Model theory of exponential maps of Abelian varieties

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Exponential Maps of Abelian Varieties

- A a complex abelian variety,
  - i.e. a complex connected projective algebraic group
- $g := \dim(A)$
- $\Lambda \hookrightarrow \mathbb{C}^g \xrightarrow{\pi} A(\mathbb{C})$
- $\Lambda \cong \mathbb{Z}^{2g}$ is a lattice in $\mathbb{C}^g$
- e.g. elliptic curve ($g = 1$) in Weierstrass form, $\pi = (\wp, \wp')$
- $A_n := \{\gamma \in A(\mathbb{C}) \mid n\gamma = 0\} \cong \left(\mathbb{Z}/n\mathbb{Z}\right)^{2g}$
- $A_\infty := \bigcup_n A_n$
- $\mathcal{O} := \{\theta \in \mathrm{End}_{\mathbb{C}}(\mathbb{C}^g) \mid \theta(\Lambda) \subseteq \Lambda\}$
- Induced maps on $A(\mathbb{C})$ are the algebraic endomorphisms, $\mathcal{O} \cong \mathrm{End}(A)$
- If A simple, i.e. no proper infinite algebraic subgroups, then $k_{\mathcal{O}} := \mathbb{Q} \otimes \mathcal{O}$ is a skew field.

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T_A

- Standing assumptions:
  - A simple
  - A defined over a number field k_0
  - All $\theta \in \mathrm{End}(A)$ defined over k_0
  - Exists "unfurled curve" $C \subseteq A$ defined over k_0
- Structure Cov_A
  - "Covering sort" V for $\mathbb{C}^g$ as a $k_{\mathcal{O}}$-vector space
  - "Field sort" F for $\mathbb{C}$ as a field with constants for k_0
  - Function $\pi : \mathbb{C}^g \to A(\mathbb{C})$, A a definable set in F
- $T_A := \mathrm{Th}(\mathrm{Cov}_A)$
- T_A has QE and is axiomatised by
  - V is a $k_{\mathcal{O}}$-vector space
  - $F \models \mathrm{ACF}^{k_0}$
  - $\pi : V \twoheadrightarrow A(F)$ is a surjective map of $\mathcal{O}$-modules
- T_A is superstable

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Classification theory

Non-elementary classification theory:

- "$A = \mathbb{G}_m$"
  - $2\pi i\mathbb{Z} \hookrightarrow \mathbb{C} \xrightarrow{\exp} \mathbb{C}^\times$
  - Zilber: Class of models of $T_{\mathbb{G}_m}$ with kernel isomorphic as a group to $\mathbb{Z}$ is Quasi-Minimal Excellent (QME), so has one model in each uncountable cardinality.
- $A = E$ an elliptic curve, $\mathcal{O} \cong \mathbb{Z}$
  - B-Gavrilovich-Zilber: QME
- In general:
  - Zilber: "arithmetic" conditions under which we have Almost Quasi-Miminal Excellence, hence uncountable categoricity

Elementary classification theory:

- $A = E$, $\mathcal{O} \cong \mathbb{Z}$
  - B-Pillay: T_E is "classifiable"
    - i.e. has $< 2^\lambda$ models of cardinality λ for arbitrarily large λ
    - i.e. has NDOP, is shallow, and has NOTOP / Primary Models Over independent Pairs

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Classification

Theorem (B-Gavrilovich-Zilber, B-Pillay)

$\mathcal{M} \models T_A$ is determined up to isomorphism by

- *isomorphism type of* $\ker(\pi)$
- *isomorphism type of* $\operatorname{im}(\pi)$
  - *(i.e. by* $\operatorname{trd}(F(\mathcal{M}))$*)*

Independent systems

Definition

- An *independent* $\mathcal{P}(N)$*-system in* ACF is a system $\mathcal{L} = (L_s)_{s\in\mathcal{P}(N)}$ such that
  - $L_s \models$ ACF
  - $s \subseteq t \implies L_s \preceq L_t$
  - there exists a system of sets $(B_s)_{s\in\mathcal{P}(N)}$ such that
    - B_s is an acl-basis of L_s
    - $B_{s\cap t} = B_s \cap B_t$
- $\mathcal{P}^-(N) := \mathcal{P}(N) \setminus \{N\}$
- The "A-boundary" of $\mathcal{L}$ is the submodule $\partial_A\mathcal{L} := A_\infty + \Sigma_{s\in\mathcal{P}^-(N)} A(L_s) \leq A(L_N)$.
  - $N = 0$: $\partial_A\mathcal{L} = A_\infty$
  - $N = 1$: $\partial_A\mathcal{L} = A(L_\emptyset)$

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Atomicity

If $\mathcal{M} \models T_A$ and $B \subseteq V(\mathcal{M})$, let $\mathrm{cl}(B) \preceq \mathcal{M}$ be the submodel such that

$$\ker(\mathrm{cl}(B)) = \ker(\mathcal{M})$$
$$F(\mathrm{cl}(B)) = \mathrm{acl}^{F(\mathcal{M})}(\pi(B)).$$

Lemma (Atomicity Lemma)

Suppose

- $\mathcal{M} \models T_A$
- $D = \partial_A \mathcal{L}$ *an* A*-boundary*
- $\overline{\alpha} \in V(\mathcal{M})$
- $B := \pi^{-1}(D) \cup \overline{\alpha} \subseteq V(\mathcal{M})$

Then

- $\mathrm{cl}(B)$ *is atomic over* B
- *If* $\mathrm{trd}(F(\mathcal{M})) \leq \aleph_0$, *follows that* $\mathrm{cl}(B)$ *is prime and minimal over* B.

N-uniqueness

Fact (N-uniqueness for ACF)

Suppose

- $\mathcal{L} = (L_s)_{s \in \mathcal{P}(N)}$ *is an independent* $\mathcal{P}(N)$*-system in* ACF
- $\sigma_s \in \mathrm{Aut}(L_s)$ *for* $s \in \mathcal{P}^-(N)$ *is a coherent system of automorphisms*
  - *i.e.* $t \subseteq s \implies \sigma_s\restriction_{L_t} = \sigma_t$

Then $\bigcup_{s \in \mathcal{P}^-(N)} \sigma_s$ *is partial elementary (and so extends to* $\sigma_N \in \mathrm{Aut}(L_N)$*).*

Unfurled curves

- We assumed:
  - Exists "unfurled curve" $C \subseteq A$ defined over k_0
- $C \subseteq A$ is "unfurled" iff $[n]^{-1}(C)$ is absolutely irreducible for all $n \in \mathbb{N}$
- Fact [Gavrilovich]: such exist over $\bar{\mathbb{Q}}$
- If $\overline{c} \in C^n$ is generic over $\bar{\mathbb{Q}}$, and $\overline{\alpha}_1, \overline{\alpha}_2 \in \pi^{-1}(\overline{c})$, then $\overline{\alpha}_1 \equiv_{\mathrm{cl}(\emptyset)} \overline{\alpha}_2$
  - (By QE, $\mathrm{tp}(\overline{\alpha}_i / \mathrm{cl}(\emptyset))$ is determined by $\mathrm{tp}^{field}((\pi(\overline{\alpha}_i/n))_{n\in\mathbb{N}})/\bar{\mathbb{Q}})$)

Proof of Classification Theorem

- Given $\mathcal{M}^1, \mathcal{M}^2$ s.t. $\ker(\mathcal{M}^1) \cong \ker(\mathcal{M}^2)$ and $\mathrm{trd}(F(\mathcal{M}^1)) = \mathrm{trd}(F(\mathcal{M}^2)) =: \lambda$
- Let $\mathcal{M}^i_\emptyset := \mathrm{cl}^{\mathcal{M}^i}(\emptyset)$
- $\mathcal{M}^i_\emptyset$ is prime and minimal over $\ker(\mathcal{M}^i)$, so $\mathcal{M}^1_\emptyset \cong \mathcal{M}^2_\emptyset$
- So assume $\mathcal{M}^1_\emptyset = \mathcal{M}^2_\emptyset =: \mathcal{M}_\emptyset$
- Take acl-bases $(c^i_j)_{j\in\lambda}$ of $C(\mathcal{M}^i)$
- By unfurledness, $(c^1_j)_j \equiv_{\mathcal{M}_\emptyset} (c^2_j)_j$
- Using N-uniqueness, inductively find a coherent system of isomorphisms

$$\sigma_s : \mathrm{cl}^{M_1}((c^1_j)_{j\in s}) \to \mathrm{cl}^{M_2}((c^2_j)_{j\in s})$$

  for $s \subseteq_{\mathrm{fin}} \lambda$
- Take limit $\sigma := \bigcup_{s\subseteq_{\mathrm{fin}}\lambda} \sigma_s : \mathcal{M}^1 \to \mathcal{M}^2$. □

Proof of Classification Theorem

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- Take limit $\sigma := \bigcup_{s\subseteq_{\mathrm{fin}}\lambda} \sigma_s : \mathcal{M}^1 \to \mathcal{M}^2$. $\square$

Kummer Pairings

Definition

- For l prime, we have the Tate module $T_l := \varprojlim A_{l^n} \cong \mathbb{Z}_l^{2g}$
- and their product $T_\infty := \varprojlim A_m \cong \Pi_l T_l \cong \hat{\mathbb{Z}}^{2g}$.
- Let $k \geq k_0$ and let $k_\infty := k(A_\infty)$.
- Define bilinear maps:

$$\langle \cdot, \cdot \rangle_n : \mathrm{Gal}(\bar{k}/k_\infty) \times A(k_\infty) \to A_n$$
$$\langle \sigma, a \rangle_n = \sigma b - b$$

  for any b such that $nb = a$.
- Taking limits, we have

$$\langle \cdot, \cdot \rangle_{l^\infty} : \mathrm{Gal}(\bar{k}/k_\infty) \times A(k_\infty) \to T_l$$
$$\langle \cdot, \cdot \rangle_\infty : \mathrm{Gal}(\bar{k}/k_\infty) \times A(k_\infty) \to T_\infty$$

Thumbtack Lemma

The Atomicity Lemma reduces to:

Lemma (Thumbtack Lemma)

Let

- *D be an A-boundary*
- *$\overline{\gamma} \in A$ be an n-tuple $\mathcal{O}$-linearly independent over D*
- *$k := k_0(D, \overline{\gamma})$*

Then $M_\infty := \langle \mathrm{Gal}(\bar{k}/k), \overline{\gamma} \rangle_\infty$ is of finite index in T_∞^n.

Kummer Theory

Theorem (Bashmakov-Ribet, Bogomolov-Serre, Faltings)

Let

- $k \geq k_0$ *a number field*
- $\overline{\gamma} \in A(k)$ *an* $\mathcal{O}$*-linearly independent n-tuple*
- $k_\infty := k(A_\infty)$

Then $M_\infty := \langle \mathrm{Gal}(\bar{k}/k_\infty), \overline{\gamma} \rangle_\infty$ *is of finite index in* T_∞^n.

Kummer Theory: sketch proof

Let l prime

- Let $M_l := \langle \mathrm{Gal}(\bar{k}/k_\infty), \overline{\gamma} \rangle_l \leq A_l^n$
- Let $G_l := \restriction_{A_l} \mathrm{Gal}(\bar{k}/k)$
- M_l is G_l-invariant: if $G_l \ni \tau =\restriction_{A_l} \tau'$ and $\sigma \in \mathrm{Gal}(\bar{k}/k_\infty)$, then

$$\begin{aligned}
\tau \langle \sigma, \overline{\gamma} \rangle_l &= \tau(\sigma\overline{\beta} - \overline{\beta}) && (l\overline{\beta} = \overline{\gamma}) \\
&= \tau'\sigma\overline{\beta} - \tau'\overline{\beta} \\
&= (\tau'\sigma\tau'^{-1})(\tau'\overline{\beta}) - \tau'\overline{\beta} \\
&= \left\langle \tau'\sigma\tau'^{-1}, \overline{\gamma} \right\rangle_l && (l\tau'\overline{\beta} = \tau'\overline{\gamma} = \overline{\gamma}),
\end{aligned}$$

  and $\tau'\sigma\tau'^{-1} \in \mathrm{Gal}(\bar{k}/k_\infty)$.
- So M_l is a $\mathbb{Z}[G_l]$-submodule of A_l^n

Kummer Theory: sketch proof

For cofinitely many l,

- we have:
  - (a) A_l is semisimple as a $\mathbb{Z}[G_l]$-module (Faltings)
  - (b) $\mathrm{End}_{G_l}(A_l) = {}^{\mathcal{O}}/_{l\mathcal{O}}$ (Faltings)
  - (c) $\overline{\gamma}$ is ${}^{\mathcal{O}}/_{l\mathcal{O}}$-linearly independent in ${}^{A(k)}/_{lA(k)}$ (Mordell-Weil)
  - (d) $H_1(\mathrm{Gal}(k_\infty/k), A_l) = 0$ (Bogomolov-Serre)
- Suppose $M_l \neq A_l^n$
- By (a) and (b), M_l is annihilated by some $\overline{\eta} \in \mathcal{O}^n \setminus l\mathcal{O}^n$
- So $\forall \sigma.\ \langle \sigma, \Sigma \eta_i a_i \rangle_l = 0$
- So $\Sigma \eta_i a_i \in lA(k_\infty)$
- By (d), $\Sigma \eta_i a_i \in lA(k)$
- By (c), $\eta_i \in l\mathcal{O}$ - contradiction.
- So $M_l = A_l^n$

It follows that $M_{l^\infty} := \langle \mathrm{Gal}(\bar{k}/k_\infty), \overline{\gamma} \rangle_{l^\infty} \leq T_l^n$ is the whole of T_l^n.

For the finitely many remaining primes, similar but l-adic argument shows M_{l^∞} is finite index in T_l^n. □

Proving the Thumbtack Lemma

Lemma (Thumbtack Lemma)

Let

- *D be an a A-boundary*
- *$\overline{\gamma} \in A$ be an n-tuple $\mathcal{O}$-linearly independent over D*
- *$k := k_0(D, \overline{\gamma})$*

Then $M_\infty := \langle \mathrm{Gal}(\bar{k}/k), \overline{\gamma} \rangle_\infty$ is of finite index in T_∞^n.

- Would like to apply Faltings
- M_∞ is G-invariant where

$$G := \restriction_{A_\infty} (N_{\mathrm{Aut}(\bar{k}/k_0(\overline{\gamma}))}(\mathrm{Aut}(\bar{k}/k_0(D)))$$

- Want that G is "large"
- Specifically, want $G \geq \restriction_{A_\infty} (\mathrm{Gal}(\bar{\mathbb{Q}}/k_1))$ for some number field k_1

Canonical bases over Independent Systems

$$
\begin{aligned}
G &= \restriction_{A_\infty} (\ N_{\mathrm{Aut}(\bar{k}/k_0(\overline{\gamma}))}(\ \mathrm{Aut}(\bar{k}/k_0(D)\ )\ ) \\
&= \restriction_{A_\infty} (\ \{\ \sigma \in \mathrm{Aut}(\bar{k}/k_0(\overline{\gamma})) \mid \sigma(k_0(D)) = k_0(D)\ \}\ ) \\
&= \restriction_{A_\infty} (\ \mathrm{Aut}(\ k_0(D)\ /\ k_0(\mathrm{Cb}(\overline{\gamma}/k_0(D)))\ )\ ),
\end{aligned}
$$

where $\mathrm{Cb}(\overline{\gamma}/k_0(D))$ is a canonical parameter for $\mathrm{locus}(\overline{\gamma}/k_0(D))$ (i.e. the minimal field of definition).

- Suppose $N = 3$
- so $k_0(D) = k_0(A(L_{12}) + A(L_{23}) + A(L_{31})) = L_{12}L_{23}L_{31}$
- Say $\mathrm{Cb}(\overline{\gamma}/k_0(D)) = \mathrm{dcl}(\overline{b}_{12}\overline{b}_{23}\overline{b}_{31})$ where $\overline{b}_{ij} \in L_{ij}$
- Let $\overline{b}_i \in L_i$ such that $\mathrm{Cb}(\overline{b}_{ij}/L_iL_j) \subseteq \mathrm{dcl}(\overline{b}_i\overline{b}_j)$
- Let $\overline{b}_\emptyset := \mathrm{Cb}(\overline{b}_1\overline{b}_2\overline{b}_3/L_\emptyset)$, and let $k_1 := k_0(\mathrm{Cb}(\overline{b}_\emptyset/\bar{\mathbb{Q}}))$
- Then any $\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/k_1)$ extends to $\sigma_\emptyset \in \mathrm{Aut}(L_\emptyset/\overline{b}_\emptyset)$
  - which extends to $\sigma_i \in \mathrm{Aut}(L_i/\overline{b}_i)$
  - 2-uniqueness gives $\sigma'_{ij} \in \mathrm{Aut}(L_iL_j/\overline{b}_i\overline{b}_j)$
  - σ'_{ij} extends to $\sigma_{ij} \in \mathrm{Aut}(L_{ij}/\overline{b}_{ij})$
  - 3-uniqueness gives $\sigma_3 \in \mathrm{Aut}(k_0(D)/\ \mathrm{Cb}(\overline{\gamma}/k_0(D))$

Local Freeness

- Remains to obtain the analogue of Mordell-Weil
- i.e. we want "bounded divisibility" in $A(k_0(D\overline{\gamma}))$ of $\langle\overline{\gamma}\rangle_{\mathcal{O}}$ for $\overline{\gamma} \in A$ linearly independent over D
- i.e. we want

$$A(k_0(D\overline{\gamma}))/_{D} \text{ is locally free}$$

- Locally free (AKA $\aleph_1$-free):
  - countable subgroups are free abelian
  - equivalently: finite rank subgroups are free abelian

Local Freeness

- N=0, i.e. $D = A_\infty$:
  - M. Larsen, 2005: If k is a finite extension of $k_0(A_\infty)$, then ${}^{A(k)}/_{A_\infty}$ is free abelian
- N=1, i.e. $D = A(L)$, $L = \mathrm{acl}(L)$:
  - by Lang-Néron's function field version of Mordell-Weil, ${}^{A(L(\overline{\gamma}))}/_{A(L)}$ is finitely generated hence free
- N>1, i.e. $D = \Sigma_{s \in \mathcal{P}^-(N)} A(L_s)$:
  - inductive argument, involving specialising horns down to the missing simplex...
  - Another story.

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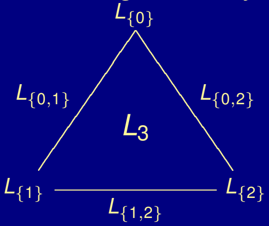
  - Another story.

Sketch Proof of ${}^{E(k)}/_{G}$ locally free

Let $G := \exp(H)$. Proceed by induction on N. $N = 1$: By Lang-Néron, ${}^{E(k)}/_{G}$ is even finitely generated.
Consider case $N = 3$. We have the independent system of algebraically closed fields:



and $k = L_{\{0,1\}}L_{\{1,2\}}L_{\{0,2\}}(\overline{\beta})$ say. We may assume $\overline{\beta} \in L_3$.
Let $\overline{b} \in E(k)^n$.

Lemma

There exists $k_1 \geq_{fin} L_{\{0,1\}}L_{\{0,2\}}(\overline{\beta}, \overline{b})$ and a place $\pi : L_3 \to_{L_{\{1,2\}}} L_{\{1,2\}}$ such that

- $\pi k_1 \subseteq k_1$
- $\pi(L_{\{0,1\}}L_{\{0,2\}}) = L_{\{1\}}L_{\{2\}}$

Sketch Proof of ${}^{E(k)}/_{G}$ locally free *cont*^{d}

Lemma

$\mathrm{pureHull}_{E(k)}(E(k_1)) = \mathrm{pureHull}_{E(k_1)+E(L_{\{1,2\}})}(E(k_1)).$

$$\begin{aligned}\mathrm{pureHull}_{E(k)/G}\left(\left\langle \bar{b}/G \right\rangle\right) &= {}^{\mathrm{pureHull}_{E(k)}(\langle \bar{b}\rangle)}/_{G} \\ &= {}^{\mathrm{pureHull}_{E(k_1)+E(L_{\{1,2\}})}(\langle \bar{b}\rangle)}/_{G} \\ &\leq {}^{\mathrm{pureHull}_{E(k_1)}(\langle \bar{b},\pi(\bar{b})\rangle)}/_{G}\end{aligned}$$

(since if $m(\alpha_{k_1} + \alpha_{L_{\{1,2\}}}) \in \langle \bar{b} \rangle$, then $\gamma := (\alpha_{k_1} + \alpha_{L_{\{1,2\}}}) - \pi(\alpha_{k_1} + \alpha_{L_{\{1,2\}}}) = \alpha_{k_1} - \pi\alpha_{k_1} \in \mathrm{pureHull}_{E(k_1)}(\langle \bar{b}, \pi\bar{b} \rangle)$, and $\gamma = \alpha_{k_1} + \alpha_{L_{\{1,2\}}}$ mod G.
So subgroup of quotient of ${}^{\mathrm{pureHull}_{E(k_1)}(\langle \bar{b},\pi\bar{b}\rangle)}/_{E(L_{\{1\}})+E(L_{\{2\}})}$, which is f.g. by induction, so f.g.