Dividing & Forking

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The content of this talk is adapted from K. Tent & M. Ziegler, *A Course in Model Theory.*

conventions We fix for all the following a language $\mathcal{L}$, a countable complete theory T with infinite models, and a monster model $\mathfrak{C}$.

We recall some results about indiscernibles:

Definition 1 (Indiscernible)**.** Let I be an infinite linear order and A a set of parameters. A sequence $(a_i)_{i\in I}$ of tuples is said to be *A-indiscernible* if for every $\mathcal{L}(A)$-formula φ and every $i_1 < \cdots < i_n$, $j_1 < \cdots < j_n \in I$:

$$\models \varphi(a_{i_1}, \cdots, a_{i_n}) \leftrightarrow \varphi(a_{j_1}, \cdots, a_{j_n})$$

Definition 2 (Ehrenfeucht-Mostowski type)**.** Let I be an infinite linear order, A a set of parameters, and $(a_i)_{i\in I}$ a sequence of tuples. The *Ehrenfeucht-Mostowski type* $\mathrm{EM}((a_i)_{i\in I}/A)$ is the set of all $\mathcal{L}(A)$-formula φ such that for all $i_1 < \cdots < i_n \in I$, $\models \varphi(a_{i_1}, \cdots, a_{i_n})$.

Lemma 3 (Standard Lemma)**.** *Let A be a set of parameters, $(a_i)_{i\in I}$ an infinite sequence of tuples and J a linear order. Then there is a sequence indexed by J of A-indiscernibles realising* $\mathrm{EM}((a_i)_{i\in I}/A)$.

Definition 4 (Dividing*)**.** We say $\varphi(x, b)$ *k-divides over A* if there is a sequence $(b_i)_{i\in\mathbb{N}}$ of realisations of $\mathrm{tp}(b/A)$ such that $\{\varphi(x, b_i) \mid i \in \mathbb{N}\}$ is k-inconsistent. We also say that φ *divides over A* if there is a k such that φ k-divides over A. Finally, we say that a set of formulas $\pi(x)$ divides over A if $\pi(x)$ implies a formula which divides over A.

*Das heißt *Teilen*

If φ divides over A, then $\{\varphi\}$ divides over A. Conversely, if $\varphi(x,b)$ implies a formula $\psi(x,b')$ which divides, then by adding dummy variables we have :

$$\vDash \forall x \varphi(x,b,b') \to \psi(x,b,b')$$

Since $\psi(x,b,b')$ divides over A, there is a sequence $(b_i,b'_i)_{i\in\mathbb{N}}$ realising $\mathrm{tp}(bb'/A)$ and such that $\{\psi(x,b_i,b'_i) \mid i \in \mathbb{N}\}$ is k-inconsistent, so $\{\varphi(x,b_i,b'_i) \mid i \in \mathbb{N}\}$ is k-inconsistent.

So φ divides over A if and only if $\{\varphi\}$ divides over A. It follows also that a set $\pi(x)$ of formulas divides over A if and only if there is a finite conjunction of formulas of $\pi(x)$ which divides over A.

Examples.

- The formula $x = b$ divides over A if and only if there is infinitely many different elements realising $\mathrm{tp}(b/A)$, which means $b \notin \mathrm{acl}(A)$.
- If a set $\pi(x)$ of formulas is consistent and defined over $\mathrm{acl}(A)$, then it doesn't divide over A.
- In $T = T_{\mathrm{DLO}}$, the formula $b_1 < x < b_2$ 2-divides over the empty set. The set $\{x > a \mid a \in \mathbb{Q}\}$ does not divide over the empty set.

Lemma 5. *A set $\pi(x,b)$ divides over A if and only if there is a sequence $(b_i)_{i\in\mathbb{N}}$ of A-indiscernibles with $\mathrm{tp}(b_0/A) = \mathrm{tp}(b/A)$ and $\bigcup_{i\in\mathbb{N}} \pi(x,b_i)$ is inconsistent.*

Proof. Let $(b_i)_{i\in\mathbb{N}}$ be a sequence of A-indiscernibles with $\mathrm{tp}(b_0/A) = \mathrm{tp}(b/A)$ and $\bigcup_{i\in\mathbb{N}} \pi(x,b_i)$ inconsistent. So, there is a conjunction $\varphi(x,b)$ of formulas from $\pi(x,b)$ such that $\Sigma(x) = \{\varphi(x,b_i) \mid i \in \mathbb{N}\}$ is inconsistent. By compactness there is a finite inconsistent subset of $\Sigma(x)$ of size k, but since the b_i are indiscernibles, $\Sigma(x)$ is k-inconsistent.

Conversely, if $\pi(x,b)$ divides over A, then there is a conjunction $\varphi(x,b)$ of formulas from $\pi(x,b)$ which divides over A. So, there is a sequence $(b_i)_{i\in\mathbb{N}}$ of realisations of $\mathrm{tp}(b/A)$ such that $\{\varphi(x,b_i) \mid i \in \mathbb{N}\}$ is k-inconsistent. By lemma 3, there is a sequence $(c_i)_{i\in\mathbb{N}}$ A-indiscernible with the same property; $\bigcup_{i\in\mathbb{N}} \pi(x,c_i)$ is then inconsistent. □

Corollary 6. *The following are equivalent:*

1) $\mathrm{tp}(a/Ab)$ *does not divide over A.*

2) For any A-indiscernible sequence $(b_i)_{i\in I}$ containing b, there exists some a' with $\operatorname{tp}(a'/Ab) = \operatorname{tp}(a/Ab)$ and such that $(b_i)_{i\in I}$ is Aa'-indiscernible.

2′) For any A-indiscernible sequence $(b_i)_{i\in I}$ containing b, there exists a sequence $(b'_i)_{i\in I}$ with $\operatorname{tp}((b'_i)_{i\in I}/Ab) = \operatorname{tp}((b_i)_{i\in I}/Ab)$ and such that $(b'_i)_{i\in I}$ is Aa-indiscernible.

2) For any A-indiscernible sequence $(b_i)_{i\in I}$ containing b, there exists a sequence $(b^*_i)_{i\in I}$ and some a^* with $\operatorname{tp}((b^*_i)_{i\in I}/Ab) = \operatorname{tp}((b_i)_{i\in I}/Ab)$, $\operatorname{tp}(a^*/Ab) = \operatorname{tp}(a/Ab)$ and such that $(b^*_i)_{i\in I}$ is Aa^*-indiscernible.*

Proof. It is immediate to see that 2)⇒2*) and that 2′)⇒2*). For the converse, since $\operatorname{tp}(a^*/Ab) = \operatorname{tp}(a/Ab)$, we can take an automorphism σ fixing Ab pointwise and taking a^* to a. Then $(b'_i)_{i\in\mathbb{N}} = (\sigma(b^*_i))_{i\in\mathbb{N}}$ suits for 2′). Choosing instead an automorphism taking each b^*_i to b_i gives us 2).

1)⇒2*): Let $(b_i)_{i\in I}$ be an infinite sequence of A-indiscernibles with $b_{i_0} = b$. Let $p(x,y) = \operatorname{tp}(ab/A)$ and consider $p(x,b) = \operatorname{tp}(a/Ab)$. Since it doesn't divide, by lemma 5 $\bigcup_{i\in I} p(x,b_i)$ is consistent. Let a^* be a realisation. By lemma 3, there is $(b'_i)_{i\in I}$ Aa^*-indiscernible realising $\operatorname{EM}((b_i)_{i\in I}/Aa^*)$. Since $\models p(a^*, b'_{i_0})$, there is an automorphism σ fixing Aa^* pointwise and taking b'_{i_0} to b. Then 2*) holds with $(b^*_i)_{i\in I} = (\sigma(b'_i))_{i\in I}$.

2)⇒1): Let $p(x,y) = \operatorname{tp}(ab/A)$ and let $(b_i)_{i\in\mathbb{N}}$ be A-indiscernible with $\operatorname{tp}(b_0/A) = \operatorname{tp}(b/A)$. By 2) there is a' with $\operatorname{tp}(a'/A) = \operatorname{tp}(a/A)$ such that $(b_i)_{i\in\mathbb{N}}$ is Aa' indiscernible. Since $\models p(a',b)$, a' realises $\bigcup_{i\in I} p(x,b_i)$, therefore $p(x,b) = \operatorname{tp}(a/Ab)$ doesn't divide over A. □

Proposition 7 (Transitivity)**.** *If $\operatorname{tp}(a/B)$ does not divide over $A \subset B$ and $\operatorname{tp}(c/Ba)$ does not divide over Aa, then $\operatorname{tp}(ac/B)$ does not divide over A.*

Proof. Let $b \in B$ be a tuple and $(b_i)_{i\in I}$ a sequence of A-indiscernibles containing b. $\operatorname{tp}(a/B)$ doesn't divide over A, so $\operatorname{tp}(a/Ab)$ doesn't divide over A, and by corollary 6 there is a sequence $(b'_i)_{i\in I}$ Aa-indiscernible such that $\operatorname{tp}((b'_i)_{i\in I}/Ab) = \operatorname{tp}((b_i)_{i\in I}/Ab)$. Now $\operatorname{tp}(c/Ba)$ doesn't divide over A, so there is a sequence $(b''_i)_{i\in I}$ Aac-indiscernible such that $\operatorname{tp}((b''_i)_{i\in I}/Aab) = \operatorname{tp}((b'_i)_{i\in I}/Aab)$. This means that $\operatorname{tp}(ac/Ab)$ does not divide over A for any b, therefore $\operatorname{tp}(ac/B)$ does not divide over A. □

Definition 8 (Forking[†])**.** A set of formulas $\pi(x)$ *forks over A* if $\pi(x)$ implies a disjunction $\bigvee_{1\leqslant j\leqslant n} \varphi_j(x,b_j)$, with each of the $\varphi_j(x,b_j)$ dividing over A.

[†]Das heißt *Forken* oder *Gabeln*

If $\pi(x)$ divides over A, then it forks over A. The converse is not true in general.

Example. We define the cyclical order in $\mathbb{Q}$ by:

$$\mathrm{cyc}(a,b,c) \Leftrightarrow (a<b<c) \vee (b<c<a) \vee (c<a<b)$$

Then, in $T = \mathrm{Th}(\mathbb{Q}, \mathrm{cyc})$, the unique type over the empty set forks but does not divide over the empty set.

By compactness, we have the following:

Proposition 9 (non-forking closeness)**.** *If $p \in S(B)$ forks over A, there is some $\varphi(x) \in p$ such that any $q \in S(B)$ containing φ forks over A.*

Corollary 10 (finite character)**.** *If $p \in S(B)$ forks over A, then there is a finite $B_0 \subset B$ such that $p|_{AB_0}$ forks over A.*

Lemma 11. *If π is finitely satisfiable in A, then it doesn't fork over A.*

Proof. If π is finitely satisfiable in A and implies a disjunction $\bigvee_{1\leqslant j\leqslant n} \varphi_j(x,b_j)$, one of the $\varphi_j(x,b_j)$ must be realised by some $a \in A$. Now for any sequence $(b_i)_{i\in\mathbb{N}}$ of realisations of $\mathrm{tp}(b_j/A)$, a realises $\{\varphi(x,b_i) \mid i \in \mathbb{N}\}$, which is therefore consistent. $\square$

Lemma 12. *Let $A \subset B$ and let π be a partial type over B. If π does not fork over A, then it can be extended to a $p \in S(B)$ which does not fork over A.*

Proof. Let $p(x)$ be a maximal non-forking over A set of $\mathcal{L}(B)$-formulas containing $\pi(x)$. p is consistent, and complete: if φ is a $\mathcal{L}(B)$-formula such that both φ and $\neg\varphi$ don't belong to p, then both $p \cup \{\varphi\}$ and $p \cup \{\neg\varphi\}$ fork over A, but then p itself forks over A. $\square$