Measurable groups of low dimension I

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D-rank 1 functional-unimodular groups

functional-unimodular

A first-order structure is *functional-unimodular* if for any parameter-definable sets X and Y in M^{eq}, if $f_1, f_2 : X \to Y$ are definable epimorphisms with fibres of constant sizes k_1 and k_2, then $k_1 = k_2$.

Previous talk:

- Measurable theories are functional-unimodular.
- *ACF* is not functional-unimodular.

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D-rank

We define the *D-rank* on formulas $\psi(x)$ as follows:

(i) $D(\psi(x)) \geq 0$ if $\psi(x)$ is consistent.

(ii) $D(\psi(x)) \geq \alpha + 1$ if there is an L-formula $\phi(x, y)$, and for every λ there are c_i for $i < \lambda$ such that $\models \phi(x, c_i) \to \psi(x)$ for all i, $\{\phi(x, c_i) \mid i < \lambda\}$ is k-inconsistent, for some $k < \omega$, and $D(\phi(x, c_i)) \geq \alpha$ for all i.

(ii) $D(\psi(x)) \geq \gamma$, where γ is a limit ordinal, if $D(\psi(x)) \geq \alpha$ for all $\alpha < \gamma$.

(ii) For a partial type $p(x)$, we define $D(p(x))$ to be the minimum of $D(\psi(x))$ for ψ a finite conjunction of formulas in $p(x)$.

Indeed, this coincides with the previous talk, as the Standard Lemma yields an indiscernible sequence (over the parameters of ψ), which is k-inconsistent.

On the other hand, an inconsistent indiscernible sequence is k-inconsistent by compactness.

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Lemma

Let G be a group. Let $B \subseteq G$ be the set of elements with finite conjugacy classes in G. Then B is a characteristic subgroup of G.

Proof: B is closed under inverses, as for $b \in B$:

$$g \in C_G(b) \implies g \in C_G(b^{-1})$$

holds. Invoke Orbit-Stabilizer.
B is closed under multiplication, as for $b_1, b_2 \in B$ it is

$$(b_1 \cdot b_2)^G \subseteq b_1^G \cdot b_2^G,$$

where the latter is finite.
Characteristic, as automorphisms map finite conjugacy classes to finite conjugacy classes.

D-rank 1 functional-unimodular groups

Definition

A group G is a *BFC group (bounded finite conjugacy-classes)* if all the conjugacy classes of G have finite size, and if there is a finite bound to their sizes.

Theorem (as a fact)

If G is a BFC group, then $[G, G]$ is finite.

Lemma (as a reminder)

If G is a group and H an abelian subgroup,

$$C_G(C_G(H))$$

is abelian.

D-rank 1 functional-unimodular groups

Theorem

Let G be a group of D-rank 1, definable in a functional-unimodular structure. Then G is definably and characteristically (finite-by-abelian)-by-finite,
i.e. there exists a characteristic, finite-index subgroup H of G and a finite characteristic subgroup A of H such that H/A is abelian.

Proof:

Claim:

If G is of finite exponent, G contains a finite non-identity conjugacy class.

Proof of the claim: Assume not, then, by D-rank 1, G is a disjoint union of finitely many infinite conjugacy classes. As otherwise the 'first two' formulas

$$x = c_{ij} \longrightarrow x \in g_i^G \longrightarrow x \in G$$

would be 2-inconsistent, where c_{ij} are elements of g_i^G.
Let n be the amount of those infinite conjugacy classes. We define an equivalence relation on those:

$$C_1 \sim C_2 \iff \exists y_1 \in C_1, y_2 \in C_2 : \ \langle y_1 \rangle \doteq \langle y_2 \rangle.$$

Suppose there are n_2 many of those conjugacy classes and let $[y^G]$ denote the equivalence class of y^G for some $y \in G$.

Continuation of the proof - claim

Fix some $y_1 \in G$.
$\langle y_1 \rangle$ is finite and has $\Phi(|\langle y_1 \rangle|)$ (Euler-totient) many generators.
$N_G(\langle y_1 \rangle)$ acts transitively on the finite set

$$\{z \in G \mid \exists g : \ (z = y_1^g) \wedge (\langle z \rangle \doteq \langle y_1 \rangle)\},$$

hence it's of size

$$\frac{|N_G(\langle y_1 \rangle)|}{|C_G(\langle y_1 \rangle)|}.$$

Moreover,

$$\langle y_1 \rangle = \langle y_2 \rangle \implies \frac{|N_G(\langle y_1 \rangle)|}{|C_G(\langle y_1 \rangle)|} = \frac{|N_G(\langle y_2 \rangle)|}{|C_G(\langle y_2 \rangle)|}$$

In conclusion,

$$\Phi(|\langle y_1 \rangle|) = [y_1^G] \cdot \frac{|N_G(\langle y_1 \rangle)|}{|C_G(\langle y_1 \rangle)|}$$

Continuation of the proof - claim

Pick representatives x_i for each conjugacy class x^G. There are maps

$$f_i : G \longrightarrow x_i^G;\ g \mapsto x_i^g$$

which all have constant fibre size of $K_i := |C_G(x_i)|$.
Put $K := lcm\{K_i \mid i \leq n\}$ and let

$$\{G_{ir} \mid 1 \leq i \leq n,\ 1 \leq r \leq \frac{K}{K_i}\}$$

be a family of pairwise disjoint copies of G.
Now, there's the map

$$f_\bullet : \bigsqcup_{i=1}^{n} \bigsqcup_{r=1}^{\frac{K}{K_i}} G_{ir} \longrightarrow G;\ g \longmapsto x_i^g$$

whose fibres are of size K.

Continuation of the proof - claim

Write $\{G_{ir} \mid 1 \leq i \leq n,\ 1 \leq r \leq \frac{K}{K_i}\}$ as

$$\{G_{ijr} \mid 1 \leq i \leq n_2,\ 1 \leq j \leq |[y_i^G]|,\ 1 \leq r \leq \frac{K}{K_i}\},$$

then $f_\bullet$ translates to

$$f_\bullet : \bigsqcup_{i=1}^{n_2} \bigsqcup_{j=1}^{|[y_i^G]|} \bigsqcup_{r=1}^{\frac{K}{K_i}} G_{ijr} \longrightarrow G;\ g \longmapsto x_i^g.$$

Furthermore,

$$f'_\bullet : \bigsqcup_{i=1}^{n_2} \bigsqcup_{j=1}^{|[y_i^G]|} \bigsqcup_{r=1}^{\frac{K}{K_i}} G_{ijr} \longrightarrow G;\ g \longmapsto g$$

has fibres of size $\sum_{i=1}^{n_2} |[y_i^G]| \cdot \frac{K}{K_i}$.

Continuation of the proof - claim

By functional-unimodularity, we get

$$K = \sum_{i=1}^{n_2} |[y_i^G]| \cdot \frac{K}{K_i}.$$

Since $\Phi(|\langle y_i \rangle|) = [y_i^G] \cdot \frac{|N_G(\langle y_i \rangle)|}{|C_G(\langle y_i \rangle)|}$, this translates to

$$1 = \sum_{i=1}^{n_2} \frac{\Phi(| < y_i > |) \cdot |C_G(< y_i >)|}{K_i \cdot |N_G(< y_i >)|}$$

If G would be of odd exponent, $|C_G(\langle y_i \rangle)|, |N_G(\langle y_i \rangle)|, K_i$ and $|\langle y_i \rangle|$ would be of odd order, but $\Phi(|\langle y_i \rangle|)$ would be even. A contradiction.

Continuation of the proof - claim

As G is of even exponent, there exists $g \in G$ such that $g^2 = 1$.
Define

$$S(g) := \{x \in G \mid x^g = x^{-1}\},$$

which is infinite since $g^G \cdot g \subseteq S(g)$.
Let k be the maximal size of a non-trivial stabiliser.
Then (note that conjugates of g are also involutions)

$$\{S(h) \mid h \in g^G\}$$

is $(k+2)$-inconsistent, as for $x \in S(h_1) \cap ... \cap S(h_{k+2})$ and $i \geq 2$:

$$x^{h_1 h_i} = (x^{-1})^{h_i} = x$$

would contradict the maximality of k.
This proves the claim, hence there is at least one finite non-identity conjugacy class.

Continuation of the proof

Let N be the union of all finite conjugacy classes.
Assume that there are *infinitely* many finite conjugacy classes.

If there would be no bound for the size of those conjugacy classes, by compactness and saturation, there would be infinitely many infinite conjugacy classes.
This would contradict D-rank 1.
Hence the sizes of the infinitely many finite conjugacy classes is bounded.
Now N is

- a BFC-group.
- of finite index in G.
- characteristic in G, by a lemma above.
- such that $[N, N]$ is finite, by a lemma above.

In other words: G is (finite-by-abelian)-by-finite.

Continuation of the proof

Assume there are *finitely* many finite conjugacy classes.

If G has finite exponent: As N is finite, G/N would be infinite and all of its conjugacy classes would be infinite, in contradiction to the claim above.

If G has infinite exponent: By saturation, there exists g such that

$$1 \neq g^n \in C_G(C_G(g)) \text{ for all } n \in \mathbb{N} \text{ and } g^G \text{ infinite.}$$

Set

$$X := \bigcup_{\sigma \in \mathrm{Aut}(G)} \{\sigma(g)\}.$$

Note that X is a union of conjugacy classes. There are only finitely many conjugacy classes, hence

$$X = \bigcup^{\text{finite}} g_i^G$$

for some suitable g_i's with g_i^G infinite.

continuation of the proof

$$X = \bigcup^{\text{finite}} g_i^G = \bigcup_{\sigma \in \mathrm{Aut}(G)} \{\sigma(g)\}, \quad g_i^G \text{ infinite}$$

As $C_G(C_G(g_i))$ are of finite index, $\{C_G(C_G(g_i))^h \mid h \in G\}$ is finite, and

$$\bigcap_{h \in G} C_G(C_G(g_i^h)) = \bigcap_{x \in X}^{\text{finite}} C_G(C_G(x)) = \bigcap_{\sigma \in \mathrm{Aut}(G)}^{\text{finite}} C_G(C_G(\sigma(g)))$$

is of finite index in G.

Also it is characteristic and abelian. □

Indecomposability and field interpretation

Zilbers Indecomposability Theorem

Let G be a group definable in a supersimple structure of finite rank. Let $\{X_i \mid i \in I\}$ be a collection of definable subsets of G. Then there exists a definable subgroup H of G such that:

(i) $H \leq \langle X_i \mid i \in I \rangle$ and every element of H is a product of a bounded finite number of elements of the X_i's and their inverses.

(ii) X_i/H is finite for each i.

If the collection of sets X_i is invariant under conjugation, then H may be chosen normal in G.

Sketch of the proof: Let k be maximal such that there's a type p of rank k such that for some $X_1, ..., X_m$ and $Y := X_1^{\pm} \cdot ... \cdot X_m^{\pm}$ the formula $x \in Y$ is contained in p.
Let A be the set of parameters of p and define

$$\mathrm{S}(p) := \{g \in G \mid p(g^{-1}x) \cup p(x) \text{ does not fork over A}\}$$

and $\mathrm{Stab}(p) := \mathrm{S}(p) \cdot \mathrm{S}(p)$.

Sketch of the proof - ZIT

$$\mathrm{S}(p) := \{g \in G \mid p(g^{-1}x) \cup p(x) \text{ does not fork over A}\}$$

Then

$$g \in S(p) \iff \exists a, c \models p : \; a = gc \text{ and } a \mathop{\smile\!\!\!\!\!\!\mid}_A g \iff g^{-1} \in S(p)$$

It follows that $\mathrm{Stab}(p) = \mathrm{S}(p) \cdot \mathrm{S}(p) \subseteq Y^2 \cdot Y^2$.

Using the Independence Theorem one can show that Stab(p) is a (type-definable) group [7.1.13, 'Simplicity Theory' by Kim, book].

Furthermore one can show that the rank of Stab(p) coincides with the rank of p [5.4.2, 'Simple Theories' by Wagner, book].

The whole proof can be found in [5.4.5, 'Simple Theories' by Wagner, book].

Sketch of the proof - ZIT

A type-definable group in a supersimple theory is an intersection of definable groups [4.4, 'Groups in simple theories' by Wagner, paper], so, by compactness, there exists a definable group H such that $H \subseteq Y^4$ and $\mathrm{Stab}(p) \subseteq H$.

The remark about H being normal is shown in [5.4.13, 'Simple Theories" by Wagner, book] □

Indecomposability and field interpretation

Definition

Let T be a first order theory. We say that *infinity is definable in* T or T *eliminates* $\exists^\infty$, if for every $M \models T$ and every formula $\varphi(\bar{x}, \bar{y})$, the set $\{\bar{a} \in M^m \mid \varphi(M^n, \bar{a} \text{ is finite}\}$ is $\emptyset$-definable.

Indecomposability and field interpretation

Zilbers Field Interpretation Theorem

Let G be a group definable in a supersimple structure of finite rank in which infinity is definable. Suppose that $G = A \rtimes H$, where A and H are each abelian, definable in G and 1-dimensional. Suppose A has no proper non-trivial G-definable subgroups which are normal in G, and $C_H(A) = \{1\}$. Then

1. The subring $K = \mathbb{Z}[H]/\operatorname{Ann}_{\mathbb{Z}[H]}(A)$ of $\operatorname{End}(A)$ is a definable field; in fact there is an integer l such that every element of K can be represented as the endomorphism $\sum_{l=1}^{l} h_i$ with $h_i \in H$.
2. $A \cong K^+$. Also H is isomorphic to a subgroup J of K^* and the conjugacion action of H on A is its multiplication action on K.
3. If G is a non-principal ultraproduct of finite groups, then K is a pseudofinite field.

Proof of ZFI

Proof: Let B be the union of the finite orbits under the action of H on A. Then $B \leq A$ is G-normal and B is definable, since infinity is definable. By assumption, B is a trivial subgroup of A.
If $B = A$ would hold, for each $a \in A$, $C_H(a)$ would be of finite index in H, as the orbit of a is finite and the latter also implies

$$C_H(a^G) = \bigcap_{g \in a^G}^{\text{finite}} C_H(a^g) \text{ of finite index in } H,$$

so $C_H(a^G) \neq \{1\}$. It is by definition

$$a^G \subseteq C_G(C_H(a^G)) \Longrightarrow a^G \subseteq C_A(C_H(a^G))$$

Also, it is $C_A(C_H(a^G))$ G-normal, since a^G is. Hence $A = C_A(C_H(a^G))$. I.o.w: A centralizes $C_H(a^G)$, but then $C_H(a^G)$ $(\neq \{1\})$ would also centralize A. This contradicts $C_H(A) = \{1\}$.

Thus $B = \{1\}$ and the action of H on $A \backslash \{0\}$ has only infinite orbits. As A is 1-dimensional, there are only finitely many of those infinite orbits.

Continuation of the proof of ZFI - first part

1. Consider $\mathbb{Z}[H]$ as a ring of endomorphisms of A, extending the conjugation action of H by linearity.
Let $r \in \mathbb{Z}[H]$, then $\mathrm{Ker}(r)$ and $\mathrm{Im}(r)$ are definable and H-normal in $G = A \rtimes H$, hence G-normal (as A is abelian) and therefore trivial.

If $\mathrm{Ker}(r) = A$, it is $r \in \mathrm{Ann}_{\mathbb{Z}[H]}(A)$.
And if $\mathrm{Ker}(r) = \{0\}$, it is $r \in \mathrm{Aut}(A)$.

Thus

$$R := \mathbb{Z}[H] / \mathrm{Ann}_{\mathbb{Z}[H]}(A)$$

is a ring of automorphisms of A.

Continuation of the proof of ZFI - first part

Let $a \in A \backslash \{0\}$.
We've seen that a^H is infinite.

By the Zilber Indecomposability Theorem, there is a definable, infinite and normal subgroup C with $C \leq \langle a^H \rangle \leq A$.

Consequently, $C = \langle a^H \rangle = A$.

By compactness there exists an integer k (not depending on a) such that

$$A = (a^H)^{\pm} + \ldots + (a^H)^{\pm} \quad (k\text{-times}).$$

Let

$$K := \{\sum_{i=1}^{k} h_i \mid h_i \in H\} / \operatorname{Ann}_{\mathbb{Z}[H]}(A),$$

then $K \subseteq R \subseteq \operatorname{End}(A)$.
Suppose $\lambda \in R \backslash \{0\}$ and let $b \in A$ such that $b = \lambda \cdot a$.
Since $A = (b^H)^{\pm} + \ldots + (b^H)^{\pm}$, there exists $\zeta \in K$ such that $\zeta \cdot b = a$.
Then

$$(\zeta\lambda - Id) \in R \text{ and } (\zeta\lambda - Id) \cdot a = 0 \overset{a \neq 0}{\Longrightarrow} \zeta\lambda = Id$$

Also, $\zeta\lambda = \lambda\zeta$ as H is abelian.
So for every $\lambda \in R$ exists $\zeta \in K$ such that $\lambda^{-1} = \zeta$, which makes R a field.
Furthermore, R is closed under inverses, so $R = K$, hence it is definable.

Continuation of the proof of ZFI - second part

2. Fix some $a \in A\backslash\{0\}$ (think of it as '1_A') and define

$$i_a: \ K \longrightarrow A; \ i_a(\lambda) = \lambda \cdot a,$$

This is onto since $A = (a^H)^{\pm} + ... + (a^H)^{\pm}$. Also it is injective and additive, hence $A \cong K^+$.
Define a multiplication $\otimes$ on A by

$$b \otimes c := i_a(\lambda\zeta),$$

where $\lambda \cdot a = b$ and $\zeta \cdot a = c$. This makes i_a an isomorphism of fields.

Furthermore it maps H to an infinite subgroup of $A^{\otimes}$, which is, indeed, of finite index as A is of rank 1.

Continuation of the proof of ZFI - third part

3. Say $G = \prod_{i \in I} g_i / \mathcal{U}$.
By Łoś' Theorem, there is $U \in \mathcal{U}$ where the formulae which define the field structure on K also do so on every G_i for all $i \in \mathcal{U}$.
So K is a non-principal ultraproduct of finite fields, and thus a pseudofinite field. □