GNS constructions, and functions of positive type

Based on Appendix C of "Kazhdan's Property T" by Bekka, Harpe, and Valeppe.

Cyclic representations

Definition:
A unitary representation $(\pi, \mathcal{H})$ is cyclic if there exists $\xi \in \mathcal{H}$ s.t. the span of the orbit of ξ is dense. Call such ξ a cyclic vector.

Remark:
If $(\pi, \mathcal{H})$ is a representation and $\xi \in \mathcal{H}$, then the closure of the span of the orbit of ξ, $< G\xi >$, is an invariant closed subspace, so yields a cyclic subrepresentation.
So irreducible $\Rightarrow$ cyclic (any $\xi \neq 0$ is cyclic),
and any unitary representation $(\pi, \mathcal{H})$ can be decomposed as an orthogonal direct sum of cyclic subrepresentations.
(Proof: take a maximal such direct sum; if not $\mathcal{H}$, consider closure of orbit of an element of the orthogonal complement for a contradiction.)

Functions of positive type

Definition:

(i) For X a topological space, a kernel of positive type is a continuous function $\Phi : X^2 \to \mathbb{C}$ such that for any finite tuple $(x_1, ..., x_n)$ from X and any $(a_1, ..., a_n) \in \mathbb{C}^n$,

$$\sum_{i,j} a_i \Phi(x_i, x_j) \overline{a}_j \geq 0.$$

(ii) For G a topological group, a function of positive type is a continuous function $\phi : G \to \mathbb{C}$ such that $(g, h) \mapsto \phi(h^{-1}g)$ is a kernel of positive type.

Remark:
Φ is of positive type iff for any finite subset $\{x_1, ..., x_n\} \subseteq X$, the matrix $(a_{ij})_{ij}$ with $a_{ij} = \Phi(x_i, x_j)$ is positive semidefinite.

Example:
$G := \langle \mathbb{R}; + \rangle$, $\phi(z) := e^{iz}$.

Lemma:

(i) If $\Phi : X^2 \to \mathbb{C}$ is of +ve type,
   (a) $\overline{\Phi(x, y)} = \Phi(y, x)$ (so $\Phi(x, x) \in \mathbb{R}$)
   (b) $|\Phi(x, y)|^2 \leq \Phi(x, x)\Phi(y, y)$

(ii) If $\phi : G \to \mathbb{C}$ is of +ve type,
   (a) $\overline{\phi(g)} = \phi(g^{-1})$ (so $\phi(e) \in \mathbb{R}$)
   (b) $|\phi(g)| \leq \phi(e)$

Proof:

(i) The matrix

$$\begin{bmatrix} \Phi(x, x) & \Phi(x, y) \\ \Phi(y, x) & \Phi(y, y) \end{bmatrix}$$

is positive semidefinite, so is Hermitian with non-negative determinant.

(ii) Follows, by considering $\Phi(g, e)$.

Remark:
Let $(\pi, \mathcal{H})$ be a unitary representation of G, let $\xi \in \mathcal{H}$.
Define $\phi_{\pi,\xi} : g \mapsto \langle g\xi, \xi \rangle$.
Then $\phi_{\pi,\xi}$ is of +ve type.
Indeed, $(g, h) \mapsto \langle h^{-1}g\xi, \xi \rangle = \langle g\xi, h\xi \rangle$ is of +ve type,
since $\sum_{ij} a_i \langle g_i\xi, g_j\xi \rangle \overline{a}_j = \sum_{ij} \langle a_i g_i\xi, a_j g_j\xi \rangle = \langle \sum_i a_i g_i\xi, \sum_i a_i g_i\xi \rangle \geq 0$.

Theorem ["GNS construction"]:
G topological group, $\phi : G \to \mathbb{C}$ of +ve type.
Then $\phi = \phi_{\pi_\phi,\xi_\phi}$ for some unitary representation $(\pi_\phi, \mathcal{H}_\phi)$ and cyclic vector $\xi_\phi \in \mathcal{H}_\phi$.

Moreover, the pair (π_ϕ, ξ_ϕ) associated to ϕ is unique up to isomorphism, in the sense that if $\phi = \phi_{\pi',\xi'}$ where ξ' is a cyclic vector in another unitary representation $(\pi', \mathcal{H}')$, then there is an intertwining isomorphism $\mathcal{H}_\phi \xrightarrow{\cong} \mathcal{H}'; \xi_\phi \mapsto \xi'$.

Remark:
So $(\pi, \xi) \mapsto \phi_{\pi,\xi}$ induces a bijective correspondence between +ve functions and cyclic vectors in unitary representations up to isomorphism.

The zero function corresponds to the trivial representation.

Cone of functions of +ve type

$\mathcal{P}(G) := \{\phi : G \to \mathbb{C} \text{ of +ve type}\}$.
$\mathcal{P}(G)$ is a cone (i.e. closed under taking positive linear combinations).

$\mathcal{P}_1(G) := \{\phi \in \mathcal{P}(G) \mid \phi(e) = 1\}$.
So $\mathcal{P}_1(G)$ is a convex subset of $\mathcal{P}(G)$.

Note $\phi_{\pi,\xi} \in \mathcal{P}_1(G)$ iff $\|\xi\| = 1$.

Definition:
A pure function of +ve type is an extreme point of $\mathcal{P}_1(G)$;
i.e. $\phi \in \mathcal{P}_1(G)$ such that whenever $\phi = t\psi_1 + (1-t)\psi_2$ with $t \in [0, 1]$ and $\psi_i \in \mathcal{P}_1(G)$, actually $t \in \{0, 1\}$.

Theorem:
$\phi \in \mathcal{P}_1(G)$ is pure iff the GNS representation π_ϕ is irreducible.

Proof:
We have $\phi = \phi_{\pi_\phi,\xi_\phi}$, with ξ_ϕ cyclic.
Suppose π_ϕ is reducible.
Say $\mathcal{H}_\phi = \mathcal{K} \oplus \mathcal{K}^\perp$ is a non-trivial invariant orthogonal decomposition;
decompose $\xi = \xi_\mathcal{K} + \xi_{\mathcal{K}^\perp}$.
Set $s := \|\xi_\mathcal{K}\|, t := \|\xi_{\mathcal{K}^\perp}\|$.
So $s^2 + t^2 = \|\xi_\phi\|^2 = \phi(e) = 1$.

Now one may calculate that $\phi = s^2\phi_{\pi_\phi,\widehat{\xi_\mathcal{K}}} + t^2\phi_{\pi_\phi,\widehat{\xi_{\mathcal{K}^\perp}}}$,
but $s \neq 0 \neq t$ since ξ is cyclic.
So ϕ is not pure.

Conversely, suppose ϕ is not pure,
say $\phi = s^2\phi_1 + t^2\phi_2$, $s^2 + t^2 = 1$, $s \neq 0 \neq t$, $\phi_i \in \mathcal{P}_1(G)$,
and suppose π_ϕ is irreducible.

By GNS, say $\phi_1 = \phi_{\pi_1,\xi_1}$ and $\phi_2 = \phi_{\pi_2,\xi_2}$;
consider the direct sum representation $\pi_1 \oplus \pi_2$ of G on $\mathcal{H}_1 \oplus \mathcal{H}_2$,
and let $\xi' := s\xi_1 + t\xi_2$.
Then $\phi_{\pi_1\oplus\pi_2,\xi'}(g) = \langle sg\xi_1 + tg\xi_2, s\xi_1 + t\xi_2 \rangle = s^2\phi_1(g) + t^2\phi_2(g) = \phi(g)$.

Consider the subrepresentation π_0 on $\mathcal{H}' := \overline{< G\xi' >}$.
By the uniqueness of GNS, this is equivalent to π_ϕ.
Since ξ_1 is cyclic,
the projection $T : \mathcal{H}' \to \mathcal{H}_1; \xi' \mapsto \xi_1$ has dense image,

so by irreducibility of π_ϕ, T is an isomorphism
(lest $(\ker T)^\perp$ be a proper subrepresentation).

So $\phi = \phi_{\pi_0,\xi'} = \phi_{\pi_1,\xi_1} = \phi_1$, contradicting impurity.
□

Fact:
If G is locally compact, the convex hull of $\text{ext}(\mathcal{P}_1(G))$ is dense in $\mathcal{P}_1(G)$ for the topology of uniform convergence on compact subsets of G.

Proof of GNS

Let ϕ be a non-zero function of +ve type on G.
Consider the translates of ϕ, $\phi_h(g) := \phi(h^{-1}g)$.
Let V be the subspace of $C(G)$ they span.
Let W be the vector space freely generated by G.
$\langle g, h\rangle := \phi(h^{-1}g)$ extends to a conjugate-symmetric sesquilinear form on W,
$\langle \sum_i a_i g_i, \sum_i b_i h_i\rangle = \sum_{ij} a_i \phi(h_i^{-1} g_i)\overline{b}_j$.
Now $\langle \sum_i a_i h_i, \sum_i a_i h_i\rangle = \sum_{ij} a_i \phi(h_i^{-1} h_j)\overline{a}_j \geq 0$ by definition of a positive function, so $\langle,\rangle$ is a semi-inner product on W.

The kernel is $\{\sum_i b_i h_i \mid \sum_i \phi_{h_i} = 0\}$,
so it induces a well-defined inner product $\langle,\rangle$ on V.

Let $\mathcal{H}$ be the Hilbert space completion of V with respect to $\langle,\rangle$;
so $\mathcal{H}$ is the space of functions on G which are pointwise limits of Cauchy sequences in V with respect to the norm induced by $<,>$.

Let π be the representation of G acting by translation on $\mathcal{H}$,
$\pi(h)f(g) := f(h^{-1}g)$,
and let $\xi := \phi \in \mathcal{H}$.
This is unitary, since it preserves $<,>$ on the ϕ_g,
and is strongly continuous since ϕ is continuous
(and so then $g \mapsto gf$ is continuous for $f = \phi$, hence for any $f \in V$, and hence for $f \in \mathcal{H}$ since a uniform limit of continuous functions is continuous),
and ξ is cyclic, and $\phi_{\pi,\xi}(h) = \langle \phi_h, \phi_e\rangle = \phi(h)$.

It remains to show the uniqueness.

Suppose $\phi_{\pi',\xi'} = \phi_{\pi,\xi}$,
with ξ' cyclic for $\pi' : G \to \mathcal{H}'$.

(*) $\| \sum_i a_i g_i \xi\|^2 = \sum_{ij} a_i \langle g_i\xi, g_j\xi\rangle \overline{a}_j = \sum_{ij} a_i \langle g_i\xi', g_j\xi'\rangle \overline{a}_j = \| \sum_i a_i g_i \xi'\|^2$

Define an isomorphism $T : \mathcal{H} \xrightarrow{\cong} \mathcal{H}'$ by $T(g\xi) := g\xi'$ and extending linearly and continuously; this is well-defined by (*).
T intertwines π and π', and $T(\xi) = \xi'$.
□

Example:
$G := \langle \mathbb{R}; +\rangle$, $\phi(z) := e^{iz}$.
$\phi_a(z) = e^{i(z-a)} = e^{-ia}\phi(z)$,
so $\mathcal{H}$ is 1-dimensional,
and the representation is $a \mapsto (e^{-ia}\cdot)$.