Pseudofinite-dimensional Schrödinger representations PRELIMINARY NOTES

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Abstract

Following Zilber [Zil16], we obtain the Schrödinger representation of the (3-dimensional) Heisenberg algebra on the tempered distributions as an ultralimit of finite-dimensional representations of certain subgroups of the Heisenberg group. Moreover, we see that the corresponding Weil representation can be obtained this way.

1 Introduction

In [Zil16], Zilber considers a certain ultralimit of finite-dimensional Hilbert spaces equipped with certain linear structure, and proposes it as a suitable replacement for the traditional functional analytic domains of quantum mechanics. Working in such a limit of finite-dimensional spaces allows one to skirt the analytic subtleties involved in traditional techniques. Zilber confirms that many calculations performed in his limit space agree with standard results.

Here, we give further confirmation that such an ultralimit of finite dimensional structures can replicate the functional analytic setup. Specifically, we show that the Schrödinger representation of the 3-dimensional Heisenberg algebra and the Weil representation of the double cover $\mathrm{Mp}_2(\mathbb{R})$ of $\mathrm{SL}_2(\mathbb{R})$ on the space $\mathcal{S}'$ of tempered distributions on $\mathbb{R}$ can be obtained as a certain ultralimit of finite dimensional subspaces of $\mathcal{S}'$ equipped with certain traces of these representations.

Although we will treat this entirely mathematically, to draw the connection with [Zil16], we briefly explain here the connection to quantum mechanics: the representation of the Heisenberg algebra provides the position and momentum observables Q and P for a particle on the line, and the Weil representation then yields the dynamics for a quadratic homogeneous Hamiltonian - such a Hamiltonian corresponds to an element of the lie algebra $\mathfrak{sl}_2(\mathbb{R})$ of $\mathrm{Mp}_2(\mathbb{R})$, and the corresponding 1-parameter subgroup gives the evolution in time of the system. These representations can be taken to be representations on the Hilbert space $L^2(\mathbb{R})$, but in the "rigged Hilbert space" presentation one extends them to representations on $\mathcal{S}'$, which is what we consider here.

In short, we argue as follows: the subspaces $\mathcal{S}'_m$ of m-periodic linear combinations of δ-functions supported on $\frac{1}{m}\mathbb{Z}$ have limit $\mathcal{S}'$ if one restricts to bounded sets; the Schrödinger representation of subgroups of the Heisenberg group restricts to $\mathcal{S}'_m$, and a formal differentiation process allows one to obtain the representation of the algebra in the limit; meanwhile, restrictions of rational points of Mp_2 restrict to partially defined operators on $\mathcal{S}'_m$ whose limits along sufficiently divisible ultrafilters are the correct totally defined operators on $\mathcal{S}'$, and the operators for real points can be approximated by the restrictions of the operators of approximate rational points. The rest of the paper provides the details.

The finite-dimensional structures we consider are essentially those considered by Zilber, but he takes quite a different ultralimit in order to have the inner product survive to the limit. Here we ignore the inner product, which can not be globally defined on $\mathcal{S}'$, and take a more simple-minded ultralimit. We leave a full understanding of the precise relation between the two constructions to future work.

2 Preliminaries

2.1 Schwartz space and tempered distributions

Let $\mathcal{S}$ be the Schwartz space of infinitely differentiable complex-valued functions on $\mathbb{R}$ with "rapidly decreasing absolute value", namely

$$\mathcal{S} = \{\phi(x) \in C^\infty(\mathbb{R}, \mathbb{C}) \mid \forall n, m.\ p_{n,m}(\phi) < \infty\}$$

where for $n, m \in \mathbb{N}$,

$$p_{n,m}(\phi) = \sup_x \mid x^n \delta^m \phi(x) \mid ,$$

and the topology on $\mathcal{S}$ is that defined by this collection of seminorms.

Recall also that the subspace $\mathcal{D} \subseteq \mathcal{S}$ of compactly supported smooth functions is dense in $\mathcal{S}$.

Let $\mathcal{S}'$ be the space of tempered distributions, the continuous dual space of $\mathcal{S}$. Write the corresponding sesquilinear pairing as $\langle T, \phi \rangle := T(\overline{\phi})$ for $T \in \mathcal{S}'$ and $\phi \in \mathcal{S}$. We have the seminorms

$$\|T\|_\phi := \mid \langle T, \phi \rangle \mid$$

for $\phi \in \mathcal{S}$. A subset B of $\mathcal{S}'$ is <u>bounded</u> iff each $\|\cdot\|_\phi$ is bounded on B. We consider bounded subsets with the topology induced from the $\|\cdot\|_\phi$. In fact this topology coincides with that induced from the strong topology on $\mathcal{S}'$, but we will not explicitly use or define the strong topology. Closed bounded subsets of $\mathcal{S}'$ are compact [Sch66, Chap.III,Thm.XII].

Recall [Rudin 7.12(d)] that every measurable function $f : \mathbb{R} \to \mathbb{C}$ of polynomial growth, i.e. bounded in absolute value by some polynomial, induces a tempered distribution

$$\langle f, \phi \rangle := \int_{\mathbb{R}} f(x)\overline{\phi}(x)dx.$$

In particular this induces embeddings of $\mathcal{S}$ and $\mathcal{D}$ into $\mathcal{S}'$, which are dense by [Sch66, Chap.III Thm.XV].

The space of distributions $\mathcal{D}'$ is the continuous dual of $\mathcal{D}$. We have topological embeddings

$$\mathcal{D} \subseteq \mathcal{S} \subseteq \mathcal{S}' \subseteq \mathcal{D}'.$$

Recall that a bilinear map $u : X \times Y \to Z$ of topological vector spaces is *hypocontinuous* if each $u(x, \cdot)$ and $u(\cdot, y)$ are continuous, and

$\{u(x,\cdot) \mid x \in B\}$ is equicontinuous for any bounded $B \subseteq X$, so in particular the restriction of the pairing to $B \times Y$ is continuous, and similarly for $B' \subseteq Y$ bounded.

By [Sch66, Chap.III,Thm.XI;Chap.VII,Sect.4], the pairings $\mathcal{D} \times \mathcal{D}' \to \mathbb{C}$ and $\mathcal{S} \times \mathcal{S}' \to \mathbb{C}$ are hypocontinuous.

Let $\mathcal{E}$ be the space of infinitely differentiable functions $\mathbb{R} \to \mathbb{C}$ with the topology of compact uniform convergence of all derivatives. By [Sch66, Chap.V,Thm.III], multiplication of functions and distributions is hypocontinuous as a function $\mathcal{E} \times \mathcal{D}' \to \mathcal{D}'$.

2.2 Metric logic structure on $\mathcal{S}'$

We view $\mathcal{S}'$ as the union of its closed bounded subsets (forming a "bornology"). $\mathcal{S}'$ is not covered by any countable set of bounded sets, nor is $\mathcal{S}'$ the union of a chain of bounded sets.

Each closed bounded $B \subseteq \mathcal{S}'$ is compact, and moreover is a complete metric space. In fact, a single metric suffices to metrise any B. Indeed, $\mathcal{S}$ is separable, so let $(\psi_i^0)_{i<\omega}$ be a countable topological basis. Concretely, we can take $\psi_i^0(x) := x^i e^{-x^2}$. Then define on $\mathcal{S}'$ the metric

$$d(T,T') := \sum_i 2^{-i} \min(1, |\left\langle T-T', \psi_i^0 \right\rangle|).$$

This does not metrise any relevant topology on $\mathcal{S}'$ itself, but it does metrise the topology on any bounded subset:

Lemma 2.1. *Let $B \subseteq \mathcal{S}'$ be closed and bounded. Then the restriction of d to B is a complete metric on B, metrising the topology of B.*

Proof. The topology on B is induced by the pseudonorms $\|\cdot\|_\phi$ and is complete with respect to them, so it suffices to see that for any $\phi \in \mathcal{S}$ and any ϵ, there exists δ such that for $T, T' \in B$,

$$d(T,T') < \delta \Rightarrow |\left\langle T-T', \phi \right\rangle| < \epsilon.$$

But indeed, the pairing is hypocontinuous, so $\langle T, . \rangle$ is equicontinuous on $B - B$; hence for a good enough approximation to ϕ as a finite linear combination ϕ_0 of the ψ_i^0, $|\langle T-T', \phi-\phi_0\rangle| < \epsilon/2$ for all $T, T' \in B$. Then take δ such that

$$d(T,T') < \delta \Rightarrow |\left\langle T-T', \phi_0 \right\rangle| < \epsilon/2.$$

□

Compact metric spaces are the analogue in continuous logic of finite structures in discrete logic, and most notions from continuous logic degenerate in such spaces. In particular, the ultraproduct of subspaces may be identified with the topological ultralimit, as we now explain.

Consider a set X and a set $\mathcal{B}$ of subsets which covers X and is closed under finite unions, and suppose d is a metric on X such that equipping each $B \in \mathcal{B}$ with the restriction of d makes it a compact metric space. The example we have in mind is $X = \mathcal{S}'$, with $\mathcal{B}$ the set of its closed bounded subsets, and d defined as above.

Call $Y \subseteq X$ <u>bounded</u> if $Y \subseteq B$ for some $B \in \mathcal{B}$. For $Y_i \subseteq B \in \mathcal{B}$ and an ultrafilter $\mathcal{U}$, recall $\lim_{i\to\mathcal{U}} Y_i = \{\lim_{i\to\mathcal{U}} y_i \mid y_i \in Y_i\} \subseteq B$. Now for $Y_i \subseteq X$, define $\lim^{\mathcal{B}}_{i\to\mathcal{U}} Y_i = \bigcup_{B\in\mathcal{B}} \lim_{i\to\mathcal{U}}(Y_i \cap B)$.

For $X = \mathcal{S}'$, this notion of "bounded ultralimit" does not agree with the usual topological ultralimit for even the strong topology on X, since unbounded ultrafilters in X can converge; for our purposes such convergence is pathological, so we restrict to bounded sets.

Now suppose $Y_i \subseteq X$ are subsets such that $\lim^{\mathcal{B}}_{i\to\mathcal{U}} Y_i = X$. If the Y_i are equipped with relations $R^{Y_i} \subseteq Y_i^n$, define the induced relation on X by $R^X := \lim^{\mathcal{B}}_{i\to\mathcal{U}} R^{Y_i} \subseteq X^n$. Then this notion of bounded ultralimit of structures agrees with the notion of ultraproduct of metric structures, in the following sense. Consider (Y_i, R^{Y_i}) as a metric structure in the language with a sort for each $B \in \mathcal{B}$, interpreted in Y_i as $Y_i \cap B$, and a real-valued metric predicate R'_B on each sort, interpreted in Y_i as the distance predicate $R'_B(x) = d(x, R^{Y_i} \cap B^n) = \inf_{y\in Y_i\cap B^n} d(x,y)$ (where we define d on X^n by $d(x,y) = \max_i d(x_i, y_i)$, say). Then the metric ultraproduct is isomorphic to (X, R^X) considered as a multisorted metric structure in the same way.

We apply this to (partial) operators $X \to X$ by considering their graphs, so we define $F^X(\lim_{i\to\mathcal{U}} y_i) := \lim_{i\to\mathcal{U}} F^{Y_i}(y_i)$ whenever $\{y_i\}_i$ is bounded, $F^{Y_i}(y_i)$ are defined, and $\{F^{Y_i}(y_i)\}_i$ is bounded, if this definition yields a well-defined partial operator.

Say f^{Y_i} are <u>provably bounded</u> if $\{F^{Y_i}(y_i)\}_i$ is bounded whenever $\{y_i\}_i$ is bounded and the $F^{Y_i}(y_i)$ are defined. Note that the composition of provably bounded operators is provably bounded.

Metric logic admits a Łos theorem, but the only consequence of this we will need is the following weak statement for exact equality of compositions of operators, which follows directly from the above definitions.

Lemma 2.2. *Suppose $F_j^{Y_i}$ are partial operators, and $F_1^{Y_i} \circ F_2^{Y_i} = F_3^{Y_i}$ holds $\mathcal{U}$-often, and suppose that the ultralimits F_1^X and F_2^X are well-defined partial operators. Suppose also that the $F_2^{Y_i}$ are provably bounded. Then F_3^X is a well-defined partial operator, and $F_1^X \circ F_2^X = F_3^X$.*

Proof. Restricting the index set, we may assume the equality holds for all i. Let $\{y_i\}_i$ be bounded with $y_i \in \mathrm{dom}(F_3^{Y_i})$. Then $\{F_2^{Y_i}(y_i)\}_i$ is bounded by provable boundedness, so $\lim_{i\to\mathcal{U}} F_1^{Y_i}(F_2^{Y_i}(y_i)) = F_1^X(\lim_{i\to\mathcal{U}} F_2^{Y_i}(y_i)) = F_1^X(F_2^X(\lim_{i\to\mathcal{U}} y_i))$. □

2.3 Schrödinger representations

The 3-dimensional real Heisenberg Lie algebra η is the central extension of the trivial Lie algebra on $\mathbb{R}^2$ via the canonical symplectic structure on $\mathbb{R}^2$; it can be defined by saying it has basis (P, Q, E), with E being central and $[Q, P] = E$. The infinitesimal Schrödinger representation (with respect to Q) of η on the Schwartz space $\mathcal{S} \subseteq L^2(\mathbb{R})$ is given by $P\phi(x) = -\phi'(x)$, $Q\phi(x) = 2\pi i x\phi(x)$, $E\phi(x) = 2\pi i\phi(x)$. The dual representation on $\mathcal{S}'$ is defined by $\langle XT, \phi\rangle := \langle T, -X\phi\rangle$; note that this is consistent with the embedding $\mathcal{S} \subseteq \mathcal{S}'$, since η acts (formally) skew-adjointly.

The corresponding real Heisenberg Lie group N is the corresponding central extension of $\mathbb{R}^2$, and can be defined as triples (u, v, e) with group structure $(u, v, e) * (u', v', e') = (u + u', v + v', e + e' + (uv' - u'v)/2)$. The corresponding exponential map is then $\exp(pP + qQ + tE) = (p, q, t)$. We have $[\exp(X), \exp(Y)] = \exp([X, Y])$. Let $\chi : \mathbb{R} \to \mathbb{C}$ be the character $\chi(x) := e^{2\pi i x}$. The corresponding Schrödinger unitary representation of

N on $\mathcal{S}$ (and even on $L^2(\mathbb{R})$) is given by

$$\exp(pP)f(x) = f(x-p)$$
$$\exp(qQ)f(x) = \chi(qx)f(x)$$
$$\exp(tE)f(x) = \chi(t)f(x).$$

This can be considered as the exponential of the infinitesimal representation of η defined above; in particular, $\frac{d}{dt}\exp(tX)f(x)|_{t=0} = Xf(x)$ for $X \in \eta$ and $f \in \mathcal{S}$. The dual representation on $\mathcal{S}'$ can be defined by $\langle XT, g\rangle := \langle T, X^{-1}g\rangle$.

3 Finite-dimensional approximations

3.1 Cyclic Schrödinger representations

Let N_m be the finite subgroup of N generated by $U_m := \exp(\frac{Q}{m})$ and $V_m := \exp(\frac{P}{m})$. This group is presented by the relations $[U_m,[U_m,V_m]] = 1 = [V_m,[U_m,V_m]]$ expressing that $[U_m,V_m]$ is central.

Let $C^*_{m^2}$ be the space of complex-valued functions on $C_{m^2} := \mathbb{Z}/m^2\mathbb{Z}$. We identify $C^*_{m^2}$ with the space of m^2-periodic functions on $\mathbb{Z}$ in the obvious way. Define a finite dimensional "cyclic" unitary representation of N_m on $C^*_{m^2}$ by

$$U_m f(k) := q^k f(k)$$
$$V_m f(k) := f(k-1),$$

where $q := \chi(\frac{1}{m^2})$. So $[U_m,V_m]$ acts centrally as multiplication by q.

We will show that the infinitesimal Schrödinger representation of η can be obtained as an ultralimit of these $C^*_{m^2}$.

We identify $C^*_{m^2}$ with the subspace $\mathcal{S}'_m$ of $\mathcal{S}'$ of m-periodic linear combinations of δ functions supported at $\frac{1}{m}\mathbb{Z}$, via the embedding θ_m defined by

$$\theta_m(f) := \sum_{k\in\mathbb{Z}} f(k)\delta_{\frac{k}{m}}.$$

This is easily verified to be an embedding of the representation of N_m on $C^*_{m^2}$ into the Schrödinger representation of $N_m \le N$ on $\mathcal{S}'$ defined in the previous section.

Define the discrete Fourier transform on $C^*_{m^2}$ by

$$\mathcal{F}_m(f)(p) = \frac{1}{m}\sum_{0\le k<m^2} f(k)\chi(\frac{pk}{m^2}).$$

Meanwhile, let $\mathcal{F}$ be the Fourier transform on $\mathcal{S}'$ defined by $\langle \mathcal{F}T, \mathcal{F}\phi\rangle = \langle T,\phi\rangle$, where $\mathcal{F}$ on $\mathcal{S}$ is the usual Fourier transform defined by $(\mathcal{F}f)(p) = \int_{\mathbb{R}} f(x)\chi(px)dx$.

Lemma 3.1. *$\mathcal{F}_m$ agrees with the restriction of $\mathcal{F}$ to $\mathcal{S}'_m$; i.e. $\mathcal{F}(\theta_m(f)) = \theta_m(\mathcal{F}_m f)$.*

Proof. Let $\text{III}_m := \sum_{l\in\mathbb{Z}}\delta_{ml}$ be the Dirac comb, consider the shifts by convolutions with δ functions $\delta_{\frac{k}{m}} * \text{III}_m = \sum_{l\in\mathbb{Z}}\delta_{ml+\frac{k}{m}}$, and note that

$$\theta_m(f) = \sum_{0\le k<m^2} f(k)(\delta_{\frac{k}{m}} * \text{III}_m).$$

Meanwhile, $\mathcal{F}\,\mathrm{III}_1 = \mathrm{III}_1$ by Poisson summation, and similarly $\mathcal{F}\,\mathrm{III}_m = \frac{1}{m}\,\mathrm{III}_{\frac{1}{m}}$. Hence

$$\begin{aligned}
\mathcal{F}\theta_m(f) &= \sum_{0\le k<m^2} f(k)\mathcal{F}(\delta_{\frac{k}{m}} * \mathrm{III}_m) \\
&= \sum_{0\le k<m^2} f(k)\left(\left(\mathcal{F}\delta_{\frac{k}{m}}\right)\cdot(\mathcal{F}\,\mathrm{III}_m)\right) \\
&= \sum_{0\le k<m^2} f(k)\left(\left(\chi\circ(\frac{k}{m}\cdot)\right)\cdot\left(\frac{1}{m}\,\mathrm{III}_{\frac{1}{m}}\right)\right) \\
&= \frac{1}{m}\sum_{0\le k<m^2} f(k)\left(\left(\chi\circ(\frac{k}{m}\cdot)\right)\cdot\left(\sum_{0\le l<m^2}(\delta_{\frac{l}{m}} * \mathrm{III}_m)\right)\right) \\
&= \sum_{0\le l<m^2}\frac{1}{m}\sum_{0\le k<m^2} f(k)\left(\left(\chi\circ(\frac{k}{m}\cdot)\right)\cdot\left(\delta_{\frac{l}{m}} * \mathrm{III}_m\right)\right) \\
&= \sum_{0\le l<m^2}\frac{1}{m}\sum_{0\le k<m^2} f(k)\left(\chi(\frac{lk}{m^2})\cdot\left(\delta_{\frac{l}{m}} * \mathrm{III}_m\right)\right) \\
&= \sum_{0\le l<m^2}\mathcal{F}_m(f)(l)\cdot\left(\delta_{\frac{l}{m}} * \mathrm{III}_m\right) \\
&= \theta_m(\mathcal{F}_m f).
\end{aligned}$$

□

Define on $C^*_{m^2}$ approximations to the derivatives of $\exp(qQ)$ and $\exp(pP)$:

$$Q_m := \frac{m}{2}(U_m - U_m^{-1}),$$

$$P_m := \frac{m}{2}(V_m - V_m^{-1}).$$

Meanwhile, consider P and Q as operators on $\mathcal{S}'$ via the infinitesimal Schrödinger representation defined above.

Lemma 3.2. *For any non-principal ultrafilter $\mathcal{U}$, identifying $C^*_{m^2}$ with $\mathcal{S}'_m$ via θ_m,*

$$\lim^{\mathcal{B}}_{m\to\mathcal{U}}(C_{m^2}, P_m, Q_m, \mathcal{F}_m) = (\mathcal{S}', P, Q, \mathcal{F}).$$

Proof.

- We first show that $\lim^{\mathcal{B}}_{m\to\mathcal{U}}\mathcal{S}'_m = \mathcal{S}'$. So let $T \in \mathcal{S}'$. By [Sch66, Chap.VI Sect.4], T is the limit of a sequence of smooth functions with compact support, say $T = \lim_{m\to\infty} g_m$ with $g_m \in \mathcal{D} \subseteq \mathcal{S} \subseteq \mathcal{S}'$. Define m-periodic approximations $T_m \in \mathcal{S}'_m$ to g_m by sampling around 0, $T_m(\frac{k}{m}) := \frac{1}{m}g_m(\frac{k}{m})$ for $\left\lfloor\frac{-m^2}{2}\right\rfloor \le k < \left\lfloor\frac{m^2}{2}\right\rfloor$. Then for any $\phi \in \mathcal{S}$, we have $\langle g_m - T_m, \phi\rangle \to_{m\to\infty} 0$. So $\lim_{m\to\infty} T_m = T$. Hence $\{T_m\}_m$ is bounded, and the ultralimit along any non-principal ultrafilter is T.
- Next, we verify that $\lim^{\mathcal{B}}_{m\to\mathcal{U}}\mathcal{F}_m = \mathcal{F}$. $\mathcal{F}$ is provably bounded and is uniformly continuous on $\mathcal{S}'$, since $\|\mathcal{F}T\|_\phi = \|T\|_{\mathcal{F}^{-1}\phi}$. So if $T_m \in \mathcal{S}'_m$ are bounded, so are $\mathcal{F}T_m$, and by Lemma 3.1, $\lim_{m\to\mathcal{U}}\mathcal{F}_m T_m = \lim_{m\to\mathcal{U}}\mathcal{F}T_m = \mathcal{F}\lim_{m\to\mathcal{U}} T_m$.

- To show $\lim^{\mathcal{B}}_{m\to\mathcal{U}} Q_m = Q$, we must show that if $T = \lim_{m\to\mathcal{U}} T_m$ with $\{T_m\}_m$ bounded, then $\{Q_m T_m\}_m$ is bounded and $\lim_{m\to\mathcal{U}} Q_m T_m = (2\pi ix)\cdot T$, i.e. for any $\phi = \phi(x) \in \mathcal{S}$,

$$\lim_{m\to\mathcal{U}} \langle Q_m T_m, \phi\rangle = \lim_{m\to\mathcal{U}} \langle T_m, -2\pi ix\phi\rangle .$$

  From the definition of Q_m, we have $\langle Q_m T_m, \phi\rangle = \langle T_m, -mi\sin(2\pi x/m)\phi\rangle$. Now $\lim_{m\to\infty} -mi\sin(2\pi x/m)\phi = -2\pi ix\phi$ in $\mathcal{S}$, since $\lim_{m\to\infty} -mi\sin(2\pi x/m) = -2\pi ix$ in $\mathcal{E}$ and multiplication is hypocontinuous. So by hypocontinuity of the pairing, $\lim_{m\to\infty}(\langle Q_m T_m, \phi\rangle - \langle T_m, -2\pi ix\phi\rangle) = 0$. This implies provable boundedness and that the ultralimit is as required.

- We verify $\lim^{\mathcal{B}}_{m\to\mathcal{U}} P_m = P$ by appeal to Lemma 2.2. One can check by direct calculation that on each $\mathcal{S}'_m$ we have $V_m = \mathcal{F}_m^{-1}\circ U_m \circ \mathcal{F}_m$, and hence by linearity $P_m = (\mathcal{F}_m^{-1}\circ Q_m\circ\mathcal{F}_m)$.
  Meanwhile, on $\mathcal{S}'$ we have $\frac{d}{\mathrm{dx}} = \mathcal{F}^{-1}\circ(-2\pi ix\cdot)\circ\mathcal{F}$.
  We have already verified provable boundedness of the $\mathcal{F}_m$ and Q_m, so we conclude by Lemma 2.2.

□

3.2 Weil representations

$\mathrm{SL}_2(\mathbb{R}) \cong \mathrm{Sp}(2,\mathbb{R})$ acts by automorphisms on η and on N, acting trivially on the centre and via the canonical action on $\mathbb{R}^2$. We will write the action in exponential notation, X^g.

Via the Stone-von Neumann theorem, this induces a projective unitary representation on S, which turns out to yield a "metaplectic" unitary representation of the double cover $\mathrm{Mp}_2(\mathbb{R})$ of $\mathrm{SL}_2(\mathbb{R})$ on $\mathcal{S}$, such that $X^g(\phi) = (\widetilde{g}\circ X\circ\widetilde{g}^{-1})(\phi)$ for $\widetilde{g}\in\mathrm{Mp}_2(\mathbb{R})$, g the image in $\mathrm{SL}_2(\mathbb{R})$, $X\in\eta$, and $\phi\in\mathcal{S}$. The dual representation on $\mathcal{S}'$ is then defined by demanding formal unitarity, $\langle\widetilde{g}T,\widetilde{g}\phi\rangle := \langle T,\phi\rangle$.

$\mathrm{SL}_2(\mathbb{R})$ is generated by $F := \begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}$ and $G^b := \begin{bmatrix}1 & b\\ 0 & 1\end{bmatrix}$ for $b\in\mathbb{R}$, so to define the metaplectic representation it suffices to define it on the lifts of these generators to $\mathrm{Mp}_2(\mathbb{R})$. Following [Fol89, p.179], we have for $\phi\in\mathcal{S}$

$$\widetilde{F}\phi(x) = \pm\chi(\frac{1}{8})\mathcal{F}\phi(x)$$
$$\widetilde{G}^b\phi(x) = \pm\chi(-b\frac{x^2}{2})\cdot\phi(x),$$

where the signs depend on the choice of lift.

We have seen that $\mathcal{F}$, and hence the action of $\widetilde{F}$ on $\mathcal{S}'$, restricts to an operator on each $\mathcal{S}'_m$. The action of $\widetilde{G}^b$ on $\mathcal{S}'$ is by multiplication by $\pm\chi(b\frac{x^2}{2})$. For $b = \frac{c}{d}$ a rational number expressed in lowest terms, this restricts via θ_m to a partial operator $\widetilde{G}^b_m$ on $\mathcal{S}'_m$ if d divides m^2, with domain $D_{m,b}$ consisting of m-periodic linear combinations of δ functions supported on $\frac{d}{m}\mathbb{Z}$.

In terms of $C^*_{m^2}$, $\widetilde{G}^b_m$ acts by $\widetilde{G}^b_m(f)(k) = \pm\chi(b\frac{k^2}{2m^2})\cdot f(k)$, defined on $d\mathbb{Z}/m^2\mathbb{Z}\subseteq\mathbb{Z}/m^2\mathbb{Z}$.

Call an ultrafilter $\mathcal{U}$ on $\mathbb{N}$ ultradivisible if for all $d\in\mathbb{N}$, $d\mathbb{N}\in\mathcal{U}$.

Lemma 3.3. *For $\mathcal{U}$ ultradivisible, $\lim^{\mathcal{B}}_{m\to\mathcal{U}} D_{m,b} = \mathcal{S}'$.*

Proof. If $d|m$,

$$D_{m,d} \supseteq \mathcal{S}'_{m/d},$$

so the result follows from ultradivisibility and Lemma 3.2. □

It follows fairly immediately that for $b \in \mathbb{Q}$ and $\mathcal{U}$ ultradivisible, $\lim^{\mathcal{B}}_{m\to\mathcal{U}} \widetilde{G}^b_m = \widetilde{G}^b$. But for $b \in \mathbb{R} \setminus \mathbb{Q}$, simply restricting to $\mathcal{S}'_m$ does not yield a nontrivial partial operator. Rather, we must simultaneously approximate b by rationals.

So given $m \in \mathbb{N}$, write $m = e^2 f$ where $e \in \mathbb{N}$ and f is a squarefree integer, and define $b_m := \frac{\lfloor be \rfloor}{e}$. So for $\mathcal{U}$ ultradivisible, $\lim_{m\to\mathcal{U}} b_m = b$. Then define $\widetilde{G}^b_m := \widetilde{G}^{b_m}_m$.

Lemma 3.4. *Let $b \in \mathbb{R}$. Then the $\widetilde{G}^b_m$ are provably bounded on the $\mathcal{S}'_m$, and for $\mathcal{U}$ ultradivisible, $\lim^{\mathcal{B}}_{m\to\mathcal{U}} \widetilde{G}^{b_m} = \widetilde{G}^b$.*

Proof. First we prove provable boundedness. So suppose $T_m \in D_{m,b_m}$ are bounded, let $\phi \in \mathcal{S}$, and let $\epsilon > 0$. Then by hypocontinuity of the pairing, for all but finitely many values of b_m,

$$\begin{aligned}
\left|\left\langle \widetilde{G}^{b_m} T_m, \phi \right\rangle - \left\langle T_m, \widetilde{G}^{-b}\phi \right\rangle\right| &= \left|\left\langle T_m, \widetilde{G}^{-b_m}\phi - \widetilde{G}^{-b}\phi \right\rangle\right| \\
&= \left|\left\langle T_m, \pm\chi((b - b_m)\frac{x^2}{2})\phi(x) \right\rangle\right| \\
&< \epsilon.
\end{aligned}$$

So since $\left\langle T_m, \widetilde{G}^{-b}\phi \right\rangle$ is bounded, also $\left\langle \widetilde{G}^{b_m} T_m, \phi \right\rangle$ is bounded.

Now we consider the ultralimit.

If $e^2|m$, we have $D_{m,b_m} = \mathcal{S}'_e$. So as in Lemma 3.3, $\lim^{\mathcal{B}}_{m\to\mathcal{U}} D_{m,b_m} = \mathcal{S}'$.

So let $T \in \mathcal{S}'$; then we may write $T = \lim_{m\to\mathcal{U}} T_m$ with $T_m \in D_{m,b_m}$ and with the T_m bounded.

Now since $\lim_{m\to\mathcal{U}} b_m = b$, $\lim_{m\to\mathcal{U}} \chi(b_m \frac{x^2}{2}) = \chi(b\frac{x^2}{2})$ in $\mathcal{E}$.

Then by hypocontinuity of multiplication,

$$\lim_{m\to\mathcal{U}} \widetilde{G}^b_m T_m = \lim_{m\to\mathcal{U}} \pm\chi(b_m \frac{x^2}{2}) \cdot T_m = \pm\chi(b\frac{x^2}{2}) \cdot T = \widetilde{G}^b T,$$

as required. □

We have already seen the analogous result for $\mathcal{F}$, and hence for $\widetilde{F}$. Now for $\widetilde{h} \in \mathrm{Mp}_2(\mathbb{R})$, write $\widetilde{h}$ as a word in the generators $\widetilde{h} = w(\widetilde{F}, \widetilde{G}^{b_1}, \ldots, \widetilde{G}^{b_n})$ and define $\widetilde{h}_m$ as the corresponding word $\widetilde{h}_m = w(\widetilde{F}_m, \widetilde{G}^{b_1}_m, \ldots, \widetilde{G}^{b_n}_m)$. Then by the provable boundedness and Lemma 2.2, $\lim^{\mathcal{B}}_{m\to\mathcal{U}} \widetilde{h}_m = \widetilde{h}$. Adding this to Lemma 3.2, we conclude:

Theorem 3.5. *For $\mathcal{U}$ ultradivisible,*

$$\lim^{\mathcal{B}}_{m\to\mathcal{U}}(\mathcal{S}'_m, Q_m, P_m, (\widetilde{h}_m)_{h\in\mathrm{Mp}_2(\mathbb{R})}) = (\mathcal{S}', Q, P, (\widetilde{h})_{\widetilde{h}\in\mathrm{Mp}_2(\mathbb{R})}),$$

where the latter structure encodes the standard Schrödinger and Weil representations on $\mathcal{S}'$ defined above.

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