Notes from a seminar in Münster on [Hrushovski-Rideau "Valued Fields, Metastable Groups"], Nov 2019.

Work in $\mathbb{U} \models ACVF$ sufficiently saturated, in geometric language (sorts K, Γ, k, S_n, T_n). Definable means definable over a small subset of $\mathbb{U}$. We write dcl resp. acl for $\mathrm{dcl}^{\mathrm{eq}}$ resp. $\mathrm{acl}^{\mathrm{eq}}$.

1 Pure imaginaries

Definition 1.1. $e \in \mathbb{U}^{\mathrm{eq}}$ is **purely imaginary** over $C \subseteq \mathbb{U}$ if $\mathrm{acl}(Ce) \cap K = \mathrm{acl}(C) \cap K$.

Lemma 1.2. *e is purely imaginary over C iff* $\mathrm{dcl}(Ce) \cap K \subseteq \mathrm{acl}(C)$.

Proof. Symmetric polynomials.

STS $\mathrm{acl}(Ce) \cap K \subseteq \mathrm{acl}(\mathrm{dcl}(Ce) \cap K)$.

But if $a \in \mathrm{acl}(Ce) \cap K$, say $a = a_1, \dots, a_n$ are the conjugates over Ce, then the coefficients of $\prod_i (x - a_i)$ are in $\mathrm{dcl}(Ce) \cap K$. □

Definition 1.3. An ∞-definable set D is **purely imaginary** if there is no definable map $D \to K$ with infinite image.

Equivalently: for any C over which D is defined, any $e \in D$ is purely imaginary over C.

Definition 1.4. A ∞-definable set D is **boundedly imaginary** if any definable map $D \to \Gamma$ is bounded.

Equivalently: for any C over which D is defined, any $e \in D$ is boundedly imaginary over C, where $e \in \mathbb{U}^{\mathrm{eq}}$ is **boundedly imaginary** over $C \subseteq \mathbb{U}$ if for every $\gamma \in \Gamma(Ce) := \Gamma \cap \mathrm{dcl}(Ce)$, $\mathrm{tp}(\gamma/C)$ is bounded (i.e. not $+\infty$ or $-\infty$, i.e. γ is in the convex hull of $\Gamma(C)$).

Lemma 1.5. *Any boundedly imaginary D is purely imaginary.*

Proof. If the image of $D \to K$ is infinite, it contains a ball B. But for $b \in B$, $x \mapsto v(x - b)$ is an unbounded map $B \to \Gamma$. □

Define $\alpha\mathcal{O} := \{x \in K : v(x) \geq \alpha\}$ and $\beta\mathfrak{m} := \{x \in K : v(x) > \beta\}$.

Lemma 1.6. *Let X be the set of closed or open balls of a fixed radius α, i.e. $X = K/\alpha\mathcal{O}$ or $X = K/\alpha\mathfrak{m}$.*

There is no definable finite correspondence $\Gamma \to X$ with infinite image, i.e. for any definable $Z \subseteq \Gamma \times X$ with $\forall \gamma \in \Gamma.\ |\pi_1^{-1}(\gamma) \cap Z| < \aleph_0$, $|\pi_2(Z)| < \aleph_0$.

Proof. Else, by Swiss cheese decomposition applied to the union of the balls, $\pi_2(Z)$ contains all but finitely many of the balls within a closed ball B of some radius $\gamma \leq \alpha$, with $\gamma < \alpha$ in the case $X = K/\alpha\mathcal{O}$.

Then Z induces a finite correspondence from Γ onto $B/\gamma\mathfrak{m}$, and hence onto an infinite subset of k. By definable Skolem functions for Γ, this yields a definable partial function $k \to \Gamma$ with infinite image, contradicting strong minimality of k. □

Lemma 1.7. *Let $\alpha \leq 0 \leq \beta \in \Gamma$ and $n \geq 1$. Then $(\alpha\mathcal{O}/\beta\mathfrak{m})^n$ is boundedly imaginary, and hence purely imaginary.*

Proof. $\underline{n=1}$: we may assume $\alpha = 0$. Suppose $f : \mathcal{O}/\beta\mathfrak{m} \to \Gamma$ is an unbounded C-definable map. Say it is unbounded above. Let $\gamma > \Gamma(C)$. Let $g : \mathcal{O} \to \mathcal{O}/\beta\mathfrak{m}$ be the quotient map. Then $g^{-1}(f^{-1}(\gamma))$ is a boolean combination of balls each of radius in $[0, \beta] \cap \Gamma(C\gamma)$. Now Γ is a pure divisible ordered abelian group, so any definable map from Γ to the bounded interval $[0, \beta]$ is eventually constant, so $[0, \beta] \cap \Gamma(C\gamma) \subseteq \Gamma(C)$.

But the balls vary with γ since f is a function, so we contradict Lemma 1.6.

$\underline{n = k+1}$: Given $f : (\alpha\mathcal{O}/\beta\mathfrak{m})^{k+1} \to \Gamma$, inductively $y \mapsto \sup_{\overline{x}} f(\overline{x}, y)$ defines a map $\alpha\mathcal{O}/\beta\mathfrak{m} \to \Gamma$, so this map is bounded, and hence so is f. □

Lemma 1.8. *$e \in \mathbb{U}^{\mathrm{eq}}$ is purely imaginary over C iff for some $\alpha \leq 0 \leq \beta$ with $\alpha, \beta \in \Gamma(Ce)$, $e \in \mathrm{dcl}(\mathrm{acl}(C), \alpha\mathcal{O}/\beta\mathfrak{m})$.*

Proof. $\Leftarrow$: Immediate from Lemma 1.7.

$\Rightarrow$: By the EI and the definition of purely imaginary, e is interdefinable with a finite tuple from $\bigcup_n S_n \cup \bigcup_n T_n \cup (\mathrm{acl}(C) \cap K)$.

Note that $K \ni x \mapsto \min(\beta, v(x))$ induces a map $\alpha\mathcal{O}/\beta\mathfrak{m} \twoheadrightarrow [\alpha, \beta]$, so $[\alpha, \beta] \subseteq \mathrm{dcl}(\alpha\mathcal{O}/\beta\mathfrak{m})$. Hence if $\alpha \leq \alpha' < 0 < \beta' \leq \beta$ then $\alpha'\mathcal{O}/\beta'\mathfrak{m} \subseteq \mathrm{dcl}(\alpha\mathcal{O}/\beta\mathfrak{m})$.

So it suffices to consider the case $e \in S_n$ or $e \in T_n$ for some n.

Let $\Lambda \leq K^n$ be a rank n free $\mathcal{O}$-submodule, say with $\mathcal{O}$-basis $(\lambda_1, \ldots, \lambda_n)$.

Let $\Lambda^- := \{\alpha : \Lambda \subseteq (\alpha\mathcal{O})^n\}$ and $\Lambda^+ := \{\beta : (\beta\mathfrak{m})^n \subseteq \Lambda\}$. Note $\sup \Lambda^- = \min_{i,j} v((\lambda_i)_j) \in \Lambda^-$.

Let $\alpha = \min\{0, \sup \Lambda^-\} \in \Lambda^-$, and $\beta = \max\{0, \inf \Lambda^+\} \in \Lambda^+$.

Then $\alpha \leq 0 \leq \beta$ and $\alpha, \beta \in \Gamma(Ce)$ and $(\beta\mathfrak{m})^n \subseteq \Lambda \subseteq (\alpha\mathcal{O})^n$. then $S_n \ni \ulcorner\Lambda\urcorner \in \mathrm{dcl}((\lambda_i/\beta\mathfrak{m})_i) \subseteq \mathrm{dcl}(\alpha\mathcal{O}/\beta\mathfrak{m})$.

Also $\beta\mathfrak{m}^n \subseteq \mathfrak{m}\Lambda$ (since $\mathfrak{m}(\beta\mathfrak{m}) = \beta\mathfrak{m}$), so $T_n \supseteq \Lambda/\mathfrak{m}\Lambda \subseteq \mathrm{dcl}(\alpha\mathcal{O}/\beta\mathfrak{m})$.

Since Λ was arbitrary, we conclude. □

Remark 1.9. It follows from the proof that each S_n and T_n is purely imaginary.

Lemma 1.10. *Let D be a ∞-definable set over C. TFAE:*

(1) D is boundedly imaginary.

(2) There exists a definable surjection $g : (\mathcal{O}/\beta\mathfrak{m})^n \twoheadrightarrow D$.

(3) For some $\alpha \leq 0 \leq \beta$ with $\alpha, \beta \in \Gamma(C)$, $D \subseteq \mathrm{dcl}(\mathrm{acl}(C), \alpha\mathcal{O}/\beta\mathfrak{m})$.

NOTE: the paper has C rather than $\mathrm{acl}(C)$, but I don't see how to get that. This statement is good enough for the application in Corollary 6.4.

Proof. (2) $\Rightarrow$ (1): by Lemma 1.7.

(3) $\Rightarrow$ (2): by compactness, we get finitely many $g_i : (\alpha_i\mathcal{O}/\beta_i\mathfrak{m})^{n_i} \to D$ with $D = \bigcup_i \mathrm{img}_i$. We can assume $\alpha_i = 0$ by multiplying, and we can assume $\beta_i = \max_i \beta_i =: \beta$, and then combine the g_i into a surjection $g : (\mathcal{O}/\beta\mathfrak{m})^{\sum_i n_i} \to D$.

(1) $\Rightarrow$ (3): By compactness, it suffices to show that if $e \in D$, then such $\alpha, \beta \in \Gamma(C)$ exist with $e \in \mathrm{dcl}(C, \alpha\mathcal{O}/\beta\mathfrak{m})$.

Now D is purely imaginary by Lemma 1.5, so by Lemma 1.8 we can find such $\alpha, \beta \in \Gamma(Ce)$.

Say $\alpha = f(e)$, where f is over C. Since D is boundedly imaginary, $\alpha \geq \inf_{D \cap \mathrm{dom} f} f =: \alpha' \in \Gamma(C)$. Similarly we find $\beta \leq \beta' \in \Gamma(C)$.

Then $\alpha\mathcal{O}/\beta\mathfrak{m} \subseteq \mathrm{dcl}(\alpha'\mathcal{O}/\beta'\mathfrak{m})$, so $e \in \mathrm{dcl}(\mathrm{acl}(C), \alpha'\mathcal{O}/\beta'\mathfrak{m})$ as required. □

Remark 1.11. Martin Hils remarks that in Lemma 1.8 we can equivalently ask just for $\alpha, \beta \in \Gamma$ rather than require them in $\Gamma(Ce)$. From this one can conclude that ∞-definable D is purely imaginary iff there is a definable surjection $f : (B^{\text{open}})^n \twoheadrightarrow D$ where B^{open} is the set of all open balls (of all radii).

Then it follows from this and Lemma 1.10, that if D is ∞-definable and purely resp. boundedly imaginary, it is contained in a purely resp. boundedly imaginary definable set.

2 Redundant

Alternative direct proof of Remark 1.9:

Lemma 2.1. *Any product of S_n's and T_n's is purely imaginary.*

So by the EI, $e \in \mathbb{U}^{\text{eq}}$ is purely imaginary over C iff it is interdefinable with a finite tuple from $\bigcup_n S_n(\mathbb{U}) \cup \bigcup_n T_n(\mathbb{U}) \cup (\operatorname{acl}(C) \cap K)$.

Proof. Let T be a completion of ACVF.

It suffices to find an uncountable model of T in which each S_n and T_n is countable, since the image of a definable map to K with infinite image contains a ball and so has the same cardinality as K.

Thanks to Martin Hils for providing the following example.

Let $L \vDash T$ be countable with $\Gamma(L) = \mathbb{Q}$. (We can take L to be an algebraic closure of $\mathbb{Q}(t)$, $\mathbb{F}_p(t)$, or $\mathbb{Q}$ with the p-adic valuation.) Consider L as a normed field (with $\|x\| := 2^{-v(x)}$).

Let $\bar{L}$ be the completion of L. Fact: $\bar{L} \vDash$ ACVF. We have $|\bar{L}| = 2^{\aleph_0}$, but $S_n(\bar{L}) = \mathrm{GL}_n(\bar{L})/\mathrm{GL}_n(\mathcal{O}(\bar{L})) \leftarrow \mathrm{GL}_n(L)/\mathrm{GL}_n(\mathcal{O}(L)) = S_n(L)$ is a bijection since $\mathrm{GL}_n(\mathcal{O}(\bar{L}))$ is an open neighbourhood of the identity and $\mathrm{GL}_n(\mathcal{O}(\bar{L})) \cap \mathrm{GL}_n(L) = \mathrm{GL}_n(\mathcal{O}(L))$. Similarly $k(\bar{L}) = \mathcal{O}(\bar{L})/\mathfrak{m}(\bar{L}) \leftarrow \mathcal{O}(L)/\mathfrak{m}(L) = k(L)$ is a bijection.

So $|S_n(\bar{L})| = \aleph_0 = |T_n(\bar{L})|$. □