Notes on *Polish topometric groups*

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1 $\mathrm{Aut}([0,1],\lambda)$

Let MALG *be the Lebesgue measure algebra*

$$\begin{aligned}\mathrm{MALG} &:= \{X \subseteq [0,1] : X\, \textit{Lebesgue measurable}\}/(\lambda(X\Delta Y) = 0)\\ &= \{X \subseteq [0,1] : X\, \textit{Borel}\}/(\lambda(X\Delta Y) = 0).\end{aligned}$$

Let $\mathrm{Aut}([0,1],\lambda) := \mathrm{Aut}((\mathrm{MALG}; \lambda, \vee, \neg))$. *This has two natural topologies:*

- $\mathrm{Aut}([0,1],\lambda)_p$*: topology of pointwise convergence, i.e. the topology inherited from the product topology on* $\mathrm{MALG}^{\mathrm{MALG}}$*;*
- $\mathrm{Aut}([0,1],\lambda)_u$*: topology of uniform convergence, defined by the metric*

$$\partial(\alpha,\beta) := \sup_{x \in M} \lambda(\alpha(x)\Delta\beta(x)).$$

Fact 1.1. $f \mapsto (X \mapsto f(X))$ *induces a group isomorphism*

$$\mathrm{Bij}_\lambda([0,1])/(\lambda(\mathrm{supp}(fg^{-1})) = 0) \xrightarrow{\sim} \mathrm{Aut}([0,1],\lambda),$$

where $\mathrm{Bij}_\lambda([0,1])$ *is the group of measure-preserving bijections.*

Theorem 1.2. *Any homomorphism from* $\mathrm{Aut}([0,1],\lambda)_p$ *to a separable topological group is continuous.*

Outline of proof.

(I) (Ryzikov,Kittrell-Tsankov,BBM): Using a result of Ryzikov that every element of $\mathrm{Aut}([0,1],\lambda)$ is a product of 3 involutions, $\mathrm{Aut}([0,1],\lambda)_u$ has the (38)-Steinhaus property and so has this automatic continuity property;

(II) (Kechris-Rosendal): The automorphism group of the countable dense measure subalgebra $\mathcal{A} \subseteq \mathrm{MALG}$ generated by dyadic rational intervals has ample generics;

(III) (BBM): Hence $\mathrm{Aut}([0,1],\lambda)$ has ample generics as a "topometric structure", and so the automatic continuity of $\mathrm{Aut}([0,1],\lambda)_u$ lifts to $\mathrm{Aut}([0,1],\lambda)_p$. □

2 Metric structures

Definition 2.1. $\mathcal{M} = (M; d, (P_i)_{i\in I}, (f_i)_{i\in I'})$ is a **metric structure** if

- $(M; d)$ is a complete bounded metric space,
- each $P_i : M^{k_i} \to \mathbb{R}$ is a uniformly continuous bounded map,
- each $f_i : M^{k'_i} \to M$ is a uniformly continous map.

$\mathcal{M}$ is **Polish** if $(M; d)$ is Polish (i.e. if $(M; d)$ is separable).

An isomorphism of metric structures is an isometry preserving the predicates and functions.

Example 2.2. MALG := (MALG; $d, \vee, \neg$) with metric $d(A, B) := \lambda(A\Delta B)$ is a Polish metric structure.

3 Topometric groups

Definition 3.1. A **topometric space** $(X; \tau, \partial)$ is a topological space $(X; \tau)$ equipped with a metric ∂ which refines the topology τ, such that ∂ is topologically lower semi-continuous, i.e. $\{(x, y) : \partial(x, y) \leq r\}$ is τ-closed for $r \in \mathbb{R}$.

A **topometric group** $(G; \tau, \partial)$ is a topometric space with a group structure such that $(G; \tau)$ is a topological group and ∂ is bi-invariant.

Fact 3.2. *If $(M; d)$ is a Polish metric space, then* $\mathrm{Iso}((M; d))$*, with the topology of pointwise convergence is a Polish group.*

Proof sketch. Say $A \subseteq M$ is countable dense. Then $\mathrm{Iso}((M; d))$ is homeomorphic to the space of isometric embeddings of A into M, which embeds as a closed subspace of M^A, which is Polish since it is the countable product of Polish spaces (explicitly: the product measure $\partial(\alpha, \beta) := \Sigma_i 2^{-1}\gamma(d(\alpha(a_i), \beta(a_i)),$ where $\gamma : \mathbb{R}_{\geq 0} \to [0, 1); x \mapsto \frac{x}{1+x}$, is a complete metrisation of M^A, and M^A is second countable by definition of the product topology). □

Fact 3.3. *Let $\mathcal{M} = (M; d, \ldots)$ be a Polish metric structure. Consider* $\mathrm{Aut}(\mathcal{M})$ *with the topology τ of pointwise convergence.*

Then $\mathrm{Aut}(\mathcal{M})$ *is a closed subgroup of* $\mathrm{Iso}((M; d))$*, so is Polish.*

Let $\partial^{\mathcal{M}}(\alpha, \beta) := \sup_{x\in M} d(\alpha(x), \beta(x))$ be the metric of uniform convergence on $\mathrm{Aut}(\mathcal{M})$*.*

Then $\partial^{\mathcal{M}}$ is bi-invariant, lower semi-continuous, and refines τ.

Definition 3.4. For $\mathcal{M}$ as in the previous fact, we consider $\mathrm{Aut}(\mathcal{M})$ as a Polish topometric group with the topometric structure $(\mathrm{Aut}(\mathcal{M}); \tau, \partial^{\mathcal{M}})$.

Example 3.5. $\mathrm{Aut}([0, 1]; \lambda) := \mathrm{Aut}(\mathrm{MALG})$ as a topometric group. (So $\mathrm{Aut}([0, 1]; \lambda)_p$ is the τ topology, $\mathrm{Aut}([0, 1]; \lambda)_u$ is the ∂ topology.

Remark 3.6. Any Polish group is isomorphic as a topological group to $\mathrm{Aut}(\mathcal{M})$ for some Polish metric structure $\mathcal{M}$ in a countable language (and $\mathcal{M}$ can be taken approximately ultrahomogeneous).

(No analagous statement given for Polish topometric groups).

4 Topometric groups with ample generics

Definition 4.1. A Polish topometric group (G, τ, ∂) has **ample generics** if for each n, the diagonal conjugacy action on G^n has an orbit with comeagre ∂-closure.

Theorem 4.2. *If $(G; \tau, \partial)$ is a topometric group with ample generics and $\phi : (G; \partial) \to H$ is a morphism of topological groups, where H is separable (or, more generally, H has uniform Souslin number[1] $\leq 2^{\aleph_0}$), then $\phi : (G; \tau) \to H$ is continuous.*

The proof is analagous to Kechris-Rosendal.

Remark 4.3. Metric version of the small index property: if $(G; \tau, \partial)$ is a Polish topometric group with ample generics, then any ∂-closed subgroup of index $< 2^{\aleph_0}$ is open.

5 Lifting ample generics

Definition 5.1. If $(X; \tau, \partial)$ is a topometric space, $A \subseteq X$, and $\epsilon > 0$, define

$$(A)_\epsilon := \{x : \partial(x, A) < \epsilon\}.$$

$A \subseteq X$ has **open enlargements** in X if $(A)_\epsilon \subseteq X$ is open for any $\epsilon > 0$.

Definition 5.2. Let $\mathcal{M} = (M; d, \ldots)$ be a Polish metric structure. A **good countable approximating substructure** is a classical countable structure $\mathcal{N}$ such that

(i) the universe N of $\mathcal{N}$ is a dense subset of $(M; d)$;

(ii) any automorphism of $\mathcal{N}$ extends (necessarily uniquely) to an automorphism of $\mathcal{M}$;

(iii) with this embedding, $\mathrm{Aut}(\mathcal{N}) \subseteq \mathrm{Aut}(\mathcal{M})$ is dense;

(iv) Consider $\mathrm{Aut}(\mathcal{N})$ with the topology of pointwise convergence where N has the discrete topology.

If $U \subseteq_{op} \mathrm{Aut}(\mathcal{N})$ is open as a subset of $\mathrm{Aut}(\mathcal{N})$, then U has open enlargements in $\mathrm{Aut}(\mathcal{M})$.

Lemma 5.3. *Let $\mathcal{A} = (A; \vee, \neg) \subseteq$ MALG be the measure subalgebra generated as a boolean algebra by the dyadic intervals, being the subintervals of $[0, 1]$ whose endpoints are dyadic rationals (i.e. of form $m2^{-n}$).*

Then $\mathcal{A}$ is a good countable approximating substructure.

Theorem 5.4. *Suppose $\mathcal{N}$ is a good countable approximating substructure of a Polish metric structure $\mathcal{M}$, and the Polish group $\mathrm{Aut}(\mathcal{N})$ has ample generics. Then the Polish topometric group $\mathrm{Aut}(\mathcal{M})$ has ample generics.*

[1] the **uniform Souslin number** of a group G is the least κ such that if $1 \in V \subseteq_{op} G$ then κ left-translates of V cover G.

5.1 Banach-Mazur

Let X be a topological space, $A \subseteq X$. The **Banach-Mazur game** *for $A \subseteq X$ is as follows:*

- *Players P1 and P2 alternate making moves.*
- *At any time, the* **game state** *is a non-empty open subset of X.*
- *a move consists of playing a non-empty open subset of the current game state; the game state is then replaced by this set.*
- *The initial game state is X, and P1 plays first.*
- *P2* **wins** *a game whose successive game states are $X = V_0 \supseteq V_1 \supseteq \dots$ if $\bigcap_{i\in\omega} V_i \subseteq A$.*

A **winning strategy** *for P2 is a deterministic choice of what to play on a given P2 move given the history of the game up to that point, such that if P2 plays according to the strategy, then P2 wins however P1 plays.*

Theorem 5.5. *P2 has a winning strategy iff A is comeagre.*

Proof in the case that X is second countable. (Adapted from a math.stackexchange.org post by Andreas Blass.)

Suppose A is comeagre, so say $A \supseteq \bigcap_{i\in\omega} U_i$ with U_i open dense. Then the following is a winning strategy for P2: if on P2's ith move the state is V, play $V \cap U_i$.

Conversely, suppose P2 has a winning strategy. Let $\mathcal{B}$ be a countable base for X; we may assume $\emptyset \notin \mathcal{B}$.

Suppose it is P1 to move, and the game state is V, so far P1 has always played sets from $\mathcal{B}$, and P2 has played according to the strategy. Let $K \subseteq V$ be the set of points $k \in V$ such that for any set from $\mathcal{B}$ which P1 could play, P2's response (playing according to P2's strategy) doesn't contain k.

Then K is nowhere dense: if it were dense in an open, then it would be dense in some $V \supseteq U \in \mathcal{B}$, but then K would intersect P2's response to U.

Now there are countably many such K, and they cover $X \setminus A$: if $x \in X \setminus A$ is not covered by the K then P1 can play such that x survives throughout the game, contradicting P2's strategy being winning. □

5.2 Lifting comeagreness

Theorem 5.6 (BBM)**.** *Let (X, τ, ∂) be a Polish topometric space. Let $Y \subseteq X$ be a dense subset equipped with a Polish topology refining the subspace topology, with complete metric d_Y. Suppose that any open (i.e. d_Y-open) $U \subseteq Y$ has open enlargements in X, and so does any open (i.e. τ-open) $U \subseteq X$.*

Let $A \subseteq Y$ be comeagre in Y. Then $\overline{A}^\partial$ is comeagre in X.

Proof. Say $A \supseteq \bigcap_{1\leq n<\omega} O_n$ with $O_n \subseteq Y$ open dense in Y. Since Y is dense in X, also O_n is dense in X.

$\overline{A}^\partial = \bigcap_n (A)_{\frac{1}{n}}$, so it suffices to show that for any $\epsilon > 0$, $(A)_{2\epsilon}$ is comeagre in X. We give a winning strategy for P2 in the Banach-Mazur game for $(A)_{2\epsilon} \subseteq X$.

Set $W_0 := Y$. On P2's ith move ($i \geq 1$), if the game state is U_i, the strategy tells P2 to choose $W_i \subseteq Y$ an open non-empty d_Y-ball of radius $< 2^{-i}$ such that $W_i \subseteq W_{i-1} \cap (U_i)_\epsilon \cap O_i$, then play $V_i := U_i \cap (W_i)_\epsilon$.

Claim 5.7. *Such a W_i exists.*

Proof. $A_i := W_{i-1} \cap (U_i)_\epsilon$ is non-empty; this holds for $i = 1$ because $W_0 = Y$ is dense, and for $i > 1$ because $\emptyset \neq U_i \subseteq (W_{i-1})_\epsilon$ by the rules of the game.

Now A_i is open in Y since U_i has open enlargements and the topology on Y refines the subspace topology, so A_i intersects the dense open O_i in a non-empty open set. □

Now V_i is open since W_i has open enlargements, and is non-empty since $\emptyset \neq W_i \subseteq (U_i)_\epsilon$.

Now say $x \in \bigcap U_i$ for a game where P2 plays according to this strategy. Then for each i, $x \in U_{i+1} \subseteq (W_i)_\epsilon$, so say $y_i \in W_i$ with $\partial(x, y_i) < \epsilon$. Let $y := \lim_i^{d_Y} y_i$. Then $A \supseteq \bigcap O_n \supseteq \bigcap W_i = \{y\}$.

Now by topological lower semi-continuity of ∂, $\partial(x, y) \leq \epsilon$. So $x \in (A)_{2\epsilon}$. So the strategy is winning. □

5.3 Proof of Theorem 5.4

Proof of Theorem 5.4. Let $A \subseteq \mathrm{Aut}(\mathcal{N})^n$ be a comeagre orbit of the diagonal conjugation action of $\mathrm{Aut}(\mathcal{N})$. Let $A' \subseteq \mathrm{Aut}(\mathcal{M})^n$ be the orbit of the diagonal conjugation action of $\mathrm{Aut}(\mathcal{M})$ containing A.

It suffices to show that the conditions of Theorem 5.6 hold for the pair $\mathrm{Aut}(\mathcal{N})^n \subseteq \mathrm{Aut}(\mathcal{M})^n$, since then $\overline{A'}^\partial \supseteq \overline{A}^\partial$ is comeagre as required.

Claim 5.8. *The topology on $\mathrm{Aut}(\mathcal{N})^n \subseteq \mathrm{Aut}(\mathcal{M})^n$ refines the subspace topology.*

Proof. Assume $n = 1$; the same proof goes through in general.

Let $U \subseteq \mathrm{Aut}(\mathcal{M})$ be a subbasic open set of the form

$$U = \{\beta \in \mathrm{Aut}(\mathcal{M}) : d(\beta(a), b) < \epsilon\}.$$

Let $\beta \in U \cap \mathrm{Aut}(\mathcal{N})$. Let $a' \in N$ such that $r := \epsilon - d(\beta(a'), b) > 0$ and $d(a, a') < \frac{r}{2}$.

Let $V := \{\beta' \in \mathrm{Aut}(\mathcal{N}) : d(\beta'(a'), \beta(a')) < \frac{r}{2}\} \subseteq_{op} \mathrm{Aut}(\mathcal{N})$. Then $\beta \in V$, and $V \subseteq U \cap \mathrm{Aut}(\mathcal{N})$ since if $\beta' \in V$ then

$$\begin{aligned} d(\beta'(a), b) &\leq d(\beta'(a), \beta'(a')) + d(\beta'(a'), \beta(a')) + d(\beta(a'), b) \\ &= d(a, a') + d(\beta'(a'), \beta(a')) + d(\beta(a'), b) \\ &< \frac{r}{2} + \frac{r}{2} + (\epsilon - r) \\ &= \epsilon. \end{aligned}$$

So $U \cap \mathrm{Aut}(\mathcal{N}) \subseteq_{op} \mathrm{Aut}(\mathcal{N})$. □

Claim 5.9. *Any open subset $U \subseteq \mathrm{Aut}(\mathcal{M})^n$ has open enlargements.*

Proof. $(U)_\epsilon = (\{1\})_\epsilon \cdot U$. □

The other conditions follow immediately from the definition of a good approximating substructure. □

6 Automatic contiuity for $\mathrm{Aut}([0,1],\lambda)_p$

Fact 6.1 (Kechris-Rosendal)**.** *$\mathcal{A}$ is the Frassé limit of the class of finite measured boolean algebras where the measure takes dyadic rational values.*

Proof of Lemma 5.3. (i) By regularity of Lebesgue measure, every $X \in \mathrm{MALG}$ is of the form $X = \bigwedge_{i<\omega} U_i$ where U_i is open, so $X = \bigwedge_{i<\omega} \bigvee_{j<\omega} I_{ij}$ where I_{ij} is a dyadic interval.

(ii) Extend $\sigma \in \mathrm{Aut}(\mathcal{A})$ to MALG by continuity, $\sigma \lim_i X_i := \lim_i \sigma X_i$ if $X_i \in \mathcal{A}$.

Then σ preserves λ, and σ preserves boolean operations since they are continuous, e.g. $\sigma\neg \lim_i X_i = \sigma \lim_i \neg X_i = \lim_i \sigma\neg X_i = \lim_i \neg\sigma X_i = \neg \lim_i \sigma X_i = \neg\sigma \lim_i X_i$.

(iii) It suffices to show that if $\beta \in \mathrm{Aut}(\mathrm{MALG})$ and $B_1, \ldots, B_n \in \mathrm{MALG}$ and $\epsilon > 0$, then there exists $\beta' \in \mathrm{Aut}(\mathcal{A})$ with $d(\beta(B_i), \beta'(B_i)) < \epsilon$.

We may assume $B_1, \ldots, B_n$ are the atoms of a finite subalgebra of MALG.

Take approximations from below $B_i \supseteq B_i' \in \mathcal{A}$ for $i > 1$ with $\lambda(B_i \setminus B_i') < \epsilon/n$. Set $B_1' := \neg(\bigvee_{i>1} B_i') \in \mathcal{A}$, so $\lambda(B_1' \setminus B_1) < \epsilon$.

Similarly obtain disjoint approximations $C_i' \in \mathcal{A}$ to $\beta(B_i)$ with $\lambda(B_i') = \lambda(C_i')$.

Then $B_i' \mapsto C_i'$ extends to an isomorphism of the generated finite dyadic rational subalgebras, so by homogeneity extends to $\beta' \in \mathrm{Aut}(\mathcal{A})$.

(iv) In fact we show this not for ∂ but for the equivalent metric ∂'

$$\partial'(\alpha, \beta) := \lambda(\mathrm{supp}(\alpha^{-1}\beta)).$$

Take $U \subseteq \mathrm{Aut}(\mathcal{A})$ a basic open neighbourhood of id,

$$U = \{\alpha : \bigwedge_{i=1}^{n} \alpha(B_i) = B_i\}.$$

WLOG $B_1, \ldots, B_n$ are the atoms of a finite subalgebra of $\mathcal{A}$.

Claim 6.2.

$$(U)_\epsilon = \{\beta : \Sigma_{i=1}^n \lambda(B_i \setminus \beta^{-1}(B_i)) < \epsilon\} =: U_\epsilon'.$$

Proof. $\subseteq$ is clear.

For the converse, note $\lambda(B_i \setminus \beta^{-1}(B_i)) = \lambda(B_i \setminus \beta(B_i))$; indeed,

$$\begin{aligned}\lambda(B_i \setminus \beta^{-1}(B_i)) &= \lambda(B_i) - \lambda(B_i \cap \beta^{-1}(B_i)) \\ &= \lambda(\beta^{-1}(B_i)) - \lambda(B_i \cap \beta^{-1}(B_i)) \\ &= \lambda(\beta^{-1}(B_i) \setminus B_i) \\ &= \lambda(B_i \setminus \beta(B_i)).\end{aligned}$$

Let $\beta \in U_\epsilon'$. Define β' setting it on $B_i \setminus \beta^{-1}(B_i)$ to be an arbitrary isomorphism with $B_i \setminus \beta(B_i)$, and otherwise to agree with β. Then $\beta' \in U$. So $\beta \in (U)_\epsilon$ by definition of U_ϵ' and of ∂'. □

So $(U)_\epsilon$ is open, as required.

□

Fact 6.3 (Kechris-Rosendal)**.** Aut($\mathcal{A}$) *has ample generics.*

Fact 6.4. *Any homomorphism from* Aut$([0,1],\lambda)_p$ *to a separable group is continuous.*

Corollary 6.5. *The topometric group* (Aut(MALG); τ, ∂') *has ample generics, hence by Theorem 4.2, any homomorphism from* Aut$([0,1],\lambda)_u$ *to a separable group is continuous.*