1 Larsen-Pink

An account of the Hrushovski-Wagner proof of Larsen-Pink.

Attention potential reader: soon after writing this note, I discovered that there's a nice streamlined account of the proof in Hrushovski's "Stable group theory and approximate subgroups", where it appears as Proposition 5.6. Most likely, you are better off reading that than this.

1.1 Statement

Let G be a simple group of finite Morley rank $\emptyset$-definable in a stable theory T.

Suppose $M = \Pi_{\mathcal{U}} M_i \models T$, a countable non-principal ultraproduct. Suppose $\Gamma_i \subseteq M_i$ are finite subsets such that $\Gamma := \Pi_{\mathcal{U}} \Gamma_i \leq G(M)$ is a definably dense subgroup, meaning that Γ is contained in no proper definable subgroup of G.

Theorem 1.1 (Hrushovski-Wagner ("Larsen-Pink")). *For any M-definable subset $X \subseteq G^n$, there exists $c \in \mathbb{R}$ such that for all i, $|X \cap \Gamma_i^n| \leq c|\Gamma_i|^{\frac{\mathrm{RM}(X)}{\mathrm{RM}(G)}}$.*

Larsen-Pink proved this in the case of G a simple algebraic group and X a subvariety, and suggested that there might be a model-theoretic proof. Hrushovski-Wagner provided such a proof, in the model-theoretically natural generality of groups of finite Morley rank.

1.2 Example / sketch proof

Suppose X is a curve in a simple algebraic group G, $n := \dim(G)$. Then (as we see below) there are $\gamma_1, \ldots, \gamma_n \in \Gamma$ such that, setting $f(x_1, \ldots, x_n) := \Pi_i x_i^{\gamma_i}$, $f : X^n \to G$ is dominant with generically finite fibres. Suppose all fibres are finite, say with fibre size bounded by k. Then since $f(\Gamma^n) \subseteq \Gamma$, $|X \cap \Gamma^n|^n = |X^n \cap \Gamma| \leq k|\Gamma|$, so $|X \cap \Gamma| \leq k|\Gamma|^{1/n}$.

In reality, some fibres might be infinite, and Γ could concentrate there... we handle this by working with complete types. When X is of higher dimension, we don't have such a clean map f, and so we work inductively, obtaining a sequence of maps with generic fibres of dimension less than $\dim(X)$.

1.3 Stable group theory

Work in a monster model $\mathbb{M} \models T$.

Systematically confuse formulae with the sets they define.

Recall

- If $A \preceq \mathbb{M}$, or more generally if $A \subseteq \mathbb{M}$ with $\mathrm{dcl}^{\mathrm{eq}}(A) = \mathrm{acl}^{\mathrm{eq}}(A)$, any $p \in S(A)$ is stationary, meaning it has a unique global non-forking extension $\mathfrak{p} \in S(\mathbb{M})$. Non-forking can here be taken to mean $\mathrm{RM}(\mathfrak{p}) = \mathrm{RM}(p)$. Define $p|_{Ag} := \mathfrak{p}|_{Ag}$, so $a \models p|_{Ag} \Leftrightarrow (a \models p \text{ and } a \mathop{\smile\!\!\!\!\!\mid}_A g)$.
- If $\mathrm{dcl}^{\mathrm{eq}}(A) = \mathrm{acl}^{\mathrm{eq}}(A)$ and $p, q \in S(A)$, their unique product type $p \otimes q \in S(A)$ is $\mathrm{tp}(a, b/A)$ where $a \models p$ and $b \models q|_{Aa}$.
- If f is a partial function definable over $A \subseteq \mathbb{M}$ and defined at $p \in S(A)$, $f_*(p) \in S(A)$ is the type $\mathrm{tp}(f(b)/A)$ where $b \models p$.

- For $A \subseteq \mathbb{M}$, $S_G(A) := \{p \in S(A) \mid p(x) \models x \in G\}$. $G \cap \mathrm{dcl}^{\mathrm{eq}}(A)$ acts on $S_G(A)$, by $g * p := (g*)_*(p)$.
- G has DCC: there is no infinite decreasing chain of definable subgroups.
- For $\mathfrak{p} \in S_G(\mathbb{M})$, $\mathrm{Stab}(\mathfrak{p}) = \{g \in G \mid g * \mathfrak{p} = \mathfrak{p}\}$ is a definable subgroup. For $p \in S_G(A)$ stationary, $\mathrm{Stab}(p) = \mathrm{Stab}(\mathfrak{p})$ where $\mathfrak{p}$ is the unique global nonforking extension; equivalently, $\mathrm{Stab}(p) = \{g \in G \mid g * p|_{Ag} = p|_{Ag}\}$.
- G is connected so has a unique generic type $\mathfrak{p}_G \in S_G(\mathbb{M})$. For $p \in S_G(A)$, $\mathrm{Stab}(p) = G \Leftrightarrow \mathrm{RM}(p) = \mathrm{RM}(G) \Leftrightarrow p = \mathfrak{p}_G|_A$.
- In G, RM is additive, meaning $\mathrm{RM}(ab/C) = \mathrm{RM}(a/bC) + \mathrm{RM}(b/C)$, and definable, meaning that if $r \in S_{G^n}(A)$ and f is a partial function definable over A and defined at r, then there exists $X \in r$ such that for all $b \in f(X)$, $\mathrm{RM}(f|_X^{-1}(b)) = \mathrm{RM}(r) - \mathrm{RM}(f_*(r))$.

For $\gamma \in \Gamma$, define $m^\gamma(g,h) := g^\gamma * h$.

Lemma 1.2 (Essentially ZIT). *Suppose $\Gamma \leq G(\mathbb{M})$ is definably dense, $\Gamma \subseteq A = \mathrm{acl}^{\mathrm{eq}}(A)$, and $p \in S_G(A)$ is non-algebraic. Then there exist $p_1, \ldots, p_n \in S_G(A)$ and $\gamma_1, \ldots, \gamma_{n-1} \in \gamma$, with $p_1 = p$, $p_n = \mathfrak{p}_G|_A$, $p_{i+1} = m_*^{\gamma_i}(p \otimes p_i)$, and such that $\mathrm{RM}(p_{i+1}) > \mathrm{RM}(p_i)$.*

Proof. WMA $A = \mathrm{acl}^{\mathrm{eq}}(\emptyset)$.

It suffices to show that if $q \in S_G(A)$ satisfies $\mathrm{RM}(m_*^\gamma(p \otimes q)) \leq \mathrm{RM}(q)$ for all $\gamma \in \Gamma$, then $q = \mathfrak{p}_G|_A$.

So let q be such, and let $S := \mathrm{Stab}(q)$. We will show that $S = G$.

Let $S' := \bigcap_{\gamma \in \Gamma} S^\gamma$. Then S' is (by the DCC) a definable subgroup, and its normaliser $N(S')$ contains Γ, so, by denseness of Γ, $N(S') = G$, so $S' \lhd G$.

So by simplicity of G, it suffices to show that $S' \neq 1$.

Let $a, b \models p$. We conclude by showing that $a^{-1}b \in S'$.

Let $e \models q|_{ab}$. So $a^\gamma e, b^\gamma e \models m_*^\gamma(p \otimes q) =: r_\gamma$.

Then $\mathrm{RM}(p) + \mathrm{RM}(q) = \mathrm{RM}(a,e) = \mathrm{RM}(a,e,a^\gamma e) = \mathrm{RM}(a, a^\gamma e) = \mathrm{RM}(a/a^\gamma e) + \mathrm{RM}(a^\gamma e)$.

But $\mathrm{RM}(p) \geq \mathrm{RM}(a/a^\gamma e)$ and $\mathrm{RM}(q) \geq \mathrm{RM}(r_\gamma) = \mathrm{RM}(a^\gamma e)$, so $\mathrm{RM}(p) = \mathrm{RM}(a/a^\gamma e)$ (and $\mathrm{RM}(q) = \mathrm{RM}(a^\gamma e)$). So $a^\gamma e \models r_\gamma|_a$. So since $e \mathop{\smile\!\!\!\!\!\mid}_a b$, $a^\gamma e \models r_\gamma|_{ab}$, i.e. $a^\gamma * q|_{ab} = r_\gamma|_{ab}$. Similarly, $b^\gamma * q|_{ab} = r_\gamma|_{ab}$.

So $(a^{-1}b)^\gamma * q|_{ab} = q|_{ab}$, so $(a^{-1}b)^\gamma \in S'$ for all γ, so $a^{-1}b \in S$. □

1.4 Pseudofinite dimensions

Recall we have $M := \Pi_{\mathcal{U}} M_i \preceq \mathbb{M}$ a countable non-principal ultraproduct, $\Gamma := \Pi_{\mathcal{U}} \Gamma_i \leq G(M)$ definably dense subgroup.

Definition 1.1.

- For X definable over M, $\delta(X) := (\log |X(\Gamma)|)/\mathrm{Fin}$ ("fine pseudofinite dimension restricted to Γ"), where $|X(\Gamma)| := \Pi_{\mathcal{U}} |X^{M_i} \cap \Gamma_i^n| \in \Pi_{\mathcal{U}} \mathbb{R}$ if $X \subseteq G^n$, and $|X(\Gamma)| := 0$ if X is on some other sort. Here Fin is the convex hull of the standard reals $\mathbb{R} \subseteq \Pi_{\mathcal{U}} \mathbb{R}$.
- For π a partial type over M, $\delta(\pi) := \inf_{\phi \in \pi} \delta(\phi)$, taking values in $\Xi \cup \{-\infty\}$ where Ξ is the formal completion of $\Pi_{\mathcal{U}} \mathbb{R}/\mathrm{Fin}$ as an ordered abelian group.

- For comparison with values of RM, embed $\mathbb{Z}$ into Ξ such that $\delta(G) = \mathrm{RM}(G)$.

Lemma 1.3.

(i) $\delta(X \cup Y) = \max(\delta(X), \delta(Y))$.

(ii) Any partial type π *over* M *admits a completion* $p \in S(M)$ *with* $\delta(p) = \delta(\pi)$.

(iii) $\delta(X \times Y) = \delta(X) + \delta(Y)$.

(iv) If $f : G^n \to G^m$ *is a function definable over* M *such that* $f(\Gamma^n) \subseteq \Gamma^m$*, if* $r \in S_{G^n}(M)$*, and if* $X \in r$ *with* $\delta(f^{-1}(b)) \leq \alpha$ *for all* $b \in f(X)^M$*, then* $\delta(r) \leq \delta(f_*(r)) + \alpha$.

Proof.

(i) $\max(|X^\Gamma|, |Y^\Gamma|) \leq |(X \cup Y)^\Gamma| \leq 2\max(|X^\Gamma|, |Y^\Gamma|)$.

(ii) Follows from (i).

(iii) Immediate from definitions.

(iv) Let $Y \in f_*(r)$, and let $X' := X \cap f^{-1}(Y)$. Then $\delta(p) \leq \delta(X') \leq \delta(Y) + \alpha$.

□

Stated in this language and generalised to partial types, Theorem 1.1 becomes:

Theorem 1.4. *For any partial type* π *over* M*,* $\delta(\pi) \leq \mathrm{RM}(p)$.

Proof. Let $k := \mathrm{RM}(\pi)$, and suppose inductively that for any π' with $\mathrm{RM}(\pi') < k$ we have $\delta(\pi') \leq \mathrm{RM}(\pi')$.

Let $p \in S_G(M)$ complete π with $\delta(p) = \delta(\pi)$. It suffices to show that $\delta(p) \leq \mathrm{RM}(p)$, since then $\delta(\pi) = \delta(p) \leq \mathrm{RM}(p) \leq \mathrm{RM}(\pi)$.

For p algebraic, clearly $\delta(p) = 0 = \mathrm{RM}(p)$.

So suppose $k > 0$, and suppose for a contradiction that $\delta(p) > k$.

Claim 1.4.1. *Suppose* $f : G^n \to G^m$ *is a function definable over* M *such that* $f(\Gamma^n) \subseteq \Gamma^m$*, and let* $r \in S_{G^n}(M)$*. Suppose* f *and* r *are as in Lemma 1.3(iii), and suppose* $\alpha := \mathrm{RM}(r) - \mathrm{RM}(f_*(r)) < k$*. Then* $\delta(r) \leq \delta(f_*(r)) + \alpha$.

Proof. By definability of RM, exists $X \in r$ such that for all $b \in f(X)$, $\mathrm{RM}(f|_X^{-1}(b)) = \alpha$.

Then since $\alpha < k = \mathrm{RM}(p)$, for all $b \in f(X)^M$, $\delta(f|_X^{-1}(b)) \leq \mathrm{RM}(f|_X^{-1}(b)) = \alpha$.

So by Lemma 1.3(iii), $\delta(r) \leq \delta(f_*(r)) + \alpha$. □

Claim 1.4.2. *For* $q \in S(M)$*,* $\delta(p \otimes q) = \delta(p) + \delta(q)$.

Proof. Let $(p \times q)(x, y) := p(x) \cup q(y)$. Let $r(x, y) \in S(M)$ complete $p \times q$ with $\delta(r) = \delta(p \times q) = \delta(p) + \delta(q)$.

Suppose for a contradiction that $r \neq p \otimes q$. So $\mathrm{RM}(r) < \mathrm{RM}(p) + \mathrm{RM}(q)$, so $\mathrm{RM}(r) - \mathrm{RM}(q) < \mathrm{RM}(p) = k$.

So by Claim 1.4.1, $\delta(r) < \delta(q) + k < \delta(q) + \delta(p)$, contradicting the choice of r. □

Suppose that $p \in S_G(M)$.

Let $p = p_1, \ldots, p_n = \mathfrak{p}_G|_M$ and $\gamma_1, \ldots, \gamma_{n-1}$ be as in Lemma 1.2.

So for $1 \leq i < n$, $m_*^{\gamma_i}(p \otimes p_i) = p_{i+1}$ and $\alpha_i := \mathrm{RM}(p \otimes p_i) - \mathrm{RM}(p_{i+1}) = \mathrm{RM}(p) + \mathrm{RM}(p_i) - \mathrm{RM}(p_{i+1}) < \mathrm{RM}(p) = k$.

So by Claims 1.4.2 and 1.4.1, $\delta(p) + \delta(p_i) = \delta(p \otimes p_i) \leq \delta(p_{i+1}) + \alpha_i$, so $\delta(p_i) \leq \delta(p_{i+1}) + \alpha_i - \delta(p)$.

So $\delta(p) = \delta(p_1) \leq \delta(p_n) + \sum \alpha_i - (n-1)\delta(p)$, so $n\delta(p) \leq \delta(\mathfrak{p}_G) + \sum \alpha_i$.

Meanwhile, $\mathrm{RM}(p_i) = \mathrm{RM}(p_{i+1}) + \alpha_i - \mathrm{RM}(p)$, so $n\,\mathrm{RM}(p) = \mathrm{RM}(\mathfrak{p}_G) + \sum \alpha_i$.

By the normalisation, $\delta(\mathfrak{p}_G) \leq \delta(G) = \mathrm{RM}(G) = \mathrm{RM}(\mathfrak{p}_G)$. So $n\delta(p) \leq n\,\mathrm{RM}(p)$, so $\delta(p) \leq \mathrm{RM}(p)$, contradicting the contrary assumption.

So if $p \in S_G(M)$, $\delta(p) \leq \mathrm{RM}(p)$.

Then by induction on n, this holds for $p \in S_{G^n}(M)$, by considering co-ordinate projections and definability of RM, as in Claim 1.4.1.

For p on other sorts, by definition $\delta(p) = 0 \leq \mathrm{RM}(p)$, so we are done. □