Introduction to geometric stability theory

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Setting
$\mathcal{L}$ a countable language, T a complete theory in $\mathcal{L}$ (with eq), M a model of T, $\mathcal{U}$ a monster model of T.

Recall that an imaginary element is an equivalence class of some $\emptyset$-definable equivalence relation on some cartesian power of $\mathcal{U}$. The typical example is the equivalence relation E defined as follows, for some formula $\varphi(x, y)$: $E(y_1, y_2) \iff \forall x\,(\varphi(x, y_1) \leftrightarrow \varphi(x, y_2))$. Given $a \in \mathcal{U}^m$, the E-equivalence class of a will sometimes be called the *code* of $\varphi(\mathcal{U}, a)$.
We say that T *eliminates imaginaries* if whenever E is a $\emptyset$-definable equivalence relation on some $\mathcal{U}^n$, then there is a $\emptyset$-definable map $f : \mathcal{U}^n \to \mathcal{U}^\ell$ whose fibers are exactly the E-equivalence classes.

Strongly minimal sets
A definable subset D of $\mathcal{U}^n$ (some $n > 0$) is *strongly minimal* if every $\mathcal{U}$-definable subset of D is either finite or cofinite.

One can define a pre-geometry on D, by: if $a \in D$, $A \subset D$, then $a \in \mathrm{cl}(A) \iff a \in \mathrm{acl}(A)$. To obtain a geometry one needs to delete $\mathrm{acl}(\emptyset)$ and quotient by the equivalence relation $a \sim b \iff \mathrm{acl}(a) = \mathrm{acl}(b)$ ($\iff a \in \mathrm{acl}(b) \setminus \mathrm{acl}(\emptyset)$). We denote by (D^*, cl) the geometry thus obtained. Note that most of the axioms of geometry are trivial, the one which matters is "Steinitz's exchange principle": if $a, b \in D$, $A \subset D$ and $a \in \mathrm{acl}(Ab) \setminus \mathrm{acl}(A)$, then $b \in \mathrm{acl}(Aa)$.

Notion of dimension of $D(M)$, for M a model of T, or a subset B of $D(\mathcal{U})$: $\dim(B)$ is the cardinality of a maximal subset I of B such that whenever $a \in I$, then $a \notin \mathrm{acl}(I \setminus \{a\})$. Using the exchange principle it is well-defined.

Morley's theorem.
Theorem (Morley). *If T is κ-categorical for some uncountable κ, then T is λ-categorical for all uncountable cardinal λ.*
Remark: κ-categorical for some uncountable κ
(1) If and only if: T is ω-stable (If $A \subset M \models T$ is countable, then $|S_n(A)| = \aleph_0$) and has no Vaughtian pair (there is no formula $\varphi(x)$ which defines in some model M of cardinality $\kappa > \aleph_0$ an infinite subset of size $< \kappa$ (or even, countable).
(2) Implies: there is a formula $\varphi(x)$ (with parameters in M) such that in every model M of T, M is algebraic over $D := \varphi(M)$, and D is a strongly minimal set. φ is defined over a tuple a of parameters which realises an isolated type over $\emptyset$.
(3) Implies: A consequence of (2) is that each model of T is uniquely determined by its dimension: the cardinality of a subset of $\varphi(M)$ consisting of algebraically independent elements. In particular, up to isomorphism, T will either have only one countable model (in which case it will be totally categorical), or $\aleph_0$. (Baldwin-Lachlan)

Stable formulas, theories

Let $\varphi(\bar{x}, \bar{y})$ be an $\mathcal{L}$-formula, with $|\bar{x}| = |\bar{y}| = n$. We say that $\varphi(\bar{x}, \bar{y})$ is *stable* iff there is no infinite $X \subset \mathcal{U}^n$ such that $\varphi(\bar{x}, \bar{y})$ defines on X a total ordering. We say that T is *stable* if every formula is stable. And otherwise we say that it is *unstable*.

Stable theories (2)
Let λ be an infinite cardinal. We say that T is λ-stable if whenever $A \subset \mathcal{U}$ is of cardinality $\leq \lambda$, then $|S_1(A)| \leq \lambda$. A result of Shelah says that exactly one of the following four cases occur:
(1) T is ω-stable, in which case it is λ-stable for all $\lambda \geq omega$.
(2) T is *superstable*, i.e., T is λ-stable for all cardinals $\geq 2^{\aleph_0}$.
(3) T is λ-stable for all cardinals satisfying $\lambda = \lambda^{\aleph_0}$.
(4) T is unstable.

The Morley rank
Let D be a definable non-empty set. One defines (the *Morley rank of D*) $\mathrm{RM}(D) \geq \alpha$ by induction on α, as follows:
– $\mathrm{RM}(D) \geq 0$,
– if α is a limit ordinal, then $\mathrm{RM}(D) \geq \alpha$ if and only if $\mathrm{RM}(D) \geq \beta$ for all $\beta < \alpha$,
– $\mathrm{RM}(D) \geq \alpha + 1$ if and only if there exists an infinite family $(D_i)_i$ of definable subsets of D which are pairwise disjoint and such that $\mathrm{RM}(D_i) \geq \alpha$ for all i.

Then $\mathrm{RM}(D)$ is the smallest α such that $\mathrm{RM}(D) \geq \alpha$, but $\mathrm{RM}(D) \not\geq \alpha + 1$ if such an α exists, and $+\infty$ if no such α exists. If $\mathrm{RM}(D) = \alpha$, then the *Morley degree* of D is the largest integer such that D can be partitioned into definable sets of Morley rank α.
If p is a type over some set A, then $\mathrm{RM}(p) = \inf\{\mathrm{RM}(D) \mid D \in p\}$.

Zilber's conjecture - Kinds of geometries
Let D be a strongly minimal set (definable in $\mathcal{U}$), and consider the associated geometry (D^*, cl). Three interesting possibilities:

(1) the geometry is *trivial* (also called *disintegrated*): for any $A \subset D$, $\mathrm{acl}(A) = \bigcup_{a \in A} \mathrm{acl}(a)$.
(2) the geometry is non-trivial, and is *locally modular*: If $A, B \subset D$ are algebraically closed and $A \cap B \neq \emptyset$, then A and B are independent over $A \cap B$.
A slightly stronger property is the property of *modularity*: whenever $A, B \subset D$ are algebraically closed, then A and B are independent over $A \cap B$.
(3) the geometry is non-modular.

Examples
Here are some examples:

(1) Consider the theory of the infinite set (in the language with only equality).

It is totally categorical, strongly minimal, and the geometry is trivial.

(2) Let Δ be a division ring, and T the theory of Δ-vector spaces (of infinite dimension if Δ is a finite field). This theory is complete and $\aleph_1$-categorical. The geometry is non-trivial and modular.

(3) Let T be a theory of algebraically closed fields (fix the characteristic). Then T is $\aleph_1$-categorical, and the geometry is non-locally modular.

Zilber's conjecture

Zilber's conjecture stated that these are essentially the only examples: i.e., if $\mathcal{U}$ is strongly minimal, then it is a finite cover of some definable set D which is bi-interpretable with one of the above.
This conjecture was disproved by E. Hrushovski, who constructed examples of strongly minimal theories which are trivial, but do not interpret a group law; and of non-locally modular strongly minimal theories in which two distinct structures of algebraically closed fields are definable. These types of constructions are commonly referred to as: "Hrushovski's constructions", or "amalgamation à la Hrushovski", or "fusion/collapse".

Apart from Hrushovski, many people have used these constructions.

The generic curve

Let me state a result which I find very amusing, and which shows that the exotic structures constructed in the counterexamples ... are not so exotic after all.

For each $d \geq 1$, let T_d the theory of the field of complex numbers, expanded by a binary relation symbol C which is interpreted as the set of points (in $\mathbb{C}^2$) of a curve of total degree d, defined by a polynomial equation $\sum_{i,j} a_{i,j} X^i Y^j = 0$ (with $i + j \leq d$) whose coefficients are algebraically independent over $\mathbb{Q}$. Then the limit theory $T_\omega = \lim_{d\to\infty} T_d$ exists, is ω-stable (of Morley rank ω) and complete, and its countable saturated model can be obtained by a Hrushovski construction.

Zariski geometry

However, Zilber's conjecture remains "morally true". He and Hrushovski isolated conditions which ensured that a strongly minimal set satisfying these axioms satisfies the conclusion of Zilber's conjecture. These are called *Zariski geometries*. They are defined for strongly minimal sets, (or, more generally, for ∞-definable sets of U-rank 1). I will not give the axioms, but let me say that they define a Noetherian topology on each cartesian power of the model, in which there is a good notion of dimension, satisfying in particular a condition on the dimension of intersections of closed sets (if $V_1, V_2 \subset D^n$, and W is an irreducible component of the intersection $V_1 \cap V_2$, then $\dim(W) \geq \dim(V_1) + \dim(V_2) - n$). At least in the case of definable sets, one has elimination of quantifiers in the language with predicates for all closed sets.

Examples
The three typical examples listed above are all Zariski geometries; the closed sets are those defined by positive quantifier-free formulas (i.e., disjunctions of conjunctions of equations). Here is a less obvious example:
The theory DCF_0 of differentially closed fields of characteristic 0. It is known that this theory is ω-stable, of rank ω. It eliminates quantifiers and imaginaries in the language $\{+,-,\cdot,D\}$ of differential fields. Hrushovski and Sokolovic showed that if D is a strongly minimal set definable in a differentially closed field K, then, taking as closed subsets of D^n the sets defined by differential equations (the so-called Kolchin closed sets) defines a Zariski topology on D. It follows that if D is non-modular, then D interprets a field, and other arguments give that this field is necessarily the field $\mathcal{C}$ of constants of K, defined by the equation $Dx = 0$.

These results can be partially generalized to the theory $\mathrm{DCF}_{0,n}$ of diffentially closed fields of characteristic 0 with n commuting derivations $D_1,\ldots,D_n$: by a result of Moosa-Pillay-Scanlon, a non-modular strongly minimal set is necessarily of order 1 (since it is non-orthogonal to the generic of a definable subgroup of $\mathbb{G}_a$ of rank 1, and those are of order 1), and therefore is non-orthogonal to the field $\mathcal{C}$ of absolute constants defined by $D_1x = D_2x = \cdots = D_nx = 0$. It is however unknown what exactly happens with other (locally modular) strongly minimal sets.

Orthogonality
There are several notions of orthogonality. It is usually used in the context where there is a good notion of independence (or *non-forking*; denoted by $\downarrow$), and this is what we will assume. By good, I in particular mean that it is symmetric and transitive.

(1) Let p and q be two complete types over A. p is *almost orthogonal to* q, $p \perp^a q$, if whenever $a \models p$ and $b \models q$, then $a \downarrow_A b$ (a and b are independent over A).
(2) Let p and q be two complete types over A and B respectively. p is *orthogonal to* q, $p \perp q$, if whenever $C \supset A \cup B$, and $a \models p$, $b \models q$, and $a \downarrow_A C$, $b \downarrow_A C$, then $a \downarrow_C b$.

Orthogonality - 2
When p and q are types of rank 1, then $a \downarrow_A C$ simply means $a \notin \mathrm{acl}(C)$, and $a \downarrow_C b$ means $a \notin \mathrm{acl}(Cb) \setminus \mathrm{acl}(C)$ (or $b \notin \mathrm{acl}(Ca) \setminus \mathrm{acl}(C)$).

If the ambient theory is stable and eliminates imaginaries, and $A = \mathrm{acl}(A)$, then the almost-orthogonality of the types $p, q \in S(A)$ can be rephrased as: $p(x) \cup q(y)$ is complete.

If $p, q \in S(A)$, one can also show that $p \not\perp q$ if and only if there are A-

independent $a_1, \dots, a_n$ realising p, A-independent $b_1, \dots, b_n$ realising q, and such that $a_1, \dots, a_n \not\downarrow_A b_1, \dots, b_n$, for some n.

Full orthogonality
In the same spirit, one will say that two definable sets D_1 and D_2 are *fully orthogonal* if whenever $S \subset D_1^m \times D_2^n$ is definable (some m, n), then S is a finite union of boxes $R_1 \times R_2$ where $R_1 \subset D_1^m$ and $R_2 \subset D_2^n$ are definable.

The trichotomy becomes a dichotomy? One-basedness
Let D be strongly minimal. If the geometry on D is trivial, then it satisfies (for trivial reasons) the local modularity axiom. Thus, as was suggested already in the last part of the example on fields with several derivations, what some people really care about is: what exactly is the structure on a non-modular strongly minimal set? Does it have more (or less) structure than a pure field?
It turns out that the property of local modularity generalises to types of arbitrary rank and to definable sets, or ∞-definable sets.

One-basedness
Let $A \subset \mathcal{U}$, and $S \subset \mathcal{U}^n$ be $\mathrm{Aut}(\mathcal{U}/A)$-invariant. Then S is *one-based* if whenever $a = (a_1, \dots, a_m) \in S^m$ and $B \supset A$, then

$$a \downarrow_C B, \text{ where } C = \mathrm{acl}^{eq}(Aa) \cap \mathrm{acl}^{eq}(B).$$

A type is *one-based* if the set of its realisations is one-based. Thus non-locally modular types of rank 1 are not one-based and conversely. The interest of one-basedness is that it is stable under unions and fibrations: if S_1 and S_2 are one-based, then so is $S_1 \cup S_2$; thus, an $\mathrm{Aut}(\mathcal{U}/A)$-invariant set S is one-based if and only if every type realised in S is one-based; if $tp(a/A)$ is one-based and $tp(b/Aa)$ is one-based, then so is $tp(ab/A)$.

Zariski geometries outside the strongly minimal context.
Assume that we have an A-definable (or ∞-definable) set D in which acl defines a pre-geometry (ie, satisfies the Steinitz' exchange principle). Then, as above, one can distinguish between the three types of geometries: trivial, non-trivial locally modular, and non locally modular. Here are two important examples:

Separably closed fields of finite degree of imperfection
Let $e \in \mathbb{N}^{>0}$, p a prime, and consider the theory $\mathrm{SCF}_{e,p}$ of separably closed fields of imperfection degree e (i.e., $[K : K^p] = p^e$. If one adds to the language e distinct constant symbols, and says that these e elements form a p-basis, then one obtains a complete theory which eliminates imaginaries, and quantifiers in a language $\mathcal{L}_\lambda$ obtained as follows:
Let $b_1, \dots, b_e$ be a p-basis of K, and let B be the set of p-monomials $\{b_1^{n_1} \cdots b_e^{n_e} \mid 0 \le n_i < p \text{ for } i = 1, \dots, e\}$. Then B forms a basis of the K^p-vector space K. To

each $b \in B$, we associate the function λ_b which to an element $a \in K$ associates the p-th root of the b-coordinate of a with respect to the basis B. I.e.,

$$a = \sum_{b \in B} \lambda_b(a)^p b.$$

SCF (ctd)
Let $\mathrm{SCF}^{\lambda}_{e,p}$ be the obvious expansion of $\mathrm{SCF}_{e,p}$ to this richer language. Then $\mathrm{SCF}^{\lambda}_{e,p}$ eliminates quantifiers. Hrushovski and Delon have shown that if p is a type of U-rank 1 (i.e., if P is the set of realisations of p in $\mathcal{U}$, then (P, acl) is a pre-geometry), then P, endowed with the topology whose closed sets are those defined by equations of the language, satisfies the (appropriately modified) axioms of a Zariski geometry. This result allowed Hrushovski to prove the Mordell-Lang conjecture in positive characteristic for certain Abelian varieties. (Sofar, his proof is the only existing proof).

Existentially closed difference fields
A difference field is a field with a distinguished automorphism σ, a structure in the language of rings expanded by a unary function symbol. Any complete theory of e.c. difference fields is supersimple of SU-rank ω. One shows that to certain types of SU-rank 1, one can associate a set of points P, in some limit structure (non first-order, model of a Robinson theory), and which satisfies an appropriate version of the axioms for Zariski geometries. This allows one to show that if p is a non-modular type of rank 1, then p is non-orthogonal to the type of a fixed field, i.e., to the field defined by $\sigma(x) = x$, or in addition, if the characteristic is $p > 0$, to a field defined by an equation $\sigma^m(x) = x^{p^n}$, where m, n are relatively prime integers, with $m > 0$.

Variations on the theme of existentially closed difference fields allow one to obtain similar results for the theory of e.c. difference differential fields (the automorphism commuting with the derivation): a non-locally modular type of SU-rank 1 will be non-orthogonal to the type defined by $\sigma(x) = x \wedge Dx = 0$.

O-minimal structures
An important example of a structure with a good independence notion, is an o-minimal structure. Recall that a totally ordered structure $(M, <, \dots)$ is *o-minimal* iff every definable subset D of M is a finite union of open intervals with extremities in M ($\cup\{-\infty, +\infty\}$ if the ordering on M has no endpoints).
This definition was first introduced by Van den Dries; the first general important results were obtained by Pillay, Steinhorn, and Knight. O-minimal structures have been the object of intensive study since then.

O-minimal structures - 2

(M, acl) is a pre-geometry, and Peterzil and Starchenko show that it satisfies a trichotomy theorem: let $a \in M$; if the geometry in an open interval around a is non-locally modular, then there is an open interval around a which inteprets a field; similar local statements for locally modular or disintegrated geometries around a.
Note that one cannot hope for bi-interpretability: it is known for instance that the field of real numbers expanded by the exponential map exp is o-minimal (Wilkie). And clearly the exponential map is not definable in the pure field structure.

Polynomially bounded expansions of the reals
Let $\mathcal{R} = (\mathbb{R}, +, -, \cdot, <, 0, 1, \dots)$ be an expansion of the field of real numbers, which is o-minimal. Then $\mathcal{R}$ is *polynomially bounded* if whenever $f : \mathbb{R} \to \mathbb{R}$ is a definable function, then for some integer $n \geq 0$, one has $\lim_{x \to +\infty} f(x) < x^n$. C. Miller has shown that $\mathcal{R}$ is polynomially bounded if and only if it does not define the exponential function. This result extends to any o-minimal expansion of an arbitrary real closed field R, with polynomially bounded being replaced by *power bounded* (a power function being a multiplicative homomorphism $f : R^\times \to R^\times$ which satisfies $f(x)/xf'(x)$ is constant).