On the automorphism groups of groups F/R'

Vladimir Tolstykh

Yeditepe University (Istanbul)

28 March 2010

Summer School of Kemerovo State University, August 1978

Seminar of Algebra and Geometry Division of KemSU, late 70s

Baumslag

Baumslag (1970s): some finitely generated relatively free groups may have very short automorphism towers.

Baumslag

Baumslag (1970s): some finitely generated relatively free groups may have very short automorphism towers.

Special cases: (absolutely) free groups,

Baumslag

Baumslag (1970s): some finitely generated relatively free groups may have very short automorphism towers.

Special cases: (absolutely) free groups, free nilpotent groups.

Complete groups

A group G is said to be if ***complete*** G is centreless and every automorphism of G is inner.

Complete groups

A group G is said to be if ***complete*** G is centreless and every automorphism of G is inner.

In particular,

$$\mathrm{Aut}(G) = \mathrm{Inn}(G) \cong G,$$

and $\mathrm{Aut}(G) \cong G$.

In the Sharpest Sense

Dyer and Formanek (1975): the automorphism group $\mathrm{Aut}(F)$ of any non-abelian free group F of ***finite rank*** is

In the Sharpest Sense

Dyer and Formanek (1975): the automorphism group Aut(F) of any non-abelian free group F of ***finite rank*** is complete.

In the Sharpest Sense

Dyer and Formanek (1975): the automorphism group $\mathrm{Aut}(F)$ of any non-abelian free group F of ***finite rank*** is complete.

This confirms Baumslag's conjecture for the case of free groups of finite rank in the ***"sharpest sense"***.

Burnside

Burnside: let G be a centerless group. Then $\mathrm{Aut}(G)$ is complete if and only if the subgroup $\mathrm{Inn}(G)$ of all inner automorphisms of G is a characteristic subgroup of the group $\mathrm{Aut}(G)$ (that is, preserved under the action of all automorphisms of the group $\mathrm{Aut}(G)$).

...Is the Only Subgroup Such That...

Formanek (1990): let F_n be a free group of finite rank $n \geqslant 2$. Then the subgroup $\mathrm{Inn}(F_n)$ is the only free normal subgroup of $\mathrm{Aut}(F)$ having rank n.

Infinitely generated free groups

Tolstykh (2000): the automorphism group of any ***infinitely generated*** and hence of any non-abelian free group is complete.

Why Inn(F) is characteristic in Aut(F)?

Let F be a nonabelian free group. Then

- the family of all inner automorphisms of F determined by powers of primitive elements of F is first-order definable in Aut(F); it follows that Inn(F) is a characteristic subgroup of Aut(F);

Why Inn(F) is characteristic in Aut(F)?

Let F be a nonabelian free group. Then

- the family of all inner automorphisms of F determined by powers of primitive elements of F is first-order definable in Aut(F); it follows that Inn(F) is a characteristic subgroup of Aut(F);
- the subgroup Inn(F) is then first-order definable in Aut(F) provided that F is of infinite rank.

Dyer and Formanek

Let $F_{n,c}$ denote a free nilpotent group of rank n and of nilpotency class c.

Dyer and Formanek

Let $F_{n,c}$ denote a free nilpotent group of rank n and of nilpotency class c.

Dyer and Formanek (1976): the automorphism group of a free nilpotent group $F_{n,2}$ of **class two** and of finite rank $n \geqslant 2$ is complete except in the case when $n = 3$.

Dyer and Formanek

Let $F_{n,c}$ denote a free nilpotent group of rank n and of nilpotency class c.

Dyer and Formanek (1976): the automorphism group of a free nilpotent group $F_{n,2}$ of ***class two*** and of finite rank $n \geqslant 2$ is complete except in the case when $n = 3$.

In the case when $n = 3$, the height of the corresponding automorphism tower is three.

T.

Tolstykh (2001): the automorphism group $\mathrm{Aut}(N)$ of any infinitely generated free nilpotent group N of ***class two*** is also complete.

Kassabov

Kassabov (2003) found an upper bound $u(n, c) \in \mathbf{N}$ for the height of the automorphism tower of $F_{n,c}$ in terms of n and c,

Kassabov

Kassabov (2003) found an upper bound $u(n, c) \in \mathbf{N}$ for the height of the automorphism tower of $F_{n,c}$ in terms of n and c, thereby finally proving Baumslag's conjecture on finitely generated free nilpotent groups.

In three steps in the most of cases

By analyzing the function $u(n, c)$ one can conclude that if c is small compared to n,

In three steps in the most of cases

By analyzing the function $u(n, c)$ one can conclude that if c is small compared to n, then the height of the automorphism tower of $F_{n,c}$ is at most three.

Infinitely generated free nilpotent groups

Tolstykh (2008): the automorphism group of any infinitely generated free nilpotent group is complete.

Infinitely generated free nilpotent groups

Tolstykh (2008): the automorphism group of any infinitely generated free nilpotent group is complete.

Thus the automorphism tower of any free nilpotent groups terminates after finitely many steps.

Residually $\mathcal{P}$

Let $\mathcal{P}$ be a property of groups.

Residually $\mathcal{P}$

Let $\mathcal{P}$ be a property of groups.

A group G is said to be ***residually*** $\mathcal{P}$ if for every nonidentity element g of G there is a homomorphism $\varphi : G \to K$ from G onto a group K with $\mathcal{P}$ such that φ does not vanish on g: $\varphi(g) \neq 1$.

Residually $\mathcal{P}$

Let $\mathcal{P}$ be a property of groups.

A group G is said to be ***residually*** $\mathcal{P}$ if for every nonidentity element g of G there is a homomorphism $\varphi : G \to K$ from G onto a group K with $\mathcal{P}$ such that φ does not vanish on g: $\varphi(g) \neq 1$.

Equivalently, the intersection of the family of all normal subgroups N of G such that the quotient group G/N has the property $\mathcal{P}$ is trivial.

Residually $\mathcal{P}$

Let $\mathcal{P}$ be a property of groups.

A group G is said to be ***residually*** $\mathcal{P}$ if for every nonidentity element g of G there is a homomorphism $\varphi : G \to K$ from G onto a group K with $\mathcal{P}$ such that φ does not vanish on g: $\varphi(g) \neq 1$.

Equivalently, the intersection of the family of all normal subgroups N of G such that the quotient group G/N has the property $\mathcal{P}$ is trivial.

For instance, free solvable groups are residually torsion-free nilpotent.

Automorphism groups of groups F_n/R'

Dyer–Formanek (1977): let F_n be a free group of finite rank $n \geqslant 2$ and let R be a characteristic subgroup of F_n such that $R \leqslant F_n'$ and the quotient group F/R is residually torsion-free nilpotent.

Automorphism groups of groups F_n/R'

Dyer–Formanek (1977): let F_n be a free group of finite rank $n \geqslant 2$ and let R be a characteristic subgroup of F_n such that $R \leqslant F_n'$ and the quotient group F/R is residually torsion-free nilpotent.

Then the automorphism group $\mathrm{Aut}(F_n/R')$ of the group F_n/R' is complete.

Automorphism groups of groups F_n/R'

Dyer–Formanek (1977): let F_n be a free group of finite rank $n \geqslant 2$ and let R be a characteristic subgroup of F_n such that $R \leqslant F_n'$ and the quotient group F/R is residually torsion-free nilpotent.

Then the automorphism group $\mathrm{Aut}(F_n/R')$ of the group F_n/R' is complete.

In particular, the automorphism group of a free solvable group of finite rank $\geqslant 2$ and of derived length $\geqslant 2$ is complete.

Why groups F/R'?

In brief: existence of derivations of groups F/R' in F/R-modules, modules over the group rings $\mathbf{Z}[F/R]$.

Fox differential calculus

A *derivation* of a given group G in a G-module M is a map $D : G \to M$ such that

$$D(ab) = D(a) + aD(b)$$

for every $a, b \in G$ (here $aD(b)$ is the result of the action of a scalar $a \in G \subseteq \mathbf{Z}[G]$ on a vector $D(b) \in M$.)

Fox differential calculus

Fox (1953): let F be a free group with a basis $(X_i : i \in I)$. Then for any prescribed elements $Y_i \in \mathbf{Z}[F]$ there is a unique derivation $D : F \to \mathbf{Z}[F]$ such that

$$D(X_i) = Y_i \qquad (i \in I).$$

Fox differential calculus

In particular, for every $i \in I$ there is a unique derivation D_i such that

$$D_i(X_j) = \delta_{ij}$$

for all $i, j \in I$.

F, R, R'

Now let R be a normal subgroup of F and let R' denote the commutator subgroup of R; the quotient group R/R' will be denoted by $\widehat{R}$.

F, R, R'

Now let R be a normal subgroup of F and let R' denote the commutator subgroup of R; the quotient group R/R' will be denoted by $\widehat{R}$.

$^{-}$ will denote the homomorphism $\mathbf{Z}[F] \to \mathbf{Z}[F/R]$ of group rings determined by the natural homomorphism $F \to F/R$;

F, R, R'

Now let R be a normal subgroup of F and let R' denote the commutator subgroup of R; the quotient group R/R' will be denoted by $\widehat{R}$.

$^{-}$ will denote the homomorphism $\mathbf{Z}[F] \to \mathbf{Z}[F/R]$ of group rings determined by the natural homomorphism $F \to F/R$;

The same symbol $^{-}$ will be used to denote the similarly defined homomorphism $\mathbf{Z}[F/R'] \to \mathbf{Z}[F/R]$.

All of 'em at once

Let M be a free F/R-module (hence an F- and an F/R'-module) with a free basis $(t_i : i \in I)$.

All of 'em at once

Let M be a free F/R-module (hence an F- and an F/R'-module) with a free basis $(t_i : i \in I)$.

Then the map

$$\partial(aR') = \sum \overline{D_i(a)} t_i$$

where a runs over F is a well-defined derivation of F/R' in M, since $\overline{D_i(b)} = 0$ for every $b \in R'$ and for every $i \in I$.

Some nice features of ∂

Moreover, ∂ is injective on F/R' (a corollary of a result by Magnus, 1939) and since

$$\partial(r_1 r_2) = \partial(r_1) + \bar{r}_1 \partial(r_2) = \partial(r_1) + \partial(r_2),$$
$$\partial(gR * r) = \partial(grg^{-1}) = \bar{g}\partial(r).$$

for all $r_1, r_2 \in R/R'$ and $g \in G$,

Some nice features of ∂

Moreover, ∂ is injective on F/R' (a corollary of a result by Magnus, 1939) and since

$$\partial(r_1 r_2) = \partial(r_1) + \bar{r}_1 \partial(r_2) = \partial(r_1) + \partial(r_2),$$
$$\partial(gR * r) = \partial(grg^{-1}) = \overline{g}\partial(r).$$

for all $r_1, r_2 \in R/R'$ and $g \in G$, R/R' and $\partial(R/R')$ can be viewed as isomorphic $\mathbf{Z}[F/R]$-modules.

Is the rank of $\mathcal{F}$ 'recognizable' by $\mathrm{Aut}(\mathcal{F})$?

Let $\mathcal{F}$ be a relatively free algebra of infinite rank $\varkappa$. When working to describe automorphisms of the group $\Gamma = \mathrm{Aut}(\mathcal{F})$ it is very helpful to know whether Γ

'***recognizes***'

the cardinality $\varkappa = \operatorname{rank} \mathcal{F}$.

Relations of cardinality $< |\Gamma|$

To achieve that one can start with relations on Γ of cardinality $< |\Gamma| = 2^{\varkappa}$.

Setwise and pointwise stabilizers

Let G be a permutation group acting on a set X, H a subgroup of G, and Y a subset of X.

Setwise and pointwise stabilizers

Let G be a permutation group acting on a set X, H a subgroup of G, and Y a subset of X.

The pointwise $H_{(Y)}$ and the setwise $H_{\{Y\}}$ stabilizers of Y in H are subgroups

$$H_{(Y)} = \{h \in H : hy = y \text{ for all } y \in Y\} \text{ and}$$
$$H_{\{Y\}} = \{h \in H : hY = Y\},$$

respectively.

Small index property

Let $\mathcal{M}$ be a countable structure. We say that $\mathcal{M}$ has the ***small index* property** if any subgroup H of the automorphism group $\Gamma = \mathrm{Aut}(\mathcal{M})$ of $\mathcal{M}$ having index $< 2^{\aleph_0}$ contains the stabilizer of $\Gamma_{(U)}$ of a finite subset U of $\mathcal{M}$:

$$|\Gamma : H| < 2^{\aleph_0} \Rightarrow \exists \text{ a finite } U \subseteq \mathcal{M} \text{ s.t. } \Gamma_{(U)} \leqslant H.$$

Small index property: examples

- Any countably infinite set Ω (Dixon, Neumann, Thomas);

Small index property: examples

- Any countably infinite set Ω (Dixon, Neumann, Thomas);
- any vector space of dimension $\aleph_0$ over a countable field (Evans);

Small index property: examples

- Any countably infinite set Ω (Dixon, Neumann, Thomas);
- any vector space of dimension $\aleph_0$ over a countable field (Evans);
- any free group of countably infinite rank (Bryant and Evans);

Small index property: examples

- Any countably infinite set Ω (Dixon, Neumann, Thomas);
- any vector space of dimension $\aleph_0$ over a countable field (Evans);
- any free group of countably infinite rank (Bryant and Evans);
- any free nilpotent group of countably infinite rank (Bryant and Evans), etc.

Another result by Dixon–Neumann–Thomas

Let Ω be an infinite set. For every subgroup H of $\Gamma = \mathrm{Sym}(\Omega)$ having index $\leqslant |\Omega|$ there is a subset U of Ω of cardinality $< |\Omega|$ such that

$$\Gamma_{(U)} \leqslant H.$$

Another result by Dixon–Neumann–Thomas

Let Ω be an infinite set. For every subgroup H of $\Gamma = \mathrm{Sym}(\Omega)$ having index $\leqslant |\Omega|$ there is a subset U of Ω of cardinality $< |\Omega|$ such that

$$\Gamma_{(U)} \leqslant H.$$

Similar result is true for the automorphism groups of some other structures.

Subgroups of small index

Proposition

Let G be a relatively free group of infinite rank $\varkappa$, $\mathcal{X}$ a basis of G and Σ a subgroup of the automorphism group $\Gamma = \mathrm{Aut}(G)$ of index at most $\mathrm{rank}\, G$*. Then there is a subset $\mathcal{U}$ of $\mathcal{X}$ of cardinality $< \varkappa$ such that*

Subgroups of small index

Proposition

Let G be a relatively free group of infinite rank $\varkappa$, $\mathcal{X}$ a basis of G and Σ a subgroup of the automorphism group $\Gamma = \mathrm{Aut}(G)$ of index at most rank G. *Then there is a subset $\mathcal{U}$ of $\mathcal{X}$ of cardinality $< \varkappa$ such that*

(i) *for every moiety $\mathcal{Z}$ of $\mathcal{X} \setminus \mathcal{U}$, Σ contains the subgroup $\Gamma_{(\mathcal{X}\setminus\mathcal{Z}),\{\langle\mathcal{Z}\rangle\}}$ of automorphisms of G which fix $\mathcal{X} \setminus \mathcal{Z}$ pointwise and preserve the subgroup $\langle\mathcal{Z}\rangle$ generated by $\mathcal{Z}$;*

Subgroups of small index

Proposition

Let G be a relatively free group of infinite rank $\varkappa$, $\mathcal{X}$ a basis of G and Σ a subgroup of the automorphism group $\Gamma = \mathrm{Aut}(G)$ of index at most rank G. *Then there is a subset $\mathcal{U}$ of $\mathcal{X}$ of cardinality $< \varkappa$ such that*
(i) *for every moiety $\mathcal{Z}$ of $\mathcal{X} \setminus \mathcal{U}$, Σ contains the subgroup $\Gamma_{(\mathcal{X}\setminus\mathcal{Z}),\{\langle\mathcal{Z}\rangle\}}$ of automorphisms of G which fix $\mathcal{X} \setminus \mathcal{Z}$ pointwise and preserve the subgroup $\langle\mathcal{Z}\rangle$ generated by $\mathcal{Z}$;*
(ii) *for every moiety $\mathcal{Z} = \{z_i : i \in I\}$ of $\mathcal{X} \setminus \mathcal{U}$, and for every subset $\{v_i : i \in I\}$ of the subgroup $\langle\mathcal{U}\rangle$ generated by $\mathcal{U}$, Σ contains an automorphism σ of G which fixes the set $\mathcal{X} \setminus \mathcal{Z}$ pointwise and takes an element z_i of $\mathcal{Z}$ to $z_i v_i$:*

$$\begin{aligned} \sigma x &= x, && x \in \mathcal{X} \setminus \mathcal{Z}, \\ \sigma z_i &= z_i v_i && i \in I. \end{aligned} \tag{1}$$

Small conjugacy classes

Consider a relatively free group G of infinite rank $\varkappa$ and let Γ denote the automorphism group $\mathrm{Aut}(G)$.

Small conjugacy classes

Consider a relatively free group G of infinite rank $\varkappa$ and let Γ denote the automorphism group $\mathrm{Aut}(G)$.

We shall say the conjugacy class σ^Γ an element $\sigma \in \mathrm{Aut}(G)$ is ***small*** if

$$|\sigma^\Gamma| \leqslant \varkappa = \mathrm{rank}\, G.$$

Small conjugacy classes

Consider a relatively free group G of infinite rank $\varkappa$ and let Γ denote the automorphism group $\mathrm{Aut}(G)$.

We shall say the conjugacy class σ^Γ an element $\sigma \in \mathrm{Aut}(G)$ is ***small*** if

$$|\sigma^\Gamma| \leqslant \varkappa = \mathrm{rank}\, G.$$

Equivalently, the index $|\Gamma : C(\sigma)|$ of the centralizer $C(\sigma)$ in Γ is at most $\varkappa$.

Small conjugacy classes

Consider a relatively free group G of infinite rank $\varkappa$ and let Γ denote the automorphism group $\mathrm{Aut}(G)$.

We shall say the conjugacy class σ^Γ an element $\sigma \in \mathrm{Aut}(G)$ is ***small*** if

$$|\sigma^\Gamma| \leqslant \varkappa = \mathrm{rank}\, G.$$

Equivalently, the index $|\Gamma : C(\sigma)|$ of the centralizer $C(\sigma)$ in Γ is at most $\varkappa$.

The family S of all automorphisms σ of G whose conjugacy class is small is a normal subgroup of $\mathrm{Aut}(G)$.

Examples

Natural examples of elements of $\mathrm{Aut}(G)$ with small conjugacy classes are of course inner automorphisms of G.

Examples

Natural examples of elements of Aut(G) with small conjugacy classes are of course inner automorphisms of G.

Let N be an infinitely generated free nilpotent group of class $c \geqslant 3$, $\mathcal{X}$ is a basis of N, $u_2, \ldots, u_c$ be some fixed elements of $\mathcal{X}$ and $\sigma \in \mathrm{Aut}(N)$ is such that

$$\sigma(x) = x[u_2, x, u_3, \ldots, u_c] \qquad (x \in \mathcal{X}).$$

Then σ has small conjugacy class in Aut(N)...

Examples

...since for every $x, y \in \mathcal{X}$ letting $w(t; \vec{u})$ denote the word $t[u_2, t, u_3, \dots, u_c]$, we have that

$$\sigma(xy) = x[u_2, x, u_3, \dots, u_c] \cdot y[u_2, y, u_3, \dots, u_c] = xy[u_2, xy, u_3, \dots, u_c],$$
$$= w(x; \vec{u}) \cdot w(y; \vec{u}) = w(xy; \vec{u})$$

Examples

...since for every $x, y \in \mathcal{X}$ letting $w(t; \vec{u})$ denote the word $t[u_2, t, u_3, \dots, u_c]$, we have that

$$\begin{aligned}\sigma(xy) &= x[u_2, x, u_3, \dots, u_c] \cdot y[u_2, y, u_3, \dots, u_c] = xy[u_2, xy, u_3, \dots, u_c], \\ &= w(x; \vec{u}) \cdot w(y; \vec{u}) = w(xy; \vec{u})\end{aligned}$$

and, in effect,

$$\sigma(z) = z[u_2, z, u_3, \dots, u_c] = w(z; \vec{u})$$

for every $z \in N$.

Examples

...since for every $x, y \in \mathcal{X}$ letting $w(t; \vec{u})$ denote the word $t[u_2, t, u_3, \ldots, u_c]$, we have that

$$\sigma(xy) = x[u_2, x, u_3, \ldots, u_c] \cdot y[u_2, y, u_3, \ldots, u_c] = xy[u_2, xy, u_3, \ldots, u_c], \\ = w(x; \vec{u}) \cdot w(y; \vec{u}) = w(xy; \vec{u})$$

and, in effect,

$$\sigma(z) = z[u_2, z, u_3, \ldots, u_c] = w(z; \vec{u})$$

for every $z \in N$. It easily follows that

$$\pi\sigma\pi^{-1}(z) = w(z; \pi\vec{u}) \qquad [z \in N]$$

and therefore the conjugacy class of σ is small.

'Homomorphic' terms/words

Proposition

Let G be a relatively free group of infinite rank $\varkappa$, $\mathcal{X}$ a basis of G. The conjugacy class of a $\sigma \in \mathrm{Aut}(G)$ is small if and only if there are finitely many elements $u_1, \ldots, u_s$ of $\mathcal{X}$ and a term $w(*; *_1, \ldots, *_s)$ of the language of group theory (a group word in symbols $*, *_1, \ldots, *_s$) such that

$$\sigma(x) = w(x; u_1, \ldots, u_s)$$

for all $x \in \mathcal{X}$ and

‘Homomorphic’ terms/words

Proposition

Let G be a relatively free group of infinite rank $\varkappa$, $\mathcal{X}$ a basis of G. The conjugacy class of a $\sigma \in \mathrm{Aut}(G)$ is small if and only if there are finitely many elements $u_1, \ldots, u_s$ of $\mathcal{X}$ and a term $w(*; *_1, \ldots, *_s)$ of the language of group theory (a group word in symbols $*, *_1, \ldots, *_s$) such that

$$\sigma(x) = w(x; u_1, \ldots, u_s)$$

for all $x \in \mathcal{X}$ and

$$w(xy; u_1, \ldots, u_s) = w(x; u_1, \ldots, u_s) \cdot w(y; u_1, \ldots, u_s)$$

for all $x, y \in \mathcal{X}$ (in effect, $\sigma(g) = w(g; u_1, \ldots, u_s)$ for all $g \in G$).

Sometimes they do recognize the rank

Proposition

Let G be a centerless relatively free group of infinite rank $\varkappa$.

Sometimes they do recognize the rank

Proposition

Let G be a centerless relatively free group of infinite rank $\varkappa$.
(i) *Suppose that the cardinality of the conjugacy class ρ^Γ of a $\rho \in \Gamma$ is less than $\varkappa$. Then ρ is the identity;*

Sometimes they do recognize the rank

Proposition

Let G be a centerless relatively free group of infinite rank $\varkappa$.
(i) *Suppose that the cardinality of the conjugacy class ρ^Γ of a $\rho \in \Gamma$ is less than $\varkappa$. Then ρ is the identity;*
(ii) *the cardinal*

$$\min\{|\pi^\Gamma| : \pi \in \Gamma, \pi \neq \mathrm{id}\}$$

is equal to $\varkappa$;

Sometimes they do recognize the rank

Proposition

Let G be a centerless relatively free group of infinite rank $\varkappa$.
(i) *Suppose that the cardinality of the conjugacy class ρ^Γ of a $\rho \in \Gamma$ is less than $\varkappa$. Then ρ is the identity;*
(ii) *the cardinal*

$$\min\{|\pi^\Gamma| : \pi \in \Gamma, \pi \neq \mathrm{id}\}$$

is equal to $\varkappa$;
(iii) *the conjugacy class of a nonidentity $\sigma \in \Gamma$ is small if and only if $|\sigma^\Gamma| \leqslant |\pi^\Gamma|$ for every nonidentity $\pi \in \Gamma$;*

Sometimes they do recognize the rank

Proposition

Let G be a centerless relatively free group of infinite rank $\varkappa$.
(i) *Suppose that the cardinality of the conjugacy class ρ^Γ of a $\rho \in \Gamma$ is less than $\varkappa$. Then ρ is the identity;*
(ii) *the cardinal*

$$\min\{|\pi^\Gamma| : \pi \in \Gamma, \pi \neq \mathrm{id}\}$$

is equal to $\varkappa$;
(iii) *the conjugacy class of a nonidentity $\sigma \in \Gamma$ is small if and only if $|\sigma^\Gamma| \leqslant |\pi^\Gamma|$ for every nonidentity $\pi \in \Gamma$;*
(iv) *the subgroup S of all elements of Γ whose conjugacy class is small is a characteristic subgroup of Γ.*

Free groups

Corollary

Let F be a free group of infinite rank $\varkappa$. Then $\sigma \in \mathrm{Aut}(F)$ has small conjugacy class if and only if σ is an inner automorphism of F. Consequently, $\mathrm{Inn}(F)$ is the largest (free) normal subgroup of $\mathrm{Aut}(F)$ of cardinality $\varkappa$.

...when their ranks are equal

Corollary

Let $\mathfrak{V}$ be a variety of groups whose free groups are centerless. Then for any infinitely generated free groups $G_1, G_2 \in \mathfrak{V}$

$$\mathrm{Aut}(G_1) \cong \mathrm{Aut}(G_2) \iff \mathrm{rank}(G_1) = \mathrm{rank}(G_2).$$

How do they act on R/R'?

Lemma

Let F be an infinitely generated free group, R a fully invariant subgroup of F and the group ring $\mathbf{Z}[F/R]$ is a domain whose units are trivial:

$$U(\mathbf{Z}[F/R]) = \pm F/R.$$

Suppose that the conjugacy class of a $\sigma \in \mathrm{Aut}(F/R')$ in $\mathrm{Aut}(F/R')$ is small. Then there is a $v \in F/R'$ such that

$$\sigma r = v\, r\, v^{-1}.$$

for every r in R/R'.

Dyer

Dyer (1974), extending Shmel'kin's result on free solvable groups (1967): if F is a free group and a normal subgroup R is such that the quotient group F/R is

(a) torsion-free and

Dyer

Dyer (1974), extending Shmel'kin's result on free solvable groups (1967): if F is a free group and a normal subgroup R is such that the quotient group F/R is

(a) torsion-free and

(b′) either is solvable, or

(b″) has nontrivial center and is not cyclic-by-periodic,

Dyer

Dyer (1974), extending Shmel'kin's result on free solvable groups (1967): if F is a free group and a normal subgroup R is such that the quotient group F/R is

(a) torsion-free and

(b$'$) either is solvable, or

(b$''$) has nontrivial center and is not cyclic-by-periodic,

then any automorphism of the group F/R' which fixes R/R' pointwise is an inner automorphism of F/R' determined by an element of R/R'.

Group rings of orderable groups

Let G be an orderable group. Then the group ring $\mathbf{Z}[G]$ of G is a domain whose units are trivial:

$$U(\mathbf{Z}[G]) = \pm G.$$

Group rings of orderable groups

Let G be an orderable group. Then the group ring $\mathbf{Z}[G]$ of G is a domain whose units are trivial:

$$U(\mathbf{Z}[G]) = \pm G.$$

For instance, if G is a residually torsion-free nilpotent group, then it is orderable:

Group rings of orderable groups

Let G be an orderable group. Then the group ring $\mathbf{Z}[G]$ of G is a domain whose units are trivial:

$$U(\mathbf{Z}[G]) = \pm G.$$

For instance, if G is a residually torsion-free nilpotent group, then it is orderable:

- Any residually orderable group is orderable;

Group rings of orderable groups

Let G be an orderable group. Then the group ring $\mathbf{Z}[G]$ of G is a domain whose units are trivial:

$$U(\mathbf{Z}[G]) = \pm G.$$

For instance, if G is a residually torsion-free nilpotent group, then it is orderable:

- Any residually orderable group is orderable;
- torsion-free nilpotent groups are orderable.

Corollary

Let F be an infinitely generated free group and R a fully invariant subgroup of F such that F, R satisfy the conditions of Dyer's theorem and the group ring $\mathbf{Z}[F/R]$ is a domain whose units are trivial. Then the group $\mathrm{Aut}(F/R')$ is complete. In particular, the automorphism groups of infinitely generated free solvable groups of derived length $\geqslant 2$ are complete.

Let F be an infinitely generated free group and R a fully invariant normal subgroup of F.

In what follows

G denotes F/R'

$\widehat{R}$ denotes R/R'.

If H is a subgroup of G, then I_H denotes the group of inner automorphisms determined by elements of H.

Describing objects related to S

Lemma

Let S be the subgroup of all automorphisms of G whose conjugacy class is small. Then

Describing objects related to S

Lemma

Let S be the subgroup of all automorphisms of G whose conjugacy class is small. Then
(i) $S = \mathrm{Inn}(G) \cdot S_{(\widehat{R})}$ *where* $S_{(\widehat{R})}$ *is the subgroup of all elements of* S *fixing* $\widehat{R}$ *pointwise;*

Describing objects related to S

Lemma

Let S be the subgroup of all automorphisms of G whose conjugacy class is small. Then

(i) $S = \mathrm{Inn}(G) \cdot S_{(\widehat{R})}$ *where* $S_{(\widehat{R})}$ *is the subgroup of all elements of S fixing* $\widehat{R}$ *pointwise;*

(ii) *the subgroup* $S_{(\widehat{R})}$ *is the Hirsch-Plotkin radical of the group* S;

Describing objects related to S

Lemma

Let S be the subgroup of all automorphisms of G whose conjugacy class is small. Then
(i) $S = \mathrm{Inn}(G) \cdot S_{(\widehat{R})}$ *where* $S_{(\widehat{R})}$ *is the subgroup of all elements of S fixing $\widehat{R}$ pointwise;*
(ii) *the subgroup* $S_{(\widehat{R})}$ *is the Hirsch-Plotkin radical of the group S;*
(iii) *if $R \leqslant F'$, then $[S, S]$ is the subgroup $I_{[G,G]}$ of all inner automorphisms of G determined by elements of $[G, G]$. In particular,*

$$I_{\widehat{R}} = S_{(\widehat{R})} \cap [S, S].$$

Describing objects related to S

Lemma

Let S be the subgroup of all automorphisms of G whose conjugacy class is small. Then

(i) $S = \mathrm{Inn}(G) \cdot S_{(\widehat{R})}$ *where* $S_{(\widehat{R})}$ *is the subgroup of all elements of S fixing $\widehat{R}$ pointwise;*

(ii) *the subgroup* $S_{(\widehat{R})}$ *is the Hirsch-Plotkin radical of the group S;*

(iii) *if $R \leqslant F'$, then $[S, S]$ is the subgroup $I_{[G,G]}$ of all inner automorphisms of G determined by elements of $[G, G]$. In particular,*

$$I_{\widehat{R}} = S_{(\widehat{R})} \cap [S, S].$$

(iv) *elements of the form $\tau_x \gamma$ where $x \in G$ whose image under the natural homomorphism $G = F/R' \to F/R$ is a primitive element of the group F/R and $\gamma \in S_{(\widehat{R})}$ form a second-order definable family of the group* $\mathrm{Aut}(G)$.

Describing inner automorphisms

Proposition

Let $R \leqslant F'$. *Suppose that the following conditions are true for an automorphism* $\sigma \in \mathrm{Aut}(G)$:

Describing inner automorphisms

Proposition

Let $R \leqslant F'$. Suppose that the following conditions are true for an automorphism $\sigma \in \mathrm{Aut}(G)$:

(a) *the conjugacy class of σ is small;*

Describing inner automorphisms

Proposition

Let $R \leqslant F'$. Suppose that the following conditions are true for an automorphism $\sigma \in \mathrm{Aut}(G)$:
(a) *the conjugacy class of σ is small;*
(b) *the image of σ under the natural homomorphism $S \to S/S_{(\widehat{R})}$ is a primitive element of the relatively free group $S/S_{(\widehat{R})} \cong F/R$;*

Describing inner automorphisms

Proposition

Let $R \leqslant F'$*. Suppose that the following conditions are true for an automorphism* $\sigma \in \mathrm{Aut}(G)$*:*
(a) *the conjugacy class of* σ *is small;*
(b) *the image of* σ *under the natural homomorphism* $S \to S/S_{(\widehat{R})}$ *is a primitive element of the relatively free group* $S/S_{(\widehat{R})} \cong F/R$*;*
(c) *the group* $L(\sigma) = \mathrm{NC}(\sigma) I_{[G,G]}$ *contains no elements of the set* $S_{(\widehat{R})} \setminus I_{\widehat{R}}$*.*

Describing inner automorphisms

Proposition

Let $R \leqslant F'$. *Suppose that the following conditions are true for an automorphism* $\sigma \in \mathrm{Aut}(G)$*:*
(a) *the conjugacy class of* σ *is small;*
(b) *the image of* σ *under the natural homomorphism* $S \to S/S_{(\widehat{R})}$ *is a primitive element of the relatively free group* $S/S_{(\widehat{R})} \cong F/R$;
(c) *the group* $L(\sigma) = \mathrm{NC}(\sigma) I_{[G,G]}$ *contains no elements of the set* $S_{(\widehat{R})} \setminus I_{\widehat{R}}$.
It follows that σ *is an inner automorphism of* G, *that* $\mathrm{NC}(\sigma) I_{[G,G]} = \mathrm{Inn}(G)$, *and that* $\mathrm{Inn}(G)$ *is a characteristic subgroup of* $\mathrm{Aut}(G)$.

Theorem (Tolstykh, 2010)

Let F be an infinitely generated free group, $R \leqslant F'$ a fully invariant subgroup of F such that the quotient group F/R is residually torsion-free nilpotent. Then the group $\mathrm{Aut}(F/R')$ is complete.