Zeros and critical points of monochromatic random waves

06-18-2018

Yaiza Canzani

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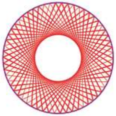

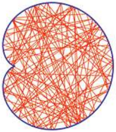

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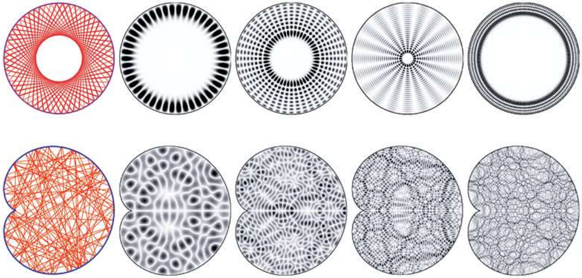

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Heuristics: $(\Delta_{\mathbb{R}^n} + lot)\, \Psi_\lambda^{x_0} = \Psi_\lambda^{x_0}$ and $\Delta_{\mathbb{R}^n}\Psi_\infty = \Psi_\infty$.

Universality. $\Psi_\lambda = \frac{1}{(\#\{\lambda_j \in [\lambda,\lambda+1)\})^{1/2}} \sum_{\lambda_j \in [\lambda,\lambda+1)} a_j \, \psi_{\lambda_j} \qquad a_j \sim N(0,1)$ **iid**

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Random wave conjecture:

$$\psi_\lambda^{x_0}(u) \quad \text{has same statistics as} \quad \Psi_\infty(u).$$

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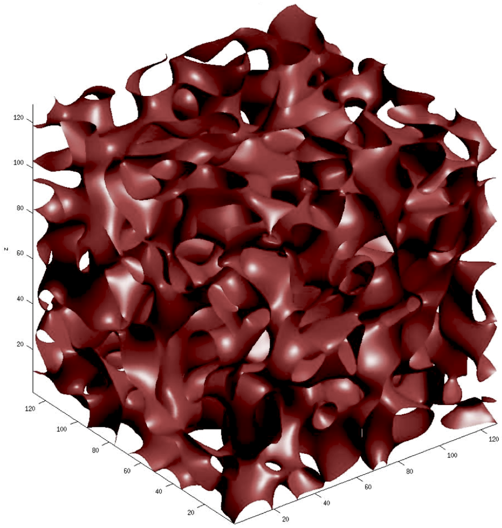
Zero set $\{\Psi_\infty = 0\}$ for $n = 3$



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- Prokhorov's Theorem:
$$\mu_\lambda^{x_0} \to \mu_\infty \quad \text{weakly.}$$
- Skorohod's Representation Theorem: there exists a coupling of $\{(\Psi_\lambda^{x_0}, \nabla \Psi_\lambda^{x_0})\}_\lambda$ and $(\Psi_\infty, \nabla \Psi_\infty)$ so that
$$(\Psi_\lambda^{x_0}, \nabla \Psi_\lambda^{x_0}) \longrightarrow (\Psi_\infty, \nabla \Psi_\infty) \quad \text{a.s}$$

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- **Obs**. We have

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In particular,

$$\mathcal{H}^{n-1}(\{\Psi_\lambda^{x_0} = 0\}) \xrightarrow{d} \mathcal{H}^{n-1}(\{\Psi_\infty = 0\}).$$

Zero sets in $\frac{1}{\lambda}$ scales

- **Obs**. We have

$$(\Psi_\lambda^{x_0}, \nabla \Psi_\lambda^{x_0}) \to (\Psi_\infty, \nabla \Psi_\infty) \qquad \text{a.s.} \quad \text{in} \quad B(0,R).$$

- **Obs**. The **zero set** $\{f = 0\}$ is stable under perturbations for all $f \in C^1(B(0,R))$.

Theorem (C-Hanin '17)

Let $x_0 \in M$. If measure{geodesic loops closing at x_0}$= 0$,

$$\delta_{\{\Psi_\lambda^{x_0}=0\}} \xrightarrow{d} \delta_{\{\Psi_\infty=0\}}.$$

In particular,

$$\mathcal{H}^{n-1}(\{\Psi_\lambda^{x_0} = 0\}) \xrightarrow{d} \mathcal{H}^{n-1}(\{\Psi_\infty = 0\}).$$

Same is true for Euler characteristic, Betti numbers, and topologies of components.

Critical points in $\frac{1}{\lambda}$ scales

$$\mathrm{Crit}_{\Psi_\lambda^{x_0}} := \frac{1}{\mathrm{Vol}(B_R)} \sum_{\substack{d\Psi_\lambda^{x_0}(u)=0 \\ u \in B_R}} \delta_u$$

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Theorem (C-Hanin '17)

Let $x_0 \in M$ with measure{geodesic loops closing at x_0}$= 0$. For every $m \in \mathbb{N}$

$$\lim_{\lambda \to \infty} \mathbb{E}\left[\text{Crit}_{\Psi_\lambda^{x_0}}\right]^m = \mathbb{E}\left[\text{Crit}_{\Psi_\infty}\right]^m$$

provided the limit is finite, which is true for $m = 1, 2$.

Critical points in $\frac{1}{\lambda}$ scales: ideas in the proof

Theorem (Kac-Rice)

Suppose that

1. $\nabla\Psi$ *is almost surely* C^2.
2. **Non-degeneracy:** *For every* $u \neq v$ *the Gaussian vector* $(\nabla\Psi(u), \nabla\Psi(v))$ *has a non-degenerate distribution.*

Then,

$$\mathbb{E}\left[\mathrm{Crit}_\Psi(\mathrm{Crit}_\Psi - 1)\right] = \int_{B\times B} Y_{\nabla\Psi}(u,v)\mathrm{Den}_{(\nabla\Psi(u),\nabla\Psi(v))}(0,0)dudv$$

where

$$Y_{\nabla\Psi}(u,v) = \mathbb{E}\left[\left|\det(\mathrm{Hess}\,\Psi(u))\right|\left|\det(\mathrm{Hess}\,\Psi(v))\right| \,\middle|\, \nabla\Psi(u) = \nabla\Psi(v) = 0\right]$$

and $\mathrm{Den}_{(\nabla\Psi(u),\nabla\Psi(v))}(0,0)$ *is the density of* $(\nabla\Psi(u), \nabla\Psi(v))$ *evaluated at* $(0,0)$.

Critical points in $\frac{1}{\lambda}$ scales: ideas in the proof

Goal:

$$\mathbb{E}\left[\mathsf{Crit}_{\Psi_\lambda^{x_0}}(\mathsf{Crit}_{\Psi_\lambda^{x_0}}-1)\right] \longrightarrow \mathbb{E}\left[\mathsf{Crit}_{\Psi_\infty}(\mathsf{Crit}_{\Psi_\infty}-1)\right]$$

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1. Non-degeneracy: If $u \neq v$, then $(\nabla\Psi_\infty(u), \nabla\Psi_\infty(v))$ is non-degenerate.
   Reduced to proving that

$$\{\omega_j e^{i\langle u,\omega\rangle}, \omega_k e^{i\langle v,\omega\rangle} : j,k = 1,\dots,n\} \quad \text{are l.i. on } S^{n-1}.$$

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2. Non-degeneracy of $(\nabla\Psi_\lambda^{x_0}(u), \nabla\Psi_\lambda^{x_0}(v))$. Use non-degeneracy for Ψ_∞ to deal with off-diagonal behavior. Use universality to deal with on-diagonal behavior.

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5. Prove that as $|u-v| \to 0$

   $$\mathrm{Den}_{(\nabla\Psi_\infty(u), \nabla\Psi_\infty(v))}(0,0) = O(|u-v|^{-n}), \qquad Y_{\Psi_\infty}(u,v) = O(|u-v|^2).$$

Global statistics

Theorem (C-Hanin'16)

If measure{geodesic loops closing at x}$= 0$ for a.e $x \in M$, then

$$\lim_{\lambda} \mathbb{E}\left[\frac{\#\{\text{critical points of } \Psi_\lambda\}}{\lambda^n}\right] = A_n$$

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If measure{geodesics joining x, y} $= 0$ for a.e. $x, y \in M$, then

$$Var\left[\frac{\#\{\text{critical points of } \Psi_\lambda\}}{\lambda^n}\right] = O\left(\lambda^{-\frac{n-1}{2}}\right)$$

$$Var\left[\frac{\mathcal{H}^{n-1}(\{\Psi_\lambda = 0\})}{\lambda}\right] = O\left(\lambda^{-\frac{n-1}{2}}\right)$$

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  2. Control Kac-Rice's integrand on Ω_λ using the local results.

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where

$$V_\lambda = \left\{ (x,y) \in M \times M : \max_{\alpha,\beta \in \{0,1\}} \{\lambda^{-\alpha-\beta} \, |\nabla_x^\alpha \nabla_y^\beta \operatorname{Cov}(\Psi_\lambda(x), \Psi_\lambda(y))|\} > \lambda^{-\frac{n-1}{4}} \right\}.$$

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  5. Control Kac-Rice's integrand on $\Omega_\lambda^c \cap {V_\lambda}^c$ by hand.

Thank you!