The Pila-Zannier proof of Manin-Mumford

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Contents

1 Manin-Mumford

- $K \leq \mathbb{C}$ number field.
- G complex abelian variety (i.e. projective connected non-trivial algebraic group) over K.

  (e.g. $G = E^g$ for E an elliptic curve over K.)
- $g := \dim(G)$.
- Let $\exp : LG \to G$ be the exponential map of $G = G(\mathbb{C})$ as a complex Lie group.
- $LG = T_0(G)$ is a g-dimensional $\mathbb{C}$-vector space.
- exp is a surjective holomorphic homomorphism.
- $\Lambda := \ker(\exp) \cong \mathbb{Z}^{2g}$ is a full lattice in LG, i.e. $\Lambda = \langle \lambda_1, \ldots, \lambda_{2g} \rangle_{\mathbb{Z}}$ where $(\lambda_i)_i$ is an $\mathbb{R}$-basis of LG.

- So as a complex Lie group, $G \cong \mathbb{C}^g/\Lambda$ is a complex torus, and is diffeomorphic to $(\mathbb{R}/\mathbb{Z})^{2g}$.
- Torsion subgroup: $G[\infty] \cong (\mathbb{Q}/\mathbb{Z})^{2g}$.
- The non-trivial connected algebraic subgroups of G are precisely the abelian subvarieties.
- G has only countably many abelian subvarieties, and each is defined over $\mathbb{Q}^{\mathrm{alg}}$.
- $G[\infty]$ is Zariski dense in G.
- A **torsion coset** is a coset $H + \xi$ of an abelian subvariety $H \leq G$ by a torsion point $\xi \in G[\infty]$.

Theorem (Manin-Mumford Conjecture; Raynaud (1983))**.** *Let $X \subseteq G$ be an irreducible (complex) subvariety and suppose $X \cap G[\infty]$ is Zariski dense in X. Then X is a torsion coset.*

Corollary. *For $X \subseteq G$ Zariski closed,*

$$X \cap G[\infty] = \bigcup_{i=0}^{n} \xi_i + H_i[\infty]$$

where $\xi_i \in G[\infty]$ and $H_i \leq G$ are abelian subvarieties.

Proof.

$$\begin{aligned} X \cap G[\infty] &= (X \cap G[\infty])^{\mathrm{Zar}} \cap G[\infty] \\ &= \left(\bigcup_{i=0}^{n} \xi_i + H_i \right) \cap G[\infty] \\ &= \bigcup_{i=0}^{n} ((\xi_i + H_i) \cap G[\infty]) \\ &= \bigcup_{i=0}^{n} \xi_i + H_i[\infty] \end{aligned}$$

□

1.1 Special Locus

- Let $X \subseteq_{\mathrm{cl}} G$.
- The **special locus** (or Ueno locus, or Kawamata locus) of X in G is

$$\mathrm{SpL}(X) := \bigcup \{\mathrm{SpL}(X, H) : H \leq G \text{ abelian subvariety}\},$$

where

$$\mathrm{SpL}(X,H) := \bigcup\{g+H : g+H \subseteq X,\ g \in G\} = \bigcap_{h\in H}(X-h).$$

We will see that $\mathrm{SpL}(X)$ corresponds to the Pila-Wilkie 'algebraic part' of $\exp^{-1}(X)$.

- The **stabiliser** of X in G is

$$\mathrm{Stab}(X) = \mathrm{Stab}_G(X) := \{g \in G : g+X = X\}.$$

Lemma. *Suppose X is irreducible.*
Then $\mathrm{SpL}(X) = X$ iff $\mathrm{Stab}(X)$ is infinite.

Proof.

- $\mathrm{Stab}(X) = \bigcap_{x\in X}(X-x)$, so $\mathrm{Stab}(X) \leq_{\mathrm{cl}} G$, so $\mathrm{Stab}(X)$ is an algebraic subgroup.
- If $\mathrm{Stab}(X)$ is infinite, then the connected component of the identity $H := \mathrm{Stab}(X)^o \leq G$ is an abelian subvariety, so $X = \bigcup\{x+H : x \in X\} = \mathrm{SpL}(X)$.
- Conversely: If $X = \mathrm{SpL}(X)$, then $X = \mathrm{SpL}(X,H)$ for some H, since each $\mathrm{SpL}(X,H)$ is closed and G has only countably many abelian subvarieties and X is irreducible.

  So $H \leq \mathrm{Stab}(X)$.

□

Lemma. $\mathrm{SpL}(X) \subseteq_{\mathrm{cl}} X$.

Proof.

- We actually prove something stronger:

  Claim. *If $(X_a)_{a\in A}$ is a constructible family of subvarieties of G,*
  then $(\mathrm{SpL}(X_a))_{a\in A}$ is a constructible family of subvarieties of G.

  Where

  - a **constructible** subset of a variety is a boolean combination of Zariski closed subsets;

    (*Fact: constructible* ⇔ *definable in* $(\mathbb{C};+,\cdot)$.)
  - $(X_a)_{a\in A}$ is a **constructible family** of subvarieties of G if
    - A is constructible,
    - $X \subseteq G \times A$ is constructible, and
    - $X_a := \{g \in G : (g,a) \in X\} \subseteq G$ is Zariski closed in G for all a.

- Since each $\mathrm{SpL}(X_a, H)$ is closed, it suffices to show that there are finitely many $H_1, \ldots, H_n$ such that for all $a \in A$, $\mathrm{SpL}(X_a) = \bigcup_i \mathrm{SpL}(X_a, H_i)$.
- Say an abelian subvariety $H \leq G$ **appears maximally** in $X \subseteq G$ if X contains some coset $\gamma + H$ which is maximal among the cosets of abelian subvarieties contained in X.
- Say H **appears maximally** in $(X_a)_a$ if it appears maximally in some X_a.
- Then it suffices to show:

  $(*)$ Only finitely many H appear maximally in $(X_a)_a$.
- We prove $(*)$ by induction on $d := \max_a \dim(X_a)$.
- We may assume that each X_a is irreducible, by:

  Fact. *There exists a constructible $(X'_{a'})_{a \in A'}$ and a constructible map $\alpha : A' \to A$ such that $(X'_{a'} : \alpha(a') = a)$ are the irreducible components of X_a.*
- We may also assume that each X_a has finite stabiliser.

  Indeed, there is (fact) a uniform bound N on the size of the finite stabilisers $\mathrm{Stab}(X_a)$, and $A' := \{a \in A : |\mathrm{Stab}(X_a)| \leq N\}$ is constructible. So to perform the reduction, it suffices to see that $(*)$ holds for $(X_a)_{a \in A \setminus A'}$.

  But for $a \in A \setminus A'$ we have $\mathrm{SpL}(X_a) = X_a$ by the previous Lemma, so by irreducibility $\mathrm{SpL}(X_a) = \bigcup_{H : H \text{ appears maximally in } X_a} \mathrm{SpL}(X_a, H)$ is equal to a single $\mathrm{SpL}(X_a, H_a)$ with H_a appearing maximally. Then if X_a contains $\alpha + H'$, it also contains $\alpha + H' + H_a$, so $H' \leq H_a$ by maximality. So H_a is the only subgroup appearing maximally in X_a. Finally, $\mathrm{SpL}(X_a, H_a) = X_a$ holds for a on a constructible subset, and $A \setminus A'$ is covered by finitely many such subsets (by logical compactness), so indeed finitely many such H_{a_i} suffice.
- Now suppose $H_0 \leq G$ appears maximally in $(X_a)_a$. Say H_0 appears maximally in X_a.
- Let $h \in H_0 \setminus \mathrm{Stab}(X_a)$.
- Then $X'_{a,h} := X_a \cap (h + X_a)$ is a proper subvariety of X_a and contains a coset of H_0.
- Since $X'_{a,h} \subseteq X_a$, actually H_0 appears maximally in $X'_{a,h}$.
- Now
  $$(X'_{a,g} : a \in A,\ g \in G,\ X'_{a,g} \neq X_a)$$
  is a constructible family of subvarieties of G each of dimension less than d. By the inductive hypothesis, only finitely many H appear maximally in it. So H_0 is one of these finitely many, as required.

□

2 Ax-Schanuel

2.1 Restricted exponentiation

- Recall $\Lambda = \ker \exp$ is freely generated by an $\mathbb{R}$-basis $\lambda_1, \ldots, \lambda_{2g}$ of $LG \cong \mathbb{C}^g$.
- We identify LG with $\mathbb{R}^{2g}$ with respect to this basis (instead of taking real and imaginary parts),

$$0 \to \mathbb{Z}^{2g} \to \mathbb{R}^{2g} \to^{\exp} G \to 0.$$

- Let $\mathcal{F} := [0,1)^{2g} \subseteq LG$ ("fundamental domain"), so the restriction $\exp_| : \mathcal{F} \to G$ is a bijection.
- $\exp_|$ is definable in $\mathbb{R}_{\text{an}}$.
- For $X \subseteq G$, $\exp_|$ yields a bijection $\exp_|^{-1}(X) \cap \mathbb{Q}^{2g} \to X \cap G[\infty]$.
- To apply Pila-Wilkie, we must determine $\exp_|^{-1}(X)^{\text{alg}}$.
- The key tool for this is the Ax-Schanuel theorem (or its "Lindemann-Weierstrass case").

2.2 Ax-Schanuel

The original Ax-Schanuel theorem concerns usual complex exponentiation (i.e. the exponential map of the multiplicative group):

Fact (Ax '71)**.** *Suppose $f_i : \Delta \to \mathbb{C}$ are holomorphic functions on the unit disc, and $f_1'(0), \ldots, f_n'(0)$ are $\mathbb{Q}$-linearly independent. Then*

$$\operatorname{trd}(f_1(t), \ldots, f_n(t), e^{f_1(t)}, \ldots, e^{f_n(t)}/\mathbb{C}) \geq n+1.$$

- Proved using differential algebra.
- Generalisations:
  - Brownawell-Kubota: for elliptic curves;
  - Kirby: for arbitrary (semi-)abelian varieties.
  - Ax '72: general analytic version for arbitrary complex algebraic groups. We will use this.

Fact (Ax '72)**.** *Let*

- *$\mathbb{H}$ be a complex algebraic group,*
- *$\Gamma \leq \mathbb{H}$ a connected analytic subgroup, and*
- *$e \in K \subseteq \Gamma$ an irreducible analytic subvariety.*

Then there is an analytic subgroup $\mathbb{H}' \leq \mathbb{H}$ containing K^{Zar} and Γ such that:

$$\dim K \leq \dim K^{\mathrm{Zar}} - (\dim \mathbb{H}' - \dim \Gamma) .$$

Idea: If K is a component of $K^{\mathrm{Zar}} \cap \Gamma$, the "expected" dimension is

$$\dim K = \dim K^{\mathrm{Zar}} - (\dim \mathbb{H} - \dim \Gamma);$$

Ax's theorem says this can be exceeded only if the intersection is really happening in a smaller group.

2.3 Algebraic part

Lemma. $\exp_{|}^{-1}(X)^{\mathrm{alg}} = \exp_{|}^{-1}(\mathrm{SpL}(X))$.

Proof.

- We show $\exp^{-1}(X)^{\mathrm{alg}} = \exp^{-1}(\mathrm{SpL}(X))$.
- Let $x \in \exp^{-1}(X)^{\mathrm{alg}}$, and suppose $x \notin \exp^{-1}(\mathrm{SpL}(X))$.
- Replacing X with $X - \exp(x)$, we may assume $x = 0$.
- So $0 \in C' \subseteq \exp^{-1}(X)$ for a semialgebraic curve C'.
- So $0 \in C \subseteq \exp^{-1}(X)$ for an irreducible algebraic curve C.
- Replace G with the smallest abelian subvariety G' containing $\exp(C)$, and X with $X \cap G'$.

  We still have $0 \notin \mathrm{SpL}(X)$.
- Consider $\Gamma_{\exp} \leq LG \times G =: \mathbb{H}$.
- Let $K \ni (0,0)$ be an analytic irreducible component of $(C \times X) \cap \Gamma_{\exp} = \Gamma_{\exp|_C}$.
- Now $\langle \exp(C)^{\mathrm{Zar}} \rangle = G$ by assumption, so $\pi_2(\langle K^{\mathrm{Zar}} \rangle) = G$, so $\langle K^{\mathrm{Zar}} \rangle \supseteq \{0\} \times G$ (by consideration of the algebraic subgroups of $LG \times G$).

  So $\langle K^{\mathrm{Zar}} \rangle + \Gamma = \mathbb{H}$.
- Meanwhile $X \neq G$ since $0 \notin \mathrm{SpL}(X)$.
- By Ax, $\dim K \leq \dim K^{\mathrm{Zar}} - (\dim \mathbb{H} - \dim \Gamma)$, so:

$$\begin{aligned} \dim(G) &= \dim(\mathbb{H}) - \dim(\Gamma) \\ &\leq \dim K^{\mathrm{Zar}} - \dim K \\ &\leq \dim(C \times X) - 1 \\ &\leq \dim(X) \\ &< \dim(G). \end{aligned}$$

  Contradiction.

□

Remark. Kawamata *On Bloch's Conjecture* (1980, Inventiones) gives an alternative proof of this (and also of the closedness of $\mathrm{SpL}(X)$).

3 Masser

- Let $\gamma \in G(\mathbb{Q}^{\mathrm{alg}})$.
- In co-ordinates according to our projective embedding (possibly permuting the co-ordinates),

  $$\gamma = [1 : \gamma_1 : \ldots . : \gamma_n] \in G \subseteq \mathbb{P}^n(\mathbb{C})$$

  with $\gamma_i \in \mathbb{Q}^{\mathrm{alg}}$.
- Then $K(\gamma) := K(\gamma_1, \ldots, \gamma_n)$.
- For $\sigma \in \mathrm{Gal}(\mathbb{Q}^{\mathrm{alg}}/K)$,

  $$\sigma(\gamma) := [1 : \sigma(\gamma_1) : \ldots : \sigma(\gamma_n)] \in G.$$

- $\deg(K(\gamma)/K)$ is the size of the orbit of γ under $\mathrm{Gal}(\mathbb{Q}^{\mathrm{alg}}/K)$.

Theorem (Masser '84)**.** *Exist $\lambda = \lambda(g) > 0$ and $C > 0$ such that for $\xi \in G[\infty]$ of order k,*

$$\deg(K(\xi)/K) \geq Ck^{\lambda}.$$

4 Pila-Zannier

4.1 Reduction

Theorem (Manin-Mumford Conjecture)**.** *If $X \subseteq_{\mathrm{cl}} G$ is irreducible and $(X \cap G[\infty])^{\mathrm{Zar}} = X$ then X is a torsion coset.*

Suffices to show:

Lemma. *Suppose $X \subseteq_{\mathrm{cl}} G$ is irreducible and defined over $\mathbb{Q}^{\mathrm{alg}}$ and suppose $\mathrm{Stab}_G(X)$ is finite.*

Then $(X \setminus \mathrm{SpL}(X)) \cap G[\infty]$ is finite.

Proof of Theorem from Lemma.

- Let H be the connected component of the stabiliser $H := \mathrm{Stab}_G(X)^o$.

  So $H \leq G$ is an abelian subvariety or the trivial subgroup.

- Then (fact) G/H is an abelian variety, and $X/H \subseteq G/H$ is an irreducible subvariety, and $\mathrm{Stab}_{G/H}(X/H)$ is finite.

  (Indeed, if $\mathrm{Stab}_{G/H}(X/H)$ contains an infinite algebraic subgroup S, then $S' = \pi_H^{-1}(S)$ stabilises $X = \pi_H^{-1}(X/H)$, but $\dim(S') = \dim(S) + \dim(H) > \dim(H)$, contradicting the choice of H).

- X is over $\mathbb{Q}^{\mathrm{alg}}$, because $G[\infty] \subseteq G(\mathbb{Q}^{\mathrm{alg}})$ and so X is $\mathrm{Aut}(\mathbb{C}/\mathbb{Q}^{\mathrm{alg}})$-invariant.

  Also H is over $\mathbb{Q}^{\mathrm{alg}}$.

  So X/H is over $\mathbb{Q}^{\mathrm{alg}}$.

- So by the Lemma, $(X/H \setminus \mathrm{SpL}(X/H)) \cap (G/H)[\infty]$ is finite.

- But:
$$\pi_H^{-1}(X/H \cap (G/H)[\infty]) \supseteq X \cap G[\infty],$$
  so
$$\pi_H^{-1}((X/H \cap (G/H)[\infty])^{\mathrm{Zar}}) \supseteq (X \cap G[\infty])^{\mathrm{Zar}} = X,$$
  so
$$(X/H \cap (G/H)[\infty])^{\mathrm{Zar}} = X/H.$$

- Now $\mathrm{Stab}_{G/H}(X/H)$ is finite, so $\mathrm{SpL}(X/H)$ is a proper closed subvariety of X/H.

- So the finite set $(X/H \setminus \mathrm{SpL}(X/H)) \cap (G/H)[\infty]$ is Zariski dense in X/H.

- So $X/H = \{\xi\}$ for some $\xi \in (G/H)[\infty]$.

- So $X = \pi^{-1}(\xi)$ is a coset of H, and it contains a torsion point since $X \cap G[\infty]$ is dense in X, so X is a torsion coset.

□

4.2 Concluding by Pila-Wilkie

Proof of Lemma.

- Let $X \subseteq_{\mathrm{cl}} G$ be irreducible and over $\mathbb{Q}^{\mathrm{alg}}$ with $\mathrm{Stab}_G(X)$ finite.

- WTS: $(X \setminus \mathrm{SpL}(X)) \cap G[\infty]$ is finite. Suppose not.

- Increasing K, we may assume X is over K.

  Then $\mathrm{SpL}(X)$ is also over K.

- So by Masser: For arbitrarily large k, there are Ck^{λ} points of order k in $X \setminus \mathrm{SpL}(X)$.

- If ξ has order k, then the height of

$$\exp_|^{-1}(\xi) \in \exp_|^{-1}(X \setminus \mathrm{SpL}(X)) \cap \mathbb{Q}^{2g} = \exp_|^{-1}(X)^{\mathrm{tr}} \cap \mathbb{Q}^{2g}$$

  is at most k.

- But by Pila-Wilkie, exists $C' > 0$ such that

$$N(\exp_|^{-1}(X)^{\mathrm{tr}}, k) \leq C' k^{\frac{\lambda}{2}}.$$

- For large enough k, these bounds contradict each other, i.e.

$$Ck^{\lambda} > C' k^{\frac{\lambda}{2}}.$$

□