Characterization of (Pseudo)finite Simple Groups

Based on Three Theorems of J.S. Wilson

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30-06-2021

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- Group theory (Jordan-Hölder theorem) tells us that "all groups are made up of simple groups" (via composition series. In particular, for finite groups these always exist).
- So is there an algebraic classification of the *simple* pseudofinite groups?

Overview

1. Finite simple groups
   - Classification of finite simple groups
   - Definability of finite simple groups
2. Pseudofinite simple groups
   - Algebraic characterization
   - Properties
3. Solvability in (pseudo)finite groups

Classification of finite simple groups

Theorem

Every finite simple group is isomorphic to one of of the following:

- A cyclic group C_p of prime order.
- An alternating group Alt_n of degree at least 5.
- A simple group of Lie type.
- One of 26 *sporadic groups*.
- The Tits group.

Classification of finite simple groups of Lie type

Theorem

Every finite simple group of Lie type is isomorphic to one of the following:

- Chevalley groups:
  - $A_n,\ n \geq 1$
  - $B_n,\ n \geq 2$
  - $C_n,\ n \geq 3$
  - $D_n,\ n \geq 4$
  - E_6, E_7, E_8, F_4, G_2
- Steinberg groups:
  - ${}^2A_n,\ n \geq 2$
  - ${}^2D_n,\ n \geq 4$
  - ${}^3D_4,\ {}^2E_6$
- Suzuki and Ree groups:
  - ${}^2B_2,\ {}^2G_2,\ {}^2F_4$

Definition

A family of finite groups indexed by prime powers is *uniformly definable* if there exist formulas ϕ, ψ such that ϕ defines a finite subset of each finite field of prime power order, ψ defines a group operation on those sets, and the family consists of these groups for the various fields.

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The Suzuki and Ree groups ${}^2B_2$, ${}^2G_2$, ${}^2F_4$ are uniformly definable in some corresponding *difference field*, i.e. the corresponding finite field enriched by a certain automorphism.

Theorem (Ryten)

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2. The Ree groups ${}^2F_4(\mathbb{F}_{2^{2k+1}})$ and the Suzuki groups ${}^2B_2(\mathbb{F}_{2^{2k+1}})$ are uniformly bi-interpretable over parameters with the difference fields $(\mathbb{F}_{2^{2k+1}}, x \mapsto x^{2^k})$. The Ree groups ${}^2G_2(\mathbb{F}_{3^{2k+1}})$ are uniformly bi-interpretable over parameters with $(\mathbb{F}_{3^{2k+1}}, x \mapsto x^{3^k})$.

Pseudofinite groups

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A group is pseudofinite if it is equivalent to an ultraproduct of finite groups, or equivalently, if every sentence holding in the group holds in some finite group.

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- $PSL_2(F)$ for a pseudofinite field F.

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- In particular, groups of Lie type over pseudofinite fields are simple.
- The theorem in fact also holds for finite fields, with a few small exceptions.

Proposition (Point)

Let $\{X(K_i) | i \in I\}$ a familiy of finite groups of the same Lie type. Then for any non-principal ultrafilter $\mathcal{U}$ it holds that

$$\prod_{i \in I} X(K_i)/\mathcal{U} \cong X(\prod_{i \in I} K_i/\mathcal{U})$$

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The Ultrapower theorem (Shelah, Keisler)

Two $\mathcal{L}$-structures are elementarily equivalent if and only if they have isomorphic ultrapowers.

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Let X be a Lie type. There is an integer k_X such that each element of each finite group $X(K)$ is a product of at most k_X commutators.

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Proposition

There is an integer k such that each element of each finite non-abelian group G is a product of k commutators.

Lemma

The finite simple groups of a fixed Lie type are *boundedly simple*, i.e. if X is a Lie type, there is an integer c_X such that for each finite group $X(K)$ and elements $g, h \in X(K)$ the element g is a product of at most c_X conjugates of h.

The proof is analogous to the previous lemma.

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A group is pseudofinite and simple if and only if it is elementarily equivalent to a simple group of Lie type over a pseudofinite field.

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Remark (Ugurlu)

In the theorem, "simple" can be replaced by "definably simple of finite centraliser dimension".

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- $$G \equiv X(F) \equiv X(F)^* \cong X(F^*) \cong X(\bar{F}^*) \cong \prod_{j \in J} X(F_j)/\mathcal{U}$$
- $\Rightarrow$ G is pseudofinite.

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- $\Rightarrow$ G is pseudofinite.
- The $X(F_j)$ are boundedly simple, so $X(F)^*$ is boundedly simple and thus G is simple.

Proof "$\Rightarrow$"

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- Recall: If $G = \prod_{j \in J} G_j/\mathcal{U}$ is any ultraproduct, and $J = J_1 \sqcup \ldots \sqcup J_n$, then $G \cong \prod_{j \in J_i} G_j/\mathcal{U}_i$ for some i where $\mathcal{U}_i = \{X \cap J_i | X \in \mathcal{U}\}$.

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- $\Rightarrow$ Can limit ourselves to ultraproducts of finite simple groups all having the same category.

Proof "$\Rightarrow$"

Remark

If P is the set of all prime numbers and $\mathcal{U}$ is some non-principal ultrafilter, then

$$\prod_{p \in P} C_p/\mathcal{U} \equiv (\mathbb{Q}, +)$$

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The sentence has the form

$$\forall x \forall y : [(x \neq 1 \wedge C_G(x,y) \neq 1) \rightarrow \bigcap_{g \in G} (C_G(x,y) C_G(C_G(x,y)))^g \neq 1]$$

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Lemma

If G is a finite group such that soc_G is a non-abelian simple group, and every element of soc_G is a product of m commutators, then every element of $[G, G]$ is a product of $m+3$ commutators.

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- Let k be the integer such that all elements of finite simple groups are products of at most k commutators.
- If G_j is non-simple, then $[G_j, G_j]$ is a proper normal subgroup which as seen above consists of all products of $k+3$ commutators.
- Thus, $\mathcal{U}$-many G_j being non-simple contradicts simplicity of G, and we can disregard any lesser amount of non-simple G_j. The result follows.

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- Suppose $G \equiv \prod_{j \in J} Alt_{n_j}/\mathcal{U}$ and let d be some integer.
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- $Alt_n \models (\exists u, w : \phi_{Alt}(u, w)) \wedge \forall u, w : \phi_{Alt}(u, w) \Rightarrow \psi_d(u, w)$ for $n \geq 4d + 5$.

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- The n_j are unbounded (otherwise the ultraproduct would be finite), so G shows the above sentence.
- Since d can be chosen arbitrarily, there would be elements u, w in G such that w is not a product of conjugates of u, so G cannot be simple.

Proposition

If G is simple and elementarily equivalent to an ultraproduct of groups $Y_n(F_n)$, where each Y is one of $A, B, C, D, {}^2A, {}^2D$, then G is elementarily equivalent to such an ultraproduct in which the integers n are bounded.

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- Then he proceeds similar to the Alt_n case, finding a formula which holds in all groups of the concerned type and deriving the result.

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Corollary (Hrushovski)

Any simple pseudofinite group has supersimple finite rank theory.

Proof

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- By the bi-interpretability result of Ryten such a group is elementarily equivalent to one either interpretable in a pseudofinite field or in an ultraproduct of the the class $\mathcal{C}_{1,2,2}$ or $\mathcal{C}_{1,2,3}$.

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- By the theorem stated above, such ultraproducts are supersimple of SU rank 1.

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- Proof "$\Rightarrow$": Uses classification of minimal finite groups which are not solvable by Thompson.

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- By Łos, such a group satisfies the sentence above.
- However, it clearly is not solvable.

Definition

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Theorem

In finite groups the radical is $\emptyset$-definable.

Thank you for your attention!

Are there questions?