1 Szemeredi-Trotter and trichotomy

Notes on various "Szemeredi-Trotter" results, and the interpretation in terms of geometric stability theory given to them by Hrushovski in his paper "On pseudo-finite dimensions".

1.1 Szemeredi-Trotter

Theorem 1.1 (Szemeredi-Trotter 1983)**.** *Given n points and m lines in $\mathbb{R}^2$, the number of point-line incidences is $\mathcal{O}(n^{2/3}m^{2/3} + n + m)$.*

Remark 1.2. In particular, given $\leq n$ points and $\leq n$ lines, the number of incidences is $\mathcal{O}(n^{4/3})$.

$4/3 = 3/2 - 1/6$.

Theorem 1.3 (Tóth 2003)**.** *Same statement, but for the complex plane $\mathbb{C}^2$, where a "line" is a coset of a 1-dimensional $\mathbb{C}$-subspace.*

Remark 1.4. The same statement for the projective plane $\mathbb{P}^2(\mathbb{C})$ follows (since by applying a Möbius transformation, WLOG none of our points or lines are at infinity).

We can think of incidences as the edges E of a bipartite graph between a set of "points" and a set of "lines". Recall Elekes-Szabo define the combinatorial dimension of a bipartite graph, with respect to a parameter b.

E *omits* $K_{b,k}$ if the common intersection of any k distinct "lines" has less than b "points". This implies that E has combinatorial dimension $\leq k$ with respect to b.

Remark 1.5. In the cases above, G omits $K_{2,2}$ - distinct lines meet in at most one point (exactly one in case of $\mathbb{P}^2(\mathbb{C})$), and dually.

Theorem 1.6 (Elekes-Szabo 2012 (symmetric version))**.** *Suppose P, L, and $I \subseteq P \times L$ are complex algebraic varieties, or just constructible sets in $\mathbb{C}$, i.e. definable in $(\mathbb{C}; +, \cdot)$.*

Let $X_P \subseteq P$ and $X_L \subseteq L$ with $|X_P|, |X_L| \leq n$, let $E := I \cap (X_P \times X_L)$, and suppose E has combinatorial dimension $\leq k$.

Then the number of incidences $|E|$ is $\mathcal{O}(n^{\frac{2k-1}{k} - \frac{(k-1)^2}{k(kD-1)} + \epsilon})$ for any $\epsilon > 0$, where $D > 0$ depends only on $\dim(L)$.

($k = 2$: $\mathcal{O}(n^{\frac{3}{2} - \frac{1}{2(2D-1)} + \epsilon})$; in the Tóth theorem, $D = 2$ and there's no ϵ.)

Theorem 1.7 (Fox-Pack-Sheffer-Suk-Zahl 2014)**.** *Suppose $I \subseteq \mathbb{R}^{d_p} \times \mathbb{R}^{d_l} =: P \times L$ is semialgebraic, i.e. definable in $(\mathbb{R}; +, \cdot)$.*

Let $X_P \subseteq P$ and $X_L \subseteq L$ with $|X_P|, |X_L| \leq n$, let $E := I \cap (X_P \times X_L)$.

(i) Suppose E omits $K_{k,k}$. Then $|E|$ is $\mathcal{O}(n^{\frac{2d_p d_l - d_l - d_p}{d_p d_l - 1} + \epsilon})$ for any $\epsilon > 0$.

(ii) Suppose I is algebraic and E omits $K_{b,k}$. Then $|E|$ is $\mathcal{O}(n^{\frac{2k-1}{k} - \frac{(k-1)^2}{k(kD-1)} + \epsilon})$ for any $\epsilon > 0$, where $D = \max(d_l, d_p)$.

Theorem 1.8 (Chernikov-Galvin-Starchenko, Dec 2016)**.** *$I \subseteq \mathbb{R}^2 \times \mathbb{R}^{d_l}$ definable in an o-minimal expansion of a field. Then (i) of the previous theorem holds, but without the ϵ.*

Theorem 1.9 (Basu-Raz, Nov 2016)**.** *Same, but only for $d_l = 2$.*

1.2 Modularity, pseudoplanes, and quasidesigns

Definition 1.10. A strongly minimal theory T is locally modular if whenever $\mathcal{M}_0 \prec \mathcal{M} \models T$, the lattice of algebraically closed subsets of M containing M_0 satisfies the modular identity: for A, B, C with $A \leq C$,

$$A \vee (B \wedge C) = (A \vee B) \wedge C.$$

Equivalently, for $A, B \subseteq M$, if $c \in \operatorname{acl}(M_0AB) \setminus \operatorname{acl}(M_0A)$, then $\operatorname{acl}(M_0Ac) \cap \operatorname{acl}(M_0B) \neq \operatorname{acl}(M_0)$.

Equivalently, $\dim(A \vee B/B) = \dim(A/A \wedge B)$ for any algebraically closed $A, B \supseteq M_0$.

T is trivial if $A \vee B = A \cup B$, i.e. $\operatorname{acl}(X) = \bigcup_{x \in X} \operatorname{acl}(x)$ for any X.

Example 1.11. The lattice of vector subspaces of a vector space is modular: $c = a + b \Rightarrow b = c - a$

Definition 1.12. A (definable) relation $I \subseteq P \times L$ is a quasidesign if all fibres $\pi_1^{-1}(p)$ and $\pi_2^{-1}(l)$ are infinite, and it omits $K_{t,2}$ for some $t \in \mathbb{N}$; it is a pseudoplane if it also omits $K_{2,s}$ for some $s \in \mathbb{N}$.

A (2,3,2)-pseudoplane is a pseudoplane with $\dim(P) = \dim(L) = 2, \dim(I) = 3$.

Theorem 1.13 (Zilber's Weak Trichotomy)**.** *T strongly minimal.*

(i) T is not locally modular iff T interprets a (2,3,2)-pseudoplane.

(ii) If T is locally modular but non-trivial, then $x = x$ is in finite-to-finite definable correspondence with a (1-based) abelian group.

Hrushovski: The above Szemeredi-Trotter statements imply that pseudofinite subsets of (algebraically closed) fields of internal characteristic 0 "have" no pseudoplane (or even quasidesign), so "are modular". Making this precise seems not to be straightforward (but even the idea seems helpful).

Let $(F, X) = \prod_i (F_i, X_i)/\mathcal{U}$ be an ultraproduct of fields equipped with distinguished finite subsets.

For $Y \subseteq F^n$, define $\delta(Y) := \operatorname{st}(\log(|Y|)/\log(|X|))$. For $A \subseteq F^n$ constructible, $A(X) := A \cap X^n$. Then $\delta(A(X)) \leq \dim(A)$. If $\delta(A(X)) = \dim(A)$, say A is "X-rich".

Corollary 1.14 (of Elekes-Szabo's Szemeredi-Trotter, k=2)**.** *Suppose* $\operatorname{char}(F_i) = 0$*, and $P, L, I \subseteq P \times L$ are constructible sets in F. Suppose $I(X) \subseteq P(X) \times L(X)$ is a quasidesign. Then*

$$\begin{aligned}\delta(I(X)) &\leq (\frac{3}{2} - \frac{1}{2(2D-1)}) \max(\delta(P(X)), \delta(L(X))) \\ &< \frac{3}{2} \max(\delta(P(X)), \delta(L(X))).\end{aligned}$$

In particular, if $\dim(P) = 2 = \dim(L)$ *and* $\dim(I) = 3$*, it can not be that P, L, I are all X-rich.*

Hrushovski goes on to define a "probability logic" structure $(F, X)^{\text{prob}}$ and a notion of modularity, such that an adaptation of the proof of the Weak Trichotomy theorem yields firstly that this lack of pseudoplanes implies modularity in internal characteristic 0, and furthermore a reproof of the following version of a theorem of Elekes-Szabo:

Theorem 1.15 (Elekes-Szabo 2012). *Suppose $R \subseteq F^3$ is an irreducible subvariety, $\dim(R) = 2$, and $\dim((\pi_i \times \pi_j)(R)) = 2$ for $i \neq j$. Suppose R is X-rich. Then R is in co-ordinatewise correspondence with the graph of the group operation of a 1-dimensional algebraic group.*

Furthermore, Hrushovski conjectures that the underlying explanation for these Szemeredi-Trotter results is the truth of the Zilber Trichotomy Conjecture in this context:

Conjecture 1.16 (Hrushovski). *If $(X, F)^{\mathrm{prob}}$ is not (locally) modular, it defines a subfield $k \subseteq F$ with $\delta(k) = 1$.*

In particular, if the ultraproduct ${}^*F_0$ of the prime fields of the F_i has $\delta({}^*F_0) = \infty$, then there's no X-rich pseudoplane. A positive characteristic version of Elekes-Szabo (previously conjectured by Bukh-Tsimerman) follows.