CCM

A sort for each compact complex manifold,
and for each reduced irreducible compact analytic space;
relations for analytic subsets of products of the sorts (locally defined by vanishing of holomorphic $\mathbb{C}$-valued functions on polydiscs).

QE, EI; tt, fRM, Z.

Chow: analytic subsets of $¶^n(\mathbb{C})$ are precisely the Zariski closed sets in the sense of AG.

Corollary:
The structure induced on $\mathbb{A}^1 := ¶^1 \setminus \infty$ is precisely that of the complex field, with constants for complex points.

TA and CCMA

Let T be a complete theory.

$T_\sigma := \mathrm{Th}(\{\langle M; \sigma\rangle \mid M \models T, \sigma : M \to Mautomorphism\})$

$TA :=$ model companion = theory of existentially closed models of T_σ if such exists.

Facts: Suppose TA exists. Then:

- $\mathrm{acl}^{TA}(C) = \mathrm{acl}_\sigma(C) := \mathrm{acl}\,\widehat{T}(\Cup_{i\in\mathbb{Z}})\sigma^i(C))$
- QE: for $C = \mathrm{acl}_\sigma(C)$, $\mathrm{qftp}^{TA}(C) \models tp^{TA}(C)$
- T (super)stable => TA (super)simple

CCMA exists,

SKIPME:
axiomatised by:

- CCM_σ;
- given $X \subseteq Y \times Y^\sigma$,
  X and Y closed irreducible, ¡— Fact: this is 1st order.
  co-ord projections dominant,
  $X' \subseteq X$ proper closed,
  then $(X \setminus X') \cap \Gamma_\sigma \neq 0$

Holomorphic dynamics and finite-dimensional types:
X CCM, $f : X \to X$ holomorphic automorphism,
$(X, f)^\sharp := \{x \in X \mid \sigma(x) = f(x)\}$ finite-dimensional definable set in CCMA.

More generally:
$(\mathcal{A}, \sigma) \models$ CCMA,
X,F closed irreducible in $\mathcal{A}$,
$F \subseteq X \times X^\sigma$,
projections dominant with finite fibres.
Then $(X, F)^\sharp := \{x \in X \mid (x, \sigma(x)) \in F\}$.

The <u>finite-dimensional</u> types are the generic types of such $(X, F)^\sharp$.

Theorem [Trichotomy]:
Let p be a finite-dimensional minimal type.

If p is not one-based,
it is non-orthogonal to $(¶^1, \mathrm{id})^\sharp$.

If p is one-based non-trivial,
it is non-orthogonal to some $(X, F)^\sharp$ with X and F definable groups.

(Proof of Zilber Dichotomy: CBP via jet spaces)

This is for *real* types, but imaginary types can come up in an analysis...

Imaginaries in TA

Suppose T superstable with EI and TA exists.

Fact [Hrushovski]: TA has gEI (imaginaries are interalgebraic with reals)

Example:
$T :=$ theory of a connected groupoid with $\pi_1 = Z/2Z$.
In TA,
$X :=$ fixed objects,
$E(x, y) \leftrightarrow Mor(x, y)$ fixed pointwise by σ;
then X/E is not eliminable.

Theorem [Hrushovski]: TFAE:

(i) TA has EI

(ii) T "eliminates finite groupoid imaginaries"

(iii) T has "3-uniqueness":
Given b,
and a_0, a_1, a_2 independent over $b \in \mathrm{acl}(a_i)$,
(i.e. $a_i \downarrow \underline{baj} a_k$) $\mathrm{acl}(a_1 a_2) \cap \mathrm{dcl}(\mathrm{acl}(a_0 a_1), \mathrm{acl}(a_0 a_2)) = \mathrm{dcl}(\mathrm{acl}(a_1), \mathrm{acl}(a_2))$

Remark: $\mathrm{acl}(b) \models T$ => 3-uniqueness (by coheiring).
So e.g. ACFA has EI.

Theorem:
CCM does not have 3-uniqueness,
so CCMA does not have EI.

Idea:
Let $X \to B$ be a principal $\mathbb{C}^*$-bundle (so have definable principal action of $\mathbb{C}^*$ on fibres).

Work in monster model $\mathbb{A}' \models CCM$.

Let $b \in B$ generic;
$a_0, a_1, a_2 \in X_b$ generic independent $/b$;
let $\phi \in (a_2/a_1)^{1/n} \in \mathbb{C}^*$.
Now $(a_2/a_1)^{1/n} = (a_2/a_0)^{1/n} * (a_0/a_1)^{1/n}$,
so $\phi \in \mathrm{dcl}(\mathrm{acl}(a_0 a_1), \mathrm{acl}(a_0 a_2))$.

So this shows non-3-uniqueness unless $\phi \in \mathrm{dcl}(\mathrm{acl}(a_1), \mathrm{acl}(a_2))$.

Now $\phi \notin \mathrm{dcl}(a_1, a_2) = \mathrm{dcl}(a_1, \phi^n)$ since $\phi \notin \mathrm{dcl}(\phi^n)$.

So STS $\mathrm{acl}(a_i) = \mathrm{dcl}(a_i)$.

So want $X \to B$ defble $\mathbb{C}^*$-bundle s.t. $a \in *X$ generic => $\mathrm{acl}(a) = \mathrm{dcl}(a)$;
i.e. any dominant generically finite $X' \to X$ has a generic section.

Finite covers of $\mathbb{C}^*$-bundles:

(I) Base change:

```
C* === C*
.       |
.       |
v  fin v
X'  ..> X
.       |
```

```
.        |
v        v
B'  -->  B
    fin
```

(II) Quotient by action of nth roots of unity on fibres:

```
    [n]
C*  -->  C*
|        .
|        .
v        v
X'  ..>  X
|        .
|        .
v        v
B   ===  B
```

Fact:
Holomorphic $\mathbb{C}^*$-bundles over B are classified by first cohomology group of sheaf of local holomorphic $\mathbb{C}^*$-valued functions,
$H^1(\mathcal{O}_B^*)$,
and (II) corresponds to multiplication by n in this group.

Fact:
Exists simply connected strongly minimal smooth compact (K3) surface B with $H^1(\mathcal{O}_B^*) \cong \mathbb{Z}$

Let $X \to B$ correspond to generator of $H^1(\mathcal{O}_B^*)$.

B s.c. s.m. => no non-trivial finite $B' \to B$;
B s.m. => X has no ramified finite covers,
=> any cover has to be as in (II) but no such exist since $[X] \in H^1(\mathcal{O}_B^*)$ not divisible.

Explicitly, the following imaginary is not eliminable:
$X \cap \mathrm{Fix}(\sigma)$ with xEx' iff $\pi(x) = \pi(x')$ and $(x'/x)^{1/2} \subseteq \mathrm{Fix}(\sigma)$

Proof:
Let $b \in B$ generic with $\sigma(b) = b$;
let $x \in X_b$ generic with $\sigma(x) = x$.

Since $\mathrm{acl}_\sigma(x) = \mathrm{acl}(x) = \mathrm{dcl}(x)$ and $\mathrm{acl}_\sigma(b) = \mathrm{acl}(b) = \mathrm{dcl}(b)$,
by the QE, $tp(x/\mathrm{acl}_\sigma(b))$ is determined by "x is generic in X_b and $\sigma(x) = x$".

So $x/E \in \mathrm{acl}^{TA,eq}(b) \setminus \mathrm{acl}^{TA}(b)$.