Oct '16 Pseudofinite-dimensional Schrödinger Rep^{ns} w/ Bradd Hart

0: Outline

(FA)
- Functional analytic framework for QM of a particle on $\mathbb{R}$:
  - $\mathbb{H}$:= Heisenberg group; Schrödinger rep^n: unitary rep^n of $\mathbb{H}$ on $L^2(\mathbb{R})$ (Stone–von Neumann: unique)
  - $SL_2(\mathbb{R}) \leq \mathrm{Aut}(\mathbb{H})$ $\rightsquigarrow$ Weil rep^n of $\widetilde{SL_2(\mathbb{R})}$ on $L^2(\mathbb{R})$

(Z)
- Zilber: findim reps V_m of subgroups of $\mathbb{H}$ with traces of $\widetilde{SL_2(\mathbb{R})}$ action; $\lim\limits_{m\to\infty} V_m$ (part of bigger picture: limit of Z-structures)

- Q^n: Relation of (Z) with (FA)?
- An Answer: With appropriate definitions
$$\lim_{m\to U}{}^{B} V_m \cong \text{Weil-Schr}^n \text{ rep}^n \text{ on } \underline{S}' \supseteq L^2(\mathbb{R})$$

1 Schrödinger rep^n & tempered dist^{ns}

η: $\mathbb{R}$-Lie algebra basis $=(P,Q,E)$

Lie bracket $[Q,P]=E$, $[E,P]=0=[E,Q]$

rep^n on $\underline{S} \subseteq L^2(\mathbb{R})$:

$Q\phi(x) = 2\pi i x\phi(x)$

$P\phi(x) = -\frac{d}{dx}\phi(x)$ $\quad QP-PQ=E$

$E\phi(x) = 2\pi i\phi(x)$

$\boxed{\chi(t) := e^{2\pi i t}}$

$\mathbb{H} = \{e^X \mid X\in\eta\}$ Lie group

$e^X \cdot e^Y = e^{X+Y+\frac{[X,Y]}{2}}$ $\quad [e^Q, e^P] = e^E$ $\quad [\text{“}e^Q e^P e^{-Q} e^{-P}\text{”}]$

unitary rep^n on $L^2(\mathbb{R})$:

$e^{qQ}f(x) = \chi(qx)f(x)$

$e^{pP}f(x) = f(x-p)$

$e^{tE}f(x) = \chi(t)f(x)$

$\underline{S}$ = Schwartz space $\subseteq L^2(\mathbb{R})$
$= \{\phi(x)\in L^2(\mathbb{R}) \mid \|\phi\|_{n,m} < \infty \ \forall n,m\in\mathbb{N}\}$

$\|\phi\|_{n,m} := \sup\limits_{x\in\mathbb{R}} |x^n\phi^{(m)}(x)|$ seminorms

$\underline{S}'$ = space of tempered dist^{ns} := cont^s dual of $\underline{S}$ $= \{T : \underline{S} \xrightarrow[\text{cont}^s]{\mathbb{C}\text{-linear}} \mathbb{C}\}$

$\langle T,\phi\rangle := T(\bar\phi)$ sesquilinear form $\underline{S}'\times\underline{S}\to\mathbb{C}$ $\quad L_2(\mathbb{R}) \hookrightarrow \underline{S}'$ $\quad \langle f,\phi\rangle := \int_{\mathbb{R}} f\bar\phi$

Dual rep^n of η on $\underline{S}'$: $\langle XT,\phi\rangle := \langle T,-X\phi\rangle$

e.g. $\delta_b\in\underline{S}'$ $\langle\delta_b,\phi\rangle := \overline{\phi(b)}$ $\quad Q\delta_b = 2\pi i b\delta_b$

$\left[\langle Q\delta_b,\phi\rangle = \langle\delta_b,-2\pi i x\phi(x)\rangle = \overline{-2\pi i b\phi(b)} = 2\pi i b\langle\delta_b,\phi\rangle\right]$

$\chi(ax)\in\underline{S}'$ $\quad \langle\chi(ax),\phi(x)\rangle := \int_{\mathbb{R}}\chi(ax)\overline{\phi(x)}\,dx$ $\quad P\chi(ax) = -2\pi i a\chi(ax)$

$$\left[\begin{aligned}\langle P\chi(ax),\phi(x)\rangle &= \langle\chi(ax),\tfrac{d}{dx}\phi(x)\rangle = \int_{\mathbb{R}}\chi(ax)\overline{\phi'(x)}\,dx \overset{\text{int}^n\text{ by parts}}{=} -\int_{\mathbb{R}}\chi'(ax)\overline{\phi(x)}\,dx\\ &= -2\pi i a\int_{\mathbb{R}}\chi(ax)\overline{\phi(x)}\,dx\\ &= -2\pi i a\langle\chi(ax),\phi(x)\rangle\end{aligned}\right]$$

Rep^n of $\mathbb{H}$ extends to $\underline{S}'$, similarly $(\langle e^X T,\phi\rangle := \langle T, e^{-X}\phi\rangle)$

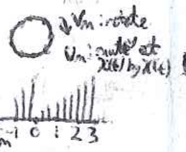
2 Fin dim Rep^{ns}

$\mathbb{H}_m \leq \mathbb{H}$ generated by $U_m := e^{Q/m}$, $V_m := e^{P/m}$ $\quad [U_m,V_m] = e^{E/m^2}$

$\underline{S}'_m := \{f : \mathbb{Z}/m^2\mathbb{Z}\to\mathbb{C}\}$ $\quad$ $\mathbb{C}$-VS dim m^2

Rep^n of $\mathbb{H}_m$ on $\underline{S}'_m$: $U_m f(k) := \chi(k/m^2)f(k)$ $\quad V_m f(k) := f(k-1)$

so $[U_m,V_m] = \chi(1/m^2)$



Embed $\underline{S}'_m$ into $\underline{S}'$ by $f\mapsto\sum\limits_{k\in\mathbb{Z}} f(k)\delta_{k/m}$

$\underline{S}'_m \hookrightarrow \underline{S}'$ subspace of m-periodic linear comb^s of δ-f^{ns} supported on $\frac{1}{m}\mathbb{Z}$

Rep^n of $\mathbb{H}_m$ on $\underline{S}'_m$ = restriction of Schr^n rep^n of $\mathbb{H}_m$ on $\underline{S}'$

Define $Q_m := \dfrac{U_m - U_m^{-1}}{2/m}$ $\quad P_m := \dfrac{V_m - V_m^{-1}}{2/m}$

Prop^n 1: Let U be a non-principle ultrafilter on $\mathbb{N}$.
Then $\lim\limits_{m\to U}{}^{B}(\underline{S}'_m; P_m, Q_m) = (\underline{S}'; P, Q)$

3 Bounded ultralimits in $\underline{S}'$

$B\subseteq\underline{S}'$ is bounded iff $\forall\phi\,\exists\delta.\ |\langle B,\phi\rangle| \leq [0,\delta]$

Weak* top^y on $\underline{S}'$: coarsest s.t. $|\langle\cdot,\phi\rangle|$ cont^s $\forall\phi\in\underline{S}$

For B bdd^d:
- $\langle\cdot,\cdot\rangle$ cont^s on $B\times\underline{S}$ (but not on $\underline{S}'\times\underline{S}$!)
- B metrisable since $\underline{S}$ sep^{ble} (e.g. $d(T,T') := \sum 2^{-i}\min(1,|\langle T-T', x^i e^{-x^2}\rangle|)$)
- B closed $\Rightarrow$ compact & complete

$\mathcal{B} := \{B\subseteq\underline{S}' \text{ closed \& bdd}^d\}$ $\quad \underline{S}' = \bigcup\mathcal{B}$

Def^n: $Y_m\subseteq\underline{S}'$, $\lim\limits_{m\to U}{}^{B} Y_m := \bigcup\limits_{B\in\mathcal{B}}\lim\limits_{m\to U}(Y_m\cap B) \subseteq \underline{S}'$ $\leftarrow$ topological ultralimit

For $R^{Y_m}\subseteq Y_m^n$ relation (e.g. graph of operator)

$\lim\limits_{m\to U}{}^{B}(Y_m; R^{Y_m}) := (\lim\limits_{m\to U}{}^{B} Y_m; \lim\limits_{m\to U}{}^{B} R^{Y_m})$

Corresponds to metric ultraproduct in lang with a sort for each $B\in\mathcal{B}$ with distance predicate for $R\cap B$.

R^k: $\underline{S}'$ has no proper el^y ext^n in this lang.

4 Fourier Transform

$F : \underline{S}\xrightarrow{\cong}\underline{S}$ $\quad F(\phi)(p) := \int_{\mathbb{R}}\chi(xp)\phi(x)\,dx$

$F : \underline{S}'\xrightarrow{\cong}\underline{S}'$ $\quad \langle FT, F\phi\rangle = \langle T,\phi\rangle$

$FPF^{-1} = Q$ $\quad FQF^{-1} = -P$

$F_m := F|_{\underline{S}'_m}$

$F_m(f)(p) = \frac{1}{m}\sum\limits_{0\leq k<m^2} f(k)\chi\left(\frac{pk}{m^2}\right)$ discrete Fourier transform

$F_m U_m F_m^{-1} = V_m$ so $F_m Q_m F_m^{-1} = -P_m$

Prop^n 1': $\lim\limits_{m\to U}{}^{B}(\underline{S}'_m; P_m, Q_m, F_m) = (\underline{S}'; P, Q, F)$

Pf sketch: $\lim\limits_{m\to U}{}^{B}\underline{S}'_m = \underline{S}'$ by sampling $\left[\underline{S}'\ni T = \lim\limits_{m\to\infty} g_m \text{ (smooth compact support)} \quad f_m(k) := \frac{1}{m}g_m(k/m) \quad \lim\limits_{m\to U} f_m = T\right]$

$\lim\limits_{m\to U}{}^{B} Q_m = Q$ since $-mi\sin(2\pi\frac{x}{m}) \xrightarrow[m\to\infty]{} -2\pi i x$

$\lim\limits_{m\to U}{}^{B} F_m = F$ since $F_m = F|_{\underline{S}'_m}$

$\lim\limits_{m\to U}{}^{B} P_m = -\lim{}^{B}(F_m Q_m F_m^{-1}) = -(\lim{}^{B} F_m)(\lim{}^{B} Q_m)(\lim{}^{B} F_m^{-1})$
$= -FQF^{-1} = P$ $\square$

5 Weil Rep^n

$SL_2(\mathbb{R})$ acts by AM^s on η and $\mathbb{H}$ $\quad \alpha(e^X) := e^{\alpha X}$

$\begin{pmatrix}P\\Q\end{pmatrix}\mapsto\begin{pmatrix}a&b\\c&d\end{pmatrix}\begin{pmatrix}P\\Q\end{pmatrix}$ $\quad E\mapsto E$

$\pi_S : \mathbb{H}\to\mathrm{Aut}(L^2(\mathbb{R}))$ Schrödinger rep^n $\quad \pi_S(e^{tE}) = \chi(t)$

$\alpha\in SL_2(\mathbb{R})$ $\quad \pi_S\circ\alpha : \mathbb{H}\to\mathrm{Aut}(L^2(\mathbb{R}))$ another unit^y rep^n, $(\pi_S\circ\alpha)(e^{tE})$ ''

Stone–von Neumann $\Rightarrow$ $(L^2(\mathbb{R}); \langle\cdot,\cdot\rangle, \pi_S) \underset{\eta}{\cong} (L^2(\mathbb{R}); \langle\cdot,\cdot\rangle, \pi_S\circ\alpha)$ $\leftarrow$ unitary equivalence

unique up to $U(1)$ by Schur

$\rightsquigarrow$ “proj^{ve} rep^n” of $SL_2(\mathbb{R})$ on $L^2(\mathbb{R})$

$\rightsquigarrow$ rep^n of $\widetilde{SL_2(\mathbb{R})}$ $\leftarrow$ univ^l cover

factors through $Mp_2(\mathbb{R})$ double cover of $SL_2(\mathbb{R})$ and restricts to rep^n on $\underline{S}$. “Weil”/“metaplectic” rep^n

$Mp_2(\mathbb{R})$ generated by $\tilde{F} := \widetilde{\begin{pmatrix}0&1\\-1&0\end{pmatrix}}$ and $\tilde{G}^b := \widetilde{\begin{pmatrix}1&b\\0&1\end{pmatrix}}$

acting on $\underline{S}$ as

$\tilde{F}\phi(x) = \pm\chi(1/8)F\phi(x)$

$\tilde{G}^b\phi(x) = \pm\chi(-b\frac{x^2}{2})\cdot\phi(x)$

hence on $\underline{S}'$, $\langle\tilde\alpha T,\tilde\alpha\phi\rangle = \langle T,\phi\rangle$

R^k: Time evolution in QM $\rightsquigarrow$ action by 1-parameter subgp of $Mp_2(\mathbb{R})$ e^{tH} for quad^c homog^s poly^l Hamiltonian $H\in sl_2(\mathbb{R})$

For $b\in\mathbb{Q}$ $\quad b = c/d$ (lowest terms)

$\tilde{G}^b_m := \tilde{G}^b|_{\underline{S}'_m}$ $\quad$ (i.e. $\tilde{G}^b_m(f)(k) = \pm\chi(b\frac{k^2}{2m^2})f(k)$)

if $d \mid m$ $\quad \mathrm{dom}\,\tilde{G}^b_m$ = m-periodic linear comb^{ns} of δ f^{ns} supported on $\frac{1}{(m/d)}\mathbb{Z}$ $\cong \underline{S}'_{m/d}$

Let U s.t. $a\mathbb{N}\in U$ $\forall a\in\mathbb{N}$

Then $\lim\limits_{m\to U}{}^{B}\mathrm{dom}\,\tilde{G}^b_m = \underline{S}'$ $\quad \lim\limits_{m\to U}{}^{B}\tilde{G}^b_m = \tilde{G}^b$ (as operators)

For $b\in\mathbb{R}$, $\tilde{G}^b_m := \tilde{G}^{b_m}_m$ where $b_m := \frac{\lfloor be\rfloor}{e}$ where $m = e^2 f$ with f squarefree

Then $\lim\limits_{m\to U}{}^{B}\tilde{G}^b_m = \tilde{G}^b$

For $\tilde\alpha\in Mp_2(\mathbb{R})$, $\tilde\alpha = w(\tilde{F}, \tilde{G}^{b_1}, \ldots, \tilde{G}^{b_n})$
$\tilde\alpha_m := w(\tilde{F}_m, \tilde{G}^{b_1}_m, \ldots, \tilde{G}^{b_n}_m)$

Then $\lim\limits_{m\to U}{}^{B}\tilde\alpha_m = \tilde\alpha$

Prop^n 1'': $a\mathbb{N}\in U$ $\forall n$ $\Rightarrow$ $\lim\limits_{m\to U}{}^{B}(\underline{S}'_m; P_m, Q_m, (\tilde\alpha_m)_{\tilde\alpha\in Mp_2(\mathbb{R})})$
$= (\underline{S}'; P, Q, (\tilde\alpha)_{\tilde\alpha\in Mp_2(\mathbb{R})})$