

$$f(A) \stackrel{\text{def}}{=} (1+A^n)^{(n-1)/n} \int_{(1+A^n)^{1/n}}^{\infty} \frac{1}{v^n-1} \frac{\mathrm{d}\hat{\Phi}(v)}{\mathrm{d}v}\,\mathrm{d}v. \tag{5.18}$$

The formula (5.17) can also be obtained in the following elementary, though perhaps not completely rigorous, way (see Ball 1981 *c*). The total energy for the deformation $R \mapsto (R^n + A^n)^{1/n}$, $A \geqslant 0$, is given by

$$\begin{aligned} I(A) &\stackrel{\text{def}}{=} \int_B W\,\mathrm{d}X - \int_{S^{n-1}} PX\cdot x(X)\,\mathrm{d}\mathscr{H}^{n-1}X \\ &= E(A) - P\omega_n(1+A^n)^{1/n}. \end{aligned} \tag{5.19}$$

(Note that we have not included a term corresponding to the potential energy of the forces acting on the cavity surface for $A > 0$, since these forces are assumed to vanish by (5.16).)

Thus

$$I'(A) = \omega_n A^{n-1}\left[\int_{(1+A^n)^{1/n}}^{\infty} \frac{1}{v^n-1}\frac{\mathrm{d}\hat{\Phi}(v)}{\mathrm{d}v}\,\mathrm{d}v - \frac{P}{(1+A^n)^{(n-1)/n}}\right], \tag{5.20}$$

where we have used (5.5) and (5.9). Setting $I'(A)\,(B-A) \geqslant 0$ for all $B \geqslant 0$ for equilibrium gives the two possibilities: (i) $A = 0$, or (ii) $A > 0$ is a root of (5.17). This calculation also proves (rigorously) that (5.16) holds for any non-trivial minimizer of I, so that $T(0) = 0$ is a natural boundary condition.

The onset of fracture (i.e. the bifurcation from the trivial solution) is governed by the behaviour of $f(A)$ as $A \to 0+$. We suppose that

$$P_{\text{cr}} \stackrel{\text{def}}{=} \lim_{\delta\to 1+} \int_{\delta}^{\infty} \frac{1}{v^n-1}\frac{\mathrm{d}\hat{\Phi}(v)}{\mathrm{d}v}\,\mathrm{d}v \tag{5.21}$$

exists and is finite. This happens whenever, for example, $\hat{\Phi}(v)$ is twice differentiable at $v = 1$ (and, in particular, if $\Phi \in C^2(\mathbb{R}^n_{++})$), since then the fact that $\mathrm{d}\hat{\Phi}/\mathrm{d}v|_{v=1} = 0$ implies that

$$\lim_{v\to 1+} \frac{1}{v^n-1}\frac{\mathrm{d}\hat{\Phi}(v)}{\mathrm{d}v} = \frac{1}{n}\frac{\mathrm{d}^2\hat{\Phi}}{\mathrm{d}v^2}\bigg|_{v=1};$$

in this case

$$P_{\text{cr}} = \int_1^{\infty} \frac{1}{v^n-1}\frac{\mathrm{d}\hat{\Phi}(v)}{\mathrm{d}v}\,\mathrm{d}v. \tag{5.22}$$

If the Baker–Ericksen inequalities hold then $P_{\text{cr}} > 0$, and since by (5.18) $f(A) > 0$ there are no roots of (5.17) when $P \leqslant 0$. In other words, *no cavity will form if the applied traction is compressive.*

Even when the Baker–Ericksen inequalities hold there may be several roots of (5.17) for a given $P > 0$; however, the following result shows that the bifurcation at $P = P_{\text{cr}}$ is locally to the right (i.e. supercritical).

Proposition 5.2. *Let $\hat{\Phi}(v)$ be twice differentiable at $v = 1$. Then $f'(0) = 0$, and if*

$$P_{\text{cr}} + \frac{1}{n(n-1)}\frac{\mathrm{d}^2\hat{\Phi}}{\mathrm{d}v^2}\bigg|_{v=1} > 0 \quad (\textit{resp.} < 0) \tag{5.23}$$

then $f'(A) > 0$ (resp. $f'(A) < 0$) for sufficiently small $A > 0$.

Remark. Since

$$\left.\frac{\mathrm{d}^2\hat{\Phi}}{\mathrm{d}v^2}\right|_{v=1} = \lim_{v\to 1+} \frac{1}{v-1}\frac{\mathrm{d}\hat{\Phi}(v)}{\mathrm{d}v} = \lim_{v\to 1+} n(n-1)\frac{v^{1-n}\Phi_{,1}-v\Phi_{,2}}{v^{1-n}-v}, \tag{5.24}$$

the Baker–Ericksen inequalities imply that $\mathrm{d}^2\hat{\Phi}/\mathrm{d}v^2|_{v=1} \geqslant 0$, so that $f'(A) > 0$ for $0 < A < \epsilon$. Also, (5.24) shows that if $\Phi \in C^2(\mathbb{R}^n_{++})$ then (5.23) can be written in the form

$$P_{\mathrm{cr}} + \Phi_{,1}(1, \ldots, 1) + \Phi_{,11}(1, \ldots, 1) - \Phi_{,12}(1, \ldots, 1) > 0 \quad (\text{resp.} < 0). \tag{5.25}$$

Proof of proposition 5.2. That $f'(0) = 0$ follows immediately from (5.18). To prove the second part, let $\theta = (1+A^n)^{(n-1)/n}$ and use (5.18) to show that

$$\lim_{\theta\to 1+} \frac{\mathrm{d}f}{\mathrm{d}\theta} = P_{\mathrm{cr}} + \frac{1}{n(n-1)}\left.\frac{\mathrm{d}^2\hat{\Phi}}{\mathrm{d}v^2}\right|_{v=1}.$$

Let $P > 0$. If $A_0 > 0$ is a root of (5.17), that is

$$P = f(A_0), \tag{5.26}$$

then from (5.20)

$$I''(A_0) = \frac{\omega_n A_0^{n-1}}{(1+A_0^n)^{(n-1)/n}} f'(A_0),$$

so that A_0 is a local minimum (resp. local maximum) of I if $f'(A_0) > 0$ (resp. $f'(A_0) < 0$).

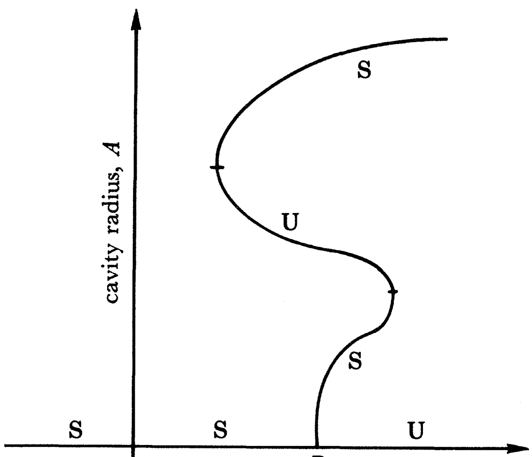

FIGURE 1. Dead-load traction problem: a typical bifurcation diagram for a material satisfying the Baker–Ericksen inequalities. S, stable; U, unstable.

To see when the trivial solution $A = 0$ is a local minimum for I note first that by (5.5), (5.9), and (5.15)–(5.17),

$$\frac{n}{\omega_n}E(A) = \frac{f(A)A^n}{(1+A^n)^{(n-1)/n}} + \Phi((1+A^n)^{(1-n)/n}, (1+A^n)^{1/n}, \ldots), \tag{5.27}$$

for all $A > 0$. Hence

$$\lim_{A\to 0+} E(A) = E(0). \tag{5.28}$$

By (5.20), if $P < P_{\mathrm{cr}}$ (resp. $P > P_{\mathrm{cr}}$), $I'(A)$ is positive (resp. negative) for sufficiently small $A > 0$, and thus by (5.19), (5.28), the trivial solution is a local minimum (resp. local maximum) of I. It follows also from (5.20) that if $P = P_{cr}$ and $f'(A) > 0$ (resp. $f'(A) < 0$) for sufficiently small $A > 0$, then the trivial solution is a local minimum (resp. local maximum).

We say that a weak solution of the traction problem is *stable* if the corresponding A is a local minimum for I, and that it is *unstable* otherwise. This terminology has a precise dynamic meaning which is discussed in §5.4.

We summarize the results of this subsection corresponding to the case when the Baker–Ericksen inequalities hold in figure 1.

5.3. *The Cauchy traction problem*

We modify the analysis of the preceding subsection to treat the case when the Cauchy stress, rather than the Piola–Kirchhoff stress, is prescribed on ∂B. The boundary condition on ∂B is now

$$T(X)X = PX, \quad X \in S^{n-1}, \tag{5.29}$$

where $P \in \mathbb{R}$ is given. The definition in general of a weak solution to this problem introduces complications that are not relevant in the context of radial solutions. We therefore make a simpler, but somewhat *ad hoc*, definition, namely that $x \in W^{1,1}(B; \mathbb{R}^n)$ is a *weak solution to the Cauchy traction boundary value problem* if it is a weak equilibrium solution such that for any $X \in S^{n-1}$

$$T(X) = \operatorname*{ess\,lim}_{Y \to X,\ Y \in B} T(Y)$$

exists and satisfies (5.29).

As before, $r(R) = R$ is a weak solution to the Cauchy traction problem for any isotropic W. Also $R \mapsto (R^n + A^n)^{1/n}$, $A > 0$, is a weak solution if and only if (5.1) holds, and then

$$p(X) = \tilde{p}(R) = (R/r)^{n-1}\Phi_1(R) - T(R),$$

where

$$T(R) = P + \int_{[1+(A/R)^n]^{1/n}}^{(1+A^n)^{1/n}} \frac{1}{v^n - 1} \frac{\mathrm{d}\hat{\Phi}}{\mathrm{d}v} \, \mathrm{d}v \tag{5.30}$$

is the radial component of the Cauchy stress. To render the problem determinate we again require that

$$T(0) = 0, \tag{5.31}$$

and so that it is possible to satisfy this condition we assume that

$$\int_\delta^\infty \frac{1}{v^n - 1} \frac{\mathrm{d}\hat{\Phi}(v)}{\mathrm{d}v} \, \mathrm{d}v$$

exists and is finite for $\delta > 1$.

The non-trivial weak solutions satisfying (5.31) correspond to positive roots of the equation

$$P = g(A), \tag{5.32}$$

where

$$g(A) \stackrel{\text{def}}{=} \int_{(1+A^n)^{1/n}}^\infty \frac{1}{v^n - 1} \frac{\mathrm{d}\hat{\Phi}(v)}{\mathrm{d}v} \, \mathrm{d}v. \tag{5.33}$$

For *radial* deformations $R \mapsto (R^n + A^n)^{1/n}$, $A \geqslant 0$, we can define a total energy functional

$$J(A) \stackrel{\text{def}}{=} \int_B W \mathrm{d}X - \frac{P}{n} \int_{x(S^{n-1})} y \cdot n(y) \, \mathrm{d}\mathscr{H}^{n-1} y,$$

where $n(y)$ is the unit outward normal to $x(S^{n-1})$ at y. But $n(y) = y/|y|$, and so

$$J(A) = E(A) - (P\omega_n/n)(1 + A^n). \tag{5.34}$$

Thus
$$J'(A) = \omega_n A^{n-1}\left[\int_{(1+A^n)^{1/n}}^{\infty} \frac{1}{v^n-1}\frac{\mathrm{d}\hat{\Phi}(v)}{\mathrm{d}v}\,\mathrm{d}v - P\right], \tag{5.35}$$
which is zero if and only if $A = 0$ or (5.32) holds.

We again suppose that $P_{\mathrm{cr}} = g(0)$ given by (5.21) exists and is finite. If the Baker–Ericksen inequalities hold then $P_{\mathrm{cr}} > 0$ and there are no roots of (5.32) when $P \leqslant 0$.

From (5.33) we see that $\lim_{A\to\infty} g(A) = 0$ and that
$$g'(A) = \frac{-A^{n-1}}{(1+A^n)^{(n-1)/n}}\left.\frac{\mathrm{d}\hat{\Phi}}{\mathrm{d}v}\right|_{v=(1+A^n)^{1/n}}. \tag{5.36}$$

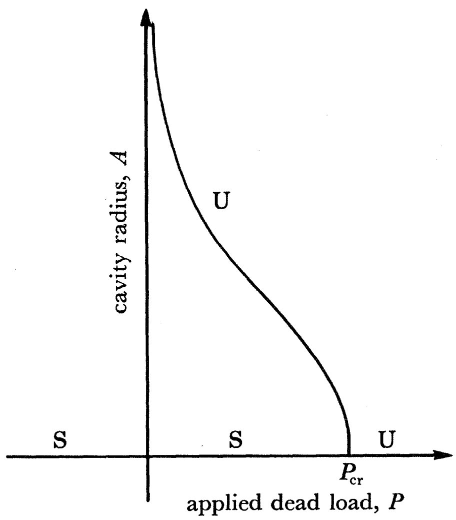

FIGURE 2. Cauchy traction problem: a typical bifurcation diagram for a material satisfying the Baker–Ericksen inequalities. S, stable; U, unstable.

Thus, provided $\hat{\Phi}(v)$ is twice differentiable at $v = 1$, $g'(0) = 0$. Also $g'(A) < 0$ for all $A > 0$ if the Baker–Ericksen inequalities hold. Similar calculations to those in §5.2 show that if $A_0 > 0$ is a root of (5.32) then A_0 is a local minimum (resp. local maximum) of J if $g'(A_0) > 0$ (resp. $g'(A_0) < 0$), and that the trivial solution $A = 0$ is a local minimum (resp. local maximum) if $P < P_{\mathrm{cr}}$ (resp. $P > P_{\mathrm{cr}}$). The dynamical significance of these statements is discussed in the next subsection.

For a material satisfying the Baker–Ericksen inequalities we thus have the bifurcation diagram sketched in figure 2.

5.4. *Stability with respect to radial motions*

We examine the Lyapunov stability of the trivial and non-trivial weak solutions discussed in the preceding subsections within the class of radial motions satisfying the equations of nonlinear elastodynamics. The points at issue are similar to those arising in the work of Calderer (1981 *a*, *b*) on the inflation under a maintained internal pressure of a thick spherical shell, except that in our work the internal radius of the shell is zero.

We consider radial motions $R \mapsto r(R, t)$ of the form
$$r^n = R^n + A(t)^n, \tag{5.37}$$
where $A(t) \geqslant 0$ is the cavity radius at time t.

The total kinetic energy is given by

$$K(t) = \frac{1}{2}\int_B \rho_R |\dot{x}(X,t)|^2 \mathrm{d}X,$$

where ρ_R (assumed to be constant and strictly positive) is the density in the reference configuration. Since $\dot{x} = (\dot{r}/R)X$ it follows that $|\dot{x}|^2 = A^{2(n-1)}\dot{A}^2/r^{2(n-1)}$, and so for a radial motion

$$K(t) = \tfrac{1}{2}\omega_n h(A(t))\dot{A}(t)^2, \tag{5.38}$$

where

$$h(A) = \begin{cases} \dfrac{\rho_R A^n}{n-2}\left[1-\dfrac{1}{(1+A^{-n})^{1-2/n}}\right] & \text{if } n > 2, \\ \tfrac{1}{2}\rho_R A^2 \ln(1+A^{-2}) & \text{if } n = 2. \end{cases} \tag{5.39}$$

Consider first the dead-load traction problem. The equation of motion, in terms of A, can be most rapidly obtained by applying Hamilton's principle to the functional

$$\int_0^\tau (K-I)\,\mathrm{d}t$$

where I is given by (5.19). Using (5.18), (5.19) we thus obtain the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}[h(A)\dot{A}] = \tfrac{1}{2}h'(A)\dot{A}^2 + \frac{A^{n-1}}{(1+A^n)^{(n-1)/n}}[P-f(A)], \tag{5.40}$$

when (5.16) holds in the presence of a cavity and $f \in C([0,\infty))$.

We say that (5.37) is a *weak solution* of the equation of motion (5.40) on the interval $[0,\tau)$, $\tau > 0$, if A is absolutely continuous on $[0,\tau)$ (so that, in particular, $\dot{A}(t)$ exists for a.e. $t \in [0,\tau)$), $h(A)\dot{A}$ is absolutely continuous on $[0,\tau)$, and (5.40) holds for a.e. $t \in [0,\tau)$. An analysis similar to that of theorem 4.3 would show that a weak solution of (5.40) generates through (5.37) a weak solution of the n-dimensional equations of nonlinear elastodynamics, but we do not attempt to render this statement precise here.

The *Cauchy problem* for (5.40) consists of finding a weak solution A on some interval $[0,\tau)$, $\tau > 0$, such that

$$A(0) = a, \tag{5.41}$$

$$\lim_{t\to 0+} A^{n-1}(t)\dot{A}(t) = b, \tag{5.42}$$

where $a \geqslant 0$ and b are given. These initial conditions correspond to specifying $x(X,0)$ and $\dot{x}(X,0)$ for all $X \in B\backslash\{0\}$, where $x(X,t) = [r(t)/R]X$ and $r(t)$ is given by (5.37). Of course, when $a > 0$, (5.42) is equivalent to $\lim_{t\to 0+}\dot{A}(t) = ba^{1-n}$. If $a = 0$, and we impose the further requirement that $\lim_{t\to 0+} K(t) < \infty$, then it follows easily from (5.39) that $\lim_{t\to 0+} A^{n-1}(t)\dot{A}(t) = 0$, so that we must take $b = 0$.

If $a > 0$, b are given, standard theory for ordinary differential equations implies that there is a unique weak solution A to the Cauchy problem defined on a sufficiently small interval $[0,\tau)$, and that $A \in C^2([0,\tau))$; to see this note that, when $A > 0$, h is smooth and positive and f is C^1. This weak solution can be extended to a maximal interval $[0,\tau^*)$, $0 < \tau^* \leqslant \infty$, such that $A \in C^2([0,\tau^*))$ and $A(t) > 0$ for $t \in [0,\tau^*)$, and A is unique. Furthermore, the energy equation

$$\tfrac{1}{2}h(A)\dot{A}^2 + I(A)/\omega_n = E_0 \tag{5.43}$$

holds for $t \in [0,\tau^*)$, where $E_0 \stackrel{\text{def}}{=} \tfrac{1}{2}h(a)b^2 + I(a)/\omega_n$.

If $P = f(A_0)$, $A_0 > 0$, the linearization of (5.40) about A_0 is

$$h(A_0)\,\ddot{u} = \frac{-A_0^{n-1}}{(1+A_0^n)^{(n-1)/n}} f'(A_0)\,u. \tag{5.44}$$

Thus A_0 is a stable (resp. unstable) rest point if $f'(A_0) > 0$ (resp. $f'(A_0) < 0$), in consonance with the terminology introduced in § 5.2.

We now study the Cauchy problem when $a = b = 0$. One solution is clearly $A \equiv 0$. We seek others satisfying $A(t) > 0$ for sufficiently small $t > 0$; such solutions must satisfy, for sufficiently small $t > 0$, the energy equation

$$\tfrac{1}{2}h(A)\dot{A}^2 = -\frac{1}{n}\left[\frac{f(A)\,A^n}{(1+A^n)^{(n-1)/n}} + \Phi((1+A^n)^{(1-n)/n}, (1+A^n)^{1/n}, \ldots)\right] + P(1+A^n)^{1/n} + E_1, \tag{5.45}$$

where E_1 is a constant, and where we have used (5.27). Thus

$$\tfrac{1}{2}h(A)\dot{A}^2 = E_1 + P - \frac{1}{n}\Phi(1, 1, \ldots) + (P - P_{\mathrm{cr}})\frac{A^n}{n} + \psi(A^n), \tag{5.46}$$

where $\lim_{\sigma\to 0+} \psi(\sigma)/\sigma = 0$. Let $k = E_1 + P - (1/n)\,\Phi(1, 1, \ldots)$. Suppose first that $P < P_{\mathrm{cr}}$. Then (5.46) takes the form

$$\tfrac{1}{2}z^2 = k - cy[1+\theta(y)], \tag{5.47}$$

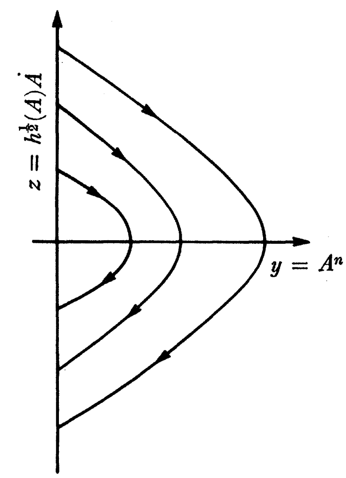

Figure 3. The phase portrait near the trivial solution for $P < P_{\mathrm{cr}}$.

where $y = A^n$, $z = h^{\frac{1}{2}}(A)\dot{A}$, $\lim_{y\to 0+} \theta(y) = 0$, and $c > 0$. Using (5.47) we may study the behaviour of (5.40) in the (y, z)-plane. By noting that

$$\int_0^{\epsilon} \frac{h^{\frac{1}{2}}(y^{1/n})\, y^{(1-n)/n}}{k - cy[1+\theta(y)]}\,\mathrm{d}y \tag{5.48}$$

is convergent for sufficiently small $\epsilon > 0$, it is easily shown that for sufficiently small $k > 0$ there is a unique solution A of (5.40) such that $A \in C^2(0, \tau(k))$, $\lim_{t\to 0+} (y(t), z(t)) = (0, (2k)^{\frac{1}{2}})$, $\lim_{t\to\tau(k)-} (y(t), z(t)) = (0, -(2k)^{\frac{1}{2}})$, $A(t) > 0$ for $0 < t < \tau(k)$, and $(y(\frac{1}{2}\tau(k)), z(\frac{1}{2}\tau(k))) = (k/c + \gamma(k), 0)$, $\gamma(k)/k \to 0$ as $k \to 0+$. Thus the phase portrait for small (y, z) is as sketched in figure 3. Each of these solutions is a weak solution of (5.40) on $[0, \tau(k))$, since $h'(A) > 0$ for small $A > 0$, and $\lim_{t\to 0+} h(A)\dot{A} = 0$. Furthermore, $\lim_{t\to 0+} A^{n-1}(t)\,\dot{A}(t) = 0$, so that each is a solution to the Cauchy problem with $a = b = 0$. *Thus solutions to the Cauchy problem are not unique.* Further, since for each of the above non-trivial solutions $\lim_{t\to 0+} K(t) = \omega_n k > 0$, *energy need not be conserved for a weak solution.* If, however, we impose the *entropy condition* (compare Lax 1957) that

$$\tfrac{1}{2}h(A)\dot{A}^2 + I(A)/\omega_n \quad \text{be non-increasing in} \quad t, \tag{5.49}$$

then the only admissible solution to the Cauchy problem with $a = b = 0$ is the trivial solution, and then figure 3 shows that the trivial solution is stable. Each of the foregoing non-trivial solutions $A = A(t; k)$ can be continued as a weak solution for all $t \geqslant \tau(k)$ in infinitely many ways without violating the entropy condition. For example, we can set

$$A(t) = 0 \quad \text{for all} \quad t \geqslant \tau(k), \tag{5.50}$$

or

$$A(t) = \begin{cases} A(t-\tau(k), k) & \text{for} \quad \tau(k) \leqslant t < 2\tau(k) \\ 0 & \text{for} \quad t \geqslant 2\tau(k) \end{cases}$$

or

$$A(t) = \begin{cases} A(t-\tau(k), \tfrac{1}{2}k) \ \text{for} \ \tau(k) \leqslant t < \tau(k)+\tau(\tfrac{1}{2}k) \\ A(t-\tau(k)-\tau(\tfrac{1}{2}k), \tfrac{1}{3}k) \ \text{for} \ \tau(k)+\tau(\tfrac{1}{2}k) \leqslant t < \tau(k)+\tau(\tfrac{1}{2}k)+\tau(\tfrac{1}{3}k) \\ 0 \ \text{for} \ t \geqslant \tau(k)+\tau(\tfrac{1}{2}k)+\tau(\tfrac{1}{3}k). \end{cases}$$

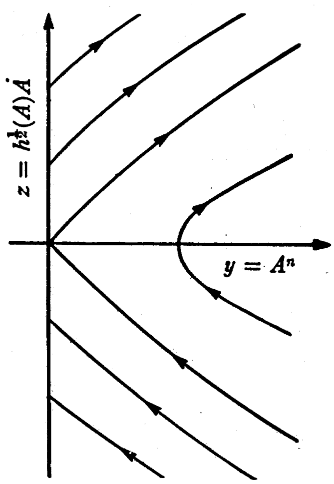

FIGURE 4. The phase portrait near the trivial solution for $P > P_{cr}$.

However, the only extension that conserves energy is given (up to alteration on a set of measure zero) by

$$A(t) = A(t-m\tau(k), k) \quad \text{for} \quad m\tau(k) \leqslant t < (m+1)\tau(k), \quad m = 1, 2, \ldots. \tag{5.51}$$

We call this the *perfect rebound* solution. (For the rebound of cavities in fluids see the photographs of Benjamin & Ellis (1966); (see also Batchelor 1967, plate 17 and pp. 481ff); our analysis ignores among other things the interesting effects of cavity contents (see Cole 1948).) The extension (5.50) is the one that loses energy fastest (compare Dafermos 1973). For these solutions the arbitrary hydrostatic pressure $p(R, t)$ becomes infinite as $t \to \tau(k)-$ for each $R \in (0, 1)$.

If $P > P_{cr}$ then (5.46) takes the form (5.47) but with $c < 0$. A similar analysis to the preceding shows that the local phase portrait in the (y, z)-plane has the form sketched in figure 4.

As for $P < P_{cr}$ we have non-uniqueness of solutions to the Cauchy problem with $a = b = 0$, but for $P > P_{cr}$ there is a solution $A(t)$ defined for sufficiently small $t > 0$ such that $A(t) > 0$ for $t > 0$ but

$$\lim_{t\to 0+} (y(t), z(t)) = (0, 0).$$

Since this solution has the same energy as the trivial solution, namely zero, there is no admissibility criterion based on energy considerations that will reject one of the solutions, and hence there is *genuine non-uniqueness of solutions to the Cauchy problem.* In particular the *trivial solution* $A \equiv 0$ *is unstable. This instability cannot be detected by formal linearization of the n-dimensional equations about the trivial solution.*

We next consider the Cauchy traction problem. The equation of motion is now

$$\frac{\mathrm{d}}{\mathrm{d}t}[h(A)\dot{A}] = \tfrac{1}{2}h'(A)\dot{A}^2 + A^{n-1}\left[P - \frac{f(A)}{(1+A^n)^{(n-1)/n}}\right]. \tag{5.52}$$

The corresponding energy equation for solutions $A(t) > 0$ is

$$\tfrac{1}{2}h(A)\dot{A}^2 = -\frac{1}{n}\left[\frac{f(A)A^n}{(1+A^n)^{(n-1)/n}} + \Phi((1+A^n)^{(1-n)/n}, (1+A^n)^{1/n}, \ldots)\right] + \frac{P}{n}(1+A^n) + E_1, \tag{5.53}$$

where E_1 is a constant. The same analysis as for the dead-load problem can be carried out, and we conclude that (i) if $P = f(A_0)$, $A_0 > 0$, then A_0 is a stable (resp. unstable) rest point if $f'(A_0) > 0$ (resp. $f'(A_0) < 0$), and (ii) the local phase portraits near the trivial solution $A = 0$ have the forms shown in figures 3 and 4, and the same inferences as in the dead-load problem can be drawn concerning the stability and instability of $A = 0$.

5.5. *Examples*

In general, explicit computation of $f(A)$, $g(A)$, or $P_{\mathrm{cr}} = f(0)$ for a given stored-energy function Φ is not possible. However, if

$$\Phi(v_1, \ldots, v_n) = \mu\left(\sum_{i=1}^{n} v_i^{\alpha} - n\right), \tag{5.54}$$

where (cf. (5.7))

$$-1 - 1/(n-1) < \alpha < n, \tag{5.55}$$

explicit formulae may be obtained.

First, setting $s = v^{-n}$, we obtain from (5.22)

$$\begin{aligned} P_{\mathrm{cr}} &= (n-1)\mu\alpha \int_1^{\infty} \frac{v^{\alpha-1}(1-v^{-n\alpha})}{v^n - 1}\,\mathrm{d}v \\ &= \frac{(n-1)\mu\alpha}{n}\int_0^1 s^{-\alpha/n}\left(\frac{1-s^{\alpha}}{1-s}\right)\mathrm{d}s, \end{aligned} \tag{5.56}$$

and hence (see Copson 1935, p. 229),

$$P_{\mathrm{cr}} = \frac{(n-1)\mu\alpha}{n}[\psi(1+\alpha-\alpha/n) - \psi(1-\alpha/n)], \tag{5.57}$$

where $\psi(z) \overset{\mathrm{def}}{=} \mathrm{d}[\ln\Gamma(z)]/\mathrm{d}z$ and $\Gamma(\cdot)$ denotes the gamma function.

If $\alpha = 0$, so that $\Phi = \text{const.}$ (the conditions for an incompressible perfect fluid) then $g(A) = f(A) = 0$, $P_{\mathrm{cr}} = 0$. If α is an integer, $0 < \alpha < n$, then by (5.33)

$$\begin{aligned} g(A) &= \frac{(n-1)\mu\alpha}{n}\int_0^{(1+A^n)^{-1}} \frac{s^{-\alpha/n}(1-s^{\alpha})}{1-s}\,\mathrm{d}s \\ &= \frac{(n-1)\mu\alpha}{n}\sum_{j=1}^{\alpha}\int_0^{(1+A^n)^{-1}} s^{j-\alpha/n-1}\,\mathrm{d}s \\ &= \frac{(n-1)\mu\alpha}{n}\sum_{j=1}^{\alpha}\frac{(1+A^n)^{-j+\alpha/n}}{j-\alpha/n}, \end{aligned} \tag{5.58}$$

and hence
$$f(A) = \frac{(n-1)\mu\alpha}{n}\sum_{j=1}^{\alpha}\frac{(1-A^n)^{-j+1+(\alpha-1)/n}}{j-\alpha/n}, \tag{5.59}$$

$$P_{\mathrm{cr}} = \frac{(n-1)\mu\alpha}{n}\sum_{j=1}^{\alpha}\frac{1}{j-\alpha/n}. \tag{5.60}$$

If $\alpha = -1$ (the only negative integer satisfying (5.55)) then
$$g(A) = (n-1)\mu(1+A^n)^{-1/n}, \tag{5.61}$$
$$f(A) = (n-1)\mu(1+A^n)^{(n-2)/n}, \tag{5.62}$$
and
$$P_{\mathrm{cr}} = (n-1)\mu. \tag{5.63}$$
If Φ is a linear combination of terms of the form (5.54) then $f(A)$, $g(A)$ and P_{cr} can be obtained by linearity.

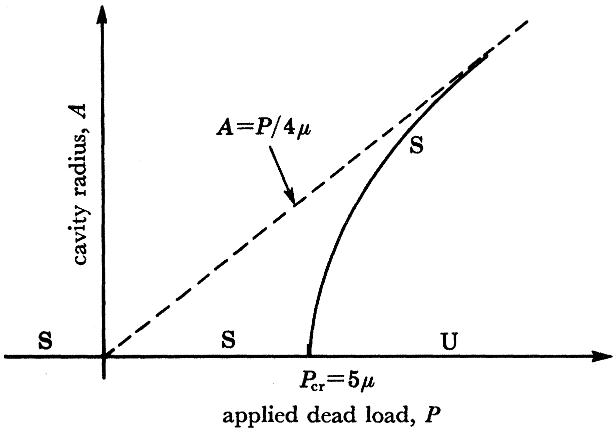

FIGURE 5. Bifurcation diagram for a neo-Hookean material in the dead-load traction problem. S, stable; U, unstable.

For a *neo-Hookean* material we take $\alpha = 2$ and $\mu > 0$ in (5.54). Suppose $n = 3$. Then from (5.58), (5.59), (5.60), we obtain
$$g(A) = \mu(1+A^3)^{-\frac{4}{3}}(5+4A^3), \tag{5.64}$$
$$f(A) = \mu(1+A^3)^{-\frac{2}{3}}(5+4A^3), \tag{5.65}$$
and
$$P_{\mathrm{cr}} = 5\mu. \tag{5.66}$$
Taking, for example, the dead-load traction problem we have the bifurcation diagram sketched in figure 5. Note that for each $P > P_{\mathrm{cr}}$ there is precisely one non-trivial equilibrium solution $A_0(P)$. The corresponding phase portraits are sketched in figure 6.

6. Radial solutions without holes in isotropic compressible elasticity

6.1. *The radial equilibrium equation*

In this section we adopt the assumptions and notations of §4 for compressible materials. We suppose from now on that $n > 1$. By theorem 4.2, if $R \mapsto r(R)$ is a weak equilibrium solution then $R^{n-1}\Phi_1(R)$ is essentially absolutely continuous for $R \in (0, 1]$ and
$$\mathrm{d}[R^{n-1}\Phi_1(R)]/\mathrm{d}R = (n-1)R^{n-2}\Phi_2(R), \quad \text{a.e.} \quad R \in (0, 1), \tag{6.1}$$
where, as before, $\Phi_i(R) \overset{\mathrm{def}}{=} \Phi_{,i}(r'(R), r(R)/R, \ldots)$.

We now suppose that $\Phi(v_1, \ldots, v_n)$ is a strictly convex function of each v_i (the tension–extension property; see proposition 3.1) and that

$$\lim_{v_1 \to 0} \Phi_{,1}(v_1, v_2, \ldots, v_2) = -\infty, \quad \lim_{v_1 \to \infty} \Phi_{,1}(v_1, v_2, \ldots, v_2) = \infty, \tag{6.2}$$

whenever $v_2 > 0$ is fixed. Then

PROPOSITION 6.1. *If $R \mapsto r(R)$ is a weak equilibrium solution then $r \in C^1((0, 1])$, $r'(R) > 0$ for all $R \in (0, 1]$, and equality holds in* (6.1) *for all $R \in (0, 1]$.*

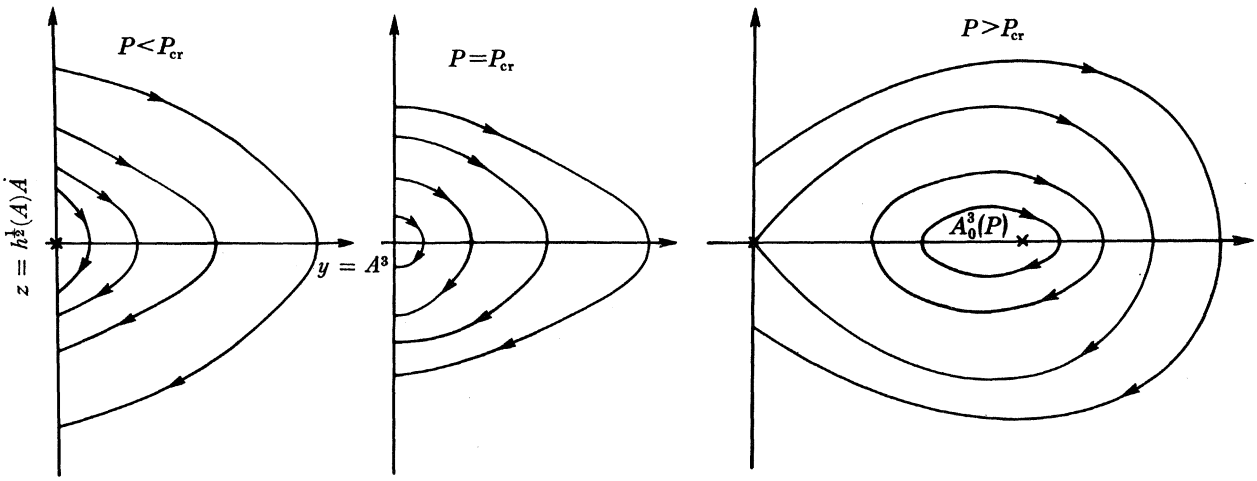

FIGURE 6. The phase portraits for a neo-Hookean material in the dead-load traction problem.

Proof. For any $R > 0$ and any $\sigma \in \mathbb{R}$ there is a unique $v_1 > 0$ such that $\Phi_{,1}(v_1, r(R)/R, \ldots) = \sigma$. Hence after possible alteration of $r'(\cdot)$ on a set of measure zero we can suppose that $\Phi_1(R)$ is continuous and $r'(R) > 0$ for all $R \in (0, 1]$. We must show that $R_j \to R \in (0, 1]$ implies $r'(R_j) \to r'(R)$. Without loss of generality we may suppose that $r'(R_j) \to l \in [0, \infty]$. If $l \in (0, \infty)$ then

$$\lim_{j \to \infty} \Phi_1(R_j) = \Phi_{,1}\left(l, \frac{r(R)}{R}, \ldots\right) = \Phi_{,1}\left(r'(R), \frac{r(R)}{R}, \ldots\right),$$

so that $l = r'(R)$ by the tension–extension property. Suppose for contradiction that $l = 0, \infty$. By passing to the limit using the continuity of $\Phi_{,1}(v_1, \ldots, v_n)$, the tension–extension property, and the fact that $\Phi_1(R_j) \to \Phi_1(R)$, we see that the inequality

$$\Phi_{,1}\left(r'(R) - \epsilon, \frac{r(R_j)}{R_j}, \ldots\right) < \Phi_{,1}\left(r'(R_j), \frac{r(R_j)}{R_j}, \ldots\right) < \Phi_{,1}\left(r'(R) + \epsilon, \frac{r(R_j)}{R_j}, \ldots\right)$$

holds for large enough j, where $\epsilon \in (0, r'(R))$ is fixed. By the tension–extension property we deduce that $|r'(R_j) - r'(R)| < \epsilon$, which is impossible.

Remark. A similar argument is given by Tonelli (1921, vol. II, p. 362).

If $\Phi \in C^2(\mathbb{R}^n_{++})$, and if we strengthen the tension–extension property by requiring that

$$\Phi_{,11}(v_1, \ldots, v_n) > 0 \quad \text{for all} \quad (v_1, \ldots, v_n) \in \mathbb{R}^n_{++} \tag{6.3}$$

(this is a consequence of the strong-ellipticity condition (3.7)), then a standard argument shows that $r \in C^2((0, 1])$ and that we can carry out the differentiation in (6.1) to obtain

$$R\Phi_{11} r'' = (n-1)\left[\Phi_2 - \Phi_1 - (r' - r/R)\,\Phi_{12}\right], \quad 0 < R \leqslant 1. \tag{6.4}$$

If, further, $\Phi \in C^m(\mathbb{R}^n_{++})$ for some $m > 2$, then $r \in C^m((0, 1])$.

6.2. *Lyapunov functions*

The radial equilibrium equation (6.1) is invariant under the transformations $(r, R) \mapsto (cr, cR)$, $c > 0$; equivalently, if $r(R)$ is a solution then so is $c^{-1}r(cR)$. This suggests the change of variables

$$v = r/R, \quad R = e^s. \tag{6.5}$$

The resulting autonomous equation is

$$d\Phi_1/ds = (n-1)(\Phi_2 - \Phi_1), \quad -\infty < s < 0, \tag{6.6}$$

where $\Phi_i = \Phi_{,i}(\dot{v}(s) + v(s), v(s), \ldots)$, $i = 1, 2$ and $\dot{v}(s) \overset{\text{def}}{=} dv(s)/ds$.

We seek functions of R, r, r' that are increasing for solutions of (6.1); such functions correspond to Lyapunov functions for (6.6). Firstly we consider the radial component of the Cauchy stress, given (see §3) by

$$T(R) \overset{\text{def}}{=} (R/r)^{n-1}\Phi_1(R). \tag{6.7}$$

From (6.1),

$$\frac{dT(R)}{dR} = -(n-1)\frac{R^{n-1}}{r^n}\left(r'\Phi_1 - \frac{r}{R}\Phi_2\right). \tag{6.8}$$

Comparing this with (3.8) and (3.10) we deduce immediately the following result, which applies in particular to any weak equilibrium solution if the hypotheses of proposition 6.1 hold, and which gives conditions under which $T(\cdot)$ is monotone on any interval where $r' \neq r/R$.

Proposition 6.2. *Let $r \in C^1((0, 1])$ be a solution of (6.1) with $r'(R) > 0$ for all $R \in (0, 1]$. If the weakened Baker–Ericksen inequalities hold then $T(\cdot) \in C^1((0, 1])$ and*

$$\frac{dT(R)}{dR}\left[r'(R) - \frac{r(R)}{R}\right] \leqslant 0 \quad \textit{for all} \quad R \in (0, 1], \tag{6.9}$$

while if the Baker–Ericksen inequalities hold then

$$\frac{dT(R)}{dR}\left[r'(R) - \frac{r(R)}{R}\right] < 0$$

whenever $R \in (0, 1]$ and $r'(R) \neq r(R)/R$.

We next consider the radial component of the *inverse Cauchy stress* (the terminology is explained later) defined by

$$\tilde{T}(R) \overset{\text{def}}{=} \Phi(R) - r'(R)\Phi_1(R), \tag{6.10}$$

where $\Phi(R) \overset{\text{def}}{=} \Phi(r'(R), r(R)/R, \ldots)$. If $r \in C^2((0, 1])$ then we obtain, using (6.1) and (6.8),

$$\frac{d\tilde{T}(R)}{dR} = -\left[\frac{r(R)}{R}\right]^n \frac{dT(R)}{dR}. \tag{6.11}$$

We record for future reference the related equation

$$\frac{d}{dR}\left\{R^n\left[\Phi - \left(r' - \frac{r}{R}\right)\Phi_1\right]\right\} = nR^{n-1}\Phi. \tag{6.12}$$

We remark that (6.12) is the specialization to radial solutions of the n-dimensional conservation law (cf. Green 1973)

$$\frac{\partial}{\partial X^\alpha}\left[X^\alpha W - \frac{\partial W}{\partial x^j_{,\alpha}}(X^\beta x^j_{,\beta} - x^j)\right] = nW. \tag{6.13}$$

From proposition 6.2 we immediately obtain the next result.

PROPOSITION 6.3. *Let $r \in C^2((0, 1])$ be a solution of* (6.1) *with $r'(R) > 0$ for all $R \in (0, 1]$. If the weakened Baker–Ericksen inequalities hold then $\tilde{T}(\cdot) \in C^1((0, 1])$ and*

$$\frac{d\tilde{T}(R)}{dR}\left[r'(R) - \frac{r(R)}{R}\right] \geqslant 0 \quad \textit{for all} \quad R \in (0, 1],$$

while if the Baker–Ericksen inequalities hold then

$$\frac{d\tilde{T}(R)}{dR}\left[r'(R) - \frac{r(R)}{R}\right] > 0,$$

whenever $R \in (0, 1]$ and $r'(R) \neq r(R)/R$.

The definition of $\tilde{T}(\cdot)$ is motivated by the following considerations (see Shield 1967, Ball 1977*b*, 1981*a*). For a homogeneous elastic body the total stored energy in a sufficiently regular invertible deformation $x(\cdot)$ is given by (cf. (3.1))

$$\begin{aligned} E(x) &= \int_\Omega W(\nabla x(X))\, dX \\ &= \int_{x(\Omega)} \hat{W}(\nabla X(x))\, dx, \end{aligned}$$

where $\hat{W}(F) \stackrel{\text{def}}{=} \det F\, W(F^{-1})$. If the material is isotropic we may correspondingly define

$$\tilde{\Phi}(v_1, \ldots, v_n) \stackrel{\text{def}}{=} v_1 \ldots v_n \Phi(v_1^{-1}, \ldots, v_n^{-1}), \tag{6.14}$$

so that $W(F) = \tilde{\Phi}(v_1(F), \ldots, v_n(F))$ for all $F \in M_+^{n\times n}$, where the $v_i(F)$ denote the singular values of F. It is thus natural to call

$$\tilde{T}(F) \stackrel{\text{def}}{=} (\det F)^{-1} \frac{\partial \hat{W}(F)}{\partial F} F^{\mathrm{T}}$$

the *inverse Cauchy stress*. When $F = \operatorname{diag}(v_1, \ldots, v_n)$, $v_i > 0$,

$$\begin{aligned} \tilde{T}(F) &= (1/v_1 \ldots v_n) \operatorname{diag}(v_1 \tilde{\Phi}_{,1}, \ldots, v_n \tilde{\Phi}_{,n}) \\ &= \operatorname{diag}(\Phi - v_1^{-1}\Phi_{,1}, \ldots, \Phi - v_n^{-1}\Phi_{,n}), \end{aligned}$$

where the arguments of Φ, $\Phi_{,i}$ are $v_1^{-1}, \ldots, v_n^{-1}$. Thus (6.10) corresponds to the case $v_1 = R'(r)$, $v_j = R(r)/r$ for $j > 1$.

It is frequently illuminating to ask whether a constitutive inequality is invariant under the involution $W \mapsto \hat{W}$. Examples of such invariant constitutive inequalities are rank 1 convexity (see Ball 1977*b*) and the Baker–Ericksen inequalities (this is easily verified).

6.3. *When are solutions without holes trivial?*

We consider the problem of finding a weak equilibrium solution $R \mapsto r(R)$ satisfying $r(1) = \lambda$, where $\lambda > 0$ is given, and such that $r(0) = 0$ (so that there is no hole). One possible solution, for any Φ, is $r(R) = \lambda R$; we call this the *trivial solution*. In this section we consider various hypotheses on Φ that guarantee that the trivial solution is the only equilibrium solution without a hole. Our first method is based on the following result.

PROPOSITION 6.4. *Let $\Phi \in C^3(\mathbb{R}^n_{++})$ and suppose that* (6.3) *holds. Let $r \in C^1((0, 1])$ be a solution of* (6.1) *with $r'(R) > 0$ for all $R \in (0, 1]$, and suppose that for some $R_0 \in (0, 1]$*

$$r(R_0)/R_0 = r'(R_0) \stackrel{\text{def}}{=} \lambda_0.$$

Then $r(R) = \lambda_0 R$ for all $R \in (0, 1]$.

Proof. Equation (6.4) has the form

$$r'' = f(R, r, r'),$$

where f is C^1 for positive arguments. Standard results on ordinary differential equations therefore imply that the solution of (6.4) with initial data $r(R_0) = \lambda_0 R_0$, $r'(R_0) = \lambda_0$ is unique. Hence $r(R) = \lambda_0 R$ for all $R \in (0, 1]$.

THEOREM 6.5. *Let $\Phi \in C^3(\mathbb{R}^n_{++})$, let (6.3) and the weakened Baker–Ericksen inequalities hold, and suppose that*

(i) *either*
$$\lim_{\substack{v_1, v_2 \to 0+ \\ v_1 > v_2}} \frac{\Phi_{,1}(v_1, v_2, \ldots, v_2)}{v_2^{n-1}} = -\infty \tag{6.15}$$

or
$$\lim_{\substack{v_1, v_2 \to 0+ \\ v_1 > v_2}} [\Phi(v_1, v_2, \ldots, v_2) - v_1 \Phi_{,1}(v_1, v_2, \ldots, v_2)] = \infty \tag{6.16}$$

and that

(ii) *either*
$$\lim_{\substack{v_1, v_2 \to \infty \\ v_1 < v_2}} \frac{\Phi_{,1}(v_1, v_2, \ldots, v_2)}{v_2^{n-1}} = \infty \tag{6.17}$$

or
$$\lim_{\substack{v_1, v_2 \to \infty \\ v_1 < v_2}} [\Phi(v_1, v_2, \ldots, v_2) - v_1 \Phi_{,1}(v_1, v_2, \ldots, v_2)] = -\infty. \tag{6.18}$$

Let $r \in C^1((0, 1])$ be a solution of (6.1) with $r'(R) > 0$ for all $R \in (0, 1]$ and suppose that $r(0) \stackrel{\text{def}}{=} \lim_{R \to 0+} r(R) = 0$. Then $r(R) = \lambda R$, $R \in [0, 1]$, for some $\lambda > 0$.

Proof. Suppose that $r(R)/R$ is not constant. By proposition 6.4, $r' - r/R$ is either strictly positive or strictly negative in $(0, 1]$. Hence r/R is either strictly increasing or strictly decreasing in $(0, 1]$, and thus $r(R)/R \to l \in [0, \infty]$ as $R \to 0+$. Since $r(0) = 0$, it follows easily from Rolle's theorem that $r'(R_j) \to l$ for some sequence $R_j \to 0+$.

Suppose $0 < l < \infty$. By proposition 6.2, $T(R)$ is monotone, and therefore

$$\lim_{R \to 0+} T(R) = \lim_{j \to \infty} T(R_j) = \frac{\Phi_{,1}(l, l, \ldots, l)}{l^{n-1}}. \tag{6.19}$$

We claim that $\lim_{R \to 0+} r'(R) = l$. If not, the continuity of $r'(\cdot)$ and the existence of the R_j would imply the existence of a further sequence $S_j \to 0+$ such that $r'(S_j) \to l_1 \neq l$, $l_1 > 0$. By (6.19),

$$\Phi_{,1}(l_1, l, \ldots, l) = \Phi_{,1}(l, l, \ldots, l),$$

which is impossible since $\Phi_{11} > 0$. Thus $r \in C^1([0, 1])$ and $r'(0) = l$. From (6.4) we now deduce that

$$\lim_{R \to 0+} \frac{R \Phi_{11} r''}{r' - r/R} = -(n-1)\, \Phi_{,11}(l, l, \ldots, l),$$

and hence r'' has the opposite sign to $r' - r/R$ for small $R > 0$, which is impossible. (The case $0 < l < \infty$ could also have been eliminated by studying the phase portrait of (6.6) near the line of rest points $\dot{v} = 0$, $v > 0$ arbitrary.)

Suppose $l = 0$. Then $r' > r/R$ for $R \in (0, 1]$, and so by proposition 6.2, $\mathrm{d}T(R)/\mathrm{d}R \leqslant 0$. But if (6.15) holds, $T(R_j) \to -\infty$, which is impossible. A similar contradiction is obtained if (6.16) holds, by using the fact that, by proposition 6.3, $\mathrm{d}\tilde{T}(R)/\mathrm{d}R \geqslant 0$.

The possibility $l = \infty$ is eliminated similarly.

Our second method is more elementary. It has the advantages of assuming less in the way of differentiability of Φ and nothing about the behaviour of Φ for small and large principal stretches, but the disadvantage of assuming that the constitutive inequality

$$(\Phi_{,i}-\Phi_{,j})/(v_i-v_j)+\Phi_{,ij} \geqslant 0 \quad \text{for all} \quad (v_1, \ldots, v_n) \in \mathbb{R}^n_{++} \tag{6.20}$$

holds whenever $v_i \neq v_j$, where $\Phi_{,i} = \Phi_{,i}(v_1, \ldots, v_n)$ etc. While it has no direct physical significance known to the author, nor apparently any obvious relation to other inequalities considered in the literature, (6.20) is satisfied for many stored-energy functions of interest. For example, it is satisfied if $n = 3$ and

$$\Phi(v_1, v_2, v_3) = \phi(v_1)+\phi(v_2)+\phi(v_3)+\psi(v_1v_2)+\psi(v_2v_3)+\psi(v_3v_1)+h(v_1v_2v_3)$$

where ϕ, ψ, h are C^2 and convex. It is also easily verified that (6.20) is invariant under the involution $\Phi \mapsto \tilde{\Phi}$ (cf. (6.14)).

THEOREM 6.6. *Let $\Phi \in C^2(\mathbb{R}^n_{++})$, and let* (6.3) *and* (6.20) *hold. If $r \in C^1((0, 1])$ is a solution of* (6.1) *with $r'(R) > 0$ for all $R \in (0, 1]$ and $r(0) = 0$, then $r(R) = \lambda R$, $R \in [0, 1]$, for some $\lambda > 0$.*

Proof. Let $\lambda = r(1)$, $\theta(R) \stackrel{\text{def}}{=} r(R)-\lambda R$. Then $\theta(0) = \theta(1) = 0$. Suppose $\theta(R_1) > 0$ for some $R_1 \in (0, 1)$. Let R_2 be the largest value of R at which θ attains its maximum. Then $r'(R_2) = \lambda < r(R_2)/R_2$, and $r''(R_2) \leqslant 0$. If $r''(R) \geqslant 0$ for $R \in [R_2, R_2+\epsilon)$, $\epsilon > 0$, then $\theta(R) \geqslant \theta(R_2)$ on this interval, contradicting the maximality of R_2. Hence there exists $R_3 \in (R_2, 1)$ such that $r'(R_3) < r(R_3)/R_3$ and $r''(R_3) < 0$. But by (6.3), (6.20) the left-hand side of (6.4) is negative at $R = R_3$, while the right-hand side is non-negative. A similar contradiction is obtained if θ takes negative values. Hence $\theta \equiv 0$ as required.

Before describing our third method we introduce some notation. Let $\boldsymbol{v} = (v_1, \ldots, v_n) \in \mathbb{R}^n_{++}$. For $1 \leqslant j \leqslant n$ write $\boldsymbol{v}^{(j)} \in \mathbb{R}^{\binom{n}{j}}$ for the vector having components $v_{i_1} v_{i_2} \ldots v_{i_j}$, where $1 \leqslant i_1 < i_2 < \ldots < i_j \leqslant n$, arranged in some definite order. We consider the following property of $\Phi: \mathbb{R}^n_{++} \to \mathbb{R}$.

(P) There exists a convex function $G: \mathbb{R}^{2^n-1}_{++} = \mathbb{R}^{\binom{n}{1}}_{++} \times \mathbb{R}^{\binom{n}{2}}_{++} \times \ldots \times \mathbb{R}^{\binom{n}{n}}_{++} \to \mathbb{R}$, symmetric with respect to permutations of the components of each of the $\mathbb{R}^{\binom{n}{j}}_{++}$, such that

$$\Phi(v_1, \ldots, v_n) = G(\boldsymbol{v}^{(1)}, \ldots, \boldsymbol{v}^{(n)}) \quad \text{for all} \quad \boldsymbol{v} \in \mathbb{R}^{n}_{++}. \tag{6.21}$$

For $n = 3$, (6.21) becomes

$$\Phi(v_1, v_2, v_3) = G(v_1, v_2, v_3; v_1v_2, v_2v_3, v_3v_1; v_1v_2v_3).$$

We say that Φ satisfies (P+) if (P) holds with G *strictly* convex.

For a radial deformation $R \mapsto r(R)$ the total stored energy is given by

$$E(r) \stackrel{\text{def}}{=} \omega_n \int_0^1 R^{n-1}\Phi\left(r'(R), \frac{r(R)}{R}, \ldots\right) \mathrm{d}R. \tag{6.22}$$

We shall consider only stored-energy functions Φ that are bounded below, and without loss of generality assume in fact that $\Phi \geqslant 0$.

PROPOSITION 6.7. *Let Φ satisfy property* (P), *and let $r \in C^1((0, 1])$ be a radial deformation with $r'(R) > 0$ for all $R \in (0, 1]$, $r(1) = \lambda > 0$, and $r(0) \stackrel{\text{def}}{=} \lim_{R\to 0+} r(R) = 0$. Then*

$$E(r) \geqslant E(\lambda R).$$

If Φ satisfies (P+) *then*

$$E(r) > E(\lambda R)$$

unless $r(R) \equiv \lambda R$.

Proof. Let $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^n_{++}$ be given. By the convexity of G,

$$G(\boldsymbol{v}^{(1)}, \ldots, \boldsymbol{v}^{(n)}) \geqslant G(\boldsymbol{u}^{(1)}, \ldots, \boldsymbol{u}^{(n)}) + \sum_{j=1}^{n} \langle A_n(\boldsymbol{u}^{(1)}, \ldots, \boldsymbol{u}^{(n)}), \boldsymbol{v}^{(j)} - \boldsymbol{u}^{(j)} \rangle \tag{6.23}$$

for certain $A_j(\boldsymbol{u}^{(1)}, \ldots, \boldsymbol{u}^{(n)}) \in \mathbb{R}^{\binom{n}{j}}$, with equality if and only if $\boldsymbol{u} = \boldsymbol{v}$ when (P+) holds. Now let $R \in (0, 1]$, $\boldsymbol{u} = (\lambda, \lambda, \ldots, \lambda)$ and $\boldsymbol{v} = (r'(R), r(R)/R, \ldots, r(R)/R)$. By (6.23) and the symmetry of G there exist real numbers $B_j(\lambda)$ such that

$$G(\boldsymbol{v}^{(1)}, \ldots, \boldsymbol{v}^{(n)}) \geqslant G(\boldsymbol{u}^{(1)}, \ldots, \boldsymbol{u}^{(n)}) + \sum_{j=1}^{n} B_j(\lambda) \sum_{l=1}^{\binom{n}{j}} (v_l^{(j)} - u_l^{(j)}), \tag{6.24}$$

where $v_l^{(j)}$, $u_l^{(j)}$ denote the components of $\boldsymbol{v}^{(j)}$, $\boldsymbol{u}^{(j)}$ respectively.

But

$$R^{n-1} \sum_{l=1}^{\binom{n}{j}} (v_l^{(j)} - u_l^{(j)}) = R^{n-1} \left[\binom{n-1}{j-1} r' \left(\frac{r}{R}\right)^{j-1} + \binom{n-1}{j} \left(\frac{r}{R}\right)^{j} - \binom{n}{j} \lambda^j \right]$$

$$= \frac{1}{n} \binom{n}{j} (r^j R^{n-j} - \lambda^j R^n)'.$$

Multiplying (6.24) by R^{n-1} and integrating, we obtain that $E(r) \geqslant E(\lambda R)$ as required, with strict inequality if (P+) holds and $r(R) \not\equiv \lambda R$.

If W is *polyconvex* (i.e. there exists a convex function $\hat{G}$ such that

$$W(F) = \hat{G}(J(F)) \quad \text{for all} \quad F \in M^{n \times n}_{+},$$

where $J(F)$ is the vector whose components are all the minors of F of all orders r, $1 \leqslant r \leqslant n$ (see Ball 1977*a*, *b*)) then clearly Φ satisfies (P). Similarly, strict polyconvexity (i.e. $\hat{G}$ strictly convex) implies (P+). If W is polyconvex then W is *quasiconvex* (see Morrey 1952, Ball 1977*a*, *b*, Ball *et al.* 1981), i.e.

$$\int_D W(F + \nabla \zeta(X)) \, \mathrm{d}X \geqslant \int_D W(F) \, \mathrm{d}X = \operatorname{meas}(D) \, W(F), \tag{6.25}$$

for all $F \in M^{n \times n}_{+}$, all bounded domains $D \subset \mathbb{R}^n$, and all $\zeta \in W^{1,\infty}_0(D; \mathbb{R}^n)$ with $F + \nabla \zeta(X) \in M^{n \times n}_{+}$ a.e. $X \in D$. When $F = \lambda 1$, the conclusion of proposition 6.7 is (6.25) specialized to radial deformations, with the exception that the proposition assumes less regularity on ζ. If W is (strictly) polyconvex then W is (strictly) rank 1 convex; the reader should bear this and the results of §3 in mind when comparing the hypotheses of theorem 6.5 and the next theorem.

THEOREM 6.8. *Let $\Phi \in C^1(\mathbb{R}^n_{++})$ satisfy* (P) *and the tension–extension property. Suppose that*

$$\liminf_{\substack{v_1, v_2 \to 0+ \\ v_1 > v_2}} [\Phi(v_1, v_2, \ldots, v_2) - (v_1 - v_2) \Phi_{,1}(v_1, v_2, \ldots, v_2)] \geqslant 0 \tag{6.26}$$

and that

$$\liminf_{\substack{v_1, v_2 \to \infty \\ v_1 < v_2}} \left[\frac{\Phi(v_1, v_2, \ldots, v_2) - (v_1 - v_2) \Phi_{,1}(v_1, v_2, \ldots, v_2)}{v_2^n} \right] \geqslant 0. \tag{6.27}$$

Let $r \in C^1((0, 1])$ be a solution of (6.1) *with $r'(R) > 0$ for all $R \in (0, 1]$ and $r(0) = 0$. Then $r(R) = \lambda R$, $R \in [0, 1]$, for some $\lambda > 0$.*

Remarks. (1) It is easily shown that (6.26), (6.27) are implied by hypotheses (i), (ii) of theorem 6.5 respectively. (2) If G is C^1 then property (P) implies that each $\Phi_{,i}$ is non-decreasing, and (P$^+$) implies the tension–extension property.

LEMMA 6.9. *Let $\Phi \in C^1(\mathbb{R}^n_{++})$ with each $\Phi_{,i}$ non-decreasing. Let $r \in C^1((0,1])$ satisfy* (6.1) *with $r'(R) > 0$ for all $R \in (0,1]$. Then* (6.12) *holds for all $R \in (0,1]$.*

Proof. Let

$$\theta(R) \overset{\text{def}}{=} \Phi(R) - r'(R)\,\Phi_1(R) - (n-1)\int_1^R \frac{r'(\rho)\,\Phi_1(\rho) - [r(\rho)/\rho]\,\Phi_2(\rho)}{\rho}\,\mathrm{d}\rho.$$

It suffices to show that $\theta'(R) = 0$ for all $R \in (0,1]$. We note that

$$\begin{aligned}\theta(R+h)-\theta(R) = &\left[\Phi\left(r'(R+h), \frac{r(R)}{R}, \ldots\right) - \Phi\left(r'(R), \frac{r(R)}{R}, \ldots\right)\right.\\ &\left.-[r'(R+h)-r'(R)]\,\Phi_{,1}\left(r'(R), \frac{r(R)}{R}, \ldots\right)\right]\\ &+\Phi\left(r'(R+h), \frac{r(R+h)}{R+h}, \ldots\right) - \Phi\left(r'(R+h), \frac{r(R)}{R}, \ldots\right)\\ &-r'(R+h)\left[\Phi_{,1}\left(r'(R+h), \frac{r(R+h)}{R+h}, \ldots\right) - \Phi_{,1}\left(r'(R), \frac{r(R)}{R}, \ldots\right)\right]\\ &-(n-1)\int_R^{R+h} \frac{r'(\rho)\,\Phi_1(\rho) - [r(\rho)/\rho]\,\Phi_2(\rho)}{\rho}\,\mathrm{d}\rho.\end{aligned}$$

The first bracket being non-negative, we obtain from (6.1) that

$$\liminf_{|h|\to 0+} \frac{\theta(R+h)-\theta(R)}{|h|} \geqslant 0.$$

Swapping $R+h$ and R, and repeating the same argument, we obtain

$$\limsup_{|h|\to 0+} \frac{\theta(R+h)-\theta(R)}{|h|} \leqslant 0,$$

which proves the lemma.

Proof of theorem 6.8. The function

$$h(R) \overset{\text{def}}{=} R^n[\Phi - (r' - r/R)\,\Phi_1]$$

is non-decreasing by (6.12) and our assumption that $\Phi \geqslant 0$. Hence $h(R) \to h_0 \in [-\infty, \infty)$ as $R \to 0+$. We show that $h_0 \geqslant 0$. First suppose that $r(R_k)/R_k \to \infty$ for some sequence $R_k \to 0+$. Define

$$\theta_k(R) = r(R) - Rr(R_k)/R_k. \tag{6.28}$$

Then $\theta_k(0) = \theta_k(R_k) = 0$, and so there exists $S_k \in (0, R_k)$ which maximizes θ_k in $[0, R_k]$. Clearly $r'(S_k) = r(R_k)/R_k \leqslant r(S_k)/S_k$. Applying (6.27) with $v_1 = r'(S_k)$, $v_2 = r(S_k)/S_k$, we see that $\liminf_{k\to\infty} h(S_k)/[r(S_k)]^n \geqslant 0$, and hence $h_0 \geqslant 0$.

Next suppose that $l = \liminf_{R\to 0+} r(R)/R \neq \infty$, and that $r(R_k)/R_k \to l$ for some sequence $R_k \to 0+$. With θ_k as in (6.28), there exists $S_k \in (0, R_k)$ which minimizes θ_k in $[0, R_k]$. Then

$$r'(S_k) = \frac{r(R_k)}{R_k} \geqslant \frac{r(S_k)}{S_k} \to l.$$

If $l = 0$, then by (6.26) $\liminf_{k\to\infty} h(S_k)/S_k^n \geqslant 0$, and hence $h_0 \geqslant 0$. If $l > 0$, then clearly $h(S_k) \to 0 = h_0$.

Now suppose that $R_1 \in (0,1]$ with $r'(R_1) \neq r(R_1)/R_1$. Integrating (6.12), which holds by lemma 6.9, we obtain

$$n\int_0^{R_1} R^{n-1}\Phi(R)\,\mathrm{d}R \leqslant R_1^n\left\{\Phi(R_1) - \left[r'(R_1) - \frac{r(R_1)}{R_1}\right]\Phi_1(R_1)\right\}.$$

By the tension–extension property

$$\Phi\left(r'(R_1), \frac{r(R_1)}{R_1}, \ldots\right) - \left[r'(R_1) - \frac{r(R_1)}{R_1}\right] \Phi_{,1}\left(r'(R_1), \frac{r(R_1)}{R_1}, \ldots\right) < \Phi\left(\frac{r(R_1)}{R_1}, \frac{r(R_1)}{R_1}, \ldots\right),$$

so that
$$\int_0^{R_1} R^{n-1}\Phi(R)\,\mathrm{d}R < \int_0^{R_1} R^{n-1}\Phi\left(\frac{r(R_1)}{R_1}, \frac{r(R_1)}{R_1}, \ldots\right)\mathrm{d}R.$$

Making the change of variables $(r, R) \mapsto (r/R_1, R/R_1)$ we obtain a contradiction to proposition 6.7.

Thus $r'(R) = r(R)/R$ for all $R \in (0, 1]$, giving the result.

We mention the following *open problem*. Consider the displacement boundary value problem for a homogeneous isotropic material under zero body force consisting of the equilibrium equations

$$\frac{\partial}{\partial X^\alpha}\left(\frac{\partial W}{\partial x^i{}_{,\alpha}}\right) = 0, \quad X \in B,$$

and boundary condition

$$x(X) = \lambda X, \quad X \in S^{n-1}.$$

Under what conditions on Φ is the only sufficiently regular (not necessarily radial) solution the trivial one $x(X) \equiv \lambda X$?

We now give an example showing that strong ellipticity is not a sufficient condition for every weak solution with $r(0) = 0$ to be trivial, and that the growth hypotheses in theorem 6.8 are essential. For simplicity we let $n = 2$, though analogous examples exist for all $n > 1$. If $n = 2$, the stored-energy function of an isotropic material can be written in the form

$$W(F) = \Phi(v_1, v_2) = g(\eta, \delta), \tag{6.29}$$

where $\eta = v_1 + v_2$, $\delta = v_1 v_2$.

Lemma 6.10. *Let $g \in C^2(\mathbb{R}^n_{++})$ satisfy $g_\eta > 0$ and*

$$\begin{bmatrix} g_{\eta\eta} & g_{\eta\delta} \\ g_{\eta\delta} & g_{\delta\delta} \end{bmatrix} > 0.$$

Then W given by (6.29) *is strongly elliptic.*

Proof. This follows from a computation with the use, for example, of Knowles & Sternberg (1978). For a related result, showing that the hypotheses imply that W is polyconvex (see earlier), the reader is referred to Ball (1977*a*, theorem 5.2).

Corollary 6.11. *Let $g(\eta, \delta) = \eta^a/\delta^b$ with $b > 0$, $a > b+1$. Then W is strongly elliptic.*

With g as in the corollary we seek a solution of (6.1) of the form

$$r(R) = R^\alpha, \quad \alpha > 0. \tag{6.30}$$

It is easily verified that this is a solution of (6.1) if either $\alpha = 1$ or

$$a \notin [2b, 2(b+1)], \quad \alpha = b[a-2(b+1)]/[(a-2b)(a-b-1)]. \tag{6.31}$$

If a, b, α satisfy (6.31) then (6.30) may or may not be a weak equilibrium solution, since for this we need also that $R\Phi_1(R), R\Phi_2(R) \in L^1(0, 1)$ (cf. theorem 4.2). Examples of values of a, b, α giving rise to weak equilibrium solutions are $a = 5$, $b = 1$, and $\alpha = \frac{1}{6}$. It is easily verified that the cases when $b+1 < a < 2b$ do not give rise to weak solutions. The total stored energy corresponding to (6.30), (6.31) is proportional to $\int_0^1 R^{1+(\alpha-1)(a-2b)}\mathrm{d}R$. Since

$$2 + (\alpha-1)(a-2b) = -(a-2b-2)(a-2b-1)/(a-b-1)$$

this energy is always *infinite*. (We have not been able to resolve the question of whether strong ellipticity implies that all solutions of (6.1) with $r(0) = 0$ and having finite energy are trivial.) The existence of these non-trivial weak solutions is not attributable to the singularity in Φ as $v_1, v_2 \to 0+$, since for them $r', r/R \to \infty$ as $R \to 0+$.

6.4. *Trivial solutions for traction boundary conditions*

We seek solutions $r(R)$ to the radial equilibrium equation (6.1) satisfying $r(0) = 0$ and a traction boundary condition at $R = 1$. Since we have shown in the last subsection that under quite general conditions related to strong ellipticity all such solutions are trivial, we consider only trivial deformations

$$r(R) = \lambda R, \quad R \in [0, 1]. \tag{6.32}$$

In the *dead-load traction problem* we thus require that

$$\Phi_1(1) = \Phi_{,1}(\lambda, \lambda, \ldots, \lambda) = P, \tag{6.33}$$

where P is the prescribed outward radial dead load on ∂B. In the *Cauchy traction problem* the boundary condition is

$$T(1) = \Phi_{,1}(\lambda, \lambda, \ldots, \lambda)/\lambda^{n-1} = P, \tag{6.34}$$

where P is the prescribed outward radial Cauchy stress on ∂B.

Even if Φ is strongly elliptic and strictly polyconvex, neither of the functions $\Phi_{,1}(\lambda, \lambda, \ldots, \lambda)$, $\Phi_{,1}(\lambda, \lambda, \ldots, \lambda)/\lambda^{n-1}$ need be increasing functions of λ. (This is easily seen by consideration of the example $\Phi(v_1, \ldots, v_n) = \mu\sum_{i=1}^n v_i^\alpha + h(v_1 \ldots v_n)$, where $\mu > 0$, $\alpha > 1$ and $h'' > 0$.) Hence for a given P there may exist several roots λ of (6.33), (6.34), and thus several trivial solutions. (It is even possible for (6.33) or (6.34) to be satisfied for an interval of values of λ.) This is true in particular when $P = 0$, when the corresponding deformations $x = \lambda X$ are *natural states* of the material.

The condition that $\Phi_{,1}(\lambda, \lambda, \ldots, \lambda)/\lambda^{n-1}$ be a strictly increasing function is known as the *pressure–compression inequality* (see Truesdell & Noll 1965), but we do not assume that it holds, or that $\Phi_{,1}(\lambda, \lambda, \ldots, \lambda)$ is increasing. This does not mean that we countenance the possibility that in an actual experiment an increase (resp. decrease) in P might produce a decrease (resp. increase) in volume, since the trivial solutions corresponding to λ on decreasing parts of the curves $\Phi_{,1}(\lambda, \lambda, \ldots, \lambda)$, $\Phi_{,1}(\lambda, \lambda, \ldots, \lambda)/\lambda^{n-1}$ are unstable for the dead-load and Cauchy traction problems respectively, in the sense that they are not local minima for the corresponding energy functionals. (These functionals are given for trivial deformations (6.32) by

$$J_1(\lambda) \stackrel{\text{def}}{=} (\omega_n/n)\,[\Phi(\lambda, \lambda, \ldots, \lambda) - nP\lambda] \tag{6.35}$$

for the dead-load traction problem, and by

$$J_2(\lambda) \stackrel{\text{def}}{=} (\omega_n/n)\,[\Phi(\lambda, \lambda, \ldots, \lambda) - P\lambda^n] \tag{6.36}$$

for the Cauchy traction problem.) Thus 'snap-through' to a value of λ on an increasing part of the curve is to be expected† together with related hysteretic effects similar to those studied by Ericksen (1975) for one-dimensional elasticity with a non-monotone stress–strain law; this presupposes, of course, that radial symmetry is preserved (see the remarks in §8) and that cavitation does not occur.

† Behaviour of this type was observed by Bridgman (1948) in his experiments on material response at high pressure (see also Bell 1973, p. 495).

7. Cavitation in compressible elasticity

7.1. *A special class of stored-energy functions*

Our aim is to study the existence of weak equilibrium solutions with cavities in isotropic compressible elasticity. To this end we consider the class of stored-energy functions $W(F) = \Phi(v_1, \dots, v_n)$ having the form $(n > 1)$

$$\Phi(v_1, \dots, v_n) = \sum_{i=1}^{n} \phi(v_i) + h(v_1 v_2 \dots v_n). \tag{7.1}$$

In our analysis we shall make use of some or all of the following assumptions on the functions ϕ and h appearing in (7.1):

(A 1) $h: (0, \infty) \to \mathbb{R}$ is C^1 and strictly convex;

(A 2) $\lim_{\delta \to 0+} h(\delta) = \lim_{\delta \to \infty} \dfrac{h(\delta)}{\delta} = \infty$;

(A 3) $\phi: (0, \infty) \to [0, \infty)$ is C^1 and convex;

(A 4) $\lim_{v \to \infty} \phi(v) = \infty$;

(A 5) there exists $\delta_0 > 0$ such that if $|\sigma - 1| < \delta_0$ then

$$|\phi'(\sigma v)| \leqslant K\phi(v)/v$$

for all $v > 0$ and some constant $K > 0$;

(A 6) $\qquad \phi(v) \geqslant K_1 v^{\gamma} \quad$ for all $\quad v > 0$,

where $\quad K_1 > 0 \quad$ and $\quad \gamma > 1$;

(A 7) $\qquad \phi(v) \leqslant K_2(1 + v^{\alpha} + v^{-\beta}) \quad$ for all $\quad v > 0$,

where $\quad K_2 > 0,\ 0 < \alpha < n,\ 0 \leqslant \beta < 1 + 1/(n-1)$.

Example 7.1 (see Ogden 1972*a*, *b*). Let

$$\Phi(v_1, \dots, v_n) = \sum_{j=1}^{M} \mu_j \left(\sum_{i=1}^{n} v_i^{\alpha_j} - n \right) + \sum_{j=1}^{N} \nu_j \left(\sum_{i=1}^{n} v_i^{-\beta_j} - n \right) + h(v_1 v_2 \dots v_n),$$

where h satisfies (A 1) and (A 2), $M \geqslant 1$, $N \geqslant 1$, $\mu_j > 0$ for $1 \leqslant j \leqslant M$, $\nu_j > 0$ for $1 \leqslant j \leqslant N$, $\alpha_1 \geqslant \alpha_2 \geqslant \dots \geqslant \alpha_M \geqslant 1$, $\beta_1 \geqslant \beta_2 \geqslant \dots \geqslant \beta_N \geqslant 0$. By setting

$$\phi(v) = \sum_{j=1}^{M} \mu_j v^{\alpha_j} + \sum_{j=1}^{N} \nu_j v^{-\beta_j}$$

and altering h by a suitable constant, it is easily verified that Φ has the form (7.1) and that (A 1)–(A 5) hold. Assumption (A 6) holds provided $\alpha_1 > 1$ and (A 7) holds provided $\alpha_1 < n$ and $\beta_1 < 1 + 1/(n-1)$.

Note that (A 1), (A 3) imply that the tension–extension property and property (P) hold and, when ϕ, h are C^2, that (6.20) holds. Assumptions (A 1)–(A 3) imply (6.2). If $v\phi'(v)$ is a strictly increasing function of v then the Baker–Ericksen inequalities hold. Examples of strongly elliptic stored-energy functions having the form (7.1) can be obtained by setting $\beta_1 = \dots = \beta_N = 0$, $\alpha_1 > 1$, in example 7.1 (cf. (5.8)). Of course, (A 6) and (A 7) are consistent only if $\alpha \geqslant \gamma$. Also, (A 6) implies (A 4).

7.2. *Existence of energy-minimizing radial deformations*

We consider first the radial displacement boundary value problem. We seek to minimize the total stored energy

$$E(r) \stackrel{\text{def}}{=} \omega_n \int_0^1 R^{n-1}\Phi\left(r'(R), \frac{r(R)}{R}, \ldots\right) \mathrm{d}R \tag{7.2}$$

(cf. (6.22)) among radial deformations $R \mapsto r(R)$ such that $r(0) \geqslant 0$, $r(R)$ is increasing, and $r(1) = \lambda$, where $\lambda > 0$ is a given constant.

THEOREM 7.1. *Suppose that* (A 1)–(A 4) *hold, and let*

$$\mathscr{A}_\lambda \stackrel{\text{def}}{=} \{r \in W^{1,1}(0,1) : r(0) \geqslant 0,\ r'(R) > 0 \text{ a.e.}, r(1) = \lambda \text{ and } E(r) < \infty\}.$$

Then E attains an absolute minimum on $\mathscr{A}_\lambda$.

Proof. We make the change of variables $u = r^n$, $\rho = R^n$ and use the notation $\dot{u} = \mathrm{d}u/\mathrm{d}\rho$. Note that $\dot{u} = r^{n-1}r'/R^{n-1} = \det \nabla x(X)$. Clearly, minimizing E on $\mathscr{A}_\lambda$ is equivalent to minimizing the functional

$$J(u) \stackrel{\text{def}}{=} \frac{\omega_n}{n} \int_0^1 f(\rho, u, \dot{u})\, \mathrm{d}\rho \tag{7.3}$$

on the set

$$\mathscr{B} \stackrel{\text{def}}{=} \{u \in W^{1,1}(0,1) : u(0) \geqslant 0,\ u(1) = \lambda,\ \dot{u}(\rho) > 0 \text{ a.e. and } J(u) < \infty\},$$

where
$$f(\rho, u, p) \stackrel{\text{def}}{=} \Phi((\rho/u)^{1-1/n}p, (u/\rho)^{1/n}, \ldots). \tag{7.4}$$

By (7.1)
$$f(\rho, u, p) = \phi((\rho/u)^{1-1/n}p) + (n-1)\,\phi((u/\rho)^{1/n}) + h(p). \tag{7.5}$$

We extend the definition of f to $(0,1) \times \mathbb{R} \times \mathbb{R}$ by setting

$$f(\rho, u, p) = +\infty \quad \text{if} \quad u \leqslant 0 \quad \text{or} \quad p \leqslant 0.$$

Then for each $\rho \in (0,1)$, $f(\rho, \cdot, \cdot)$ is a continuous function from $\mathbb{R} \times \mathbb{R}$ to $\overline{\mathbb{R}}$; this follows from (7.5) and (A 1)–(A 4). Our minimization problem is now equivalent to minimizing J on the set

$$\mathscr{C} \stackrel{\text{def}}{=} \{u \in W^{1,1}(0,1) : u(1) = \lambda,\ J(u) < \infty\}.$$

Let $u^{(r)}$ be a minimizing sequence for J in $\mathscr{C}$. By (7.5),

$$\int_0^1 h(|\dot{u}^{(r)}(\rho)|)\, \mathrm{d}\rho \leqslant \text{const.},$$

so that by (A 2) and the de la Vallée Poussin criterion there exists a subsequence $u^{(\mu)}$ of $u^{(r)}$ such that $u^{(\mu)} \rightharpoonup u_\infty$ in $W^{1,1}(0,1)$. Since $f(\rho, u, \cdot)$ is convex we may apply a standard type of lower semi-continuity theorem (see, for example, Ball *et al.* 1981, theorem 5.4 or the references therein) to conclude that

$$J(u_\infty) \leqslant \liminf_{\mu \to \infty} J(u^{(\mu)}).$$

Since $u^{(\mu)}(1) = \lambda \to u_\infty(1)$, $u_\infty \in \mathscr{C}$ and is the desired minimizer.

We now consider the radial dead-load traction problem in which the boundary condition is (cf. (5.11), (5.13))

$$T_R(X)X = PX, \quad X \in S^{n-1}, \tag{7.6}$$

and P is a given constant.

Corresponding to (7.6) we seek to minimize

$$E_P(r) \stackrel{\text{def}}{=} \omega_n \left[\int_0^1 R^{n-1}\Phi\left(r'(R), \frac{r(R)}{R}, \ldots\right) \mathrm{d}R - Pr(1)\right] \tag{7.7}$$

among radial deformations $R \mapsto r(R)$ such that $r(0) \geqslant 0$ and $r(R)$ is increasing.

THEOREM 7.2. *Suppose that* (A 1)–(A 3), (A 6) *hold, and let*

$$\mathscr{A}_P \stackrel{\text{def}}{=} \{r \in W^{1,1}(0,1) : r(0) \geqslant 0, r'(R) > 0 \textit{ a.e., and } E_P(r) < \infty\}.$$

Then E_P attains an absolute minimum on $\mathscr{A}_P$.

Proof. Making the same change of variables as in the proof of theorem 7.1, we note that for any admissible u we have

$$\begin{aligned}
J_1(u) &\stackrel{\text{def}}{=} J(u) - \omega_n P u(1)^{1/n} \\
&\geqslant \omega_n \left\{\int_0^1 [(n-1)K_1(u/\rho)^{\gamma/n} + h(\dot u)]\,\mathrm{d}\rho - Pu(1)^{1/n}\right\} \\
&\geqslant \omega_n \left\{\int_0^1 \{(n-1)K_1[u(0)/\rho]^{\gamma/n} + \tfrac{1}{2}h(\dot u) + \dot u\}\,\mathrm{d}\rho - Pu(1)^{1/n} + C\right\} \\
&= \omega_n \left\{\frac{1}{2}\int_0^1 h(\dot u)\,\mathrm{d}\rho + C_1 u(0)^{\gamma/n} + [u(1) - Pu(1)^{1/n}] - [u(0) - Pu(0)^{1/n}] - Pu(0)^{1/n} + C\right\} \\
&\geqslant \omega_n \left\{\frac{1}{2}\int_0^1 h(\dot u)\,\mathrm{d}\rho + \tfrac{1}{2}C_1 u(0)^{\gamma/n} + C_2,\right\}
\end{aligned}$$

where $C_1 > 0$, and where we have used (A 2) and (A 6). Hence from any minimizing sequence $u^{(r)}$ for J_1 we can extract a subsequence $u^{(\mu)}$ such that $u^{(\mu)} \rightharpoonup u$ in $W^{1,1}(0,1)$. The remainder of the proof is the same as for theorem 7.1.

In the Cauchy traction problem the boundary condition is (cf. (5.29))

$$T(X) = PX, \quad X \in S^{n-1}, \tag{7.8}$$

where P is a given constant.

Corresponding to (7.8) we may seek to minimize

$$\hat{E}_P(r) \stackrel{\text{def}}{=} \omega_n \left[\int_0^1 R^{n-1}\Phi\left(r'(R), \frac{r(R)}{R}, \ldots\right) \mathrm{d}R - \frac{P}{n}\, r(1)^n\right] \tag{7.9}$$

among radial deformations $R \mapsto r(R)$ such that $r(0) \geqslant 0$ and $r(R)$ is increasing. However, if ϕ satisfies (A 7) and $P > 0$ then $\hat{E}_P$ is not bounded below, as may be seen by setting $u = a + \rho$, $u = r^n$, $\rho = R^n$ in (7.9) and letting $a \to \infty$. If ϕ satisfies (A 6) with $\gamma > n$, $\hat{E}_P$ is bounded below (and attains its minimum) as can be proved similarly to theorem 7.2; however, this hypothesis implies that the deformation x corresponding to any r with $\hat{E}_P(r) < \infty$ belongs to $W^{1,\gamma}(B; \mathbb{R}^n)$ and is thus continuous, so that $r(0) = 0$. Thus minimizing $\hat{E}_P$ is not generally of interest for the static study of cavitation in the Cauchy traction problem.

7.3. *The Euler–Lagrange equation*

In this subsection we show in particular that the minimizers whose existence are guaranteed by theorems 7.1, 7.2 satisfy the radial equilibrium equation (cf. (6.1))

$$\mathrm{d}[R^{n-1}\Phi_1(R)]/\mathrm{d}R = (n-1)R^{n-2}\Phi_2(R) \tag{7.10}$$

and the relevant boundary conditions. This is not completely straightforward owing to the constraint $r'(R) > 0$ a.e.; this difficulty was first encountered and analysed in one-dimensional elasticity problems by Antman (1970, 1976) (see also Antman & Brezis 1978). We begin with the displacement boundary value problem.

THEOREM 7.3. *Suppose that* (A 1)–(A 5) *hold, and let r be any minimizer of E on* $\mathscr{A}_\lambda$. *Then* $r \in C^1((0,1])$, $r'(R) > 0$ *for all* $R \in (0,1]$, $R^{n-1}\Phi_1(R) \in C^1((0,1])$ *and* (7.10) *holds for all* $R \in (0,1]$. *If* $r(0) > 0$ *then* $R^{n-2}\Phi_2(R) \in L^1(0,1)$ *and*

$$\lim_{R\to 0+} T(R) = 0,$$

where $T(R)$ *is the radial component of the Cauchy stress* (*cf.* (6.7)).

Proof. We first show that r satisfies (7.10) for $R > 0$ using a technique in Ball (1981*b*); this part of the argument does not make direct use of the convexity of Φ with respect to r'. For $k = 1, 2, \ldots$ let $S_k \overset{\text{def}}{=} \{\rho \in (1/k, 1) : 1/k \leqslant \dot{u}(\rho) \leqslant k\}$, where u is the corresponding minimizer of J on $\mathscr{B}$ (cf. (7.3)). Let χ_k denote the characteristic function of S_k, and let $v \in L^\infty(0,1)$ be such that $\int_{S_k} v\,\mathrm{d}\rho = 0$. For ϵ sufficiently small define

$$u_\epsilon(\rho) = u(\rho) + \epsilon\int_0^\rho \chi_k(\tau)\,v(\tau)\,\mathrm{d}\tau. \tag{7.11}$$

Note that $\dot{u}_\epsilon(\rho) = \dot{u}(\rho)$ if $\rho \leqslant 1/k$ or $\dot{u}(\rho) \notin [1/k, k]$, and that $u_\epsilon(0) = u(0)$. Thus for any $\rho \in (0,1)$ we have by the mean value theorem that

$$\begin{aligned}\left|\frac{f(\rho, u_\epsilon(\rho), \dot{u}_\epsilon(\rho)) - f(\rho, u(\rho), \dot{u}(\rho))}{\epsilon}\right| &\leqslant \left|\frac{\phi((\rho/u_\epsilon)^{1-1/n}\dot{u}_\epsilon) - \phi((\rho/u)^{1-1/n}\dot{u}_\epsilon)}{\epsilon}\right| \\ &\quad + \left|\frac{\phi((\rho/u)^{1-1/n}\dot{u}_\epsilon) - \phi((\rho/u)^{1-1/n}\dot{u})}{\epsilon}\right| \\ &\quad + (n-1)\left|\frac{\phi((u_\epsilon/\rho)^{1/n}) - \phi((u/\rho)^{1/n})}{\epsilon}\right| + \left|\frac{h(\dot{u}_\epsilon) - h(u)}{\epsilon}\right| \\ &\leqslant \left|\frac{\phi((\rho/u_\epsilon)^{1-1/n}\dot{u}_\epsilon) - \phi((\rho/u)^{1-1/n}\dot{u}_\epsilon)}{\epsilon}\right| + C \\ &= \left|\frac{\phi((u/u_\epsilon)^{1-1/n}(\rho/u)^{1-1/n}\dot{u}_\epsilon) - \phi((\rho/u)^{1-1/n}\dot{u}_\epsilon)}{\epsilon}\right| + C.\end{aligned} \tag{7.12}$$

At points ρ where $\rho \leqslant 1/k$ or $\dot{u}(\rho) \in [1/k, k]$ the right-hand side of (7.12) is bounded by a constant independent of ϵ. At points $\rho > 1/k$ where $\dot{u}(\rho) \notin [1/k, k]$ we have $\dot{u}_\epsilon(\rho) = \dot{u}(\rho)$ and the right-hand side of (7.12) equals

$$\left|\phi'(g_\epsilon(\rho)\,(\rho/u)^{1-1/n}\dot{u})\left[\frac{(u/u_\epsilon)^{1-1/n} - 1}{\epsilon}\right](\rho/u)^{1-1/n}\dot{u}\right| + C,$$

where for $|\epsilon|$ sufficiently small $|g_\epsilon(\rho) - 1| < \delta_0$. By (A 5) this expression is bounded by

$$C[\phi((\rho/u)^{1-1/n}\dot{u}) + 1],$$

which is an integrable function of ρ since $J(u) < \infty$. From (7.12) we deduce that, for $|\epsilon|$ sufficiently small, $u_\epsilon \in \mathscr{B}$ (defined after (7.3)) and that by the dominated convergence theorem $J(u_\epsilon)$ is differentiable with respect to ϵ. Since $J(u_\epsilon)$ is minimized at $\epsilon = 0$ we deduce that

$$\frac{\mathrm{d}}{\mathrm{d}\epsilon}J(u_\epsilon)\bigg|_{\epsilon=0} = \left[\int_{1/k}^1 f_u\int_0^\rho \chi_k(\tau)\,v(\tau)\,\mathrm{d}\tau + f_p\,\chi_k(\rho)\,v(\rho)\right]\mathrm{d}\rho = 0.$$

Using (A 5) as above we see that $f_u \in L^1(1/k, 1)$, and hence integrating by parts we obtain

$$\int_{S_k} \left(f_p - \int_1^\rho f_u\right) v(\rho)\,\mathrm{d}\rho = 0. \tag{7.13}$$

Since (7.13) holds for all $v \in L^\infty(0, 1)$ with $\int_{S_k} v\,\mathrm{d}\rho = 0$, and since f_p is bounded in S_k, we deduce that

$$f_p - \int_1^\rho f_u = c_k \quad \text{a.e. } \rho \in S_k,$$

where c_k is a constant. Since meas $((0, 1)\backslash \bigcup_{k=1}^\infty S_k) = 0$ it follows that all the c_k are equal and

$$f_p - \int_1^\rho f_u = \text{const.} \quad \text{a.e. } \rho \in (0, 1). \tag{7.14}$$

In particular $$f_p(\rho, u, \dot{u}) = \phi'((\rho/u)^{1-1/n}\dot{u})\,(\rho/u)^{1-1/n} + h'(\dot{u})$$

is uniformly bounded in every interval $(\epsilon, 1)$ where $\epsilon > 0$. By (A 1)–(A 4) this implies that for each $\epsilon > 0$ there is a number $m(\epsilon) > 0$ such that

$$m(\epsilon)^{-1} \leqslant \dot{u}(\rho) \leqslant m(\epsilon) \quad \text{for all} \quad \rho \in (\epsilon, 1).$$

Hence $r \in W^{1,\infty}(\epsilon, 1)$ and the usual argument applied directly to $E_\lambda(r)$ shows that

$$R^{n-1}\Phi_1(R) = (n-1)\int_1^R \rho^{n-2}\Phi_2(\rho)\,\mathrm{d}\rho + \text{const.}, \quad \text{a.e. } R \in (0, 1). \tag{7.15}$$

By proposition 6.1, $r \in C^1((0, 1])$, $r'(R) > 0$ for all $R \in (0, 1]$, $R^{n-1}\Phi_1(R) \in C^1((0, 1])$ and (7.10) holds for all $R \in (0, 1]$.

Suppose $r(0) > 0$. Then $u(0) > 0$. Thus (A 5) implies that $f_u \in L^1(0, 1)$.

Let $w \in C^\infty(0, 1)$ satisfy $w(\rho) = 1$ for $\rho \in (0, \frac{1}{3})$, $w(\rho) = 0$ for $\rho \in (\frac{2}{3}, 1)$. Define

$$u_\epsilon(\rho) = u(\rho) + \epsilon w(\rho).$$

Then $$\left.\frac{\mathrm{d}}{\mathrm{d}\epsilon} J(u_\epsilon)\right|_{\epsilon=0} = \int_0^1 (f_u w + f_p \dot{w})\,\mathrm{d}\rho$$
$$= -\lim_{\rho\to 0+} f_p = 0,$$

where we have used (7.14). Since

$$f_p = (R/r)^{n-1}\Phi_1(R)$$

this proves that

$$\lim_{R\to 0+} T(R) = \lim_{R\to 0+} \left[\left(\frac{R}{r}\right)^{n-1} \phi'(r') + h'\left(\frac{r^{n-1}r'}{R^{n-1}}\right)\right] = 0. \tag{7.16}$$

Finally, by (7.16),

$$R^{n-2}\Phi_2 = R^{n-2}\left[(n-1)\,\phi'\left(\frac{r}{R}\right) + r'\left(\frac{r}{R}\right)^{n-2} h'\left(\frac{r^{n-1}r'}{R^{n-1}}\right)\right]$$
$$= g(R)\,r^{n-2}r' + \frac{R^{n-1}}{r}\left[(n-1)\,\frac{r}{R}\,\phi'\left(\frac{r}{R}\right) - r'\phi'(r')\right],$$

where g is bounded, and hence by (A 5) $R^{n-2}\Phi_2(R) \in L^1(0, 1)$.

Corollary 7.4. *Suppose that* (A 1)–(A 5) *hold, and let r be any minimizer of E on $\mathscr{A}_\lambda$ such that $r(0) > 0$. Then $x(X) = [r(R)/R]\,X$ is a weak equilibrium solution.*

Proof. By (A 3), (A 4), $\int_0^1 R^{n-1}r'(R)\,\mathrm{d}R < \infty$, and so $x \in W^{1,1}(B; \mathbb{R}^n)$ by lemma 4.1. The result follows from theorems 4.2 and 7.3.

For the dead-load traction problem we have

THEOREM 7.5. *Suppose that* (A 1)–(A 6) *hold, and let* r *be any minimizer of* E_P *on* $\mathscr{A}_P$. *Then* $r \in C^1((0, 1])$, $r'(R) > 0$ *for all* $R \in (0, 1]$, $R^{n-1}\Phi_1(R) \in C^1((0, 1])$, (7.10) *holds for all* $R \in (0, 1]$, *and*

$$\lim_{R \to 1-} \Phi_1(R) = P. \tag{7.17}$$

If $r(0) > 0$ *then* $R^{n-2}\Phi_2(R) \in L^1(0, 1)$ *and* $\lim_{R\to 0+} T(R) = 0$.

Proof. Since we can regard r as a minimizer for E subject to fixed displacement boundary conditions the conclusions of theorem 7.3 hold, and it therefore suffices to prove (7.17). But this follows by considering variations of the form

$$u_\epsilon(\rho) = u(\rho) + \epsilon w(\rho),$$

where $w \in C^\infty(0, 1)$ satisfies $w(\rho) = 0$ for $\rho \in (0, \frac{1}{3})$, $w(\rho) = 1$ for $\rho \in (\frac{1}{3}, 1)$.

In the dead-load traction problem one can show that a minimizer r of E_P in $\mathscr{A}_P$ with $r(0) > 0$ generates a *weak equilibrium solution* $x(X) = [r(R)/R]\,X$ to the n-dimensional equations, in the sense that $x \in W^{1,1}(B; \mathbb{R}^n)$, $\det \nabla x(X) > 0$ a.e. $X \in B$, $\partial W(\nabla x(\cdot))/\partial F \in L^1(B; \mathbb{R}^{n^2})$ and

$$\int_B \frac{\partial W}{\partial x^i_{,\alpha}} \psi^i_{,\alpha}\, \mathrm{d}X - P\int_{S^{n-1}} X\cdot\psi(X)\, \mathrm{d}\mathscr{H}^{n-1}X = 0 \tag{7.18}$$

for all $\psi \in C^\infty(\bar{B}; \mathbb{R}^n)$. This follows by suitably modifying the proof of theorem 4.2.

7.4. *Cavitation and energy minimizers*

Our aim is to show that for λ or P large enough the energy minimizers $r(R)$ satisfy $r(0) > 0$. The first step is to observe that (A 1)–(A 4) imply that the growth conditions (6.26), (6.27) hold. Hence theorem 6.8 implies that $r(R) = r(1)R$ whenever $r(0) = 0$. (We could equally well have used theorem 6.6 here, but to have done so would have meant making the extra hypotheses that ϕ, h be C^2 with either $\phi'' > 0$ or $h'' > 0$.) Thus to show that $r(0) > 0$ it suffices to produce an admissible deformation with less energy than any given admissible trivial deformation; this is the content of the following result.

PROPOSITION 7.6. *Let* (A 1)–(A 3), (A 7) *hold. Then*
(*a*) *for any* $\lambda > 0$ *sufficiently large*

$$\inf_{\bar{r}\in\mathscr{A}_\lambda} E(\bar{r}) < E(\lambda R);$$

(*b*) *for any* $P > 0$ *sufficiently large*

$$\inf_{\bar{r}\in\mathscr{A}_P} E_P(\bar{r}) < \inf_{\mu>0} E_P(\mu R).$$

Proof.
(*a*) For fixed ϵ, $0 < \epsilon < 1$, define $r_\epsilon(R) = \lambda[(1-\epsilon)R^n + \epsilon]^{1/n}$. Then

$$\begin{aligned} E(r_\epsilon) - E(\lambda R) &= \omega_n \int_0^1 R^{n-1}\left[\Phi(r'_\epsilon, \frac{r_\epsilon}{R}, \ldots) - \Phi(\lambda, \lambda, \ldots)\right] \mathrm{d}R \\ &= \omega_n \int_1^\infty g(z)\,[\Phi(\lambda(1-\epsilon)\,z^{1-n}, \lambda z, \ldots) - \Phi(\lambda, \lambda, \ldots)]\,\mathrm{d}z, \end{aligned}$$

where $g(z) \overset{\text{def}}{=} \epsilon z^{n-1}/(z^n - 1 + \epsilon)^2$. Thus, by (A 3), (A 7),

$$\begin{aligned} E(r_\epsilon) - E(\lambda R) &\leqslant \omega_n \int_1^\infty g(z)\,[K_2(n + \lambda^\alpha(1-\epsilon)^\alpha z^{\alpha(1-n)} + (n-1)\,\lambda^\alpha z^\alpha + \lambda^{-\beta}(1-\epsilon)^{-\beta} z^{\beta(n-1)} \\ &\qquad + (n-1)\,\lambda^{-\beta} z^{-\beta}) + h(\lambda^n(1-\epsilon)) - h(\lambda^n)]\,\mathrm{d}z \\ &\leqslant C_1 + C_2\lambda^\alpha + C_3\lambda^{-\beta} + (\omega_n/n)\,[h(\lambda^n(1-\epsilon)) - h(\lambda^n)] \end{aligned}$$

since $g(z) \sim \epsilon z^{-1-n}$ as $z \to \infty$. But by (A 1),

$$h(\lambda^n) \geqslant h(\lambda^n(1-\epsilon)) + \lambda^n \epsilon h'(\lambda^n(1-\epsilon)),$$

and so
$$E(r_\epsilon) - E(\lambda R) \leqslant C_1 + C_2\lambda^\alpha + C_3\lambda^{-\beta} - (\omega_n/n)\,\lambda^n\epsilon h'(\lambda^n(1-\epsilon)).$$

Letting $\lambda \to \infty$, we see that since $\alpha < n$, $\lim_{\delta\to\infty} h'(\delta) = \infty$,

$$E(r_\epsilon) < E(\lambda R) \quad \text{for} \quad \lambda \quad \text{sufficiently large,}$$

as required.

(*b*) Defining r_ϵ as above, we have that

$$E_P(r_\epsilon) - E_P(\mu R) \leqslant C_1 + C_2\lambda^\alpha + C_3\lambda^{-\beta} + (\omega_n/n)\,[h(\lambda^n(1-\epsilon)) - h(\mu^n)] + \omega_n P(\mu - \lambda).$$

Fix M sufficiently large for $h'(M^n(1-\epsilon)) > 0$, and set $\lambda = \mu + M$. Then

$$\begin{aligned} E_P(r_\epsilon) - E_P(\mu R) &\leqslant C_1 + C_2(\mu+M)^\alpha + C_3 M^{-\beta} \\ &\quad - (\omega_n/n)\,[\mu^n - (\mu+M)^n(1-\epsilon)]\,h'((\mu+M)^n(1-\epsilon)) - \omega_n PM \\ &\leqslant C_4 - \omega_n PM. \end{aligned}$$

Hence for sufficiently large P,

$$E_P(r_\epsilon) < \inf_{\mu>0} E_P(\mu R)$$

as required.

The discussion preceding proposition 7.6 thus gives

THEOREM 7.7.
(*a*) *Let* (A 1)–(A 5), (A 7) *hold. Then for sufficiently large* $\lambda > 0$ *any minimizer* r *of* E *on* $\mathscr{A}_\lambda$ *satisfies* $r(0) > 0$.
(*b*) *Let* (A 1)–(A 7) *hold. Then for sufficiently large* $P > 0$ *any minimizer* r *of* E_P *on* $\mathscr{A}_P$ *satisfies* $r(0) > 0$.

There now follow conditions guaranteeing that compressive boundary data do not give rise to cavitation.

THEOREM 7.8. *Let* (A 1)–(A 6) *hold, and suppose further that* $\phi, h \in C^3(0,\infty)$, *that* $\phi'' > 0$ *or* $h'' > 0$, *and that* $v\phi'(v)$ *is non-decreasing for all* $v > 0$. *Then*
(*a*) *if* $\lambda < \bar\lambda$, *where* $\bar\lambda$ *is a root of* $\Phi_{,1}(\bar\lambda, \bar\lambda, \ldots) = 0$ (*so that* $x = \bar\lambda X$ *is a natural state*), *then* $r(R) = \lambda R$ *is the unique minimizer of* E *on* $\mathscr{A}_\lambda$;
(*b*) *if* $P < 0$, *then each minimizer* r *of* E_P *on* $\mathscr{A}_P$ *has the form* $r(R) = \bar\lambda R$, *where* $\Phi_{,1}(\bar\lambda, \bar\lambda, \ldots) = P$.

Proof. Suppose r is a solution of (7.10) with $r(0) > 0$ and $\lim_{R\to 0+} T(R) = 0$. By proposition 6.4, $r' - r/R$ does not vanish in $(0, 1)$. Since $\lim_{R\to 0+} r/R = \infty$ this implies that $r' < r/R$ throughout $(0, 1)$. Since $v\phi'(v)$ is non-decreasing the weakened Baker–Ericksen inequalities hold, and so by proposition 6.2 $T(R)$ is a non-decreasing function of R. In particular, $T(R) \geqslant 0$ for all $R \in [0, 1]$.

If r minimizes E_P on $\mathscr{A}_P$, therefore, we have $\Phi_1(1) \geqslant 0$, which is impossible by (7.17) if $P < 0$. This proves (*b*).

Suppose r minimizes E on $\mathscr{A}_\lambda$ and that $\lambda < \bar\lambda$, where $\Phi_1(\bar\lambda, \bar\lambda, \ldots) = 0$. Then $\lim_{R\to 0+} r/R = \infty$, $\lim_{R\to 1} r/R = \lambda$, so that $r(R_0)/R_0 = \bar\lambda$ for some $R_0 \in (0, 1)$.

Thus
$$0 = (1/\bar\lambda)^{n-1}\Phi_{,1}(\bar\lambda, \bar\lambda, \ldots) > (1/\lambda)^{n-1}\Phi_{,1}(r'(R_0), \bar\lambda, \ldots) = T(R_0) \geqslant 0,$$

which is a contradiction.

7.5. Cavitation: the onset of fracture

In this subsection we study the bifurcation of equilibrium solutions with cavities from the trivial solution as λ or P increases. In particular we seek to determine the bifurcation points λ_{cr} and P_{cr}. Our approach is to rewrite the equilibrium equation as a first-order differential equation for $T(w)$, the radial component of the Cauchy stress, as a function of the inverse $w \overset{\text{def}}{=} R/r$ of the circumferential strain, and to solve this equation with initial condition $T(0) = 0$.

We suppose throughout this subsection that (A 1)–(A 7) hold, that $\phi, h \in C^3$, that $\limsup_{v\to 0+} \phi''(v) < \infty$ (so that in particular (A 7) holds with $\beta = 0$), that $v\phi'(v)$ is strictly increasing, and that $h'' > 0$. These hypotheses imply in particular that the relation (cf. (6.7))

$$T = w^{n-1}\Phi_{,1}(r', w^{-1}, \ldots) \tag{7.19}$$

can be solved for r' to yield

$$r' = \theta(T, w), \tag{7.20}$$

where $\theta: \mathbb{R} \times (0, \infty) \to (0, \infty)$ is C^2. We can thus rewrite (6.8) in the form

$$\frac{\mathrm{d}T}{\mathrm{d}R} = -(n-1)\frac{w^{n-1}}{r}[\theta(T, w)\Phi_{,1}(\theta(T, w), w^{-1}, \ldots) - w^{-1}\Phi_{,2}(\theta(T, w), w^{-1}, \ldots)]$$

$$= -(n-1)\frac{w^{n-1}}{r}[\theta(T, w)\phi'(\theta(T, w)) - w^{-1}\phi'(w^{-1})]. \tag{7.21}$$

Since

$$\frac{\mathrm{d}w}{\mathrm{d}R} = -\frac{w}{r}[\theta(T, w) - w^{-1}], \tag{7.22}$$

we can combine (7.21), (7.22) to obtain

$$\frac{\mathrm{d}T}{\mathrm{d}w} = G(T, w), \tag{7.23}$$

where

$$G(T, w) \overset{\text{def}}{=} \begin{cases} (n-1)w^{n-2}\left[\dfrac{\theta(T, w)\phi'(\theta(T, w)) - w^{-1}\phi'(w^{-1})}{\theta(T, w) - w^{-1}}\right] & \text{if } \theta(T, w) \neq w^{-1}, \\ (n-1)w^{n-2}[\phi'(w^{-1}) + w^{-1}\phi''(w^{-1})] & \text{if } \theta(T, w) = w^{-1}. \end{cases} \tag{7.24}$$

(We shall shortly see that the values of G when $\theta(T, w) = w^{-1}$ are not relevant for us.)

Let $R_0 > 0$, and let $r \in C^1((0, R_0))$ be a solution of (7.10) satisfying $r(0) > 0$, $\lim_{R\to 0+} T(R) = 0$, and $r'(R) > 0$ for $R \in (0, R_0)$. Then from the proof of theorem 7.8 we see that $r' < r/R$ for $R \in (0, R_0)$ so that r/R is strictly decreasing. Thus r generates a solution to the initial-value problem

$$\left.\begin{aligned} \mathrm{d}T/\mathrm{d}w &= G(T, w), \\ T(0) \overset{\text{def}}{=} \lim_{w\to 0+} T(w) &= 0 \end{aligned}\right\} \tag{7.25}$$

defined on the interval $(0, R_0/r(R_0))$ and satisfying $\theta(T(w), w) < w^{-1}$ there. This solution is *unique*. To prove this note that since

$$T(R) = \left(\frac{R}{r}\right)^{n-1}\phi'(r') + h'\left[r'\left(\frac{r}{R}\right)^{n-1}\right],$$

it follows from our hypotheses that $\lim_{R\to 0+} r'(R) = 0$, and hence that

$$\lim_{R\to 0+} r'\left(\frac{r}{R}\right)^{n-1} = \eta, \tag{7.26}$$

where η is the unique root of $h' = 0$. Hence any solution $T(w)$ of (7.25) satisfies

$$\tfrac{1}{2}\eta \leqslant \theta(T, w)/w^{n-1} \leqslant 2\eta \tag{7.27}$$

for sufficiently small $w > 0$. From (7.19), (7.20) and (7.24) we deduce that

$$\frac{\partial G(T, w)}{\partial T} = \frac{(n-1)\, w^{2n-3}}{(\theta - w^{-1})^2} \left\{ \frac{\theta\phi''(\theta)\,(\theta - w^{-1}) - w^{-1}[\phi'(\theta) - \phi'(w^{-1})]}{w^{2(n-1)}\phi''(\theta) + h''(\theta w^{1-n})} \right\},$$

and hence that, for sufficiently small w,

$$\left| \frac{\partial G(T, w)}{\partial T} \right| \leqslant C \tag{7.28}$$

whenever (7.27) holds, where we have used (A 5), (A 7), $\limsup_{v \to 0+} \phi''(v) < \infty$ and $h'' > 0$.

Let T_1, T_2 be two solutions of (7.25). Then

$$\frac{\mathrm{d}}{\mathrm{d}w}(T_1 - T_2)(w) = \frac{\partial G(\tilde{T}(w), w)}{\partial T}(T_1 - T_2)(w), \tag{7.29}$$

where $\tilde{T}(w) \in [T_1(w), T_2(w)]$. Since $\partial\theta/\partial T \geqslant 0$, $\tfrac{1}{2}\eta \leqslant \theta(\tilde{T}(w), w)/w^{n-1} \leqslant 2\eta$ for sufficiently small $w > 0$, and so by (7.28), (7.29) and Gronwall's inequality we deduce that $T_1 = T_2$ as required.

Let $\lambda > 0$ be such that a minimizer $r_\lambda(R)$ of E on $\mathscr{A}_\lambda$ exists with $r_\lambda(0) > 0$. (Such λ exist by theorem 7.7 (*a*).) Since by (6.4) we have $r_\lambda''(R) > 0$, there follows the *a priori* estimate

$$0 < r_\lambda'(R) < r_\lambda'(S) < r_\lambda(S)/S < r_\lambda(R)/R \quad \text{for} \quad 0 < R < S,$$

which implies that r_λ can be extended uniquely to the whole of $(0, \infty)$ as a solution of (7.10). Define

$$\lambda_{\mathrm{cr}} \stackrel{\mathrm{def}}{=} \lim_{R\to\infty} \frac{r_\lambda(R)}{R} = \lim_{R\to\infty} r_\lambda'(R). \tag{7.30}$$

Then $T(w)$ is defined on $[0, \lambda_{\mathrm{cr}}^{-1}]$, and by (7.20), (7.30), λ_{cr} is the largest root of the equation

$$\mu^{-1} = \theta(T(\mu^{-1}), \mu^{-1}); \tag{7.31}$$

equivalently, λ_{cr} is the largest root of

$$T(\mu^{-1}) = \mu^{1-n}\Phi_{,1}(\mu, \mu, \ldots, \mu). \tag{7.32}$$

Since $\theta(T(w), w) < w^{-1}$ for all $w \in (0, \lambda_{\mathrm{cr}}^{-1})$, the uniqueness of T implies that λ_{cr} is independent of λ, provided there is a solution of (7.10) on $(0, 1]$ with $r_\lambda(0) > 0$ and $T(0) = 0$. Note that T is increasing by the Baker–Ericksen inequalities, and hence that $\lambda_{\mathrm{cr}} > \max \{\bar{\lambda} : \Phi_{,1}(\bar{\lambda}, \bar{\lambda}, \ldots) = 0\}$.

THEOREM 7.9. *A solution r_λ of (7.10) on $(0, 1]$ with $r_\lambda(0) > 0$, $r_\lambda(1) = \lambda$ and $T(0) = 0$ exists if and only if $\lambda > \lambda_{\mathrm{cr}}$. The solution r_λ is the unique minimizer of E on $\mathscr{A}_\lambda$, and can be extended uniquely as a solution of (7.10) to the whole of $(0, \infty)$. If $\lambda > \mu > \lambda_{\mathrm{cr}}$, then*

$$r_\mu(R) = r_\lambda(c(\lambda, \mu) R)/c(\lambda, \mu) \quad \textit{for all} \quad R \geqslant 0, \tag{7.33}$$

where $c(\lambda, \mu)$ is the unique number such that $r_\lambda(c(\lambda, \mu))/c(\lambda, \mu) = \mu$. Furthermore $r_\lambda(R) \to \lambda R$ uniformly on $[0, 1]$ as $\lambda \downarrow \lambda_{\mathrm{cr}}$.

Proof. Let λ be such that a minimizer r_λ of E on $\mathscr{A}_\lambda$ exists with $r_\lambda(0) > 0$. Then r_λ is the unique solution of the initial value problem

$$\left.\begin{aligned} \frac{\mathrm{d}r_\lambda(R)}{\mathrm{d}R} &= \theta\left[T\left(\frac{R}{r_\lambda(R)}\right), \frac{R}{r_\lambda(R)}\right], \quad 0 < R < \infty, \\ r_\lambda(1) &= \lambda. \end{aligned}\right\} \tag{7.34}$$

If $\mu > \lambda_{\mathrm{cr}}$ and r_μ is defined by (7.33), then

$$\left.\begin{aligned} \frac{\mathrm{d}r_\mu(R)}{\mathrm{d}R} &= \theta\left[T\left(\frac{R}{r_\mu(R)}\right), \frac{R}{r_\mu(R)}\right], \quad 0 < R < \infty, \\ r_\mu(1) &= \mu. \end{aligned}\right\} \tag{7.35}$$

The uniqueness of solutions to (7.34) shows that any solution r_μ of (7.10) with $r_\mu(0) > 0$, $r_\mu(1) = \mu$ and $T(0) = 0$ has the form (7.33). We have already shown that no such solution exists for $\mu \leqslant \lambda_{\mathrm{cr}}$.

We now show that r_μ given by (7.33) minimizes E in $\mathscr{A}_\mu$. We have that

$$\begin{aligned} E(r_\mu) - E(\mu R) &= \omega_n \int_0^1 R^{n-1}\left[\Phi\left(r_\mu', \frac{r_\mu}{R}, \ldots\right) - \Phi(\mu, \mu, \ldots)\right] \mathrm{d}R \\ &= \frac{\omega_n}{c(\lambda, \mu)^n} \int_0^{c(\lambda,\mu)} R^{n-1}\left[\Phi\left(r_\lambda'(R), \frac{r_\lambda(R)}{R}, \ldots\right) - \Phi(\mu, \mu, \ldots)\right] \mathrm{d}R \\ &= \frac{\omega_n}{n}\{\Phi(r_\lambda'(c(\lambda, \mu)), \mu, \ldots) - [r_\lambda'(c(\lambda, \mu)) - \mu]\,\Phi_{,1}(r_\lambda'(c(\lambda, \mu)), \mu, \ldots) \\ &\quad - \Phi(\mu, \mu, \ldots)\} - \frac{1}{nc(\lambda, \mu)^n} \lim_{R\to 0+} R^n\left[\Phi - \left(r_\lambda' - \frac{r_\lambda}{R}\right)\Phi_1\right], \end{aligned} \tag{7.36}$$

where we have used (6.12).

But since $r_\lambda'(0) = 0$ it follows that

$$\lim_{R\to 0+} R^n\left(r_\lambda' - \frac{r_\lambda}{R}\right)\Phi_1 = 0. \tag{7.37}$$

From (7.36) we deduce that $\lim_{R\to 0+} R^n\Phi(r_\lambda', r_\lambda/R, \ldots)$ exists, and thus, since $E(r_\lambda) < \infty$, that

$$\lim_{R\to 0+} R^n\Phi\left(r_\lambda', \frac{r_\lambda}{R}, \ldots\right) = 0. \tag{7.38}$$

Since $\Phi(v_1, \ldots, v_n)$ is strictly convex in v_1 we deduce from (7.36)–(7.38) that

$$E(r_\mu) < E(\mu R).$$

Thus, from the discussion in §7.4, we see that r_μ minimizes E in $\mathscr{A}_\mu$.

Finally, the fact that $r_\lambda(R) \to \lambda R$ uniformly follows from (7.33) and the fact that $\lim_{\lambda\to\lambda_{\mathrm{cr}}} c(\lambda_1, \lambda) = \infty$.

Remarks

(i) The formula (7.33) shows that as $\lambda \downarrow \lambda_{\mathrm{cr}}$ the value of $r_\lambda(R)/R$ differs significantly from λ_{cr} only in a boundary layer $0 < R < d_\lambda$ whose thickness d_λ is proportional to the cavity radius $r_\lambda(0)$.

(ii) Note that, for each $\lambda > \lambda_{\mathrm{cr}}$, $r_\lambda(R)$ extended to $(0, \infty)$ solves the radial equilibrium problem with zero Cauchy stress on the cavity surface and radial Cauchy stress $\lambda_{\mathrm{cr}}^{1-n}\Phi_{,1}(\lambda_{\mathrm{cr}}, \lambda_{\mathrm{cr}}, \ldots)$ at infinity.

We now discuss the stability of the equilibrium solutions to the displacement boundary value problem. We say that a solution $\bar{r} \in \mathscr{A}_\lambda$ of (7.10) with $T(0) = 0$ if $r(0) > 0$ is *stable* if there exists $\epsilon > 0$ such that $E(r) > E(\bar{r})$ whenever $r \in \mathscr{A}_\lambda$ and $0 < \max_{R \in [0,1]} |r(R) - \bar{r}(R)| < \epsilon$. (In the terminology of the calculus of variations $\bar{r}$ is stable if it is a strong relative minimum; it is not obvious that this implies Lyapunov stability of $\bar{r}$ with respect to, for example, the equations of nonlinear elastodynamics, even within the class of radial motions.) We say that $\bar{r}$ is *unstable* if it is not stable.

THEOREM 7.10. *The trivial solution* $r = \lambda R$ *is stable if* $\lambda \leqslant \lambda_{\mathrm{cr}}$ *and unstable if* $\lambda > \lambda_{\mathrm{cr}}$. *If* $\lambda > \lambda_{\mathrm{cr}}$ *then* r_λ *is stable.*

Proof. We have seen in theorems 7.1, 7.3 and 7.9 that $r = \lambda R$ is the unique absolute minimizer of E when $\lambda \leqslant \lambda_{\mathrm{cr}}$, and that r_λ is the unique absolute minimizer when $\lambda > \lambda_{\mathrm{cr}}$. It therefore suffices to show that $r = \lambda R$ is unstable if $\lambda > \lambda_{\mathrm{cr}}$. For $\lambda > \lambda_{\mathrm{cr}}$, $\nu > \lambda_{\mathrm{cr}}$, define

$$r^{(\nu)}(R) = \begin{cases} r_\nu(R) & \text{if } 0 \leqslant R \leqslant R_\nu \\ \lambda R & \text{if } R_\nu < R \leqslant 1, \end{cases}$$

where $R_\nu = 1/c(\lambda, \nu)$. Note that $r_\nu(R_\nu) = r_\lambda(1)/c(\lambda, \nu) = \lambda R_\nu$, so that $r^{(\nu)} \in \mathscr{A}_\lambda$. Also $R_\nu \to 0$ as $\nu \to \lambda_{\mathrm{cr}}$, so that $r^{(\nu)} \to \lambda R$ uniformly. But

$$\begin{aligned} E(r^{(\nu)}) - E(\lambda R) &= \omega_n \int_0^{R_\nu} R^{n-1}\left[\Phi\left(r_\nu', \frac{r_\nu}{R}, \ldots\right) - \Phi(\lambda, \lambda, \ldots)\right] \mathrm{d}R \\ &= \frac{\omega_n}{c(\lambda, \nu)^n} \int_0^1 R^{n-1}\left[\Phi\left(r_\lambda'(R), \frac{r_\lambda(R)}{R}, \ldots\right) - \Phi(\lambda, \lambda, \ldots)\right] \mathrm{d}R \\ &= \frac{1}{c(\lambda, \nu)^n}\,[E(r_\lambda) - E(\lambda R)] < 0, \end{aligned}$$

and hence $r = \lambda R$ is unstable.

We now study the bifurcation in the dead-load traction problem. Let $\lambda_1 > \lambda_{\mathrm{cr}}$ be fixed. The non-trivial equilibrium solutions with $T(0) = 0$ and $\Phi_1(1) = P$ are exactly those $r_\lambda(R)$ for which

$$\Phi_{,1}(r_\lambda'(1), \lambda, \ldots) = P. \tag{7.39}$$

Thus bifurcation from the trivial solution occurs at $P = P_{\mathrm{cr}}$, where

$$P_{\mathrm{cr}} = \Phi_1(\lambda_{\mathrm{cr}}, \lambda_{\mathrm{cr}}, \ldots). \tag{7.40}$$

The direction of bifurcation is given by $\mathrm{d}\Phi_{,1}(r_\lambda'(1), \lambda, \ldots)/\mathrm{d}\lambda|_{\lambda=\lambda_{\mathrm{cr}}}$. But

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\lambda}\Phi_{,1}(r_\lambda'(1), \lambda, \ldots) &= \frac{\mathrm{d}}{\mathrm{d}\lambda}\Phi_{,1}(r_{\lambda_1}'(c(\lambda_1, \lambda)), \lambda, \ldots) \\ &= \Phi_{11} r_{\lambda_1}''(c(\lambda_1, \lambda))\frac{\mathrm{d}c(\lambda_1, \lambda)}{\mathrm{d}\lambda} + (n-1)\,\Phi_{12} \\ &= \Phi_{11} r_{\lambda_1}''(c(\lambda_1, \lambda))\frac{c(\lambda_1, \lambda)}{r_\lambda'(1) - \lambda} + (n-1)\,\Phi_{12} \\ &= -(n-1)\left[\frac{\Phi_{,1}(r_\lambda'(1), \lambda, \ldots) - \Phi_{,2}(r_\lambda'(1), \lambda, \ldots)}{r_\lambda'(1) - \lambda}\right], \end{aligned} \tag{7.41}$$

where we have used the definition of $c(\lambda_1, \lambda)$ and (6.4). Thus

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\Phi_{,1}(r_\lambda'(1), \lambda, \ldots)|_{\lambda=\lambda_{\mathrm{cr}}} = -(n-1)\,[\Phi_{,11}(\lambda_{\mathrm{cr}}, \lambda_{\mathrm{cr}}, \ldots) - \Phi_{,12}(\lambda_{\mathrm{cr}}, \lambda_{\mathrm{cr}}, \ldots)]. \tag{7.42}$$

Thus the bifurcation at $P = P_{cr}$ is to the right (resp. left) if

$$\Phi_{,11}(\lambda_{cr}, \lambda_{cr}, \ldots) < (\text{resp.} >) \; \Phi_{,12}(\lambda_{cr}, \lambda_{cr}, \ldots). \tag{7.43}$$

Bifurcation to the left would mean that as P is increased past P_{cr} the body 'snaps through' to an equilibrium solution with finite cavity radius; in bifurcation to the right the cavity radius would initially grow continuously from zero.

We do not attempt a study of the stability of the various equilibrium solutions here, except to remark that it follows immediately from theorem 7.10 that if $\lambda > \lambda_{cr}$ then $r = \lambda R$ is unstable (i.e. it is not a strong relative minimum for E_p on $\mathscr{A}_p$, where P is given by (7.39)).

In the Cauchy traction problem the non-trivial equilibrium solutions with $T(0) = 0$ and $T(1) = P$ are exactly those $r_\lambda(R)$ for which

$$\lambda^{1-n}\Phi_{,1}(r'_\lambda(1), \lambda, \ldots) = 0. \tag{7.44}$$

Thus bifurcation from the trivial solution occurs at $P = P^*$, where

$$P^* = \lambda_{cr}^{1-n}\Phi_{,1}(\lambda_{cr}, \lambda_{cr}, \ldots). \tag{7.45}$$

Note that by (7.41), for all $\lambda > \lambda_{cr}$,

$$\frac{d}{d\lambda}\lambda^{1-n}\Phi_{,1}(r'_\lambda(1), \lambda, \ldots) = -(n-1)\lambda^{-n}\left[\frac{r'_\lambda(1)\,\Phi_{,1}(r'_\lambda(1), \lambda, \ldots) - \lambda\Phi_{,2}(r'_\lambda(1), \lambda, \ldots)}{r'_\lambda(1) - \lambda}\right], \tag{7.46}$$

which is negative by the Baker–Ericksen inequalities. Hence the bifurcation is always to the left. If $\lambda = 1$ is a natural state then $\lambda_{cr} > 1$ and so $P_{cr} > P^*$.

Finally, we make an observation analogous to proposition 5.1 concerning the existence of equilibrium solutions with infinite energy.

Proposition 7.11. *If r is a solution of* (7.10) *on* $(0, 1]$ *with* $r(0) > 0$, *then* $E(r) < \infty$ *if and only if* $T(0) \overset{\text{def}}{=} \lim_{R\to 0+} T(R)$ *is finite.*

Proof. By the Baker–Ericksen inequalities $T(0)$ exists. From (6.12)

$$\tau^n\Phi(\tau) + n\int_\tau^1 R^{n-1}\Phi(R)\,dR = C + \tau^n\left[r'(\tau) - \frac{r(\tau)}{\tau}\right]\Phi_1(\tau), \tag{7.47}$$

for any $\tau > 0$. Since r' is bounded it follows immediately that $E(r) < \infty$ implies that $T(0) = -\infty$. Conversely, if $T(0) = -\infty$, then the left-hand side of (7.47) tends to infinity as $\tau \to 0+$, which implies that $E(r) = \infty$.

7.6. *An example*

Consider the stored-energy function

$$\Phi(v_1, \ldots, v_n) = \mu\left[\sum_{i=1}^{n} v_i^2 - n + H(v_1 \ldots v_n)\right], \tag{7.48}$$

where $\mu > 0$ and $H\colon (0, \infty) \to \mathbb{R}^+$ is a C^3 function satisfying $\lim_{\delta\to 0+} H(\delta) = \infty$, $H''(\delta) > 0$ for all $\delta > 0$, and

$$H(\delta) = k(\delta - 1 - k^{-1})^2 \quad \text{for} \quad \delta \geqslant \tfrac{1}{2}, \tag{7.49}$$

where $k > 1$.

The function Φ is a compressible version of the incompressible neo-Hookean material having stored-energy function $\mu(\sum_{i=1}^n v_i^2 - n)$, this material being obtained formally by letting $k \to \infty$; Φ is strongly elliptic.

Suppose $n \geqslant 3$; then on setting $\phi(v) = \mu v^2$, $h(\delta) = \mu[H(\delta) - n]$, it is easily verified that (A 1)–(A 7) and the hypotheses of §7.5 hold.

The natural states of Φ are determined by solving the n equations

$$\Phi_1 = \ldots = \Phi_n = 0. \tag{7.50}$$

We obtain $v_1 = \ldots = v_n = v$, where v is a root of

$$2 + v^{n-2} H'(v^n) = 0. \tag{7.51}$$

Since $k > 1$, $v = 1$ is the largest root of (7.51).

Let $r(R)$ be a solution of (7.10) on $(0, 1]$ with $r(0) > 0$; $T(0) = 0$. Then by (7.26)

$$\lim_{R \to 0+} r' \left(\frac{r}{R}\right)^{n-1} = 1 + k^{-1}. \tag{7.52}$$

Also, substitution of (7.48) into (6.4) shows that

$$\frac{\mathrm{d}}{\mathrm{d}R}\left[r'\left(\frac{r}{R}\right)^{n-1}\right] \geqslant 0, \quad R > 0. \tag{7.53}$$

Hence $r'(r/R)^{n-1} > 1 + k^{-1}$, $R > 0$, so that when considering non-trivial equilibrium solutions we can assume that H is given by (7.49).†

We thus have that

$$T = 2\mu[r'(w^{n-1} + kw^{1-n}) - 1 - k], \tag{7.54}$$

so that

$$\theta(T, w) = \frac{(T/2\mu + 1 + k)\, w^{n-1}}{w^{2(n-1)} + k}. \tag{7.55}$$

Thus (7.23) becomes

$$\begin{aligned}\frac{\mathrm{d}T}{\mathrm{d}w} &= 2(n-1)\,\mu w^{n-2}[\theta(T, w) + w^{-1}] \\ &= 2(n-1)\,\mu w^{n-3}\left[\left(\frac{T}{2\mu} + 1 + k\right)\frac{w^n}{w^{2(n-1)} + k} + 1\right].\end{aligned} \tag{7.56}$$

Writing (7.56) in the form

$$\frac{\mathrm{d}}{\mathrm{d}w}\left[\frac{T + 2\mu(1+k)}{(w^{2(n-1)} + k)^{\frac{1}{2}}}\right] = 2(n-1)\,\mu\,\frac{w^{n-3}}{(w^{2(n-1)} + k)^{\frac{1}{2}}}, \tag{7.57}$$

we obtain the solution of the initial-value problem (7.25), namely

$$\frac{T(w) + 2\mu(1+k)}{(w^{2(n-1)} + k)^{\frac{1}{2}}} = 2\mu\,\frac{(1+k)}{k^{\frac{1}{2}}} + 2(n-1)\,\mu \int_0^w \frac{\tau^{n-3}\,\mathrm{d}\tau}{(\tau^{2(n-1)} + k)^{\frac{1}{2}}}. \tag{7.58}$$

Therefore λ_{cr} is the largest root of the equation

$$\lambda(1 + k\lambda^{2(n-1)})^{\frac{1}{2}} = \frac{1+k}{k^{\frac{1}{2}}} + (n-1)\int_\lambda^\infty \frac{\mathrm{d}s}{(1 + ks^{2(n-1)})^{\frac{1}{2}}}. \tag{7.59}$$

In fact, since the left- and right-hand sides of (7.59) are increasing and decreasing functions of $\lambda > 0$ respectively, λ_{cr} is the only root of (7.59).

The values of P_{cr} and P^* are given by

$$P_{\mathrm{cr}} = 2\mu\lambda_{\mathrm{cr}}^{n-1}[\lambda_{\mathrm{cr}}^{2-n} - 1 + k(\lambda_{\mathrm{cr}}^n - 1)], \tag{7.60}$$

and

$$P^* = 2\mu[\lambda_{\mathrm{cr}}^{2-n} - 1 + k(\lambda_{\mathrm{cr}}^n - 1)]. \tag{7.61}$$

† In particular, the existence of equilibrium solutions with cavities is not attributable to the singularity in $W(F)$ as $\det F \to 0+$, and such solutions may exist even if $H(\delta)$ is not convex for large δ.

We now study the behaviour of λ_{cr} as $k \to \infty$. From (7.59) we see that as $k \to \infty$

$$\lambda_{cr}(1+k\lambda_{cr}^{2(n-1)})^{\frac{1}{2}} \leqslant \frac{1+k}{k^{\frac{1}{2}}}+\frac{C}{k^{\frac{1}{2}}},$$

and hence that

$$k(\lambda_{cr}^{2n}-1) \leqslant C.$$

In particular, since $\lambda_{cr} > 1$,

$$\lim_{k\to\infty} \lambda_{cr} = 1. \tag{7.62}$$

Expanding (7.59) by the binomial theorem we deduce that

$$\lambda_{cr}^{n} k^{\frac{1}{2}}\left[1+\frac{1}{2k\lambda_{cr}^{2(n-1)}}+o(k^{-1})\right] = \frac{1+k}{k^{\frac{1}{2}}}+\frac{(n-1)}{k^{\frac{1}{2}}}\int_{\lambda_{cr}}^{\infty}\frac{1}{s^{n-1}}\left[1-\frac{1}{2ks^{2(n-1)}}+o(k^{-1})\right]\mathrm{d}s,$$

and it follows that

$$\lim_{k\to\infty} k(\lambda_{cr}^{n}-1) = \frac{3n-4}{2(n-2)}. \tag{7.63}$$

From (7.60)–(7.63) we obtain

$$\lim_{k\to\infty} P_{cr} = \lim_{k\to\infty} P^{*} = \left(\frac{3n-4}{n-2}\right)\mu. \tag{7.64}$$

Also we have that

$$\lim_{k\to\infty}\,[\Phi_{,11}(\lambda_{cr},\lambda_{cr},\ldots)-\Phi_{,12}(\lambda_{cr},\lambda_{cr},\ldots)] = \lim_{k\to\infty} 2\mu[1+\lambda_{cr}^{n-2}-k\lambda_{cr}^{n-2}(\lambda_{cr}^{n}-1)] = \mu\left(\frac{n-4}{n-2}\right), \tag{7.65}$$

so that bifurcation in the dead-load traction problem is locally to the right when $n = 3$ and to the left when $n > 4$. The results (7.62), (7.64), (7.65) are in accord with those for the incompressible problem (see (5.60), (5.66) and proposition 5.2), except that in the incompressible problem bifurcation is to the right for all $n \geqslant 3$.

We now consider the case $n = 2$. Recall that in the incompressible case non-trivial deformations for the neo-Hookean material have infinite energy (example 5.1). From (7.57) we see immediately that for (7.40) there is no non-trivial equilibrium solution satisfying $T(0) = 0$. However, it is possible for a non-trivial equilibrium solution to exist with $T(0) = -\infty$; such a solution must have infinite energy by proposition 7.11. We suppose that for sufficiently small $\delta > 0$,

$$H(\delta) = -\ln\delta. \tag{7.66}$$

Writing $w = R/r$, $\delta = rr'/R$, we note that the radial equilibrium equation (7.10) can be written in the form

$$\frac{\mathrm{d}\delta(w)}{\mathrm{d}w} = \frac{2(1-\delta w^{2})}{w[2w^{2}+H''(\delta)]}. \tag{7.67}$$

We seek to solve (7.67) for a non-zero $\delta(\cdot)$ satisfying $\delta(0) = 0$. It will suffice to do this for small w, and hence we can insert (7.66) into (7.67) to obtain

$$\frac{\mathrm{d}\delta}{\mathrm{d}w} = \frac{2\delta^{2}(1-\delta w^{2})}{w(1+2w^{2}\delta^{2})}. \tag{7.68}$$

For small δ, w the right-hand side of (7.68) is like δ^{2}/w, and this motivates the change of variables $v = -1/2\ln w$ after which (7.68) becomes

$$\frac{\mathrm{d}\delta(v)}{\mathrm{d}v} = \frac{\delta^{2}}{v^{2}}\left(\frac{1-\delta\,\mathrm{e}^{-1/v}}{1+2\delta^{2}\,\mathrm{e}^{-1/v}}\right). \tag{7.69}$$

We claim that (7.69) has a strictly positive solution on $(0, v_0)$ satisfying $\lim_{v\to 0+} \delta(v) = 0$, provided $v_0 > 0$ is sufficiently small; to prove this we set

$$K = \{\delta \in C([0, v_0]) : \delta(0) = 0,\ 1+\tfrac{1}{4}e^{-1/v} \leqslant \delta(v)/v \leqslant 1+\tfrac{1}{2}e^{-1/v} \quad \text{for} \quad 0 < v \leqslant v_0\}$$

and
$$(Q\delta)(v) = \int_0^v \left[\frac{\delta(s)}{s}\right]^2 \left[\frac{1-\delta(s)\,e^{-1/s}}{1+2\delta(s)^2\,e^{-1/s}}\right] ds. \tag{7.70}$$

Then it is not hard to show that K is a bounded, closed and convex subset of $C([0, v_0])$ and that for sufficiently small $v_0 > 0$ Q maps K into K and is compact and continuous. The existence of the desired solution $\delta(v)$ now follows from Schauder's theorem. Let $\delta(w)$ be the corresponding solution of (7.68). Given $\epsilon > 0$ let $r(R)$ denote the solution of the initial value problem

$$\left.\begin{aligned} \frac{dr}{dR} &= \frac{R}{r}\,\delta\left(\frac{R}{r}\right), \\ r(0) &= \epsilon \end{aligned}\right\} \tag{7.71}$$

defined for $0 \leqslant R < R_0$, where R_0 is sufficiently small. The solution of (7.71) exists since $|\delta'(w)|$ is bounded for small $w \geqslant 0$. Clearly r satisfies (7.10) on $(0, R_0)$. This solution can be extended to $[0, 1]$ by the monotonicity of r' and r/R. Since

$$T(R) = \mu\left[2\,\frac{R}{r}\,r'(R) + \frac{R}{rr'(R)}\right]$$

it follows that $T(0) = -\infty$. Note also that since $\delta(w)\ln w \sim -\frac{1}{2}$ as $w \to 0+$ it follows easily that $R\Phi_1(R)$, $R\Phi_2(R) \in L^1(0, 1)$, and hence that by theorem 4.2 $x = [r(R)/R]X$ is a weak equilibrium solution.

8. Concluding remarks

We begin by relating our analysis to the work of Gent & Lindley (1958) described in the introduction. As explained there, Gent & Lindley calculated the radial dead load traction P_c at infinity required to expand a spherical cavity of initial radius R_0 indefinitely, and found that P_c was independent of R_0 and given (for their materials) by (5.22). To prove that $P_c = P_{cr}$ for an incompressible material we note that the arguments leading to (4.16) or (5.35) show that, for a spherical shell of internal and external radii R_0, R_1 in the reference configuration, the radial component $P(R_1)$ of the Piola–Kirchhoff stress at $R = R_1$ in the deformation $r^n = R^n + A^n$ is given by

$$P(R_1) = \left[1+\left(\frac{A}{R_1}\right)^n\right]^{1-1/n} \int_{[1+(A/R_1)^n]^{1/n}}^{[1+(A/R_0)^n]^{1/n}} \frac{1}{v^n-1}\,\frac{d\hat{\Phi}(v)}{dv}\,dv, \tag{8.1}$$

provided $T(R_0) = 0$. The value of P_c is obtained by setting $R_1 = \infty$ and letting $r(R_0)/R_0$ tend to infinity in (8.1), and this shows that $P_c = P_{cr}$. For compressible material $P_c = P_{cr}$ also (cf. remark (ii) after theorem 7.9). The underlying reason is the invariance of the equilibrium equation and boundary conditions under the transformation $(X, x) \mapsto (cX, cx)$; *an infinitesimal hole in a finite piece of material behaves like a finite hole in an infinite expanse of material.* We remark that the actual failure criterion proposed by Gent & Lindley was that failure occurs at the least dead-load traction $\tilde{P}$ that, when applied at infinity, will cause the circumferential stretch at $R = R_0$ to reach a specified value v_0. Gent & Lindley took v_0 to lie between 4 and 10, and noted that then $\tilde{P}$ differs little from P_c.

The boundary condition $T(0) = 0$ supposes the cavity to be empty; it is easy, however, to modify our static analysis to incorporate other assumptions concerning cavity contents (for example that the cavity fills with air at atmospheric pressure; Gent & Lindley found this correction to be small). It is also possible to account for the effects of surface energy; we indicate how to do this for an incompressible material. The total energy of the deformation

$$R \mapsto (R^n + A^n)^{1/n}, \quad A \geqslant 0, \quad \text{is now} \quad (\text{cf. } (5.19))$$

$$I_1(A) \overset{\text{def}}{=} E(A) - P\omega_n(1 + A^n)^{1/n} + \omega_n e(A), \tag{8.2}$$

where $\omega_n e(A)$ is the energy of the cavity surface. For equilibrium $I_1'(A)\,(B-A) \geqslant 0$ for all $B \geqslant 0$, and thus $A = 0$ or

$$P = (1+A^n)^{1-1/n}\left[\int_{(1+A^n)^{1/n}}^{\infty} \frac{1}{v^n - 1}\frac{\mathrm{d}\hat{\Phi}(v)}{\mathrm{d}v}\,\mathrm{d}v + \frac{e'(A)}{A^{n-1}}\right]. \tag{8.3}$$

If $\lim_{A\to 0+} e'(A)/A^{n-1} = 0$ then the non-trivial solutions bifurcate from the trivial solution $r = R$ at $P = P_{\mathrm{cr}}$, as before. However, if we take the standard model in which

$$e(A) = KA^{n-1} \tag{8.4}$$

is proportional to surface area, then $P \to \infty$ as $A \to 0+$ and no bifurcation occurs. The effect of surface energy on cavitation for liquids has been studied by Fisher (1948) and Irwin (1958), and for solids such as rubber by Gent & Tompkins (1969) and Williams & Schapery (1965).

The equilibrium solutions with cavities can be obtained as limits of non-singular deformations in various ways. For example, consider the mixed boundary value problem for a spherical shell of initial internal and external radii $R_0 > 0$, $R_1 = 1$ in which $r(1) = \lambda$ and $T(R_0) = 0$. If $\lambda > \lambda_{\mathrm{cr}}$ then the energy minimizer for this problem (for a compressible material) tends to $r_\lambda(R)$ as $R_0 \to 0+$. (The corresponding result for an incompressible material holds trivially.) However, certain other approximation schemes never converge to a solution with a cavity. Suppose, for example, that we minimize

$$I_\epsilon(x) = \int_B \left[W(\nabla x(X)) + \epsilon \sum_{\substack{|I|=k \\ i=1,\dots,n}} x_I^i(X)\, x_I^i(X)\right] \mathrm{d}X, \tag{8.5}$$

subject to the boundary condition $x(X) = \lambda X$ for $X \in \partial B$, where $\epsilon > 0$ and x_I^i denotes a typical kth-order derivative of x^i. Then if k is sufficiently large, any x with $I(x) < \infty$ is C^1 (provided W is bounded below). Therefore if W is quasiconvex, $x(X) = \lambda X$ minimizes I_ϵ absolutely for any $\epsilon > 0$, since it minimizes $\int_B W(\nabla x)\,\mathrm{d}X$ and makes the second term zero. Clearly as $\epsilon \to 0$ these minimizers do not converge to a solution with a cavity, however large λ may be.

A possible application of our theory of cavitation in two-dimensional compressible elasticity (suggested to me by J. E. Marsden) is to the bursting of a balloon under an internal pressure. Suppose that the balloon is incompressible and spherical with radius 1 and thickness $\delta \ll 1$ in the reference configuration. If the balloon remains spherical and has deformed radius λ then locally the surface will be in two-dimensional extension with principal stretches both equal to λ. The corresponding (compressible) two-dimensional stored-energy function is

$$\Psi(v_1, v_2) = \delta\Phi(1/v_1 v_2, v_1, v_2). \tag{8.6}$$

If bursting occurs initially by two-dimensional cavitation then the radius λ at bursting should equal the value of λ_{cr} corresponding to Ψ.

The growth conditions assumed in the existence theory of Ball (1977*a*, *b*) and Ball *et al.* (1981) exclude the stored-energy functions shown here to allow the existence of discontinuous equilibrium solutions of finite energy. Essentially this is because for deformations with cavities the distributional determinant Det ∇x is a measure. Our examples have implications for attempts to prove regularity of weak equilibrium solutions for strong materials; in particular, example 5.1 and the example constructed at the end of §7 show that even for strong materials there may exist discontinuous equilibria of infinite energy.

Koiter (1976) has drawn attention to the question of whether positivity of the second variation of the total energy at an equilibrium solution is a sufficient condition for stability. In a displacement problem for a compressible material the second variation at the trivial solution $x = \lambda X$ is given by the expression

$$L\phi = \int_B \frac{\partial^2 W(\lambda \mathbf{1})}{\partial F^i_\alpha \partial F^j_\beta} \phi^i{}_{,\alpha}(X)\,\phi^j{}_{,\beta}(X)\,\mathrm{d}X, \tag{8.7}$$

where the smooth functions ϕ are required to vanish on ∂B. If W is strongly elliptic then it is well known that $L\phi > 0$ unless $\phi = 0$. The examples of non-uniqueness to the incompressible Cauchy problem in §5.4 and the results of §7 then strongly suggest that $x = \lambda X$ is unstable if $\lambda > \lambda_{cr}$; a proof of this would require a detailed study of radial solutions to the equations of compressible elastodynamics (the related problem of collapse of a spherical bubble in a compressible elastic fluid has been studied by Hunter (1960)). Without hypotheses preventing cavitation the criterion of the second variation thus seems unlikely to be generally valid. We remark that no information about λ_{cr} is obtained by linearizing (1.2) around the trivial solutions $x = \lambda X$; the information obtained by linearization and application of the implicit function theorem in Banach space is merely that $x = \lambda X$ is isolated in various spaces of smooth functions from other equilibria (for the technique see Marsden & Hughes (1978)).

In the dead-load traction problem, bifurcation into a regular non-symmetric equilibrium solution may occur before the critical load P_{cr} is reached. Consider, for example, an incompressible material. Then it is known (see Hill 1957; Beatty 1967) that the trivial solution is a weak relative minimum of the total energy provided $0 < P < P_0$, where

$$P_0 = (\Phi_{,11} - \Phi_{,12} + \Phi_{,1})\big|_{v_1=v_2=v_3=1}. \tag{8.8}$$

(Formula (8.8) may be obtained by setting the second variation of

$$I(v_1, v_2) = \Phi(v_1, v_2, 1/v_1 v_2) - P(v_1 + v_2 + 1/v_1 v_2)$$

at $v_1 = v_2 = 1$ equal to zero.) At $P = P_0$ bifurcation into a non-trivial homogeneous deformation in general occurs. Since P_0 depends only on the behaviour of Φ near $v_1 = v_2 = v_3 = 1$, while P_{cr} depends on the global behaviour of Φ, there is in general no relation between their values. For a neo-Hookean material (see Rivlin 1948),

$$P_0 = 4\mu < 5\mu = P_{cr},$$

while for other materials $P_0 > P_{cr}$. For more information on the bifurcating homogeneous solutions see Ball & Schaeffer (1982). For remarks on loss of spherical symmetry for elastic shells under pressure see Haughton & Ogden (1978).

Finally we remark that it would be interesting to extend our results by relaxing the hypothesis of spherical symmetry. The work of Varley & Cumberbatch (1980) may be relevant here.

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