Definably Amenable Approximate Subgroups

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Contents

The note was written for a talk given in the seminar course *Topological Dynamics and Model Theory* at University of Münster. It is based on [MW15].

Background

To solve the Hilbert's fifth problem, Gleason and Yamabe proved the following theorem:

Theorem 1.1 (Gleason-Yamabe)**.** Let G be a locally compact group. Then there is an open subgroup G' of G such that for every open neighborhood U of the identity of G', there is a compact open subgroup $K' \trianglelefteq G'$ such that G/K' is isomorphic to a Lie group.

Inspired by this theorem, Breuillard, Green and Tao classfied the finite approximate subgroups of local groups:

Theorem 1.2. Let A be a K-approximate subgroup. Then there is an approximate subgroup A^* and an A^*-invariant subgroup H^* such that

- finitely many left translates of A^* cover A,
- $\langle A^* \rangle / H^*$ is nilpotent.

The existence of $\langle A^* \rangle / H^*$ as a Lie model was firstly shown by Hrushovski, which involves additional parameters when defining the Lie model. Our goal of this talk is to show Massicot and Wagner's result in [MW15] that this can be done without additional parameters:

Theorem 1.3. In a local group G, a definable amenable approximate subgroup A gives rise

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to a type-definable subgroup $H \subseteq A^4$ such that finitely many left translates of H cover A.

The proof essentially relies on techniques from Sander's theorem.

The main part

Definition 2.1. (1) (Local groups) A set closed under inverse and endowed with an associative multiplication operation defined for up to 100 elements is called a local group.

(2) (Symmetric sets) A subset A of a (local) group G is *symmetric*, if $1 \in A$ and A is closed under inverse.

(3) (Approximate subgroups) A symmetric subset A of G is a K-*approximate subgroup* for some $K < \omega$, if $A^2 := \{aa' \mid a, a' \in A\}$ is contained in K left cosets of A. A is called an *approximate subgroup* if it is a K-approximate subgroup for some $K < \omega$.

(4) (Definably amenable subgroups) A definable approximate subgroup A is *definably amenable*, if there is a left translate-invariant finitely additive measure μ on the definable subsets of $\langle A \rangle$ with $\mu(A) = 1$.

(5) (Wide definable sets) A definable subset $B \subseteq \langle A \rangle$ is *wide* in A, if A is covered by finitely many left translates of B.

(6) (Equivalent approximate subgroups) Two approximate subgroups are called *equivalent*, if each one is wide in the other.

Remark 2.2. If A is a K-approximate subgroup of a (local) group G, then there is some $E \subseteq G$ of cardinality less than K such that $A^2 \subseteq EA$. Inductively, $A^n \subseteq E^{n-1}A$ for all $n < \omega$.

From now on, we assume that:

- The ambient structure $\mathcal{G}$ is ω^+-saturated and has domain G as a (local) group.
- A is definably amenable with a fixed measure μ witnessing the definable amenability.
- A is a K-approximate subgroup of G, so there exists a finite set $E \subseteq G$ with $|E| \leqslant K$ such that $A^2 \subseteq EA$.

We will repeatedly apply ω^+-saturation in this section. In fact, the consequences of ω^+-saturation are the only "model theory" that will be used.

Remark 2.3. Let G be a (local) group.

(1) Assume that A is a definable symmetric subset of G. Every definable subset $X \subseteq \langle A \rangle$ is contained in A^n for some $n < \omega$.

(2) Assume that $(A_i)_{i<\omega}$ is a sequence of definable subsets of G. Then $\bigcap_{i<\omega} A_i$ is nonempty, as long as $\bigcap_{i \in I_0} A_i$ is nonempty for any finite $I_0 \subseteq \omega$.

Proof. (1) Since A is definable, A^n is definable for any $n < \omega$. Then the type defining $x \notin X \backslash \langle A \rangle = X \backslash (\bigcup_{n<\omega} A^n)$ is not realizable, so by ω^+-saturation, the type is inconsistent, which implies that $X \subseteq A^{n_1} \cup \cdots \cup A^{n_k}$ for some $k < \omega$. Note that $A^i \subseteq A^j$ for $i < j$, since $1 \in A$. Hence X is contained in A^n for some $n < \omega$.

(2) Since $\bigcap_{i\in I_0} A_i$ is nonempty, for any finite $I_0 \subseteq \omega$, $\bigcap_{i<\omega} A_i$ is consistent by compactness. Then by ω^+-saturation, $\bigcap_{i<\omega} A_i$ must be realizable, i.e. nonempty. $\square$

Stated from the model-theoretic point of view, under the ω^+-saturation assumption, a definable approximate subgroup is just a symmetric *generic* subset of $\langle A\rangle$, i.e. every definable subset of $\langle A\rangle$ is covered by finitely many left translates of A.

Remark 2.4. Assume that X is a definable subset of $\langle A\rangle$. By Remark 2.3, $X \subseteq A^n$ for some $n < \omega$. Since A is K-approximate for some $K < \omega$,

$$\mu(X) \leqslant \mu(A^n) \leqslant K^{n-1}\mu(A) < \infty.$$

The second inequality follows from Remark 2.2.

Remark 2.5. If $\lim_{n\to\infty} \mu(A^n) < \infty$, then there is some $n < \omega$ with $A^n = \langle A\rangle$.

Proof. Suppose not. Then we can take $a_n \in A^{n+1}\backslash A^n$ for any $n < \omega$. In fact, $a_nA \cap a_mA = \varnothing$ for $n < m-2$. Assume that $x \in a_nA \cap a_mA$. Then $x = a_na = a_ma'$ for a, a' in A, so $a_m = a_naa'^{-1} \in a_nA^2 \subseteq A^{n+2} \subseteq A^{m-1}$, contradicted to that $a_m \in A^m\backslash A^{m-1}$. Fix $n < \omega$. Then $(a_{3k}A \mid k \leqslant n)$ is a sequence of disjoint left translates of A inside A^{3n+2}. We have

$$\mu(A^{3n+2}) \geqslant \mu(\bigcup_{k\leqslant n} a_{3k}A) = \sum_{k\leqslant n}\mu(a_{3k}A) = \sum_{k\leqslant n}\mu(A) \overset{\mu(A)=1}{=} n+1.$$

Since n is arbitrary here, $\lim_{n\to\infty}\mu(A^n) = \infty$, contradiction. $\square$

By Remark 2.5, we can assume that $\lim_{n\to\infty}\mu(A^n) = \infty$.

Fact 2.6 (Ruzsa's covering lemma)**.** Let $X, Y \subseteq G$ be definable such that $\mu(XY) \leqslant K\mu(Y)$. Then $X \subseteq ZYY^{-1}$ for some finite set $Z \subseteq X$ with $|Z| \leqslant K$.

Proof. Assume WLOG that $X \neq \varnothing$, since otherwise the conclusion is trivial. Lt Z be a finite subset $Z \subseteq X$ such that $zY \cap z'Y = \varnothing$ for all $z \neq z' \in Z$. Then

$$|Z|\mu(Y) = \mu(ZY) \leqslant \mu(XY) \leqslant K\mu(Y),$$

so $|Z| \leqslant K$. Hence, we can choose Z to be a <u>maximal</u> such set. For $x \in Z$, clearly, $x \in Z \subseteq ZYY^{-1}$. For $x \in X\backslash Z$, there exists $z \in Z$ such that $zY \cap xY \neq \varnothing$, otherwise it contradicts the maximality of Z. Then $x \in ZYY^{-1}$. Therefore, $X \subseteq ZYY^{-1}$. $\square$

Lemma 2.7. Let $B \subseteq \langle A\rangle$ be definable.

(1) If $\mu(B) > 0$, then BB^{-1} is wide in A and symmetric.

(2) If B is wide in A and symmetric, then B is an approximate subgroup equivalent to A.

Proof. (1) Clearly, BB^{-1} is symmetric. Since AB is a definable subset of $\langle A\rangle$, $\mu(AB) < \infty$ by Remark 2.4. Since B is a definable subset of $\langle A\rangle$, $\mu(B) < \infty$. Hence there exists some $L > 0$ such that $\mu(AB) \leqslant L\mu(B)$. By Fact 2.6, there exists some finite set $Z \subseteq A$ with $|Z| \leqslant L$

such that $A \subseteq ZBB^{-1}$, so BB^{-1} is wide in A.

(2) Assume that B is wide in A and symmetric. By (1) in Remark 2.3, there exists some $n < \omega$ such that $B \subseteq A^n$, so $B^2 \subseteq A^{2n} \subseteq E^{2n-1}A$. Since B is wide in $\langle A \rangle$, there exists some finite set $Y \subseteq \langle A \rangle$ such that $A \subseteq YB$, so $B^2 \subseteq A^2 \subseteq E^{2n-1}A \subseteq E^{2n-1}YB$, where $E^{2n-1}Y$ is a finite set. Hence B is an approximate subgroup. Moreover, since $B \subseteq A^n \subseteq E^{n-1}A$, A is wide in B. Therefore, A and B are equivalent. □

Lemma 2.8 (Sander)**.** Let $f : (0,1] \to [1,K]$ and $\epsilon > 0$. Then there exists $n < \omega$ depending on K, ϵ, and $t > \frac{1}{(2K)^{2^n-1}}$ such that $f(\frac{1}{(2K)^{2^n-1}}) \geqslant (1-\epsilon)f(t)$.

Proof. Assume WLOG that $0 < \epsilon < 1$, since otherwise $(1-\epsilon)f(t) \leqslant 0$, so the inequality holds trivially. Construct $(t_n) \subseteq (0,1]$ inductively by letting $t_0 = 1$ and $t_{n+1} = \frac{t_n^2}{2K}$ for $n < \omega$. Then $t_n = \frac{1}{(2K)^{2^n-1}}$ for all $n < \omega$. For a fixed $n < \omega$, if for all $i < n$, $f(t_{i+1}) < (1-\epsilon)f(t_i)$, then $f(t_n) < (1-\epsilon)^n f(t_0) \leqslant (1-\epsilon)^n K$. But since $f(t_n) \geqslant 1$, $K \geqslant 1$, there exists some $n < \omega$ such that $(1-\epsilon)^n K < 1 \leqslant f(t_n)$, which means that there exists some $i < n$ such that $f(t_{i+1}) \geqslant (1-\epsilon)f(t_i)$. Since $t_i = \frac{1}{(2K)^{2^i-1}} > \frac{1}{(2K)^{2^n-1}}$ by $i < n$, t_i is the desired t in the conclusion. □

Theorem 2.9. For every $m < \omega$, there is a definable L-wide (in A) approximate subgroup S with $S^m \subseteq A^4$, where L depends only on K and m.

Proof. Recall that A is a K-approximate subgroup of G. Firstly we prove this claim.

Claim 2.10. If $B \subseteq A$ is definable with $\mu(B) \geqslant t\mu(A)$ for some $0 < t \leqslant 1$ and $s = \frac{t}{2K}$, then A is covered by $N := \lfloor \frac{1}{s} \rfloor$ many translates of $X = \{g \in A^2 \mid \mu(gB \cap B) \geqslant st\mu(A)\}$ by elements of A.

Proof. Suppose not. We can find $(g_i : i \leqslant N) \subseteq A$ such that $g_j \in (A \setminus \bigcup_{i<j} g_i X)$, so for all $j \leqslant N$ and $i < j$, $g_j \notin g_i X$ implies $\mu(g_i B \cap g_j B) < st\mu(A)$. Then

$$
\begin{aligned}
K\mu(A) \geqslant \mu(A^2) &\geqslant \mu(\bigcup_{i \leqslant N} g_i B) \\
&\geqslant (N+1)\mu(B) - \sum_{i<j\leqslant N} \mu(g_i B \cap g_j B) \\
&> (N+1)t\mu(A) - \frac{N(N+1)}{2} st\mu(A) \\
&= (1 - N\frac{s}{2})(N+1)t\mu(A) \\
&\geqslant (1 - \frac{1}{s}\frac{s}{2})\frac{1}{s}t\mu(A) = \frac{1}{2s}t\mu(A) = K\mu(A),
\end{aligned}
$$

contradiction. □

Since the measure μ is not supposed to be definable, X need not be definable neither, but we would like to get a definable one.

Let $P_n(X)$ ($n < \omega$, $0 < t \leqslant 1$) be a predicate on definable subsets of A defined recursively: for a definable subset B of A

- $P^t_0(B)$ if $B \neq \varnothing$,
- $P^t_{n+1}(B)$ if $P^t_n(B)$ and A is covered by $\lfloor \frac{2K}{t} \rfloor$ many translates of

$$X^t_{n+1}(B) = \{g \in A^2 \mid P^{t^2/(2K)}_n(gB \cap B) \wedge P^{t^2/(2K)}_n(g^{-1}B \cap B)\}.$$

If $(B_x)_x$ is uniformly definable by $\psi(y;x)$, then $P^t_n(B_x)$ is defined by some formula $\theta^t_n(x)$, which can be seen inductively. For $0 < t \leqslant 1$, let $\mathfrak{B}_t$ be the family of definable subsets B of A with $P^t_n(B)$ for all $n < \omega$. Then the following properties hold:

(1) If B is a definable subset of A and $\mu(B) \geqslant t\mu(A)$, then $P^t_n(B)$ for all $n < \omega$, so $B \in \mathfrak{B}_t$.

(2) From (1), $A \in \mathfrak{B}_t$, so $\mathfrak{B}_t \neq \varnothing$.

(3) $\mathfrak{B}_{t_1} \subseteq \mathfrak{B}_{t_2}$ for $t_1 \geqslant t_2$.

For (1), we will prove by induction on n that for all B definable in $\langle A \rangle$ and $0 < t \leqslant 1$, $\mu(B) \geqslant t\mu(A)$ implies $P^t_n(B)$.

- $n = 0$: $\mu(B) \geqslant t\mu(A) \Leftrightarrow B \neq \varnothing \Leftrightarrow P^t_0(B)$.
- $n+1$ when assuming it holds for n: Assume that $\mu(B) \geqslant t\mu(A)$. By the induction hypothesis and $\mu(B) \geqslant t\mu(A)$, we have $P^t_n(B)$. Let $N = \lfloor \frac{2K}{t} \rfloor$. By Claim 2.10, A is covered by N-translates of

$$X := \{g \in A^2 \mid \mu(gB \cap B) \geqslant st\mu(A)\}.$$

  Hence, there exist $(g_i \mid 1 \leqslant i \leqslant N)$ in A such that for all $a \in A$, there exists some $g_i \in A$ for $1 \leqslant i \leqslant N$ such that $g_i^{-1}a \in X$. So $\mu((g_i^{-1}a) \cdot B \cap B) \geqslant st\mu(A)$ and by induction hypothesis, $P^{st}_n((g_i^{-1}a) \cdot B \cap B)$ holds. Similarly, $P^{st}_n((a^{-1}g_i) \cdot B \cap B)$ holds. Then the same $(g_i \mid 1 \leqslant i \leqslant N)$ satisfies that for all $a \in A$, there exists some g_i for $1 \leqslant i \leqslant N$ such that

$$g_i^{-1}a \in X^t_{n+1}(B) = \{g \in A^2 \mid P^{st}_n(gB \cap B) \wedge P^{st}_n(g^{-1}B \cap B)\}.$$

  Hence A is covered by N-translates of $X^t_{n+1}(B)$ in A, so $P^t_{n+1}(B)$ holds.

For (3), we can prove by induction on n that $P^{t_1}_n(gB \cap B)$ implies $P^{t_2}_n(gB \cap B)$ if $t_1 \geqslant t_2$, because the induction hypothesis will imply $X^{t_1}_{n+1}(B) \subseteq X^{t_2}_{n+1}(B)$ due to $\lfloor \frac{2K}{t_1} \rfloor \leqslant \lfloor \frac{2K}{t_2} \rfloor$ at the induction step. Then it follows that $\mathfrak{B}_{t_1} \subseteq \mathfrak{B}_{t_2}$ for $t_1 \geqslant t_2$.

Define $f : (0,1] \to \mathbb{R}$ by $f(t) = \inf\{\frac{\mu(BA)}{\mu(A)} \mid B \in \mathfrak{B}_t\}$. Fix $\epsilon > 0$. Note that $1 \leqslant f(t) \leqslant K$ for $0 < t \leqslant 1$. By Lemma 2.8, there is some $t > 0$ depending on K and ϵ such that $f(\frac{t^2}{2K}) \geqslant (1-\epsilon)f(t)$. Choose $B \in \mathfrak{B}_t$ with $\frac{\mu(BA)}{\mu(A)} \leqslant (1+\epsilon)f(t)$. Let

$$X_n = X^t_n(B) = \{g \in A^2 \mid P^{st}_n(gB \cap B) \wedge P^{st}_n(g^{-1}B \cap B)\}.$$

Let $X = \bigcap_{n<\omega} X_n$. Note that each X_n is definable, so X is type-definable. For all $n < \omega$,

- X_n is symmetric by definition.
- $X_{n+1} \subseteq X_n$, because $P^{st}_{n+1}(B)$ implies $P^{st}_n(B)$ for all definable $B \subseteq A$.
- N-translates of X_n cover A for all $n < \omega$, because $B \in \mathfrak{B}_t$ and $X_n = X^t_n(B)$.

Every finite intersection of $\{X_n\}_{n<\omega}$ is nonempty, because $X_{n+1} \subseteq X_n$ for all $n < \omega$ implies that every finite intersection is just some element in $\{X_n\}_{n<\omega}$. By (2) in Remark 2.3, X is nonempty. Moreover, for $g \in X$, we have $gB \cap B \in \mathfrak{B}_{st}$, so

$$\begin{aligned}\mu(gBA \cap BA) &\geqslant \mu((gB \cap B) \cdot A) \geqslant f(\frac{t^2}{2K})\mu(A) \\ &\geqslant (1-\epsilon)f(t)\mu(A) \geqslant \frac{1-\epsilon}{1+\epsilon}\mu(BA).\end{aligned}$$

Hence, for $g \in X$, we have

$$\begin{aligned}\mu(gBA \triangle BA) &= \mu(gBA \cup BA) - 2\mu(gBA \cap BA) \\ &\leqslant 2\mu(BA) - \frac{2(1-\epsilon)}{1+\epsilon}\mu(BA) \\ &= \frac{4\epsilon}{1+\epsilon}\mu(BA) < 4\epsilon\mu(BA).\end{aligned}$$

Hence, for $g_1, \cdots, g_m \in X$,

$$\begin{aligned}&\mu(g_1 \cdots g_m BA \triangle BA) \\ \leqslant &\mu((BA \triangle g_1 BA) \cup g_1(BA \triangle g_2 BA) \cup \cdots \cup g_1 \cdots g_{m-1}(BA \triangle g_m BA)) \\ \leqslant &\mu(BA \triangle g_1 BA) + \mu(BA \triangle g_2 BA) + \cdots + \mu(BA \triangle g_m BA) \\ < &4m\epsilon\mu(BA),\end{aligned}$$

where the first inequality is because for any sets C, D, E, $C \triangle E \subseteq (C \triangle D) \cup (D \triangle E)$ and it can be generalized to arbitrarily many sets by induction. In particular, if $\epsilon \leqslant \frac{1}{4m}$, then

$$\mu(g_1 \cdots g_m BA \triangle BA) < \mu(BA),$$

so $g_1 \cdots g_m BA \cap BA \neq \varnothing$, because otherwise $\mu(g_1 \cdots g_m BA \triangle BA) = 2\mu(BA)$, contradiction. Whence, $X^m \subseteq A^4$, because for any $g_1, \cdots, g_n$ from X, if x is in $g_1 \cdots g_m BA \cap BA$ and $g_1 \cdots g_m x' = x$, then $g_1 \cdots g_m = x(x')^{-1} \in A^4$. By ω^+-saturation, $X^m \subseteq A^4$ implies that $X_n^m \subseteq A^4$ for some $n < \omega$, because by compactness, we have $\bigcap_{n<\omega} X_n^m = (\bigcap_{n<\omega} X_n)^m$ and by compactness again we get that $X_n^m \subseteq A^4$ for some $n < \omega$. $S := X_n$ is N-wide in A, symmetric and definable, so it is an approximate subgroup equivalent to A by Lemma 2.7. □

Lemma 2.11. Let $X_1, \cdots, X_n$ be definable subsets of A with $N_i\mu(X_i) \geqslant \mu(A)$ for some $N_i < \omega$. Then there is a definable subset $D \subseteq A$ such that

- $D^{-1}D \subseteq (X_1^{-1}X_1) \cap (X_2^{-1}X_2)^2 \cdots \cap (X_n X_n^{-1})^2$,
- $K^{n-1}N_1 \cdots N_n\mu(D) \geqslant \mu(A)$.

Proof. Since $\mu(AX_2) \leqslant K\mu(A) \leqslant KN_2\mu(X_2)$, by Fact 2.6, there are $g_1, \cdots, g_{KN_2}$ such that $A \subseteq \bigcup_{i=1}^{KN_2} g_i X_2 X_2^{-1}$. Then there is some $i < \omega$ such that

$$KN_1N_2\mu(X_1 \cap g_i X_2 X_2^{-1}) \geqslant \mu(A),$$

since otherwise for all $1 \leqslant i \leqslant KN_2$, $KN_1N_2\mu(X_1 \cap g_i X_2 X_2^{-1}) < \mu(A)$, so

$$\mu(A) \leqslant N_1\mu(X_1) \leqslant N_1KN_2\mu(g_i X_2 X_2^{-1} \cap X_1) < \mu(A),$$

contradiction. The second inequality comes from $A \subseteq \bigcup_{i=1}^{KN_2} g_i X_2 X_2^{-1}$ intersecting X_1 on both side. Observe that Fact 2.6 can be applied generally:

> If B is a definable subset of A with $\mu(A) \leqslant N\mu(B)$ and C is a definable subset of A with $\mu(A) \leqslant N'\mu(C)$, then
> - $KNN'\mu(C \cap g_i BB^{-1}) \geqslant \mu(A)$ for some $i \leqslant KN$.
> - Let D be $C \cap g_i BB^{-1}$. Then $D^{-1}D \subseteq (C^{-1}C) \cap (BB^{-1})^2$ and $KNN'\mu(D) \geqslant \mu(A)$.

Now, we iterately use our observation:

- $B := X_2$, $C := X_1$: the observation yields D_0 such that $D_0^{-1}D_0 \subseteq X_1^{-1}X_1 \cap (X_2X_2^{-1})^2$ and $KN_1N_2\mu(D_0) \geqslant \mu(A)$.

Suppose D_i has been constructed with $K^{i+1}N_1 \cdots N_{i+2}\mu(D_i) \geqslant \mu(A)$ and $D_i^{-1}D_i \subseteq (X_1^{-1}X_1) \cap (X_2X_2^{-1})^2 \cap \cdots \cap (X_{i+2}X_{i+2}^{-1})^2$.

- $B := X_{i+3}$, $C := D_i$: the observation yields D_{i+1} such that $K^{i+2}N_1 \cdots N_{i+3}\mu(D_i) \geqslant \mu(A)$ and $D_{i+1}^{-1}D_{i+1} \subseteq (X_1^{-1}X_1) \cap (X_2X_2^{-1})^2 \cap \cdots \cap (X_{i+3}X_{i+3}^{-1})^2$.

This procedure ends when we get D_{n-2}. Then let $D = D_{n-2}$ and it satisfies the requirement for D in the statement. □

Theorem 2.12. Let R be a definable N-wide symmetric subsets with $R^4 \subseteq A^4$. Then there exists a definable L-wide symmetric subset $(S^8)^A \subseteq R^4$, where L depends only on K and N.

Proof. If $A \subseteq XR$, then $R^2 \subseteq A^4 \subseteq E^3A \subseteq E^3XR$, so R is a K^3N-approximate subgroup. By Theorem 2.9, there is some definable approximate subgroup $T \subseteq R^4$ equivalent to R with $T^{48} \subseteq R^4$ and T is $\{a_1, \cdots, a_n\}$-wide in A, which means that $A \subseteq \bigcup_{i=1}^n a_iT$. Define $\overline{\mu}$ on definable subsets of $\langle A \rangle$ by $\overline{\mu}(X) := \frac{1}{n}\sum_{i=1}^n \mu(Xa_i)$. Then

- $\overline{\mu}$ is left-invariant,
- $\overline{\mu}(A) = \frac{1}{n}\sum_{i=1}^n \mu(Aa_i) \leqslant \mu(A^2) \leqslant K\mu(A)$,
- $\overline{\mu}(a_iTa_i^{-1}) \geqslant \frac{1}{n}\mu(T) \geqslant \frac{1}{n^2}\mu(A) \geqslant \frac{1}{Kn^2}\overline{\mu}(A)$, where the second inequality is because $\mu(A) \leqslant \sum_{i=1}^n \mu(a_iT) = n\mu(T)$ and the last inequality is because $\overline{\mu}(A) \leqslant K\mu(A)$ from the last clause.

Since all the $a_iTa_i^{-1}$ $(1 \leqslant i \leqslant n)$ are subsets of A^6. Because $A^6 \subseteq E^5A$ and $|E| \leqslant K$, we have

$$K^6n^2\overline{\mu}(a_iTa_i^{-1}) \geqslant K^5\overline{\mu}(A) \geqslant \overline{\mu}(A^6).$$

Apply Lemma 2.11 to $a_iTa_i^{-1} \subseteq A^6$ $(1 \leqslant i \leqslant n)$ with $K^6n^2\overline{\mu}(a_iTa_i^{-1}) \geqslant K^5\overline{\mu}(A) \geqslant \overline{\mu}(A^6)$. Let X_i be $a_iTa_i^{-1}$. There exists $D \subseteq A^6$ such that $S := D^{-1}D \subseteq (X_1^{-1}X_1) \cap (X_2X_2^{-1})^2 \cap \cdots \cap (X_nX_n^{-1})^2$ and $K^{n-1}N_1 \cdots N_n\mu(D) \geqslant \mu(A)$. Since $S \subseteq X_i^{-1}X_i$ for any $1 \leqslant i \leqslant n$, $S^{a_i} = a_i^{-1}Sa_i \subseteq T^4$ for any $1 \leqslant i \leqslant n$. Since $A \subseteq \bigcup_{i=1}^n a_iT$, $S \subseteq T^6$, so $(S^8)^A \subseteq T^{48} \subseteq R^4$. □

Corollary 2.13. There is a type-definable normal subgroup H of $\langle A \rangle$ contained in A^4 such that every definable superset of H contained in $\langle A \rangle$ is wide in A.

Proof. Recall that as long as

- A is K-approximate,
- R is a definable N-wide symmetric subset with $R^4 \subseteq A^4$,

there exists some definable L-wide symmetric subset S with $(S^8)^A \subseteq R^4$. Take $S_0 = R$. Assume S_i has been constructed. Let S_{i+1} satisfy that $(S_{i+1}^8)^A \subseteq S_i^4$ by Theorem 2.12. Let $H = \bigcap_{i<\omega}(S_i^4)^A$. H is normal, type-definable, and every definable superset of H contained in $\langle A \rangle$ is wide in A. □

Reference

[MW15] Jean-Cyrille Massicot and Frank O. Wagner. "Approximate subgroups". In: *Journal de l'École polytechnique* 2 (2015), pp. 55–63. DOI: 10.5802/jep.17.