Some model theory of covers
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Introduction

- The talk will discuss recent and current pieces of work with Hrushovski-Peterzil, Bays, and Berarducci-Peterzil.
- It will be about group covers in various senses. I will concentrate on the first order context, but also make some not very deep comments on infinitary categoricity.
- Among the inspirations are the works of Zilber and his students around the model theory of universal covers, which I guess were themselves inspired and motivated by the problem of understanding the complex exponential field.

Commutative algebraic groups I

- Let G be the group $(\mathbb{C}, +)$ as a structure, and consider the 2-sorted structure consisting of G, $(\mathbb{C}, +, \cdot)$ and $\pi : G \to \mathbb{C}^*$ the exponential map from the first sort into the second sort.
- Equivalently we could consider the 3-sorted structure which in addition has a sort for the kernel of π, which we identify with $(\mathbb{Z}, +)$ and a symbol for the inclusion of the kernel in G.
- Or just consider the one sorted structure $(G, +)$ equipped with a predicate for the kernel of π and with the field structure on the quotient. (So this a group in the sense of model theory.)
- More generally we could take G to be the universal cover of any commutative complex algebraic group H.
- In any case let M denote this structure.

Commutative algebraic groups II

- From QE results of Boris we know that M, or $Th(M)$, is a superstable group of U-rank 2, in which both the kernel $(\mathbb{Z},+)$ and the quotient $(\mathbb{C},+,\cdot)$ are "stably embedded", i.e. acquire no additional structure.
- The kernel and the quotient are groups of U-rank 1, Morley rank 1, respectively, and their generic types are the only regular types up to nonorthogonality: $Th(M)$ is "2-dimensional".
- Also the theory of the kernel, $Th(\mathbb{Z},+)$, although not ω-stable, has a "classifiable" class of models: any model is of the form "elementary substructure of the profinite completion of $\mathbb{Z}$" direct sum a $\mathbb{Q}$-vector space.

Commutative algebraic groups III

- Let us remark that M is NOT interpretable in the 2-sorted structure $((\mathbb{Z},+),(\mathbb{C},+,\cdot))$, consisting of the kernel and quotient with no relations between them.
- Because if it were, then the basic model theory of finite rank superstable groups would imply that G would definably split as a direct sum of the kernel and quotient, which it does not even do as an abstract group. However:

Theorem 0.1

M is (naturally) interpretable in the 2-sorted structure $((\mathbb{Z},+),(\mathbb{R},+,\cdot))$, where we identify $\mathbb{C}$ with $\mathbb{R}\times\mathbb{R}$.

This is a special case of the following result:

Theorem 0.2

(HPP) Let M_0 be an o-minimal expansion of the real field, H a connected Lie group definable in M_0, and $\pi : G \to H$ its universal covering group. Then the structure $((G, \cdot), M_0, \pi)$ is naturally interpretable in the 2-sorted structure $((\Gamma, +), M_0)$, where we denote the group operation of the (central) kernel Γ additively.

- The proof makes use of results of Edmundo and Eleftheriou.
- Essentially $G \to H$ is isomorphic to an Ind-definable (in M_0) $G_1 \to H$ and by Skolem functions the latter has a definable (in M_0) section s, yielding a cocycle from $H \times H$ to Γ definable in (Γ, M_0), from which we define a copy of G in (Γ, M_0).

Commutative algebraic groups V

- Let us return to the superstable group M: $((\mathbb{C},+),(\mathbb{C},+,.),\pi))$ from the beginning of the talk.
- Zilber, Gavrilovich, Bays,.. investigated the uncountable categoricity of the $L_{\omega_1,\omega}$ sentence consisting of the first order theory of M together with a sentence pinning down the isomorphism type of the kernel as $(\mathbb{Z},+)$, as well as analogues with elliptic curves in place of $\mathbb{G}_m$.
- As Martin mentioned in his talk, the methods involving the action of Galois on torsion points and division sequences, apertain to the first order theory of M, and one obtains, in spite of the non-interpretability result mentioned earlier, the following:

Commutative algebraic groups VI

Theorem 0.3

(BP) ANY model of $Th(M)$ is determined by the isomorphism type of the kernel and the isomorphism type of the quotient (as an algebraically closed field). In particular $Th(M)$ has $NOTOP$,or equivalently in this situation $PMOP$, existence of prime models over independent pairs.

- Analogous results hold for suitable abelian varieties in place of $\mathbb{G}_m$.
- So for these universal cover issues (but NOT for the full structure $(\mathbb{C}, +, \cdot, exp)$) the relative categoricity results are really at the first order level.

Lie groups I

- Given the work on infinitary categoricity of universal covers of algebraic groups, we were naturally curious about the Lie group, or o-minimal analogues.
- The situation, as already referred to above, consists of an an o-minimal expansion M_0 of the real field, a connected Lie group H definable in M_0, $\pi : G \to H$ the universal cover of H (as a topological group) and M the structure $((G, \cdot), M_0, \pi)$ with parameters if necessary from M_0 which are needed to define H. Let Γ denote $ker(\pi)$.
- What can be said regarding (i) the categoricity of the $L_{\omega_1,\omega}$ theory of M, (ii) the categoricity, relative to the sorts of the kernel and field sort, of the first order theory of M?

Lie groups II

- One might think that (ii) has a positive answer because of the interpretability result Theorem 0.1 that M is interpretable in $((\Gamma, +), M_0)$. But this is NOT not a bi-interpretability result.
- However one can prove that there is, in M, an $L_{\omega_1,\omega}$-definable (over $\emptyset$) section s of the cover $G \to H$.
- This makes use again of the existence of an ind-definable, in M_0, copy $G_1 \to H$ of $G \to H$, as well as the definability in M of the induced isomorphism between G/Γ' and G_1/Γ' for each finite index subgroup Γ' of Γ.
- From the section s one obtains a definable bijection $H \times \Gamma \to G$, and thus:

Theorem 0.4

(BPP) The $L_{\omega_1,\omega}$-theory of M is outright categorical.

Lie groups III

- A special case of relative categoricity of the first order theory of M, would be relative categoricity with respect to the M_0-sort of the $L_{\omega_1,\omega}$-sentence consisting of $Th(M)$ plus a sentence fixing the isomorphism type of $(\Gamma, +)$ which is the precise analogue of the situation studied by Zilber et al.
- This is FALSE, even in the simplest case where H is $\mathbb{S}^1$, and so G is $(\mathbb{R}, +)$ and Γ is $(\mathbb{Z}, +)$.
- The reason is rather boring. This "real" context is the opposite of the "algebraic" context. There is no Galois action on torsion: the torsion points are in $dcl(\emptyset)$. Moreover the structure M_0 is itself rigid (no automorphism).
- The map $\hat{\pi}$ from G to H^ω taking g to $(\pi(g/n))_n$ is an embedding, and the isomorphism type of a model of $Th(M)$ with same M_0 and Γ, is determined by this image

Lie groups IV

One concluses

Theorem 0.5

(BPP) There are many (at least continuum) rigid nonisomorphic models of $Th(M)$ with kernel (isomorphic to) Γ and the field sort (isomorphic to) M_0, all of which are, moreover, rigid.

- A final remark in this section is that the results also hold with suitable modifications for arbitrary (non Archimedean, possibly saturated) o-minimal structures M_0' in place of M_0.
- H will be a group definable in M_0', and now G will be its "o-minimal universal cover".
- Again one has $L_{\omega_1,\omega}$-definability of a section $s : H \to G$ yielding versions of Theorem 0.4.
- And rigidity of torsion is enough to yield versions of Theorem 0.5.

Finite central extensions I

- Let H be a connected compact Lie group, and let $\pi : G \to H$ be a finite central extension of H as an *abstract group*.
- Namely $\pi : G \to H$ is a surjective homomorphism whose kernel is finite and in the centre of G.
- The problem or question, is whether any such G can be realized as a topological cover of H, namely whether G can be equipped with the structure of a not necessarily connected (Hausdorff) topological group such that the topology on H is precisely the quotient topology of G. (In which case G will itself be compact Lie.)
- We have been recently informed that this is a remaining open case of Milnor's conjecture.

Finite central extensions II

- Before being aware of this we tried to prove it, obtaining some conditional results. I will restrict myself to here to giving some easy model-theoretic equivalents of the statement/problem.
- Again we take us our structure the 2-sorted structure $M = ((G, .), M_0, \pi)$ where M_0 is an o-minimal expansion of the real field in which H is definable, and G a finite central extension of H as an abstract group. Let Γ denote the finite kernel.
- There is no harm in choosing M_0 to be the real field $(\mathbb{R}, +, \cdot)$ itself.
- Given this structure M, we let M^* denote a saturated elementary extension, and $\pi^* : G^* \to H^*$ (with of course the same kernel Γ as π).
- Note that $H^*/(H^*)^{00} = H$ under the standard part map.

Finite central extensions III

Theorem 0.6

(BPP) With the above notation, the following are equivalent:

- *(i) G can be equipped with the structure of a topological cover of H.*
- *(ii) The structure M_0 is stably embedded in M (i.e. no additional structure is acquired).*
- *(iii) M_0 with structure induced from M is o-minimal.*
- *(iv) M is (naturally) interpretable in M_0.*
- *(v) For some type-definable bounded index subgroup $(G^*)^{00}$ of G^*, π^* induces an isomorphism between $(G^*)^{00}$ and $(H^*)^{00}$.*

Discussion of proof I

- Assuming (i) that G IS a topological cover, there is no harm in assuming it to be connected, so a quotient of the universal cover. The earlier discussion around the ind-interpretability of the universal cover of H in M_0 yields the interpretability of $G \to H$ in M_0 (over H) which essentially implies all other conditions.
- Without any assumptions, we always have a 0-definable in M section $s : H \to G$ of π, using the fact that $ker(\pi)$ is finite, of size n say, and that every element of H has an nth root (and just finitely many).
- Hence assuming (ii), stably embeddability of M_0 in M, we see that again M is interpretable in M_0, G is equipped with a Lie group structure by o-minimality and everything follows.
- Likewise, if we start with assumption (iii)

Discussion of proof II

- The case where we assume (v) is amusing.
- $\pi^* : G^* \to H^*$ induces $f : G^*/(G^*)^{00} \to H^*/(H^*)^{00}$.
- The embeddings of G in G^* and H in H^* induce isomorphisms i_G of G with $G^*/(G^*)^{00}$ and i_H of H with $H^*/(H^*)^{00}$ (as mentioned earlier), in such a way that f coincides with $\pi : G \to H$.
- $G^*/(G^*)^{00}$ and $H^*/(H^*)^{00}$ are equipped with the logic topology, i_H is also a homeomorphism, and i_G equips G with topological group structure such that $\pi : G \to H$ is a topological covering. End of proof.

Discussion of proof III

- In fact we know in general precisely what $(G^*)^{00}$ *should be*: it should be (with multiplicative notation), the set of nth powers of $(\pi^*)^{-1}((H^*)^{00})$, where n is the cardinality of Γ.
- With this "definition" of $(G^*)^{00}$, π^* always induces a bijection between $(G^*)^{00}$ and $(H^*)^{00}$.
- However we want (as (v) states) $(G^*)^{00}$ to be a *subgroup* of G^*, and that is among the technical problems.
- When H is commutative, one sees quickly that G is also commutative, and so $(G^*)^{00}$ as defined above IS a subgroup.
- So a conclusion is that for H commutative the conditions (i) to (v) all hold.