Sum-Product Th^m [Bourgain-Katz-Tao '06]:

$\forall \delta>0.\ \exists c>0,\epsilon>0\ \forall p$ prime.

if $A \subseteq \mathbb{F}_p$ with $\underbrace{p^{\delta} \leq |A|}$ $\leq p^{1-\delta}$ (not needed (see last week))

then $\max(|A+A|, |A\cdot A|) \geq c|A|^{1+\epsilon}$

Cor^y [BGT '06] (Szemerédi-Trotter for $\mathbb{F}_p$):

$\forall \alpha<2.\ \exists c,\epsilon>0.\ \forall p$ prime.

if L resp. P is a set of lines resp points in $\mathbb{P}^2(\mathbb{F}_p)$ and $|L|,|P| \leq N \leq p^{\alpha}$

then $|I| \leq cN^{3/2-\epsilon}$ where $I = \{(p,\ell) \in P \times L \mid p \in \ell\}$

Open Problem: Analogue of the general "alg^c quasidesign" Sz-Tr results we got in char 0 by appeal to $\mathbb{R}$.

Hrushovski conjectures such follow from sum-product and some form of trichotomy.

Non-standard formulation of Cor^y 1:

(After Hrushovski) (slightly copying)

Let $F = \prod_p \mathbb{F}_p / \mathcal{U}$ in $\mathcal{L}$ ($\mathcal{U} \ni P(p$ prime$)$) ultrafilter non-princ) ← $\mathcal{L}$ ctble exp^n of Lring

Let $^*\mathbb{R} := \mathbb{R}^{\mathcal{U}} = \prod_p \mathbb{R} / \mathcal{U}$

Let $\xi_0 \in {}^*\mathbb{R}$ define $\delta = \delta_{\xi_0}(\phi) := st\left(\frac{\log|\phi(F)|}{\log \xi_0}\right) \in \mathbb{R} \cup \{-\infty, \infty\}$ for ϕ / F

(Then δ is cont^s)

Cor^y 1': If L resp. P is a def^ble set of lines resp points in $\mathbb{P}^2(F)$ and $0<\delta(L), \delta(P) \leq 2$ and $\delta(F) > 1$

then $\delta(I) < 3$

(If $\neg$(Cor^y 1) ultra product of ctrs $\Rightarrow \neg$(Cor^y 1'))

partial (Hrushovski's version of BKT proof)

pf: WLOG, $\delta(L), \delta(P) = 2$

Suppose $\delta(I) \geq 3$

Lemma: If ... then exists ... Pf: ... by compactness & $\aleph_1$-sat^n ... so done $\square$

So say $(p,\ell) \in I$, $\delta(p,\ell) = \delta(I) \geq 3$

Pf.

Let $\ell' \underset{p}{\equiv} \ell$, $\ell' \underset{p}{\overset{\delta}{\perp}} \ell$ i.e. $\delta(\ell'/p) = \delta(\ell'/p\ell)$

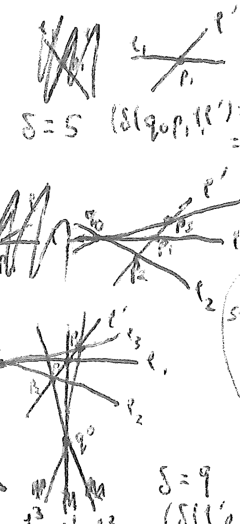
(exists by Lemma 1 - complete tp(ℓ/p) to $p\ell$ with same δ)

Then $4 \geq \delta(\ell) + \delta(\ell') \geq \delta(\ell\ell') = \delta(\ell\ell'p) = \delta(\ell'/p\ell) + \delta(\ell/p) + \delta(p)$
$= 2\delta(\ell p) - \delta(p) \geq 6 - 2 = 4$

since $\ell \neq \ell'$ so p unique int^n of ℓ, ℓ' ($p \in dcl(\ell, \ell')$)

So $\boxed{\delta(\ell p) = 3 \\ \delta(\ell) = \delta(p) = 2}$

Rename: $p_1 := p$, $\ell_1 := \ell$



$\delta = 4$

Let $q_0 \underset{\ell_1 p}{\overset{\delta}{\equiv}} p_1$ (with $\ell_1 p_1$)

$\delta = 5$ $(\delta(q_0 p_1 \ell_1 \ell') = \delta(q_0 / p_1 \ell_1 \ell') + \delta(p_1 \ell_1 \ell') = \delta(q_0/\ell_1) + 4 = (3-2)+4)$

$p_2, \ell_2 \underset{q_0 \ell'}{\overset{\delta}{\equiv}} p_1, \ell_1$ with $p_1 \ell_1$

$\delta = 7$

($\delta(q_0 \ell') = 4$ since $4 \geq \delta(q_0) + \delta(\ell') \geq \delta(q_0 \ell')$
$= 5 - \delta(\ell_1 / q_0 \ell')$
$\geq 5 - \delta(\ell_1 / q_0)$
$= 4$)

$p_3, \ell_3 \underset{q^0 \ell' p_1 \ell_2}{\overset{\delta}{\equiv}} p_1, \ell_1$

$q^0 \ell' \ell^2 \ell^3 \underset{\ell' p_1 p_2 p_3}{\overset{\delta}{\equiv}} q_0 \ell, \ell_2 \ell_3$

$\delta = 9$

($\delta(\ell' p_1 p_2 p_3) = 5$ by sim arg^t ($\delta(q_0 \ell' p_1 p_2 p_3) = 7$)

Then $\delta(q_0 q^0 p_1 \ell_1 \ell_2 \ell_3 \ell' \ell^2 \ell^3 \ldots / q_0 q^0 \ell, \ell_2)$
$= 9 - 6 = 3$

PTO

and for Ψ a partial type $/F$ (i.e. a deductively closed set of fmlae $/F$),

$\delta(\Psi) := \inf_{\phi \in \Psi} \delta(\phi)$.

Close $\mathcal{L}$ under "cardinality comparison quantifiers" wrt ξ_0 s.t. $\mathcal{L}$ is still ctble and $\{a \mid \log|\phi(F,a)| < q\,\xi_0\}$ is ϕ-def^ble $(= \Theta^q_{\phi(x,y)}(F))$, $\Theta^q_{\phi(x,y)} \in \mathcal{L}$ for any $q \in \mathbb{Q}_{>0}$ and $\phi \in \mathcal{L}$.

Lemma 1: If $E \underset{ctble}{\subseteq} F$ and Ψ is a partial type $/E$,

then exists $a \in F$ s.t. $\Psi \subseteq tp(a/E)$ and $\delta(a/E) = \delta(\Psi)$ ($:= \delta(tp(a/E))$)

pf: $\Psi \cup \{\neg\phi \mid \delta(\phi) < \delta(\Psi)\}$ is consistent,

else $\Psi \vdash \bigvee_{i=1}^{n} \phi_i$, but $\delta(\bigvee_{i=1}^{n} \phi_i) = \max \delta(\phi_i) < \delta(\Psi)$ ✗

So realised in F by $\aleph_1$-sat^n $\square$

Lemma 2: $\delta(ab/E) = \delta(a/Eb) + \delta(b/E)$

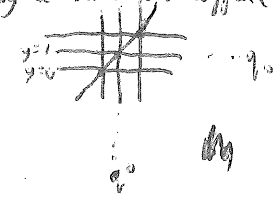
pf: Using card^y comp^n, given $\epsilon>0$, find $\phi(x,y) \in tp(ab/E)$

s.t. $|\delta(\phi) - \delta(ab/E)| < \epsilon$
$|\delta(\phi(x,b)) - \delta(a/Eb)| < \epsilon$
$|\delta(\exists x.\phi(x,y)) - \delta(b/E)| < \epsilon$
$b' \models \exists x.\phi(x,y) \Rightarrow |\delta(\phi(x,b')) - \delta(\phi(x,b))| < \epsilon$

so $|\delta(\phi) - (\delta(\phi(x,b)) + \delta(\exists x\,\phi(x,y)))| < \epsilon$

so $|\delta(ab/E) - (\delta(a/Eb) + \delta(b/E))| < 4\epsilon$ $\square$

Applying a projective linear transformation,
$q_0 \to [1,0,0]$ $\quad q^0 \to [0,1,0]$
and a vertical affine linear trans^n, $l_1 \to \{y=1\}$
$l_2 \to \{y=0\}$



Then if $l_3 \to \{y=t\}$, $l^1 \to \{x=x_1\}$, $l^2 \to \{x=x_0\}$
$l^3 \to \{x=\alpha\}$,

then $\alpha = x_0 + (x_1 - x_0)t$
$= x_1 t + x_0(1-t)$

NA^n: "$\delta'(t)$" means $\delta(l_3/q_0q^0 l_1 l_2)$ etc

Now $\delta'(x_0 x_1 \alpha t) = 3 = \delta'(x_0 x_1 \alpha)$ etc

$\delta'(x_0 x_1 \alpha / t) = 2$ ①

$\delta'(\alpha x_1 t / x_0) = 2$ ②

Let $\varepsilon > 0$

Say X_0, X_1, A def^{ble} $1 \leq \delta(X_0), \delta(X_1), \delta(A) < 1+\varepsilon$

$2 \leq \delta(\{(x_0 x_1 \alpha) \in X_0 \times X_1 A \mid \alpha = t x_1 + (1-t)x_0\})$

Then in $\mathbb{F}_p$, for arb^{ly} large p,

$|\{(x_0, x_1) \in X_0 \times X_1 \mid t x_1 + (1-t)x_0 \in A\}| >> N^{-\varepsilon} |X_0||X_1|$

(ε arb^y small) where "$\delta(N) = 1$" i.e. N ... ξ_0 i.e. $[N_p]_\mathcal{U} = \xi_0$

Let $\varepsilon > 0$,

$\delta(l^2/q_0 q^0 l_1 l_2) = 1$ so take def^{ble} $\Lambda^2_\cdot \ni l^2$ over $q_0 q^0 l_1 l_2$ s.t. $\delta(\Lambda^2_\cdot) < 1+\varepsilon$, then X_0 its image under the transformation in F

take def^{ble} $X_0 \ni x_0$ s.t. $\delta(X_0) < 1+\varepsilon$, sim X_1, A, T

Then $\delta(\{(x_0, x_1, \alpha, t) \in X_0 \times X_1 \times A \times T \mid \alpha = t x_1 + (1-t)x_0\}) \geq 3$

① $\delta(\{(x_0, x_1, \alpha) \in X_0 \times X_1 \times A \mid \alpha = t x_1 + (1-t)x_0\}) \geq 2$

② $\delta(\{x_1, \alpha, t) \in X_1 \times A \times T \mid$ " " $\quad$ " "

Then for any $\varepsilon > 0$, in $\mathbb{F}_p$, for arb^y large p (on a $\mathcal{U}$-large set of p):

$|X_0|, |X_1|, |A|, |T| << N^{1+\varepsilon}$ $\quad$ where $N = N_p$, $\xi_0 = (N_p)/\mathcal{U}$
$\delta(F) > 1$, say $\delta(F) > \frac{1}{1-\delta} \Rightarrow N \ll p^{1-\delta}$

$|\{(x_0, x_1, \alpha) \in X_0 \times X_1 \times A \mid \alpha = t x_1 + (1-t)x_0\}| >> N^{2-\varepsilon}$

($t = t_p$, $A = A_p$ etc)

So $|\{x_0, x_1) \in X_0 \times X_1 \mid t x_1 + (1-t)x_0 \in A\}| >> N^{-c_1\varepsilon} |X_0||X_1|$

<u>Th^m</u> [Balog-Szemerédi, Gowers' version]:

G ab^n group, $A, B \subseteq G$, $|A| = |B|$,
$X \subseteq A \times B$, $|X| \geq \frac{|A||B|}{K}$, $|\{a+b \mid (a,b) \in X\}| \leq K|A|$

then exist $A' \subseteq A$, $B' \subseteq B$,
$|A'| \geq cK^{-c}|A|$, $|B'| \geq cK^{-c}|B|$
$|A' - B'| \leq cK^c|A|$

So get X_0', X_1' $\quad$ $|X_i'| >> N^{1-c_2\varepsilon}$
s.t. $|(1-t)X_0' + tX_1'| << N^{1+c_2\varepsilon}$

<u>Lemma</u> [sumsets estimates]:
$A, B \subseteq G$ ab^n gp^p $\quad$ $|A+B| \leq K \min(|A|, |B|)$
then $|A+A| \leq cK^c|A|$

So $|tX_1' + tX_1'| << N^{1+c_3\varepsilon}$
so $|X_1' + X_1'| << N^{1+c_3\varepsilon}$

Now using ②, translating horizontally so $x_0 = 0$,
$|\{(t, x_1) \in T \times X_1' \mid t x_1 \in A\}| >> N^{-c\varepsilon} |T||X_1'|$
($0 \notin T, X_1'$)

so sim, get $X_1'' \subseteq X_1'$ s.t. $|X_1''| >> N^{1-c\varepsilon}$, $|X_1'' \cdot X_1''| << N^{1+c_5\varepsilon}$
$N < p^{1-\delta}$ $\quad$ ✗ sum-product.