Notes from a seminar on the Keisler order; primarily an exposition of the proof of VI.3.12 in Shelah "Classification Theory"

0.1 Statements

Definition. For $\mathcal{U}$ an ultrafilter on λ,

$$\mu(\mathcal{U}) := \min_{\alpha \in \omega^{\mathcal{U}} \setminus \omega} \left|\{\beta \in \omega^{\mathcal{U}} : \beta < \alpha\}\right|,$$

$$\mathrm{lcf}(\mathcal{U}) := \min\{|A| : A \subseteq \omega^{\mathcal{U}} \setminus \omega,\ \forall \beta \in \omega^{\mathcal{U}} \setminus \omega.\ \exists \alpha \in A.\ \alpha < \beta\}.$$

Fact. *(i) The nfcp theories form the minimal class in the Keisler order.*

(ii) Let T be a countable stable fcp theory. Then a regular ultrafilter $\mathcal{U}$ on $\lambda \geq \aleph_0$ saturates T iff $\mu(\mathcal{U}) > \lambda$.

Theorem 1 (Shelah VI.4.8)**.** *Let T be a countable unstable theory. Let $\mathcal{U}$ be a regular ultrafilter on $\lambda \geq \aleph_0$ with* $\mathrm{lcf}(\mathcal{U}) \leq \lambda$*. Then $\mathcal{U}$ does not saturate T.*

Theorem 2 (Shelah VI.3.12)**.** *For any $\lambda \geq \aleph_0$, there is a regular ultrafilter $\mathcal{U}$ on λ with*

$$\mathrm{lcf}(\mathcal{U}) \leq \alpha_1,\ \mu(\mathcal{U}) > \lambda.$$

Corollary. *If T_1 is stable and T_2 is unstable, then $T_1 < T_2$ in the Keisler order.*

Proof. If T_1 is nfcp, we are done by the minimality of nfcp and the fact that nfcp $\Rightarrow$ stable. So suppose T_1 is stable fcp.

By Theorem 2, there is a regular ultrafilter $\mathcal{U}$ on $\lambda := \aleph_1$ such that

- $\mathrm{lcf}(\mathcal{U}) \leq \lambda$, hence $\mathcal{U}$ does not saturate T_2;
- $\mu(\mathcal{U}) > \lambda$, hence $\mathcal{U}$ saturates T_1.

□

Proof of Theorem 1

By instability, there is a countable $M \vDash T$ and tuples $b_i \in M^{<\omega}$ for $i \in \omega$ and a formula $\phi(x, y)$ such that

$$M \vDash \phi(b_i, b_j) \Leftrightarrow i \leq j.$$

For $\alpha \in \omega^{\mathcal{U}}$, let $b_\alpha := (t \mapsto b_{f(t)})/\mathcal{U}$ where $f \in \omega^\lambda$, $\alpha = f/\mathcal{U}$. Then by Łos,

$$M^{\mathcal{U}} \vDash \phi(b_\alpha, b_\beta) \Leftrightarrow \alpha \leq \beta.$$

Now suppose $M^{\mathcal{U}}$ is λ^+-saturated. Since $\mathrm{cf}(\mathcal{U}) \leq \lambda$, there is a coinitial subset $\{\alpha_j : j \in \lambda\} \subseteq \omega^{\mathcal{U}} \setminus \omega$. Let $a \in M^{\mathcal{U}}$ realise $\{\phi(b_i, x) : i \in \omega\} \cup \{\neg\phi(b_{\alpha_j}, x) : j \in \lambda\}$. So

$$\{i \in \omega^{\mathcal{U}} : \forall j < i.\ M^{\mathcal{U}} \vDash \phi(b_j, a)\} = \omega.$$

This contradicts the overspill lemma of non-standard analysis, which states that there is no infinite internal set of standard elements.

Alternatively (pointed out by Martin Hils): regularity of $\mathcal{U}$ implies $\aleph_1$-saturation, so there can't be a countable internal set.

Alternatively: if $a = g/\mathcal{U}$, $g \in M^\lambda$, then by Łos, $\omega = \prod_{t\in\lambda}\{i \in \omega : \forall j < i.\ M \vDash \phi(b_j, a(t))\}$. But any subset of ω has the property that if it is bounded then it has a maximal element, so by Łos this also holds of any ultraproduct of subsets of ω (apply Łos to $\exists x.\forall y.(P(y) \to y < x) \to \exists x.(P(x) \land \forall y.(P(y) \to y < x))$. But $\omega \subseteq \omega^{\mathcal{U}}$ is bounded and has no maximal element. □

Proof of Theorem 2.

Let $\alpha := 2^\lambda \cdot \aleph_1$, so $|\alpha| = 2^\lambda$ and $\mathrm{cf}(\alpha) = \aleph_1$.

Notation. For $\sigma = \phi(g_1, \ldots, g_k)$ a (possibly infinitary) $\{<\}$-sentence with parameters $g_1, \ldots, g_k \in \omega^\lambda$, define $[[\sigma]] := \{t : \omega \vDash \phi(\overline{g}(t))\}$. For D a filter on λ, write $\omega^D \vDash \sigma$ to mean $[[\sigma]] \in D$.

Note that for any $A \subseteq \lambda$, $A = [[\chi_A = 1]]$.

(So really this is just a convenient substitute for talking about subsets of λ, letting us use the notation of propositional logic.)

For Σ a set of sentences, let $D\langle\Sigma\rangle$ be the filter generated by $D \cup \{[[\sigma]] : \sigma \in \Sigma\}$.

For a sentence σ, let $D\langle\sigma\rangle := D\langle\{\sigma\}\rangle$.

Lemma. *For D a filter and σ, τ sentences, $\omega^{D\langle\sigma\rangle} \vDash \tau$ iff $\omega^D \vDash \sigma \to \tau$.*

Proof. $\omega^{D\langle\sigma\rangle} \vDash \tau$ iff $[[\tau]] \in D\langle\sigma\rangle$
iff $[[\tau]] \supseteq A \cap [[\sigma]]$ for some $A \in D$
iff $(\lambda \setminus [[\sigma]]) \cup [[\tau]] \supseteq A$ for some $A \in D$
iff $[[\sigma \to \tau]] \in D$
iff $\omega^D \vDash \sigma \to \tau$. □

Definition. Say sentences $(\tau_i)_i$ partition modulo a filter D if $\omega^D \vDash \neg(\tau_i \land \tau_j)$ for $i \neq j$ and for any sentence σ,

$$\big(\forall i.\ \omega^D \vDash \tau_i \to \sigma\big) \Rightarrow \omega^D \vDash \sigma.$$

Claim 1. *There exist $f_i \in \omega^\lambda$ and regular filters D_i for $i \leq \alpha$ such that*

(1) $(D_i)_i$ is an increasing chain;

(2) if $i < j < \alpha$ and $n \in \omega$, then $\omega^{D_j} \vDash f_i \neq n$;

(3) if $j \leq i < \alpha$, then $(f_i = n)_{n\in\omega}$ partitions mod D_j.

(4) $\mathcal{U} := \cup_{i<\alpha} D_i$ is an ultrafilter.

Claim. *(i) Then $f_i/\mathcal{U}$ are strictly decreasing, i.e. $i < j \leq \alpha \Rightarrow f_j/\mathcal{U} < f_i/\mathcal{U}$.*

(ii) If $g \in \omega^\lambda$ and $g/\mathcal{U} \in \omega^{\mathcal{U}} \setminus \omega$, then for some $i < \alpha$ we have $f_j/\mathcal{U} < g/\mathcal{U}$ for all $i \leq j < \alpha$.

Proof. (i) $\forall n \in \omega.\ \omega^{D_j} \vDash f_i \neq n$, so $\forall n \in \omega.\ \omega^{D_j} \vDash f_i > n$, so $\forall n \in \omega.\ \omega^{D_j} \vDash (f_j = n \to f_i > f_j)$. But $(f_j = n)_n$ partitions mod D_j, so $\omega^{D_j} \vDash f_i > f_j$. Then also $\omega^{\mathcal{U}} \vDash f_i > f_i$, i.e. $f_i/\mathcal{U} > f_j/\mathcal{U}$, as required.

(ii) We have $\forall n \in \omega.\ \omega^{\mathcal{U}} \models g > n$. Since $\mathrm{cf}(\alpha) = \aleph_1 > \aleph_0$, already $\forall n \in \omega.\ \omega^{D_i} \models g > n$ for some $i < \alpha$. We then conclude exactly as in (i).

□

So if $g/\mathcal{U} \in \omega^{\mathcal{U}} \setminus \omega$, for some $\beta < \alpha$ we have $f_i/\mathcal{U} \in g/\mathcal{U}$ for $\beta < i < \alpha$, so

$$\left|\{\beta \in \omega^{\mathcal{U}} : \beta < g/\mathcal{U}\}\right| \geq \left|\{i : \beta < i < \alpha = 2^\lambda \cdot \aleph_1\}\right| = 2^\lambda > \lambda.$$

So $\mu(\mathcal{U}) > \lambda$.

Furthermore, by the Claim, $(f_i/\mathcal{U})_{i<\alpha}$ is a coinitial decreasing sequence in $\omega^{\mathcal{U}} \setminus \omega$, so

$$\mathrm{lcf}(\mathcal{U}) \leq \mathrm{cf}(\alpha) = \aleph_1,$$

as required.

This concludes the proof of Theorem 2 modulo Claim 1.

Proof of Claim 1

Definition. For $\mathcal{G} \subseteq \omega^\lambda$, define $\mathrm{FI}(\mathcal{G}) := \{h : G_0 \to \omega \mid G_0 \subseteq_{\mathrm{fin}} \mathcal{G}\}$.

For $h \in \mathrm{FI}(\mathcal{G})$, let σ_h be the sentence $\bigwedge_{g \in \mathrm{dom}(h)} g = h(g)$. We write just h for σ_h.

For D a filter and sentences τ, σ, say τ **decides** σ mod D if either $\omega^D \models h_i \to \sigma$ or $\omega^D \models h_i \to \neg\sigma$.

Say σ is **supported** by $\mathcal{G}$ mod D if there are $h_i \in \mathrm{FI}(\mathcal{G})$ for $i \in \omega$ such that each h_i decides σ mod D and $(h_i)_i$ partitions mod D.

Say $\mathcal{G}$ is **independent** modulo D if for all $h \in \mathrm{FI}(\mathcal{G})$,

$$\omega^D \not\models \neg h.$$

We build D_i such that (1) and (2) hold and also

(i) $D_i = D_0 \langle \Sigma_i \rangle$ where each $\sigma \in \Sigma_i$ is supported by $f_{<i}$ mod D_0;

(ii) $f_{\geq i}$ is independent mod D_i;

(iii) D_i is maximal such that (i) and (ii) hold.

(3) and (4) will then follow.

First we find the f_i and D_0.

Lemma. *There are functions $g_i : \lambda \to \lambda$ for $i \in 2^\lambda$ such that for any $I_0 \subseteq_{\mathrm{fin}} 2^\lambda$ and $h : I_0 \to \lambda$, there exists $\gamma \in \lambda$ such that $\forall i \in I_0.\ g_i(\gamma) = h(i)$.*

Proof. Enumerate the pairs $(A_\gamma, F_\gamma)_{\gamma\in\lambda}$ of finite subsets $A \subseteq \lambda$ and functions $F : \mathcal{P}(A) \to \lambda$. For $B \subseteq \lambda$, let $g_B(\gamma) := F_\gamma(B \cap A_\gamma)$. Now given $h : I_0 \to \lambda$, for some finite $A \subseteq \lambda$ the $(B \cap A)_{B \in I_0}$ are distinct, and so for some γ we have $A_\gamma = A$ and $\forall B \in I_0.\ F_\gamma(B \cap A) = h(B)$, as required. □

Applying this lemma, we can find $(f_i : \lambda \to \omega)_{i<\alpha}$ and $g : \lambda \to S_{\aleph_0}(\lambda)$ such that any finite set of values for $((f_i)_i, g)$ occurs somewhere on λ. Let D_0' be the filter generated by $R := \{\{t \in \lambda : i \in g(t)\} : i \in \lambda\}$. Then D_0' is regular since R is a regularising family, and $f_{<\alpha}$ is independent mod D_0'. Extend D_0 to a maximal filter D_0 such that $f_{<\alpha}$ is independent mod D_0. Then (i)-(iii) hold for D_0.

Let $\mathrm{FI} := \mathrm{FI}(f_{<\alpha})$.

Lemma 1. *Suppose $\mathcal{G} \subseteq \omega^\lambda$ is independent modulo a filter D, and $\mathcal{G}' \subseteq \mathcal{G}$, and $h \in \mathcal{G}$, and τ is supported by $\mathcal{G}'$ mod D, and $\omega^D \models h \to \tau$. Then $\omega^D \models h|_{\mathcal{G}'} \to \tau$.*

Proof. Say $h_i \in \mathrm{FI}(\mathcal{G}')$, $i \in \omega$, partition mod D and each h_i decides τ mod D.

Let $i \in \omega$. If $\omega^D \models h_i \to \tau$ then clearly $\omega^D \models h_i \to (h|_{\mathcal{G}'} \to \tau)$. If $\omega^D \models h_i \to \neg\tau$ then $\omega^D \models h_i \to \neg h$, so $\omega^D \models \neg(h_i \wedge h)$, so by independence $h_i \cup h \notin \mathrm{FI}(\mathcal{G})$, but then already $h_i \cup h|_{\mathcal{G}'} \notin \mathrm{FI}(\mathcal{G})$, so $\omega^D \models h_i \to \neg h|_{\mathcal{G}'}$, so again $\omega^D \models h_i \to (h|_{\mathcal{G}'} \to \tau)$.

We conclude since the h_i partition mod D. □

Lemma. *For any σ, if $\omega^{D_0} \not\models \neg\sigma$ then $\omega^{D_0} \models h \to \sigma$ for some $h \in \mathrm{FI}$.*

Proof. Otherwise $f_{\geq 0}$ is independent modulo $D_0 \langle \neg\sigma \rangle$, contradicting the maximality of D_0. □

Fact. *If $C \subseteq S_{\aleph_0}(\lambda) := \{A : A \subseteq_{\mathrm{fin}} \lambda\}$ and $|C| > \aleph_0$, there exists $C' \subseteq C$ with $|C'| > \aleph_0$ and $A_0 \subseteq_{\mathrm{fin}} \lambda$ such that $\forall A_1, A_2 \in C'.\ (A_1 \neq A_2 \Rightarrow A_1 \cap A_2 = A_0)$.*

Proof. Omitted. See [Shelah-classificationTheory Appendix Theorem 1.4]. □

Lemma 2. *For any σ there is $\beta < \alpha$ such that σ is supported by $f_{<\beta}$ mod D_0.*

Proof. Let $H = \{h_i\}_i \subseteq \mathrm{FI}$ be maximal such that each h_i decides σ mod D_0 and $i \neq j \Rightarrow \omega^{D_0} \models \neg(h_i \wedge h_j)$.

Suppose $|H| > \aleph_0$. By the previous Fact, uncountably many have domains intersecting pairwise in a fixed finite set, so in particular there exist $h \neq h' \in H$ with $h \cup h' \in \mathrm{FI}$. But then $\omega^{D_0} \models \neg(h \wedge h')$ contradicts independence.

So $|H| = \aleph_0 < \mathrm{cf}(\alpha)$, so there exists $\beta < \alpha$ such that each $h_i \in \mathrm{FI}(f_{<\beta})$. It remains to show that H partitions mod D_0.

Suppose $\forall h \in H.\ \omega^{D_0} \models h \to \tau$ but $\omega^{D_0} \not\models \tau$. Then $\omega^{D_0} \not\models \sigma^\epsilon \to \tau$ for some ϵ, i.e. $\omega^{D_0} \not\models \neg(\sigma^\epsilon \wedge \neg\tau)$.

Applying the previous Lemma to $\sigma^\epsilon \wedge \neg\tau$ yields $h' \in \mathrm{FI}$ which decides σ and $\omega^{D_0} \models h' \to \neg\tau$ so h' is inconsistent with each $h \in H$, contradicting maximality of H. □

Given D_i, let $D'_{i+1} := D_i \langle \{f_i \neq n : n < \omega\} \rangle$. Then D'_{i+1} satisfies (i) since D_i does, and satisfies (ii) since D_i does and by Lemma 1. Extend D'_{i+1} to D_{i+1} satisfying the maximality condition (iii).

For $\beta < \alpha$ a limit ordinal, $D'_\beta := \cup_{i<\beta} D_i$ clearly satisfies (i) and (ii). Extend it to D_β satisfying the maximality condition (iii).

(1) and (2) are clear from the construction.

We check (4). To show that $\mathcal{U}$ is an ultrafilter, let σ be arbitrary, suppose $\omega^{\mathcal{U}} \not\models \sigma$, and we show $\omega^{\mathcal{U}} \models \neg\sigma$.

By Lemma 2, say σ is $f_{<\beta}$-supported mod D_0. Now $\omega^{D_\beta} \not\models \sigma$, so by (iii) for D_β, $f_{\geq\beta}$ is not independent modulo $D_\beta \langle \sigma \rangle$. But then say $h \in \mathrm{FI}(f_{\geq\beta})$ and $\omega^{D_\beta \langle \sigma \rangle} \models \neg h$. Then $\omega^{D_\beta} \models \sigma \to \neg h$, so $\omega^D \models (\tau \wedge \sigma) \to \neg h$ for some $f_{<\beta}$-supported τ with $\omega^{D_\beta} \models \tau$. So $\omega^D \models h \to \neg(\tau \wedge \sigma)$, and supportedness is closed under boolean combinations (exercise), so by Lemma 1 $\omega^D \models \neg(\tau \wedge \sigma)$, so $\omega^{D_\beta} \models \neg\sigma$, so $\omega^{\mathcal{U}} \models \neg\sigma$.

Finally, we check (3). Suppose $\beta \leq \gamma < \alpha$, let $f := f_\gamma$, and suppose $\forall n \in \omega.\ \omega^{D_\beta} \models (f = n \to \sigma)$. Say $\omega^{D_\beta} \models \tau_n$ and $\omega^{D_0} \models \tau_n \to (f = n \to \sigma)$ and τ_n is supported by $f_{<\beta}$ mod D_0.

By Lemma 2, say $h_i^\epsilon \in \mathrm{FI}$, $i \in \omega$, $\epsilon \in \{\top, \bot\}$, partition mod D_0 and $\omega^{D_0} \models h_i^\epsilon \to \sigma^\epsilon$.

Let $h_{i,\beta}^\bot := h_i^\bot|_{f<\beta}$. Now $\omega^{D_0} \models \neg\sigma \to (f = n \to \neg\tau_n)$ so $\omega^{D_0} \models (h_i^\bot \wedge f = n) \to \neg\tau_n$.

Now $\omega^{D_\beta} \models h_i^\bot \to f \neq n$ for all n, so clearly $f \notin \mathrm{dom}(h_i^\bot)$, so $h_i^\bot \cup (f \mapsto n) \in \mathrm{FI}$, and so by Lemma 1, $\omega^D \models h_{i,\beta}^\bot \to \neg\tau_n$.

Let $\theta := \bigvee_i h_{i,\beta}^\bot$. Then $\omega^{D_0} \models h_i^\epsilon \to (\neg\theta \to \sigma)$ for all i, ϵ, so $\omega^{D_0} \models \neg\theta \to \sigma$.

Let H be a maximal antichain in $\{h \in \mathrm{FI} : (\exists i. h \supseteq h_{i,\beta}^\bot) \vee (\forall i. h \cup h_{i,\beta}^\bot \notin \mathrm{FI})\}$.

Claim. *H partitions mod D_0.*

Proof. By Lemma 2, it suffices to show that for $h' \in \mathrm{FI}$, if $\forall h \in H.\ \omega^{D_0} \models h \to \neg h'$ then $\omega^D \models \neg h'$. Suppose not. Now either $\forall i.\ h' \cup h_{i,\beta}^\bot \notin \mathrm{FI}$, or h' extends to some $h'' = h' \cup h^\bot i, \beta \in \mathrm{FI}$. Then by maximality, h' resp. $h'' \in H$. But then $\omega^D \models \neg h'$, contradiction. □

If $\forall i. h \cup h_{i,\beta}^\bot \notin \mathrm{FI}$ then $\omega^D \models h \to \neg\theta$, and meanwhile $\omega^D \models h_{i,\beta}^\bot$ for each i, so for $h \in H$, $\omega^{D_0} \models h \to (\theta \to \neg\tau_n)$, so $\omega^{D_0} \models \theta \to \neg\tau_n$.

So $\omega^{D_\beta} \models \sigma$ as required. □

– Martin Bays 2019