Some basic results from field theory and algebraic geometry

Rosario Mennuni

WWU Münster
Reading group on model theory of pseudofinite structures

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References

I have mainly looked at: (but there is plenty of literature on the subject: books, notes, stackexchange...)

- Z. Chatzidakis' notes (see learnweb page).
- S. Lang, *Introduction to Algebraic Geometry*. Also *Algebra*.
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Plan of the talk:

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- One can also speak of R-algebras for R a ring. (the first definition does not work anymore)
- Lots of things also have more abstract (i.e. categorical) definitions, but this will not concern us here.

Tensor product: construction

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9. Fact: up to isomorphism, $A \otimes_F B$ does not depend on $\mathcal{E}_A, \mathcal{E}_B$.

Tensor product: properties and examples

Some properties of tensor products: (recall: $ca \otimes b = a \otimes cb$)

1. $\otimes$ is associative (up to isomorphism).
2. Bilinear maps $A \times B \to C$ "are the same as" linear maps $A \otimes B \to C$.
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4. Enough to check: some F-basis of some $R \subseteq E$ with $E = \mathrm{Quot}(R)$ is L-lin.ind.
5. If $F \subseteq E \subseteq K$ then $L \underset{F}{\downarrow}^{\text{l.d.}} K \iff (L \underset{F}{\downarrow}^{\text{l.d.}} E \wedge LE \underset{E}{\downarrow}^{\text{l.d.}} K)$.

More on linear disjointness

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9. So if L/F is Galois, then $E \underset{F}{\downarrow}^{\text{l.d.}} L \iff E \cap L = F \iff \restriction L\colon \mathrm{Gal}(LE/E) \to \mathrm{Gal}(L/F)$ is an iso.

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12. Non-example: with $p = \operatorname{char} F$, let $T \in F$ have no p-th root and $E \coloneqq F(T^{1/p})$.
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6. Let $\mathfrak{p}$ be a prime ideal of $F[\bar{X}]$. Then $\mathfrak{p}F^{\text{alg}}[\bar{X}]$ is prime if and only if $\text{Quot}(F[\bar{X}]/\mathfrak{p})$ is a regular extension of F. (Keep this in mind for later!)

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3. Maximal ideals must come from points: every maximal ideal of $k[X_1, \dots, X_n]$ is of the form $(X_1 - a_1, \dots, X_n - a_n)$. Contrast with $(X^2 + 1)$ in $\mathbb{R}[X]$.
4. $\mathcal{I}\mathcal{V}$ is as small as possible: $\forall n, \forall A \subseteq k[X_1, \dots, X_n] \left(\mathcal{I}(\mathcal{V}(A)) = \sqrt{(A)} \right)$.

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5. Corollary of the Nullstellensatz: if $k \vDash \mathsf{ACF}$, then $\mathcal{IV}$ and $\mathcal{VI}$ are bijections between the radical ideals of $k[X_1, \ldots, X_n]$ and the Zariski closed subsets of k^n.

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10. If $X = X_1 \cup \ldots \cup X_n$, each X_i closed irreducible, and $X_i \not\subseteq X_j$, then the X_i are the irreducible components of X.

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11. Aside: the Ω-Zariski subspace topology on k^n equals the k-Zariski topology.

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   - 3.4 Some $a \in V(\Omega)$ such that $V(\Omega)$ is the closure of $\{a\}$ in the k-Zariski topology on Ω.

     Warning: this is not even T0.
   - 3.5 Some $a \in V(\Omega)$ with $\operatorname{trdeg}(k(a)/k) = \dim V$.
   - 3.6 Some $a \in V(\Omega)$ with $\operatorname{tp}(a/k)$ of the same Morley rank as V.
   - 3.7 At any rate: a point satisfying the equations in $\mathfrak{p}$ and no other equation over k.
4. Any point $a \in \Omega^n$ is a generic point over k of $\mathcal{I}_k(\{a\})$. (may have $a \notin k^n$!)
5. b is a *specialisation* of a over k iff $\mathcal{I}_k(\{a\}) \subseteq \mathcal{I}_k(\{b\})$.
6. Specialisations correspond to surjective morphisms between coordinate rings.

Generic points

1. Work inside a saturated $\Omega \models \mathsf{ACF}$. Variety = irreducible Zariski closed set.
2. Let V be a variety defined over $k \subseteq \Omega$ and $\mathfrak{p} := \mathcal{I}_k(V) \subseteq k[X_1, \ldots, X_n]$.
3. A *generic point of V* is, equivalently:
   - 3.1 "The element $(X_1, \ldots, X_n) + \mathfrak{p}$ of $k[X_1, \ldots, X_n]/\mathfrak{p}$."
   - 3.2 More precisely, any image $a = (a_1, \ldots, a_n)$ of it under some k-embedding in Ω.
   - 3.3 In other words, $k[a_1, \ldots, a_n] \cong_k k[X_1, \ldots, X_n]/\mathfrak{p}$.
   - 3.4 Some $a \in V(\Omega)$ such that $V(\Omega)$ is the closure of $\{a\}$ in the k-Zariski topology on Ω.

     Warning: this is not even T0.
   - 3.5 Some $a \in V(\Omega)$ with $\operatorname{trdeg}(k(a)/k) = \dim V$.
   - 3.6 Some $a \in V(\Omega)$ with $\operatorname{tp}(a/k)$ of the same Morley rank as V.
   - 3.7 At any rate: a point satisfying the equations in $\mathfrak{p}$ and no other equation over k.
4. Any point $a \in \Omega^n$ is a generic point over k of $\mathcal{I}_k(\{a\})$. (may have $a \notin k^n$!)
5. b is a *specialisation* of a over k iff $\mathcal{I}_k(\{a\}) \subseteq \mathcal{I}_k(\{b\})$.
6. Specialisations correspond to surjective morphisms between coordinate rings.
7. The points of a variety are exactly the specialisations of its generic points.